Hindawi Publishing Corporation
Journal of Complex Systems
Volume 2015, Article ID 832467, 7 pages
http://dx.doi.org/10.1155/2015/832467

Research Article

Some Fixed Point Theorems in Complex Valued b-Metric Spaces

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Received 11 September 2014; Revised 26 December 2014; Accepted 30 December 2014

Academic Editor: Yang Tang

Recently, Azam et al. introduced the notion of complex valued metric spaces and proved fixed point theorems under the contraction condition. Rao et al. introduced the notion of complex valued b-metric spaces. In this paper, we obtain some fixed point results for the mapping satisfying rational expressions in complex valued b-metric spaces. Also, an example is given to illustrate our obtained result.

1. Introduction

Banach contraction principle [1] is one of the most important results in fixed point theory. Later, a large number of articles have been devoted to the improvement and generalization of the Banach contraction principle by using different form of contraction condition in various spaces. Bakhtin [2] introduced the notion of b-metric space which is a generalized form of metric spaces. Azam et al. [3] introduced the notion of complex valued metric spaces which is more general than well-known metric spaces and also proved common fixed point theorems for mappings satisfying rational expression. Afterwards, the concept of complex valued b-metric spaces was introduced in 2013 by Rao et al. [4]. In the sequel, Mukheimer [5] proved some common fixed point theorems in complex valued b-metric spaces.

The aim of this paper is to prove some fixed point theorems for map with different type of rational expressions in complex valued b-metric spaces. Our results unify, generalize, and complement the comparable results from the current literature.

2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows.

$z_1 \precsim z_2$ if and only if $\mathrm{Re}(z_1) \leq \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) \leq \mathrm{Im}(z_2)$. Thus $z_1 \precsim z_2$ if one of the following holds:

(1) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$,

(2) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$,

(3) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,

(4) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$.

We will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (2), (3), and (4) is satisfied; also we will write $z_1 \prec z_2$ if only (4) is satisfied.

It follows that

(i) $0 \precsim z_1 \precnsim z_2$ implies $|z_1| < |z_2|$,

(ii) $z_1 \precsim z_2$ and $z_2 \prec z_3$ imply $z_1 \prec z_3$,

(iii) $0 \precsim z_1 \precsim z_2$ implies $|z_1| \leq |z_2|$,

(iv) $a, b \in \mathbb{R}$ and $a \leq b$ imply $az \precsim bz$ for all $z \in \mathbb{C}$.

The following definitions and results [4] will be needed in the sequel.

Definition 1 (see [4]). Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{C}$ is called a complex valued b-metric on X if, for all $x, y, z \in X$, the following conditions are satisfied:

(i) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$;

(iii) $d(x,y) \precsim s[d(x,z) + d(z,y)]$.

The pair (X,d) is called a complex valued b-metric space.

Example 2 (see [4]). Let $X = [0,1]$. Define the mapping $d : X \times X \to \mathbb{C}$ by $d(x,y) = |x-y|^2 + i|x-y|^2$ for all $x, y \in X$. Then (X,d) is a complex valued b-metric space with $s = 2$.

Definition 3 (see [4]). Let (X,d) be a complex valued b-metric space.

(i) A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x,r) = \{y \in X : d(x,y) \prec r\} \subseteq A$.

(ii) A point $x \in X$ is called limit point of a set A whenever, for every $0 \prec r \in \mathbb{C}$, $B(x,r) \cap (A - \{x\}) \neq \emptyset$.

(iii) A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A.

(iv) A subset $A \subseteq X$ is called closed whenever each element of A belongs to A.

(v) A subbasis for Hausdorff topology τ on X is a family $F = \{B(x,r)\colon x \in X \text{ and } 0 \prec r\}$.

Definition 4 (see [4]). Let (X,d) be a complex valued b-metric space; let $\{x_n\}$ be a sequence in X and $x \in X$.

