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Research Article

Successive Complementary Expansion Method for Solving Troesch's Problem as a Singular Perturbation Problem

Süleyman Cengizci[1] and Aytekin Eryılmaz[2]

[1] *Insitute of Applied Mathematics, Middle East Technical University, 06800 Ankara, Turkey*
[2] *Department of Mathematics, Nevsehir Haci Bektas Veli University, 50300 Nevsehir, Turkey*

Correspondence should be addressed to Aytekin Eryılmaz; eryilmazaytekin@gmail.com

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A simple and efficient method that is called *Successive Complementary Expansion Method (SCEM)* is applied for approximation to an unstable two-point boundary value problem which is known as Troesch's problem. In this approach, Troesch's problem is considered as a singular perturbation problem. We convert the hyperbolic-type nonlinearity into a polynomial-type nonlinearity using an appropriate transformation, and then we use a basic zoom transformation for the boundary layer and finally obtain a nonlinear ordinary differential equation that contains SCEM complementary approximation. We see that SCEM gives highly accurate approximations to the solution of Troesch's problem for various parameter values. Moreover, the results are compared with Adomian Decomposition Method (ADM) and Homotopy Perturbation Method (HPM) by using tables.

1. Introduction

Troesch's highly sensitive problem arises from a system of a nonlinear ordinary differential equations which occur in the investigation of the confinement of a plasma column by radiation pressure [1]. It also arises in the theory of gas porous electrodes [2, 3]. The problem is defined by

$$y'' = \lambda \sinh(\lambda y), \quad 0 \le x \le 1, \tag{1}$$

with the boundary conditions

$$\begin{aligned} y(0) &= 0, \\ y(1) &= 1, \end{aligned} \tag{2}$$

where $y = y(x)$ and $\lambda > 0$. Roberts and Shipman [4] have shown that the closed form solution to problem (1) with the boundary conditions (2) in terms of the Jacobi elliptic function $\mathrm{sc}(n \mid m)$ is as follows:

$$y(x) = \frac{2}{\lambda}\sinh^{-1}\left[\frac{y'(0)}{2}\mathrm{sc}\left(\lambda x \mid 1 - \frac{1}{4}\left(y'(0)\right)^2\right)\right], \tag{3}$$

where $y'(0) = 2\sqrt{1-m}$. The constant m satisfies the transcendental equation

$$\mathrm{sc}(\lambda \mid m) = \frac{\sinh(\lambda/2)}{\sqrt{1-m}}, \tag{4}$$

where $\mathrm{sc}(\lambda \mid m)$ is the Jacobi function defined by $\mathrm{sc}(\lambda \mid m) = \sin\phi/\cos\phi = \tan\phi$. Here ϕ, λ, and m are related by the integral

$$\lambda = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}. \tag{5}$$

It has been shown in [4, 5] that $y(x)$ has a singularity approximately located at

$$x_s \approx \frac{1}{2\lambda}\ln\left(\frac{16}{1-m}\right) = \frac{1}{\lambda}\ln\left(\frac{8}{y'(0)}\right). \tag{6}$$

This singularity makes the problem very difficult to solve for large λ values.

The first explanation of Troesch's problem was given and solved by Weibel [1]. Because of its physical significance, Troesch's problem has always been attractive to scientists and various methods have been implemented to solve it such as Monte Carlo method [6], Modified Newton Method [7, 8], Transformation Groups Method [9], Invariant Embedded Method [10], Shooting Method [5], Inverse Shooting Method [11], The quasilinearization Method [12, 13], and Matched Asymptotic Expansions Method [14]. Recently, approximate solution techniques such as Adomian Decomposition Method [15, 16], Laplace transform and a modified decomposition technique [16], Variational Iteration Method [17, 18], Homotopy Perturbation Method [19], Differential Transform Method [20], B-spline collocation approach [21], The Sinc-Galerkin Method [22], Wavelet Analysis Method [23], Sinc-Collocation Method [24], and Jacobi Collocation Method [25] have been successfully applied to Troesch's problem by various researchers. In [26–29] more detailed information and references on Troesch's problem may also be seen.

