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Research Article

Variational Problem Involving Operator Curl in a Multiconnected Domain

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We shall study the problem of minimizing a functional involving the curl of vector fields in a three-dimensional, bounded multiconnected domain with prescribed tangential component on the boundary. The paper is an extension of L^2 minimization problem of the curl of vector fields. We shall prove the existence and the estimate of minimizers of more general functional which contains L^p norm of the curl of vector fields.

1. Introduction

In this paper, we consider the following problem which was proposed by Pan [1, p. 9].

Problem A. Minimize the L^p norm of the curl of vector fields in a given space with tangential trace on the boundary being prescribed.

The problem is related to the mathematical theory of liquid crystal, of superconductivity, and of electromagnetic field. When $p = 2$ and Ω is a simply connected domain without holes, Bates and Pan [2, 3] showed the existence of minimizer. For the multiconnected domain, the author of [1] obtained the existence of a minimizer of the Problem A in the case $p = 2$.

In the present paper we shall extend the results to more general functional containing Problem A.

More precisely, let $S(x,t)$ be a Carathéodory function on $\Omega \times [0,\infty)$ and $S(x,t^2)$ is a convex function with respect to t; moreover assume that for a.e. $x \in \Omega$, $S(x,t) \in C^1((0,\infty))$, and there exist $1 < p < \infty$ and $\lambda, \Lambda > 0$ such that for a.e. $x \in \Omega$ and all $t > 0$:

$$\lambda t^{(p-2)/2} \le S_t(x,t) := \frac{\partial}{\partial t} S(x,t) \le \Lambda t^{(p-2)/2}. \tag{1}$$

Without loss of generality, we may assume that $S(x,0) = 0$. We furthermore assume the following structure condition:

$$\left(S_t\left(x,|\mathbf{a}|^2\right)\mathbf{a} - S_t\left(x,|\mathbf{b}|^2\right)\mathbf{b}\right)\cdot(\mathbf{a}-\mathbf{b}) > 0 \quad \text{for any } \mathbf{a},\mathbf{b} \in \mathbb{R}^3 \text{ with } \mathbf{a} \ne \mathbf{b}. \tag{2}$$

Under (1) with $S(x,0) = 0$, we have

$$\frac{2}{p}\lambda t^{p/2} \le S(x,t) \le \frac{2}{p}\Lambda t^{p/2}. \tag{3}$$

For example, the function $S(x,t) = \nu(x)t^{p/2}$ where $\nu(x)$ is a measurable function satisfying $0 < \nu_* \le \nu(x) \le \nu^* < \infty$ for a.e. $x \in \Omega$ satisfies (1)-(2).

Let Ω be a bounded domain in $\mathbb{R}^3$ with C^2 boundary $\partial\Omega$. Let $\mathscr{H}_T$ be a given tangential vector field on $\partial\Omega$. Let $W^{1,p}(\Omega,\mathbb{R}^3)$ be the standard Sobolev space of vector fields. From now, we denote the tangential component of a vector field $\mathbf{u}$ by $\mathbf{u}_T$; that is, $\mathbf{u}_T = \mathbf{u} - (\mathbf{u}\cdot\boldsymbol{\nu})\boldsymbol{\nu}$, where $\boldsymbol{\nu}$ is the outer normal unit vector to the boundary $\partial\Omega$. For any given tangential vector field on $\partial\Omega$

$$\mathscr{H}_T \in W^{1-1/p,p}\left(\partial\Omega,\mathbb{R}^3\right), \tag{4}$$

define a space of vector fields

$$W_t^{1,p}\left(\Omega,\mathbb{R}^3,\mathscr{H}_T\right) = \left\{\mathbf{u}\in W^{1,p}\left(\Omega,\mathbb{R}^3\right);\mathbf{u}_T=\mathscr{H}_T \text{ on } \partial\Omega\right\}. \tag{5}$$