(i) If, for every $c \in \mathbb{C}$, with $0 \prec r$, there is $N \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x, and x is the limit point of $\{x_n\}$. One denotes this by $\lim_{n\to\infty} x_n = x$ or $\{x_n\} \to x$ as $n \to \infty$.

(ii) If, for every $c \in \mathbb{C}$, with $0 \prec r$, there is $N \in \mathbb{N}$ such that, for all $n > N$, $d(x_n, x_{n+m}) \prec c$, where $m \in \mathbb{N}$, then $\{x_n\}$ is said to be Cauchy sequence.

(iii) If every Cauchy sequence in X is convergent, then (X,d) is said to be a complete complex valued b-metric space.

Lemma 5 (see [4]). *Let (X,d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$.*

Lemma 6 (see [4]). *Let (X,d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

3. Main Results

Theorem 7. *Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a mapping satisfying*

$$d(Tx,Ty) \precsim \frac{\lambda d^2(x,y)}{1 + d(x,y)} + \mu d(y,Ty), \tag{1}$$

for all $x, y \in X$, where λ, μ are nonnegative reals with $s\lambda + \mu < 1$. Then T has a unique fixed point in X.

Proof. For any arbitrary point, $x_o \in X$. Define sequence $\{x_n\}$ in X such that

$$x_{2n+1} = Tx_{2n} \quad \text{for } n = (0,1,2,3,\ldots). \tag{2}$$

Now, we show that the sequence $\{x_n\}$ is Cauchy:

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Tx_{2n}, Tx_{2n+1}) \\ &\precsim \frac{\lambda d^2(x_{2n}, x_{2n+1})}{1 + d(x_{2n}, x_{2n+1})} + \mu d(x_{2n+1}, Tx_{2n+1}) \\ &= \frac{\lambda d^2(x_{2n}, x_{2n+1})}{1 + d(x_{2n}, x_{2n+1})} + \mu d(x_{2n+1}, x_{2n+2}) \end{aligned} \tag{3}$$

which implies that

$$\begin{aligned} &|d(x_{2n+1}, x_{2n+2})| \\ &\quad \leq \lambda \frac{|d(x_{2n}, x_{2n+1})|}{|1 + d(x_{2n}, x_{2n+1})|} |d(x_{2n}, x_{2n+1})| \\ &\qquad + \mu |d(x_{2n+1}, x_{2n+2})|. \end{aligned} \tag{4}$$

Since $|1 + d(x_{2n}, x_{2n+1})| > |d(x_{2n}, x_{2n+1})|$, we get

$$\begin{aligned} |d(x_{2n+1}, x_{2n+2})| &\leq \lambda |d(x_{2n}, x_{2n+1})| \\ &\quad + \mu |d(x_{2n+1}, x_{2n+2})| \end{aligned} \tag{5}$$

and hence

$$|d(x_{2n+1}, x_{2n+2})| \leq \frac{\lambda}{1-\mu} |d(x_{2n}, x_{2n+1})|. \tag{6}$$

Similarly, we obtain

$$|d(x_{2n+2}, x_{2n+3})| \leq \frac{\lambda}{1-\mu} |d(x_{2n+1}, x_{2n+2})|. \tag{7}$$

Since $s\lambda + \mu < 1$ and $s \geq 1$, we get $\lambda + \mu < 1$.

Therefore, with $\delta = \lambda/(1-\mu) < 1$ and for all $n \geq 0$ and consequently, we have

$$\begin{aligned} |d(x_{2n+1}, x_{2n+2})| &\leq \delta |d(x_{2n}, x_{2n+1})| \leq \delta^2 |d(x_{2n-1}, x_{2n})| \\ &\leq \cdots \leq \delta^{2n+1} |d(x_0, x_1)|, \\ |d(x_{n+1}, x_{n+2})| &\leq \delta |d(x_n, x_{n+1})| \leq \delta^2 |d(x_{n-1}, x_n)| \\ &\leq \cdots \leq \delta^{n+1} |d(x_0, x_1)|. \end{aligned} \tag{8}$$