In this paper, we consider Troesch's problem as a singular perturbation problem. By using appropriate transformations, we apply Successive Complementary Expansion Method (SCEM) to it. The results obtained by SCEM are compared with HPM and ADM and the analytic solutions.

2. The Successive Complementary Expansion Method

In this section, we first give a short overview of asymptotic approximations. One can consult [30] for some definitions on the asymptotic approximations and singular perturbations in more detail.

SCEM was first introduced by Mauss and Cousteix (see [30–34]). It has always been applied to partial differential equations except for some introductory examples in [30]. SCEM is based on the knowledge of generalized asymptotic expansions and the Method of Matched Asymptotic Expansions (MMAE). At first, a structure of the uniformly valid approximation (UVA) must be assumed and then the method to construct the UVA is deduced [32, 33]. In this method, any matching procedure is required in contrast to MMAE. Furthermore, the boundary conditions are satisfied exactly but not asymptotically.

Let us consider a sequence of real numbers $\{f(n)\}_{n=1}^{\infty}$. We are familiar with the limit notation $\lim_{n\to\infty} f(n)$. If this limit exists we can denote it as $\lim_{n\to\infty} f(n) = f \in \mathbb{R}$. We can give the following definition: given any $\varepsilon > 0$ there exists a number $n_0(\varepsilon)$ such that $|f(n) - f| < \varepsilon$ for any $n \geqslant n_0(\varepsilon)$. This definition contains information about the behaviour of $f(n)$ as $n \to \infty$ but not about how $f(n)$ approaches f. In order to describe the behaviour of sequences in a more precise way, we need some other definitions, Bachmann-Landau notations. Let us consider two continuous functions of real numbers that depend on a small parameter $\varepsilon f(\varepsilon) = O(g(\varepsilon))$ for $\varepsilon \to 0$ if there exists positive constants C and ε_0 such that, in $(0, \varepsilon_0]$, $|f(\varepsilon)| \leqslant C|g(\varepsilon)|$ for $\varepsilon \to 0$. $f(\varepsilon) = o(g(\varepsilon))$ for $\varepsilon \to 0$ if $\lim_{\varepsilon\to 0}(f(\varepsilon)/g(\varepsilon)) = 0$. $f(\varepsilon) = O_S(g(\varepsilon))$ if $f(\varepsilon) = O(g(\varepsilon))$ and $f(\varepsilon) \neq o(g(\varepsilon))$ as $\varepsilon \to 0$. Let E be a set of real functions that depend on ε, strictly positive and continuous in $(0, \varepsilon_0]$, such that $\lim_{\varepsilon\to 0}\delta(\varepsilon)$ exists and if δ_1 and $\delta_2 \in E$, $\delta_1\delta_2 \in E$. A function that satisfies these conditions is called *order function*. Given two functions $\phi(x,\varepsilon)$ and $\phi_a(x,\varepsilon)$ defined in a domain Ω, they are asymptotically identical to order $\delta(\varepsilon)$ if their difference is asymptotically smaller than $\delta(\varepsilon)$, where $\delta(\varepsilon)$ is an order function; that is,

$$\phi(x,\varepsilon) - \phi_a(x,\varepsilon) = o(\delta(\varepsilon)), \tag{7}$$

where ε is small parameter arising from the physical problem under consideration. The function $\phi_a(x,\varepsilon)$ is named as *asymptotic approximation* of the function $\phi(x,\varepsilon)$. Asymptotic approximations in general form are defined by

$$\phi_a(x,\varepsilon) = \sum_{i=1}^{n} \delta_i(\varepsilon)\varphi_i(x,\varepsilon), \tag{8}$$

where the asymptotic sequence of order functions $\delta_i(\varepsilon)$ is an asymptotic sequence such that $\delta_{i+1}(\varepsilon) = o(\delta_i(\varepsilon))$, as $\varepsilon \to 0$. In these conditions approximation (8) is named as *generalized asymptotic expansion*. If expansion (8) is written in the form of

