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Research Article

On Stability of Vector Nonlinear Integrodifferential Equations

Michael Gil'

Department of Mathematics, Ben-Gurion University of the Negev, P.O. Box 653, 84105 Beersheba, Israel

Correspondence should be addressed to Michael Gil'; gilmi@bezeqint.net

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Let Ω be a bounded domain in a real Euclidean space. We consider the equation $\partial u(t,x)/\partial t = C(x)u(t,x) + \int_\Omega K(x,s)u(t,s)ds + [F(u)](t,x)$ $(t > 0;\ x \in \Omega)$, where $C(\cdot)$ and $K(\cdot,\cdot)$ are matrix-valued functions and $F(\cdot)$ is a nonlinear mapping. Conditions for the exponential stability of the steady state are established. Our approach is based on a norm estimate for operator commutators.

1. Introduction and Statement of the Main Result

Throughout this paper, $\mathbb{C}^n$ is the complex n-dimensional Euclidean space with a scalar product $(\cdot,\cdot)_n$ and norm $\|\cdot\|_n = \sqrt{(\cdot,\cdot)_n}$; $\mathbb{C}^{n\times n}$ is the set of $n \times n$-matrices; I is the unit operator in corresponding space; Ω is a bounded domain with a smooth boundary in a real Euclidean space; $L^2(\mathbb{C}^n,\Omega) = L^2$ is the Hilbert space of functions defined on Ω with values in $\mathbb{C}^n$, the scalar product

$$(v,w)_{L^2} = \int_\Omega (v(x), w(x))_n\, dx \quad \left(w, v \in L^2\right), \tag{1}$$

and the norm $\|\cdot\|_{L^2} = \sqrt{(\cdot,\cdot)_{L^2}}$.

Our main object in this paper is the equation

$$\frac{\partial u(t,x)}{\partial t} = C(x)\,u(t,x) + \int_\Omega K(x,s)\,u(t,s)\,ds + [F(u)](t,x) \quad (t > 0;\ x \in \Omega), \tag{2}$$

where $C(\cdot)$ and $K(\cdot,\cdot)$ are matrix-valued functions defined on Ω and $\Omega \times \Omega$, respectively, with values in $\mathbb{C}^{n\times n}$, and $F(\cdot) : L^2 \to L^2$ satisfy conditions pointed out below, and $u(\cdot,\cdot)$ is unknown.

Traditionally, (2) is called the Barbashin type integrodifferential equation or simply the Barbashin equation. It plays an essential role in numerous applications, in particular, in kinetic theory [1], transport theory [2], continuous mechanics [3], control theory [4], radiation theory [5, 6], and the dynamics of populations [7]. Regarding other applications, see [8]. The classical results on the Barbashin equation are represented in the well-known book [9]. The recent results about various aspects of the theory of the Barbashin equation can be found, for instance, in [10–14] and the references given therein. In particular, in [11], the author investigates the solvability conditions for the Cauchy problem for a Barbashin equation in the space of bounded continuous functions and in the space of continuous vector-valued functions with the values in an ideal Banach space. The stability and boundedness of solutions to a linear scalar nonautonomous Barbashin equation have been investigated in [15].

The literature on the asymptotic properties of integrodifferential equations is rather rich (cf. [16–22] and the references given therein), but the stability of nonlinear vector integrodifferential equations is almost not investigated. It is at an early stage of development.

A solution of (2) is a function $u(t,\cdot) : [0,\infty) \to L^2$ having a measurable derivative bounded on each finite interval.

It is assumed that under consideration F provides the existence and uniqueness of solutions (e.g., it is Lipschitz continuous). The zero solution of (2) is said to be exponentially stable, if there are constants $m_0 \geq 1$, $\delta_0 > 0$, and $\alpha > 0$, such that $\|u(t)\|_{L^2} \leq m_0\|u(0)\|_{L^2}e^{-\alpha t}$ $(t \geq 0)$, provided $\|u(0)\|_{L^2} \leq \delta_0$. It is globally exponentially stable if $\delta_0 = \infty$.

Suppose that, for a positive $r \leq \infty$,

$$\|F(h)\|_{L^2} \leq q\,\|h\|_{L^2} \quad \left(h \in L^2;\ \|h\|_{L^2} \leq r\right). \tag{3}$$

For example, for an integer $p > 1$, let $[F(h)](x) = (Th(x))^p$. Here,

$$Th(x) = \int_\Omega b(x,s)\, h(s)\, ds \tag{4}$$

with a matrix kernel $b(x,s)$ satisfying

$$J_p = \left(\int_\Omega \left(\int_\Omega \|b(x,s)\|_n^2\, ds \right)^p dx \right)^{1/2} < \infty. \tag{5}$$

Then, by the Schwarz inequality,

$$\|Th(x)\|_n^2 \le \int_\Omega \|b(x,s)\|_n^2\, ds\, \|h\|_{L^2}^2. \tag{6}$$

Thus,

$$\begin{aligned} &\int_\Omega \|[F(h)](x)\|^2\, dx \\ &\le \int_\Omega \left(\int_\Omega \|b(x,s)\|_n^2\, ds \right)^p dx\, \|h\|_{L^2}^{2p}. \end{aligned} \tag{7}$$

Hence, for any $r < \infty$, we have condition (3) with $q = r^{p-1} J_p$.