Thus for any $m > n, m, n \in \mathbb{N}$ and since $s\delta = s\lambda/(1-\mu) < 1$, we get

$$\begin{aligned}
&|d(x_n, x_m)| \\
&\quad \le s|d(x_n, x_{n+1})| + s|d(x_{n+1}, x_m)| \\
&\quad \le s|d(x_n, x_{n+1})| + s^2|d(x_{n+1}, x_{n+2})| + s^2|d(x_{n+2}, x_m)| \\
&\quad \le s|d(x_n, x_{n+1})| + s^2|d(x_{n+1}, x_{n+2})| \\
&\qquad + s^3|d(x_{n+2}, x_{n+3})| + s^3|d(x_{n+3}, x_m)| \\
&\vdots \\
&|d(x_n, x_m)| \\
&\le s|d(x_n, x_{n+1})| + s^2|d(x_{n+1}, x_{n+2})| \\
&\quad + s^3|d(x_{n+2}, x_{n+3})| + \cdots + s^{m-n-1}|d(x_{m-2}, x_{m-1})| \\
&\quad + s^{m-n}|d(x_{m-1}, x_m)|.
\end{aligned} \tag{9}$$

By using (8), we get

$$\begin{aligned}
&|d(x_n, x_m)| \\
&\quad \le s\delta^n|d(x_0, x_1)| + s^2\delta^{n+1}|d(x_0, x_1)| \\
&\qquad + s^3\delta^{n+2}|d(x_0, x_1)| + \cdots + s^{m-n-1}\delta^{m-2}|d(x_0, x_1)| \\
&\qquad + s^{m-n}\delta^{m-1}|d(x_0, x_1)| = \sum_{i=1}^{m-n} s^i\delta^{i+n-1}|d(x_0, x_1)|.
\end{aligned} \tag{10}$$

Therefore,

$$\begin{aligned}
|d(x_n, x_m)| &\le \sum_{i=1}^{m-n} s^{i+n-1}\delta^{i+n-1}|d(x_0, x_1)| \\
&= \sum_{t=n}^{m-1} s^t\delta^t|d(x_0, x_1)| \\
&\le \sum_{t=n}^{\infty} (s\delta)^t|d(x_0, x_1)| \\
&= \frac{(s\delta)^n}{1-s\delta}|d(x_0, x_1)|
\end{aligned} \tag{11}$$

and hence

$$|d(x_n, x_m)| \le \frac{(s\delta)^n}{1-s\delta}|d(x_0, x_1)| \longrightarrow 0 \quad \text{as } m, n \longrightarrow \infty. \tag{12}$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists some $u \in X$ such that $x_n \to u$ as $n \to \infty$. Suppose this is not possible; then there exists $z \in X$ such that

$$|d(u, Tu)| = |z| > 0. \tag{13}$$

Now,

$$\begin{aligned}
z = d(u, Tu) &\precsim sd(u, x_{2n+2}) + sd(x_{2n+2}, Tu) \\
&= sd(u, x_{2n+2}) + sd(Tx_{2n+1}, Tu) \\
&\precsim sd(u, x_{2n+2}) + \frac{s\lambda d^2(x_{2n+1}, u)}{1 + d(x_{2n+1}, u)} + s\mu d(u, Tu)
\end{aligned} \tag{14}$$

which implies that

$$\begin{aligned}
|z| = |d(u, Tu)| &\le s|d(u, x_{2n+2})| \\
&\quad + \frac{s\lambda|d^2(x_{2n+1}, u)|}{|1 + d(x_{2n+1}, u)|} + s\mu|d(u, Tu)|.
\end{aligned} \tag{15}$$

Taking the limit of (15) as $n \to \infty$, we obtain that $|z| = |d(u, Tu)| \le 0$, a contradiction with (13).

So $|z| = 0$. Hence $Tu = u$.