Then it is clear that $W_t^{1,p}(\Omega,\mathbb{R}^3,\mathscr{H}_T)$ is a closed convex set in $W^{1,p}(\Omega,\mathbb{R}^3)$. We consider the minimization problem

$$R_t^p\left(\mathscr{H}_T\right) = \inf_{\mathbf{u}\in W_t^{1,p}(\Omega,\mathbb{R}^3,\mathscr{H}_T)} \int_\Omega S\left(x,|\operatorname{curl}\mathbf{u}|^2\right)dx. \tag{6}$$

When $p=2$, $S(x,t)=t$, and Ω is a simply connected domain without holes, the authors of [2, 3] showed that (6) is achieved, and then in the case where $p=2$, $S(x,t)=T$, and Ω is bounded multiconnected domain, the author of [1] succeeded to show the existence of a minimizer of (6).

Since we allow Ω to be a multiconnected domain in $\mathbb{R}^3$, throughout this paper, we assume that the domain Ω satisfies the following (O1) and (O2) (cf. Dautray and Lions [4] and Amrouche and Seloula [5]).

(O1) Ω is a bounded domain in $\mathbb{R}^3$ with C^2 boundary $\partial\Omega$. Ω is locally situated on one side of $\partial\Omega$; $\partial\Omega$ has a finite number of connected components $\Gamma_1,\dots,\Gamma_{m+1}$ $(m\geq 0)$ and Γ_{m+1} denoting the boundary of the infinite connected component of $\mathbb{R}^3\setminus\overline{\Omega}$.

(O2) There exist n manifolds of dimension 2 and of class C^2 denoted by $\Sigma_1,\dots,\Sigma_n$ $(n\geq 0)$ such that $\Sigma_i\cap\Sigma_j=\emptyset$ $(i\neq j)$ and they are nontangential to $\partial\Omega$ and such that $\Omega\setminus(\bigcup_{i=1}^n\Sigma_i)$ is simply connected and pseudo $C^{1,1}$.

The number n is called the first Betti number and m the second Betti number of Ω. We say that Ω is simply connected if $n=0$, and Ω has no holes if $m=0$. If we define the spaces

$$\begin{aligned}\mathbb{K}_N^p(\Omega) &= \left\{\mathbf{u}\in W^{1,p}\left(\Omega,\mathbb{R}^3\right);\operatorname{curl}\mathbf{u}=\mathbf{0},\operatorname{div}\mathbf{u}=0 \text{ in } \Omega,\boldsymbol{\nu}\cdot\mathbf{u}=0 \text{ on } \partial\Omega\right\},\\ \mathbb{K}_T^p(\Omega) &= \left\{\mathbf{u}\in W^{1,p}\left(\Omega,\mathbb{R}^3\right);\operatorname{curl}\mathbf{u}=\mathbf{0},\operatorname{div}\mathbf{u}=0 \text{ in } \Omega,\mathbf{u}_T=\mathbf{0} \text{ on } \partial\Omega\right\},\end{aligned} \tag{7}$$

then it is well known that $\dim\mathbb{K}_N^p(\Omega)=n$ and $\dim\mathbb{K}_T^p(\Omega)=m$. We note that $\mathbb{K}_N^p(\Omega)$ and $\mathbb{K}_T^p(\Omega)$ are contained in $W^{1,p}(\Omega,\mathbb{R}^3)$; moreover, $\mathbb{K}_N^p(\Omega)$ and $\mathbb{K}_T^p(\Omega)$ are closed subspaces of $W^{1,p}(\Omega,\mathbb{R}^3)$. Also it will be shown in Lemma 4 that $\mathbb{K}_N^p(\Omega)$ and $\mathbb{K}_T^p(\Omega)$ are closed subspaces of $L^p(\Omega,\mathbb{R}^3)$. Thus since $\mathbb{K}_T^p(\Omega)$ is a finite-dimensional closed subspace of $L^p(\Omega,\mathbb{R}^3)$, $\mathbb{K}_T^p(\Omega)$ has a complement $\mathbb{L}^p$ in $L^p(\Omega,\mathbb{R}^3)$; that is, $\mathbb{L}^p$ is a closed subspace of $L^p(\Omega,\mathbb{R}^3)$, $\mathbb{L}^p\cap\mathbb{K}_T^p(\Omega)=\{\mathbf{0}\}$, and $L^p(\Omega,\mathbb{R}^3)=\mathbb{L}^p\oplus\mathbb{K}_T^p(\Omega)$ (the direct sum). Therefore, for any $\mathbf{w}\in L^p(\Omega,\mathbb{R}^3)$, there exist uniquely $\mathbf{v}\in\mathbb{L}^p$ and $\mathbf{u}\in\mathbb{K}_T^p(\Omega)$ such that $\mathbf{w}=\mathbf{v}+\mathbf{u}$. We denote the projection $P:L^p(\Omega,\mathbb{R}^3)\to\mathbb{L}^p$ by $P\mathbf{w}=\mathbf{v}$.