$$\phi_a(x,\varepsilon) = E_0\phi = \sum_{i=1}^{n} \delta_i^{(0)}(\varepsilon)\varphi_i^{(0)}(x), \tag{9}$$

then it is called *regular asymptotic expansion*, where the special operator E_0 is *outer expansion operator* at a given order $\delta(\varepsilon)$. Thus $\phi - E_0\phi = o(\delta(\varepsilon))$. For more detailed information about the asymptotic approximations, [10, 11, 13, 19] can be studied. Interesting cases occur when the function is not regular in Ω so (8) or (9) is valid only in a restricted region $\Omega_0 \in \Omega$ called the outer region. We are faced with a singular perturbation problem and we must introduce boundary layer domains. We introduce an inner domain which can be formally denoted as $\Omega_1 = \Omega_0 - \Omega$ and located near the point $x = 1$ (for Troesch's problem). The boundary layer variable is $\overline{x} = (x-1)/\xi(\varepsilon)$, with $\xi(\varepsilon)$ being the order of thickness of this boundary layer. If a regular expansion can be constructed in Ω_1, we can write down

$$\phi_a(x,\varepsilon) = E_1\phi = \sum_{i=0}^{n} \delta_i^{(1)}(\varepsilon)\varphi_i^{(1)}(\overline{x}), \tag{10}$$

where the *inner expansion operator* E_1 is defined in Ω_1 at the same order $\delta(\varepsilon)$ as the outer expansion operator E_0; thus, $\phi - E_1\phi = o(\delta(\varepsilon))$ and

$$\phi_a = E_0\phi + E_1\phi - E_1E_0\phi \tag{11}$$

is clearly uniformly valid approximation (UVA) [35, 36]. In the Method of Matched Asymptotic Expansions (MMAE) [14], two distinct solutions of the singular perturbation problem are found for two distinct regions (inner and outer) and then to obtain uniformly valid approximation over the whole domain the solutions are matched using limit process

$$\lim_{x\to 1} y_0(x) = \lim_{\overline{x}\to -\infty} Y(\overline{x}), \tag{12}$$

where $Y(\overline{x})$ is boundary layer approximation. Finally uniformly valid approximation is obtained by simply adding the inner and outer solutions and subtracting the common limit. That is, using the procedure

$$y \approx y_0(x) + Y_0(\overline{x}) - Y_0(-\infty), \tag{13}$$

or equivalently

$$y \approx y_0(x) + Y_0(\overline{x}) - y_0(1^-), \tag{14}$$

one can reach the composite solution which is uniformly valid over the whole domain. The uniformly valid SCEM approximation is in the regular form

$$y_n^{\text{scem}}(x, \overline{x}, \epsilon) = \sum_{i=1}^{n} \delta_i(\varepsilon)\left[y_i(x) + \Psi_i(\overline{x})\right], \tag{15}$$

where δ_i is an asymptotic sequence and $\Psi_i(\overline{x})$ are the complementary functions that depend on $\overline{x}$. Functions $y_i(x)$ are the outer approximation functions that have been found by MMAE and they only depend on x, not also ε. If the functions $y_i(x)$ and $\Psi_i(\overline{x})$ depend also on ε, the uniformly valid SCEM approximation is called *generalized SCEM approximation* that is in the following form [30, 31]:

$$y_{ng}^{\text{scem}}(x, \overline{x}, \epsilon) = \sum_{i=1}^{n} \delta_i(\varepsilon)\left[y_i(x, \varepsilon) + \Psi_i(\overline{x}, \varepsilon)\right]. \tag{16}$$

For the sake of simplicity, we adopt just one-term approximation in this study. That is, we look for an approximation in the form of

$$y_0^{\text{scem}}(x, \overline{x}, \varepsilon) = y_0(x, \varepsilon) + \Psi_0(\overline{x}, \varepsilon). \tag{17}$$

To improve the accuracy of approximation, (17) can be iterated using (16). It means that successive complementary terms will be added to the approximation.