Now, we show that T has a unique fixed point in X. To show this, assume that u^* is another fixed point of T. Then,

$$d(u, u^*) = d(Tu, Tu^*) \precsim \frac{\lambda d^2(u, u^*)}{1 + d(u, u^*)} + \mu d(u^*, Tu^*). \tag{16}$$

So

$$|d(u, u^*)| \le \frac{\lambda|d(u, u^*)|}{|1 + d(u, u^*)|}|d(u, u^*)| + \mu|d(u^*, Tu^*)|, \tag{17}$$

since

$$|1 + d(u, u^*)| > |d(u, u^*)|. \tag{18}$$

Therefore,

$$\begin{aligned}
|d(u, u^*)| &< \lambda|d(u, u^*)| + \mu|d(u^*, u^*)| \\
&= \lambda|d(u, u^*)|, \text{ a contradiction.}
\end{aligned} \tag{19}$$

So $u = u^*$, which proves the uniqueness of fixed point in X. This completes the proof. □

Corollary 8. *Let (X, d) be a complete complex valued b-metric space with the coefficient $s \ge 1$ and let $T : X \to X$ be a mapping satisfying (for some fixed n)*

$$d(T^n x, T^n y) \precsim \frac{\lambda d^2(x, y)}{1 + d(x, y)} + \mu d(y, T^n y), \tag{20}$$

for all $x, y \in X$, where λ, μ are nonnegative reals with $s\lambda + \mu < 1$. Then T has a unique fixed point in X.

Proof. From Theorem 7, we obtain $u \in X$ such that

$$T^n u = u. \tag{21}$$

The uniqueness follows from

$$\begin{aligned}
d(Tu, u) = d(TT^n u, T^n u) &= d(T^n Tu, T^n u) \\
&\precsim \frac{\lambda d^2(Tu, u)}{1 + d(Tu, u)} + \mu d(u, T^n u).
\end{aligned} \tag{22}$$

By taking modulus (22), we have

$$|d(Tu,u)| \leq \frac{\lambda |d(Tu,u)|}{|1+d(Tu,u)|} |d(Tu,u)| + \mu |d(u,u)|, \quad (23)$$

since

$$|1+d(Tu,u)| > |d(Tu,u)|. \quad (24)$$

Therefore,

$$|d(Tu,u)| < \lambda |d(Tu,u)|, \text{ a contradiction.} \quad (25)$$

So $Tu = u$. Hence $Tu = T^n u = u$.

Therefore, the fixed point of T is unique. This completes the proof. □

Theorem 9. *Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a mapping satisfying*

$$d(Tx,Ty) \precsim \lambda d(x,y) + \frac{\mu d(x,Tx)\, d(y,Ty)}{d(x,Ty)+d(y,Tx)+d(x,y)}, \quad (26)$$

for all $x, y \in X$ such that $x \neq y$, $d(x,Ty)+d(y,Tx)+d(x,y) \neq 0$, where λ, μ are nonnegative reals with $\lambda + s\mu < 1$ or $d(Tx,Ty) = 0$ if $d(x,Ty)+d(y,Tx)+d(x,y) = 0$. Then T has a unique fixed point in X.

Proof. For any arbitrary point, $x_o \in X$. Define sequence $\{x_n\}$ in X such that

$$x_{2n+1} = Tx_{2n} \quad \text{for } n = (0,1,2,3,\ldots). \quad (27)$$

Now, we show that the sequence $\{x_n\}$ is Cauchy:

$$\begin{aligned} &d(x_{2n+1}, x_{2n+2}) \\ &\quad = d(Tx_{2n}, Tx_{2n+1}) \\ &\quad \precsim \lambda d(x_{2n}, x_{2n+1}) \\ &\qquad + \frac{\mu d(x_{2n}, Tx_{2n})\, d(x_{2n+1}, Tx_{2n+1})}{d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Tx_{2n}) + d(x_{2n}, x_{2n+1})} \\ &\quad = \lambda d(x_{2n}, x_{2n+1}) \\ &\qquad + \frac{\mu d(x_{2n}, x_{2n+1})\, d(x_{2n+1}, x_{2n+2})}{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1}) + d(x_{2n}, x_{2n+1})} \end{aligned} \quad (28)$$