Define

$$\begin{aligned}H^p(\Omega,\operatorname{curl},\operatorname{div}0) &= \left\{\mathbf{u}\in L^p\left(\Omega,\mathbb{R}^3\right);\operatorname{curl}\mathbf{u}\in L^p\left(\Omega,\mathbb{R}^3\right),\operatorname{div}\mathbf{u}=0 \text{ in } \Omega\right\},\\ H_t^p(\Omega,\operatorname{curl},\operatorname{div}0,\mathscr{H}_T) &= \left\{\mathbf{u}\in H^p(\Omega,\operatorname{curl},\operatorname{div}0);\mathbf{u}_T=\mathscr{H}_T \text{ on } \partial\Omega\right\}.\end{aligned} \tag{8}$$

Note that if $\mathbf{u}\in L^p(\Omega,\mathbb{R}^3)$ and $\operatorname{curl}\mathbf{u}\in L^p(\Omega,\mathbb{R}^3)$, then the tangent trace $\mathbf{u}_T$ is well defined as an element of $W^{-1/p,p}(\partial\Omega,\mathbb{R}^3)$ (cf. [5, p. 45]), and

$$H^p(\Omega,\operatorname{curl},\operatorname{div}0)\cap W^{1,p}\left(\Omega,\mathbb{R}^3\right) = \left\{\mathbf{u}\in W^{1,p}\left(\Omega,\mathbb{R}^3\right);\operatorname{div}\mathbf{u}=0 \text{ in } \Omega\right\}. \tag{9}$$

Moreover, we note that if $\mathscr{H}_T\in W^{1-1/p,p}(\partial\Omega,\mathbb{R}^3)$, then

$$H_t^p(\Omega,\operatorname{curl},\operatorname{div}0,\mathscr{H}_T)\subset W_t^{1,p}\left(\Omega,\mathbb{R}^3,\mathscr{H}_T\right). \tag{10}$$

(cf. Amrouche and Seloula [6, Theorem 2.3]). We will see, in Lemma 2 of Section 2, that

$$R_t^p\left(\mathscr{H}_T\right) = \inf_{\mathbf{v}\in H_t^p(\Omega,\operatorname{curl},\operatorname{div}0,\mathscr{H}_T)} \int_\Omega S\left(x,|\operatorname{curl}\mathbf{v}|^2\right)dx. \tag{11}$$

We are in a position to state the main theorem.

Theorem 1. *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain satisfying (O1) and (O2), and let $\mathscr{H}_T\in W^{1-1/p,p}(\partial\Omega,\mathbb{R}^3)$ be a tangential vector field on $\partial\Omega$. Then $R_t^p(\mathscr{H}_T)$ is achieved, and the minimizer $\mathbf{A}$ of $R_t^p(\mathscr{H}_T)$ in $H_t^p(\Omega,\operatorname{curl},\operatorname{div}0,\mathscr{H}_T)$ satisfies the following estimate. There exists a constant $C=C(\Omega)>0$ independent of $\mathscr{H}_T$ such that*

$$\|P\mathbf{A}\|_{W^{1,p}(\Omega)}\leq C\left\|\mathscr{H}_T\right\|_{W^{1-1/p,p}(\partial\Omega)}. \tag{12}$$

2. Preliminaries

In this section, we shall give some lemmas as preliminaries.