3. Application of SCEM to Troesch's Problem

In this section, we solve Troesch's problems for different values of the parameter λ using the computer algebra system Matlab and make a comparison between our results and those ones reported in the literature to confirm the efficiency and accuracy of our method. In first step, let us turn Troesch's problem into singular perturbation problem as follows:

$$y'' = \lambda \sinh(\lambda y). \tag{18}$$

Letting $\lambda = 1/\varepsilon$ in (18), we obtain

$$\begin{aligned} y'' &= \frac{1}{\varepsilon}\sinh\left(\frac{y}{\varepsilon}\right), \\ \varepsilon y'' - \sinh\left(\frac{y}{\varepsilon}\right) &= 0. \end{aligned} \tag{19}$$

Now, in order to avoid overflow or excessive error growth during numerical integration, removing the hyperbolic-type nonlinearity using the variable transformation, which was originally proposed in [36], $y(x) = 4\varepsilon \tanh^{-1}(u(x))$ or equivalently $u(x) = \tanh(y(x)/4\varepsilon)$, we get

$$\begin{aligned} y' &= \frac{4\varepsilon}{(1-u^2)}u', \\ y'' &= \frac{4\varepsilon}{(1-u^2)}u'' + \frac{8\varepsilon u}{(1-u^2)^2}\left(u'\right)^2 \end{aligned} \tag{20}$$

or

$$\left(1-u^2\right)u'' + 2u\left(u'\right)^2 = \frac{u\left(1+u^2\right)}{\varepsilon^2} \tag{21}$$

and finally Troesch's problem has the form

$$\varepsilon^2 u'' + \frac{2\varepsilon^2 u\left(u'\right)^2}{(1-u^2)} - \frac{u\left(1+u^2\right)}{(1-u^2)} = 0 \tag{22}$$

with the boundary conditions

$$\begin{aligned} u(0) &= 0, \\ u(1) &= \tanh\left(\frac{1}{4\varepsilon}\right). \end{aligned} \tag{23}$$

We reach polynomial-type nonlinearity instead of hyperbolic-type nonlinearity. The problem has the outer solution (near the point $x = 0$) $y_0(x) = 0$ (or equivalently $u_0(x) = 0$). So the complementary solution directly produces approximation to Troesch's problem. If we substitute the inner variable $\overline{x} = (x-1)/\varepsilon$ into (22) and boundary conditions (23), using the chain rule, we reach

$$\Psi_0'' + \frac{2\Psi_0\left(\Psi_0'\right)^2 - \Psi_0\left(1+\Psi_0^2\right)}{1-\Psi_0^2} = 0 \tag{24}$$

and equivalently

$$\Psi_0'' + \frac{\Psi_0\left(2\left(\Psi_0'\right)^2 - \Psi_0^2 - 1\right)}{1-\Psi_0^2} = 0 \tag{25}$$

with the boundary conditions

$$\begin{aligned} \Psi_0\left(\frac{-1}{\varepsilon}\right) &= 0, \\ \Psi_0(0) &= \tanh\left(\frac{1}{4\varepsilon}\right), \end{aligned} \tag{26}$$

where Ψ_0 is the complementary function. Here, the complementary function of SCEM, Ψ_0, is calculated numerically by *Matlab bvp4c* routine. Results which are obtained using SCEM are presented in the following section. Applying SCEM, we reach really highly accurate approximations to Troesch's problem.

Table 1: Solution of Troesch's problem for $\varepsilon = 2$.

x	Exact solution	SCEM	ADM [13]	HPM [16]	Abs. err. in SCEM
0.1	0.09594435	0.09594435	0.09593835	0.09593956	$3.13441692e(-9)$
0.2	0.19212874	0.19212875	0.19211805	0.19211932	$7.32982755e(-9)$
0.3	0.28879440	0.28879441	0.28878032	0.28878069	$1.30811447e(-8)$
0.4	0.38618485	0.38618487	0.38616870	0.38616754	$1.98750830e(-8)$
0.5	0.48454716	0.48454719	0.48453029	0.48452741	$2.60076459e(-8)$
0.6	0.58413325	0.58413328	0.58411697	0.58411278	$2.88212714e(-8)$
0.7	0.68520115	0.68520118	0.68518684	0.68518224	$2.55610896e(-8)$
0.8	0.78801652	0.78801654	0.78800556	0.78800183	$1.51078886e(-8)$
0.9	0.89285422	0.89285422	0.89284802	0.89284621	$9.280242130e(-10)$