which implies that

$$\begin{aligned} &|d(x_{2n+1}, x_{2n+2})| \\ &\quad \leq \lambda |d(x_{2n}, x_{2n+1})| + \frac{\mu |d(x_{2n+1}, x_{2n+2})|}{|d(x_{2n}, x_{2n+2})| + |d(x_{2n}, x_{2n+1})|} \\ &\qquad \times |d(x_{2n}, x_{2n+1})|, \end{aligned} \quad (29)$$

since

$$|d(x_{2n+1}, x_{2n+2})| \leq |d(x_{2n+1}, x_{2n})| + |d(x_{2n}, x_{2n+2})|. \quad (30)$$

Therefore,

$$\begin{aligned} |d(x_{2n+1}, x_{2n+2})| &\leq \lambda |d(x_{2n}, x_{2n+1})| + \mu |d(x_{2n}, x_{2n+1})| \\ &= (\lambda+\mu) |d(x_{2n}, x_{2n+1})|. \end{aligned} \quad (31)$$

Similarly, we obtain

$$|d(x_{2n+2}, x_{2n+3})| \leq (\lambda+\mu) |d(x_{2n+1}, x_{2n+2})|. \quad (32)$$

Since $\lambda + s\mu < 1$ and $s \geq 1$, we get $\lambda + \mu < 1$.

Therefore, with $\delta = \lambda + \mu < 1$ and for all $n \geq 0$ and consequently, we have

$$\begin{aligned} &|d(x_{2n+1}, x_{2n+2})| \\ &\quad \leq \delta |d(x_{2n}, x_{2n+1})| \leq \delta^2 |d(x_{2n-1}, x_{2n})| \\ &\quad \leq \cdots \leq \delta^{2n+1} |d(x_0, x_1)|, \\ &|d(x_{n+1}, x_{n+2})| \\ &\quad \leq \delta |d(x_n, x_{n+1})| \leq \delta^2 |d(x_{n-1}, x_n)| \\ &\quad \leq \cdots \leq \delta^{n+1} |d(x_0, x_1)|. \end{aligned} \quad (33)$$

Thus, for any $m > n$, $m, n \in \mathbb{N}$, we have

$$\begin{aligned} |d(x_n, x_m)| &\leq s|d(x_n, x_{n+1})| + s|d(x_{n+1}, x_m)| \\ &\leq s|d(x_n, x_{n+1})| + s^2 |d(x_{n+1}, x_{n+2})| \\ &\quad + s^2 |d(x_{n+2}, x_m)| \\ &\leq s|d(x_n, x_{n+1})| + s^2 |d(x_{n+1}, x_{n+2})| \\ &\quad + s^3 |d(x_{n+2}, x_{n+3})| + s^3 |d(x_{n+3}, x_m)| \\ &\vdots \\ &|d(x_n, x_m)| \\ &\quad \leq s|d(x_n, x_{n+1})| + s^2 |d(x_{n+1}, x_{n+2})| \\ &\qquad + s^3 |d(x_{n+2}, x_{n+3})| + \cdots \\ &\qquad + s^{m-n-1} |d(x_{m-2}, x_{m-1})| \\ &\qquad + s^{m-n} |d(x_{m-1}, x_m)|. \end{aligned} \quad (34)$$

By using (33), we get

$$\begin{aligned}
&|d(x_n,x_m)| \\
&\quad \le s\delta^n |d(x_0,x_1)| + s^2\delta^{n+1}|d(x_0,x_1)| \\
&\qquad + s^3\delta^{n+2}|d(x_0,x_1)| + \cdots + s^{m-n-1}\delta^{m-2}|d(x_0,x_1)| \\
&\qquad + s^{m-n}\delta^{m-1}|d(x_0,x_1)| \\
&\quad = \sum_{i=1}^{m-n} s^i\delta^{i+n-1}|d(x_0,x_1)|.
\end{aligned} \tag{35}$$