The following notations are introduced: for a linear operator A, A^* is the adjoint operator, $\|A\|$ is the operator norm, and $\sigma(A)$ is the spectrum. For $n \times n$-matrix C, put

$$g(C) = \left[N_2^2(C) - \sum_{k=1}^n |\lambda_k(C)|^2 \right]^{1/2}, \tag{8}$$

where $\lambda_k(C)$, $k = 1, \ldots, n$, are the eigenvalues of C, counted with their multiplicities; $N_2(C) = (\text{Trace } CC^*)^{1/2}$ is the Frobenius (Hilbert-Schmidt) norm of C. The following relations are checked in [23, Section 2.1]: $g^2(C) \le N_2^2(C) - |\text{Trace } C^2|$,

$$\begin{gathered} g\left(e^{i\tau} C + zI\right) = g(C) \quad (\tau \in \mathbb{R},\ z \in \mathbb{C}), \\ g^2(C) \le \frac{N_2^2(C - C^*)}{2}. \end{gathered} \tag{9}$$

If C is a normal matrix, $CC^* = C^*C$, then $g(C) = 0$. Furthermore, denote

$$\begin{aligned} \xi &:= \frac{1}{2} \sup_{h \in L^2, \|h\|_{L^2}=1} \int_\Omega \int_\Omega \left((K(x,s) + K^*(s,x))\, h(s), \right. \\ &\quad \left. h(x) \right)_n ds\, dx, \\ \gamma &= \left(\int_\Omega \int_\Omega \|C(x) K(x,s) - K(x,s) C(s)\|_n^2\, ds\, dx \right)^{1/2} \end{aligned} \tag{10}$$

and assume that

$$\begin{aligned} \alpha_0 &:= \xi + \sup_x \Re\sigma(C(x)) = \xi + \sup_x \max_k \Re\lambda_k(C(x)) \\ &< 0. \end{aligned} \tag{11}$$

In addition, with the notation $g_0 = \sup_x g(C(x))$, put

$$\begin{aligned} \chi &= \sum_{j,k=0}^{n-1} \frac{g_0^{j+k} (k+j)!}{2^{j+k} |\alpha_0|^{j+k+1} (j!k!)^{3/2}}, \\ p(t) &= \sum_{k=0}^{n-1} \frac{t^k g_0^k}{(k!)^{3/2}}, \\ \zeta_0 &= 2 \int_0^\infty e^{2\alpha_0 t} p(t) \int_0^t p(t-s)\, p(s)\, ds\, dt. \end{aligned} \tag{12}$$

This integral is simply calculated. If $C(x)$ is a normal matrix for all x, then

$$\begin{aligned} g(C(x)) &= 0, \\ p(t) &\equiv 1, \\ \chi &= \frac{1}{|\alpha_0|}, \\ \zeta_0 &= \frac{1}{2|\alpha_0|^2}. \end{aligned} \tag{13}$$

Now, we are in a position to formulate our main result.

Theorem 1. *Let conditions (3), (11), and*

$$\gamma\zeta_0 + \chi q < 1 \tag{14}$$

hold. Then, the zero solution to (2) is exponentially stable. If, in addition, $r = \infty$ in (3), then the zero solution is globally exponentially stable.

This theorem is proved in the next 3 sections. It gives us "good" results when γ is "small," that is, if matrices $K(x,s)$ and $C(x)$ "almost commute" and $\sup_{x,s}\|C(x) - C(s)\|_n$ is "small." If (2) is scalar, then $g_0 = 0$,

$$\begin{aligned} \chi &= \frac{1}{|\alpha_0|}, \\ \gamma &= 0, \\ p(t) &\equiv 1, \\ \zeta_0 &= \frac{1}{2|\alpha_0|^2}. \end{aligned} \tag{15}$$

So, in the scalar case, condition (14) takes the form

$$q < |\alpha_0|. \tag{16}$$

This condition is similar to the stability test derived in [24] for scalar integrodifferential equations.