Table 2: Solution of Troesch's problem for $\varepsilon = 1$.

x	Exact solution	SCEM	ADM [13]	HPM [16]	Abs. err. in SCEM
0.1	0.08466141	0.08466143	0.08424876	0.08438170	$2.18232920e(-8)$
0.2	0.17017167	0.17017172	0.16943070	0.16962076	$5.16987398e(-8)$
0.3	0.25739438	0.25739447	0.25641450	0.25659292	$8.76146832e(-8)$
0.4	0.34722348	0.34722359	0.34608572	0.34621073	$1.09822865e(-7)$
0.5	0.44060059	0.44060067	0.43940198	0.43944227	$7.88214280e(-8)$
0.6	0.53853523	0.53853518	0.53736570	0.53733006	$5.56789394e(-8)$
0.7	0.64212944	0.64212911	0.64108380	0.64101046	$3.25396753e(-7)$
0.8	0.75260880	0.75260812	0.75178800	0.75173354	$6.76312419e(-7)$
0.9	0.87136294	0.87136211	0.87090870	0.87088353	$8.29947601e(-7)$

Table 3: Absolute error of SCEM approximation for $\varepsilon = 0.1$.

x	Exact solution	SCEM	Abs. err. in SCEM
0.10	0.000004211	0.000042111	$3.79006047e(-5)$
0.20	0.000129963	0.000129964	$1.57300000e(-10)$
0.30	0.000358977	0.000358978	$4.20000000e(-10)$
0.40	0.000977901	0.000977902	$1.08799999e(-9)$
0.50	0.002659017	0.002659019	$2.60240000e(-9)$
0.60	0.007228924	0.007228929	$4.58729999e(-9)$
0.70	0.019664060	0.019664057	$2.41529999e(-9)$
0.80	0.053730329	0.053730314	$1.46944000e(-8)$
0.90	0.152114078	0.152114033	$4.55675999e(-8)$
0.95	0.276267734	0.276267643	$9.12888999e(-8)$
0.99	0.574076498	0.574076080	$4.17938999e(-7)$

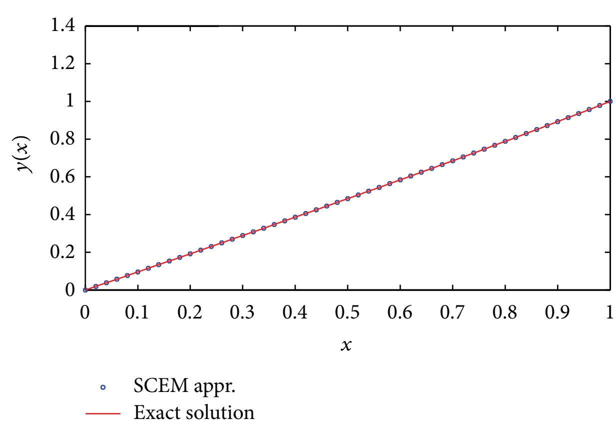

Figure 1: Exact solution and SCEM approximation of Troesch's problem for $\varepsilon = 2$.