Therefore,

$$\begin{aligned}
|d(x_n,x_m)| &\le \sum_{i=1}^{m-n} s^{i+n-1}\delta^{i+n-1}|d(x_0,x_1)| \\
&= \sum_{t=n}^{m-1} s^t\delta^t|d(x_0,x_1)| \\
&\le \sum_{t=n}^{\infty}(s\delta)^t|d(x_0,x_1)| \\
&= \frac{(s\delta)^n}{1-s\delta}|d(x_0,x_1)|
\end{aligned} \tag{36}$$

and hence

$$|d(x_n,x_m)| \le \frac{(s\delta)^n}{1-s\delta}|d(x_0,x_1)| \longrightarrow 0 \quad \text{as } m,n \longrightarrow \infty. \tag{37}$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists some $u \in X$ such that $x_n \to u$ as $n \to \infty$. Suppose this is not possible; then there exists $z \in X$ such that

$$|d(u,Tu)| = |z| > 0. \tag{38}$$

Now,

$$\begin{aligned}
z &= d(u,Tu) \\
&\precsim sd(u,x_{2n+2}) + sd(x_{2n+2},Tu) \\
&= sd(u,x_{2n+2}) + sd(Tx_{2n+1},Tu) \\
&\precsim sd(u,x_{2n+2}) + s\lambda d(x_{2n+1},u) \\
&\quad + \frac{s\mu d(x_{2n+1},Tx_{2n+1})\,d(u,Tu)}{d(x_{2n+1},Tu)+d(u,Tx_{2n+1})+d(x_{2n+1},u)} \\
&= sd(u,x_{2n+2}) + s\lambda d(x_{2n+1},u) \\
&\quad + \frac{s\mu d(x_{2n+1},x_{2n+2})\,d(u,Tu)}{d(x_{2n+1},Tu)+d(u,x_{2n+2})+d(x_{2n+1},u)}
\end{aligned} \tag{39}$$

which implies that

$$\begin{aligned}
&|z| \\
&= |d(u,Tu)| \le s|d(u,x_{2n+2})| + s\lambda|d(x_{2n+1},u)| \\
&\quad + \frac{s\mu|d(x_{2n+1},x_{2n+2})|}{|d(x_{2n+1},Tu)|+|d(u,x_{2n+2})|+|d(x_{2n+1},u)|} \\
&\quad \times |d(u,Tu)|.
\end{aligned} \tag{40}$$

Taking the limit of (40) as $n \to \infty$, we obtain that $|z| = |d(u,Tu)| \le 0$, a contradiction with (38).

So $|z| = 0$. Hence $Tu = u$.

Now, we show that T has a unique fixed point in X. To show this, assume that u^* is another fixed point of T. Then,

$$\begin{aligned}
d(u,u^*) &= d(Tu,Tu^*) \\
&\precsim \lambda d(u,u^*) \\
&\quad + \frac{\mu d(u,Tu)\,d(u^*,Tu^*)}{d(u,Tu^*)+d(u^*,Tu)+d(u,u^*)}
\end{aligned} \tag{41}$$

so that

$$\begin{aligned}
|d(u,u^*)| &\le \lambda|d(u,u^*)| \\
&\quad + \frac{\mu|d(u,Tu)|\,|d(u^*,Tu^*)|}{|d(u,Tu^*)|+|d(u^*,Tu)|+|d(u,u^*)|} \\
&< \lambda|d(u,u^*)|, \text{ a contradiction.}
\end{aligned} \tag{42}$$

So $u = u^*$, which proves the uniqueness of fixed point in X.