2. Auxiliary Results

Let $\mathscr{H}$ be a Hilbert space with a scalar product $(\cdot,\cdot)_{\mathscr{H}}$ and the norm $\|\cdot\|_{\mathscr{H}} = \sqrt{(\cdot,\cdot)_{\mathscr{H}}}$; $\mathscr{B}(\mathscr{H})$ denotes the set of bounded linear operators in $\mathscr{H}$ and $[A_1, A_2] = A_1A_2 - A_2A_1$ is the commutator of $A_1, A_2 \in \mathscr{B}(\mathscr{H})$.

Lemma 2. *Let $A, B \in \mathscr{B}(\mathscr{H})$ and $C = [A, B]$. Then,*

$$\left[e^{At}, B\right] = \int_0^t e^{As} C e^{A(t-s)} ds. \tag{17}$$

Proof. Put $J(t) = \int_0^t e^{As} C e^{A(t-s)} ds$. Then, $(d/dt)(J(t)e^{-tA}) = e^{At} C e^{-At}$. On the other hand,

$$\frac{d}{dt}\left(\left[e^{At}, B\right] e^{-tA}\right) = \frac{d}{dt}\left(e^{At} B e^{-At} - B\right) = e^{At} C e^{-At}. \tag{18}$$

So, $[e^{At}, B] = J$, as claimed. □

Let

$$\alpha(A) := \sup \Re\sigma(A) < 0. \tag{19}$$

Then, the Lyapunov equation

$$WA + (WA)^* = -2I \tag{20}$$

has a unique solution $W \in \mathscr{B}(\mathscr{H})$ and it can be represented as

$$W = 2\int_0^\infty e^{A^* t} e^{At} dt \tag{21}$$

(cf. [25]). Denote $\Lambda_B = (1/2)\sup\sigma(B + B^*)$,

$$\begin{gathered} \zeta(A) = 2\int_0^\infty \left\|e^{At}\right\| \int_0^t \left\|e^{As}\right\| \left\|e^{A(t-s)}\right\| ds\, dt, \\ \psi(W, B) = \begin{cases} \Lambda_B \|W\| & \text{if } \Lambda_B > 0, \\ \Lambda_B \lambda_W & \text{if } \Lambda_B \le 0, \end{cases} \end{gathered} \tag{22}$$

where $\lambda_W = \inf\sigma(W)$.

Lemma 3. *Under condition (19), one has*

$$\begin{aligned} \operatorname{Re}(WB) &= \frac{1}{2}\left((WB) + (WB)^*\right) \\ &\le \left(\psi(W, B) + \|C\|\zeta(A)\right) I. \end{aligned} \tag{23}$$

Proof. Making use of (21), we can write

$$\operatorname{Re}(WB) = \int_0^\infty \left(e^{A^* t} e^{At} B + B^* e^{A^* t} e^{At}\right) dt. \tag{24}$$

But $e^{At} B = B e^{At} + [e^{At}, B]$, $B^* e^{A^* t} = e^{A^* t} B^* + [B^*, e^{A^* t}] = e^{A^* t} B^* + [e^{At}, B]^*$. So $2\operatorname{Re}(WB) = J_1 + J_2$, where

$$\begin{aligned} J_1 &= \int_0^\infty e^{A^* t}\left(B + B^*\right) e^{At} dt, \\ J_2 &= \int_0^\infty \left(e^{A^* t}\left[e^{At}, B\right] + \left(e^{A^* t}\left[e^{At}, B\right]\right)^*\right) dt. \end{aligned} \tag{25}$$

We have

$$J_1 \le 2\Lambda_B \int_0^\infty e^{A^* t} e^{At} dt = \Lambda_B W. \tag{26}$$

If $\Lambda_B > 0$, then $J_1 \le \Lambda_B \|W\| I$. If $\Lambda_B < 0$, then $J_1 \le \Lambda_B \lambda_W I$. So $J_1 \le 2\psi(W, B) I$.

In addition, by Lemma 2,

$$\begin{aligned} \|J_2\| &\le 2\int_0^\infty \left\|e^{At}\right\| \left\|\left[e^{At}, B\right]\right\| dt \\ &\le 2\int_0^\infty \left\|e^{At}\right\| \|C\| \int_0^t \left\|e^{As}\right\| \left\|e^{A(t-s)}\right\| ds\, dt \\ &= \|C\| \zeta(A). \end{aligned} \tag{27}$$

This proves the lemma. □

3. Equations in a Hilbert Space

In this section, for simplicity, we put $\|\cdot\|_{\mathscr{H}} = \|\cdot\|$. Put $\omega(r) = \{v \in \mathscr{H},\ \|v\| \le r\}$ $(0 < r \le \infty)$. Consider in $\mathscr{H}$ the equation

$$\frac{d}{dt} u = (A + B) u + F(u) \quad (t \ge 0), \tag{28}$$

where $A, B \in \mathscr{B}(\mathscr{H})$ and F continuously maps $\omega(r)$ into $\mathscr{H}$ and satisfies

$$\|Fv\| \le q\|v\| \quad (v \in \omega(r)). \tag{29}$$

The solution and stability are defined as in Section 1. The existence and uniqueness of solutions are assumed. Recall that W is a solution of (20).