4. Conclusions

In this study, an efficient method called Successive Complementary Expansion Method (SCEM) is used to solve the nonlinear two-point boundary value problem with application to Troesch's equation. SCEM is an easy-applicable and efficient method. It does not require any matching procedure in contrast to MMAE. Moreover, the boundary conditions are satisfied exactly, not asymptotically. In Tables 1–3, the absolute errors in solutions obtained by the presented method for $\varepsilon = 2$, $\varepsilon = 1$, and $\varepsilon = 0.1$, respectively, are shown. In Figures 1–5, we compare exact and SCEM solutions. The results obtained here were compared with the exact solution, ADM [15], and HPM [19]. It is shown that the method is computationally attractive and the presented method is much

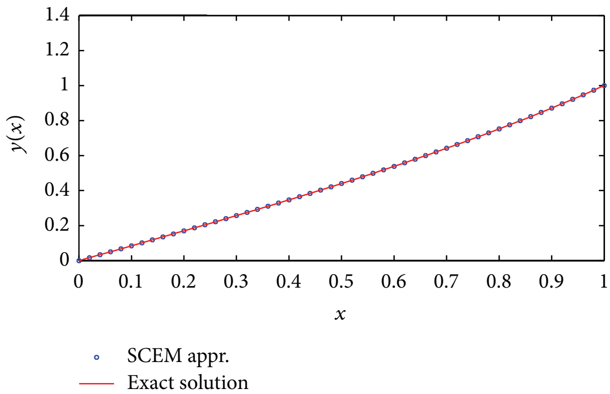

Figure 2: Exact solution and SCEM approximation of Troesch's problem for $\varepsilon = 1$.

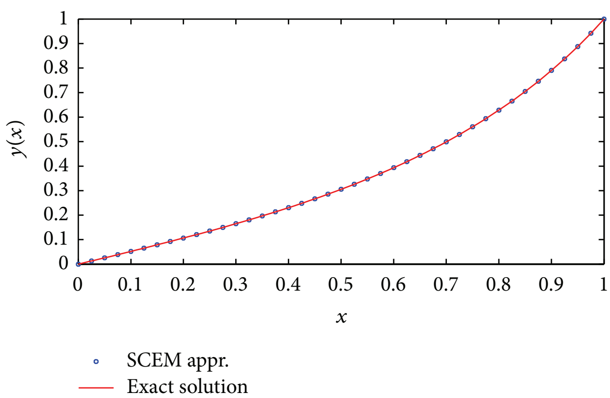

Figure 3: Exact solution and SCEM approximation of Troesch's problem for $\varepsilon = 0.5$.

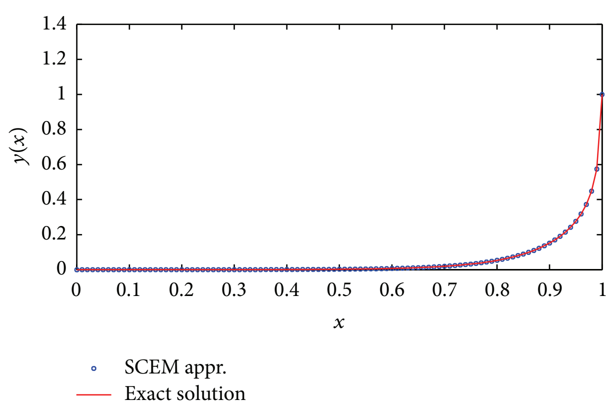

Figure 4: Exact solution and SCEM approximation of Troesch's problem for $\varepsilon = 0.1$.

better than other reported ones in the literature in the sense of accuracy and efficiency.

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

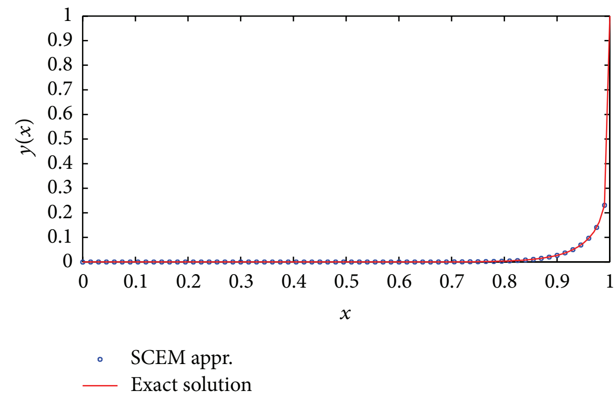

Figure 5: Exact solution and SCEM approximation of Troesch's problem for $\varepsilon = 0.05$.

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