Now, we consider the following case: $d(x_{2n},Tx_{2n+1}) + d(x_{2n+1},Tx_{2n}) + d(x_{2n},x_{2n+1}) = 0$ (for any n) implies $d(Tx_{2n},Tx_{2n+1}) = 0$, so that $x_{2n} = Tx_{2n} = x_{2n+1} = Tx_{2n+1} = x_{2n+2}$. Thus we have $x_{2n+1} = Tx_{2n} = x_{2n}$, so there exist K_1 and l_1 such that $K_1 = Tl_1 = l_1$. Using foregoing arguments, one can also show that there exist K_2 and l_2 such that $K_2 = Tl_2 = l_2$. As $d(l_1,Tl_2)+d(l_2,Tl_1)+d(l_1,l_2) = 0$ (due to definition) implies $d(Tl_1,Tl_2) = 0$, $K_1 = Tl_1 = Tl_2 = K_2$, which in turn yields that $K_1 = Tl_1 = TK_1$. Similarly, one can also have $K_2 = TK_2$. As $K_1 = K_2$ implies $TK_1 = K_1$, therefore $K_1 = K_2$ is fixed point of T.

We now prove that T has unique fixed point. For this, assume that K_1^* in X is another fixed point of T. Then we have $TK_1^* = K_1^*$. As $d(K_1,TK_1^*) + d(K_1^*,TK_1) + d(K_1,K_1^*) = 0$, therefore $d(K_1,K_1^*) = d(TK_1,TK_1^*) = 0$.

This implies that $K_1^* = K_1$. This completes the proof of the theorem. □

Corollary 10. *Let (X,d) be a complete complex valued b-metric space with the coefficient $s \ge 1$ and let $T : X \to X$ be a mapping satisfying (for some fixed n):*

$$\begin{aligned}
&d(T^nx,T^ny) \\
&\quad \precsim \lambda d(x,y) + \frac{\mu d(x,T^nx)\,d(y,T^ny)}{d(x,T^ny)+d(y,T^nx)+d(x,y)},
\end{aligned} \tag{43}$$

for all $x,y \in X$ such that $x \ne y$, $d(x,Ty) + d(y,Tx) + d(x,y) \ne 0$, where λ,μ are nonnegative reals with $\lambda + s\mu < 1$ or

$d(T^n x, T^n y) = 0$ *if* $d(x, T^n y) + d(y, T^n x) + d(x, y) = 0$. *Then* T *has a unique fixed point in* X.

Proof. From Theorem 9, we obtain $u \in X$ such that

$$T^n u = u. \tag{44}$$

The uniqueness follows from

$$\begin{aligned} d(Tu, u) &= d(TT^n u, T^n u) = d(T^n Tu, T^n u) \\ &\precsim \lambda d(Tu, u) \\ &\quad + \frac{\mu d(Tu, T^n Tu)\, d(u, T^n u)}{d(Tu, T^n u) + d(u, T^n Tu) + d(Tu, u)} \\ &= \lambda d(Tu, u). \end{aligned} \tag{45}$$

By taking modulus (45), we get

$$|d(Tu, u)| < \lambda |d(Tu, u)|, \text{ a contradiction.} \tag{46}$$

So $Tu = u$. Hence $Tu = T^n u = u$.

Therefore, the fixed point of T is unique. This completes the proof. □

Example 11. Let $X = \mathbb{C}$ be the set of complex numbers.

Define $d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ as

$$d(z_1, z_2) = |x_1 - x_2|^2 + i|y_1 - y_2|^2, \tag{47}$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

Obviously, (X, d) is complete complex valued b-metric space with $s = 2$. Define $T : \mathbb{C} \to \mathbb{C}$ as

$$T(x + iy) = \begin{cases} 0; & x, y \in Q, \\ 2 + 2i; & x, y \in Q^c, \\ 2; & x \in Q^c,\ y \in Q, \\ 2i; & x \in Q,\ y \in Q^c. \end{cases} \tag{48}$$

Now, for $x = 1/\sqrt{2}$ and $y = 0$, we get

$$d\left(T\left(\frac{1}{\sqrt{2}}\right), T(0)\right) = d(2, 0) = 4 \precsim \lambda d\left(\frac{1}{\sqrt{2}}, 0\right) = \lambda \frac{1}{\sqrt{2}}. \tag{49}$$

Thus $\lambda \geq \sqrt{2}$ which is a contradiction as $0 \leq \lambda < 1$.