Lemma 4. *Let conditions (19) and (29) with $r = \infty$ hold. Then, any solution of (28) satisfies the inequality*

$$\begin{gathered} \|u(t)\| \le \left(\frac{\|W\|}{\lambda_W}\right)^{1/2} \|u(0)\| e^{-\nu t}, \quad t \ge 0, \\ \text{where } \nu := 1 - \psi(W, B) - \zeta(A)\|C\| - q\|W\|. \end{gathered} \tag{30}$$

Proof. For brevity, we write $[Fu](t) = Fu(t)$. Multiplying (28) by W and doing the scalar product, we get

$$\begin{aligned} \left(Wu'(t), u(t)\right) &= \left(W(A + B) u(t), u(t)\right) \\ &\quad + \left(WFu(t), u(t)\right). \end{aligned} \tag{31}$$

Since $(d/dt)(Wu(t), u(t)) = (Wu'(t), u(t)) + (u(t), Wu'(t))$, due to (20) and Lemma 3, it can be written that

$$\begin{aligned} &\frac{d}{dt}\left(Wu(t), u(t)\right) \\ &\quad = 2\operatorname{Re}\left(W(A + B) u(t), u(t)\right) \\ &\qquad + 2\operatorname{Re}\left(WFu(t), u(t)\right) \\ &\quad \le 2\left(-1 + \psi(W, B) + \zeta(A)\|C\|\right)\left(u(t), u(t)\right) \\ &\qquad + 2\operatorname{Re}\left(WFu(t), u(t)\right). \end{aligned} \tag{32}$$

Taking into account the fact that due to (29)

$$\begin{aligned} \left(WFu(t), u(t)\right) &\le \|W\| \|Fu(t)\| \|u(t)\| \\ &\le \|W\| q \|u(t)\|^2, \end{aligned} \tag{33}$$

we get

$$\frac{d}{dt}(Wu(t), u(t)) \leq -2\nu(u(t), u(t)). \tag{34}$$

From this inequality, we have $(Wu(t), u(t)) \leq (Wu(0), u(0))e^{-2\nu t}$. Hence,

$$\lambda_W(u(t), u(t)) \leq \|W\|(u(0), u(0))e^{-2\nu t}, \tag{35}$$

as claimed. □

Lemma 5. *Let conditions (29) and $\nu < 0$ hold. Then, the zero solution to (28) is exponentially stable. If $r = \infty$ in (29), then the zero solution to (28) is globally exponentially stable.*

Proof. If $r = \infty$, then the required result is due to the previous lemma. If $r < \infty$, then, taking $\|u(0)\| < r(\lambda_W/\|W\|)^{1/2}$ due to the previous lemma, $\|u(t)\| < r$. Hence, we easily obtain the required result. □

4. Proof of Theorem 1

Take

$$\begin{aligned} Ah(x) &= (C(x) + \xi I)h(x), \\ Bh(x) &= \int_\Omega K(x,s)h(s)\,ds - \xi h(x) \\ &\qquad \left(h \in L^2\right). \end{aligned} \tag{36}$$

Then, $\Lambda(B) \leq 0$ and

$$\begin{aligned} &[A, B]h(x) \\ &\quad = \int_\Omega [C(x)K(x,s) - K(x,s)C(s)]h(s)\,ds. \end{aligned} \tag{37}$$

So $\|[A, B]\|_{L^2} \leq \gamma$. Due to [23, Example 1.7.3],

$$\left\|e^{C(x)t}\right\|_n \leq e^{\alpha(C(x))t}\sum_{k=0}^{n-1}\frac{t^k g^k(C(x))}{(k!)^{3/2}} \quad (t \geq 0), \tag{38}$$

where $\alpha(C(x)) = \sup \operatorname{Re} \sigma(C(x))$. Hence,

$$\left\|e^{At}\right\|_{L^2} \leq \sup_x \left\|e^{(C(x)+\xi)t}\right\|_n \leq e^{\alpha_0 t}p(t) \quad (t \geq 0), \tag{39}$$

since $g(C(x) + \xi I) = g(C(x))$. Consequently, $\zeta(A) \leq \zeta_0$. In addition,

$$\|W\|_{L^2} \leq 2\int_0^\infty \left\|e^{At}\right\|_{L^2}^2 dt \leq 2\int_0^\infty e^{2\alpha_0 t}p^2(t)\,dt = \chi. \tag{40}$$

Now, the required result is due to Lemma 5.

Competing Interests

The author declares that there are no competing interests regarding the publication of this paper.

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