However, notice that $T^n z = 0$ for $n > 1$, so

$$\begin{aligned} &d(T^n x, T^n y) \\ &\quad = 0 \\ &\quad \precsim \lambda d(x, y) + \frac{\mu d(x, T^n x)\, d(y, T^n y)}{d(x, T^n y) + d(y, T^n x) + d(x, y)}, \end{aligned} \tag{50}$$

for all $x, y \in X$ and $\lambda, \mu \geq 0$ with $\lambda + 2\mu < 1$. So all conditions of Corollary 10 are satisfied to get a unique fixed point 0 of T.

Example 12. Let $X = [0, 1]$. Consider a b-metric $d : X \times X \to \mathbb{C}$ defined as

$$d(x, y) = \frac{2}{3}\left[|x - y|^2 + i|x - y|^2\right], \quad \forall x, y \in X. \tag{51}$$

To verify that (X, d) is a complete complex valued b-metric space with $s = 2$, it is enough to verify the triangular inequality condition:

$$\begin{aligned} d(x, y) &= \frac{2}{3}\left[|x - y|^2 + i|x - y|^2\right] \\ &= \frac{2}{3}\left[|x - z + z - y|^2 + i|x - z + z - y|^2\right] \\ &\precsim \frac{2}{3}\left[\left\{|x - z|^2 + |z - y|^2 + 2|x - z||z - y|\right\}\right. \\ &\qquad \left. + i\left\{|x - z|^2 + |z - y|^2 + 2|x - z||z - y|\right\}\right] \\ &\precsim \frac{2}{3}\left[\left\{|x - z|^2 + |z - y|^2 + |x - z|^2 + |z - y|^2\right\}\right. \\ &\qquad \left. + i\left\{|x - z|^2 + |z - y|^2 + |x - z|^2 + |z - y|^2\right\}\right] \\ &= 2\left[\frac{2}{3}\left\{|x - z|^2 + i|x - z|^2\right\}\right. \\ &\qquad \left. + \frac{2}{3}\left\{|z - y|^2 + i|z - y|^2\right\}\right] \\ &= 2\left[d(x, z) + d(z, y)\right]. \end{aligned} \tag{52}$$

That is,

$$d(x, y) \precsim s\left[d(x, z) + d(z, y)\right]. \tag{53}$$

Therefore, $s = 2$. Define $T : X \to X$ as $Tx = x/2$, $Ty = y/2$ for all $x, y \in X$. Then

$$\begin{aligned} d(Tx, Ty) &= d\left(\frac{x}{2}, \frac{y}{2}\right) \\ &= \frac{2}{3}\left[\left|\frac{x}{2} - \frac{y}{2}\right|^2 + i\left|\frac{x}{2} - \frac{y}{2}\right|^2\right] \\ &= \frac{1}{6}\left[|x - y|^2 + i|x - y|^2\right] = \frac{1}{6} d(x, y), \\ &d(Tx, Ty) \\ &\precsim \frac{3}{8} d(x, y) + \frac{(1/5)\, d(x, x/2)\, d(y, y/2)}{d(x, y/2) + d(y, x/2) + d(x, y)}. \end{aligned} \tag{54}$$

Here

$$\lambda + s\mu = \left(\frac{3}{8}\right) + (2)\left(\frac{1}{5}\right) = \frac{31}{40} < 1. \tag{55}$$

It is easily and clearly verified that the map T satisfies contractive condition (26) of Theorem 9 with the coefficients $s = 2$, $\lambda = 3/8$, and $\mu = 1/5$. Observe that the point $0 \in X$ remains fixed under T and is indeed unique.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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