Lemma 2. *Let $\mathscr{H}_T\in W^{1-1/p,p}(\partial\Omega,\mathbb{R}^3)$ be a tangential vector field on $\partial\Omega$. Then one has*

$$R_t^p\left(\mathscr{H}_T\right) = \inf_{\mathbf{v}\in H_t^p(\Omega,\operatorname{curl},\operatorname{div}0,\mathscr{H}_T)} \int_\Omega S\left(x,|\operatorname{curl}\mathbf{v}|^2\right)dx. \tag{13}$$

Proof. Put

$$\begin{aligned}\alpha &= \inf_{\mathbf{u}\in W_t^{1,p}(\Omega,\mathbb{R}^3,\mathscr{H}_T)} \int_\Omega S\left(x,|\operatorname{curl}\mathbf{u}|^2\right)dx,\\ \beta &= \inf_{\mathbf{v}\in H_t^p(\Omega,\operatorname{curl},\operatorname{div}0,\mathscr{H}_T)} \int_\Omega S\left(x,|\operatorname{curl}\mathbf{v}|^2\right)dx.\end{aligned} \tag{14}$$

Since $H_t^p(\Omega,\operatorname{curl},\operatorname{div}0,\mathscr{H}_T)\subset W_t^{1,p}(\Omega,\mathbb{R}^3,\mathscr{H}_T)$, it is trivial that $\alpha\leq\beta$. For any $\mathbf{u}\in W^{1,p}(\Omega,\mathbb{R}^3,\mathscr{H}_T)$, the problem

$$\begin{aligned}\Delta\varphi &= \operatorname{div}\mathbf{u} \quad \text{in } \Omega,\\ \varphi &= 0 \quad \text{on } \partial\Omega\end{aligned} \tag{15}$$

has a unique solution $\varphi \in W^{2,p}(\Omega)$ (cf. Girault and Raviart [7, Theorem 1.8]). If we define $\mathbf{v} = \mathbf{u} - \nabla\varphi \in W^{1,p}(\Omega, \mathbb{R}^3)$, then $\operatorname{curl}\mathbf{v} = \operatorname{curl}\mathbf{u}$, $\operatorname{div}\mathbf{v} = \operatorname{div}\mathbf{u} - \Delta\varphi = 0$ in Ω and $\mathbf{v}_T = \mathbf{u}_T - (\nabla\varphi)_T = \mathbf{u}_T = \mathscr{H}_T$. Thus $\mathbf{v} \in H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathscr{H}_T)$. So we have

$$\int_\Omega S\left(x, |\operatorname{curl}\mathbf{u}|^2\right) dx = \int_\Omega S\left(x, |\operatorname{curl}\mathbf{v}|^2\right) dx \geq \beta. \tag{16}$$

Thus we have $\alpha \geq \beta$. □

By Lemma 2, the minimization problem (1) reduces to the following problem.

Problem B. Find the minimizer $\mathbf{u} \in H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathscr{H}_T)$ such that

$$R_t^p\left(\mathscr{H}_T\right) = \inf_{\mathbf{v}\in H_t^p(\Omega,\operatorname{curl},\operatorname{div} 0,\mathscr{H}_T)} \int_\Omega S\left(x, |\operatorname{curl}\mathbf{v}|^2\right) dx. \tag{17}$$

In the later, we frequently use the following lemma.

Lemma 3. (i) *If* $\mathbf{u} \in L^p(\Omega, \mathbb{R}^3)$, $\operatorname{curl}\mathbf{u} \in L^p(\Omega, \mathbb{R}^3)$, $\operatorname{div}\mathbf{u} \in L^p(\Omega)$, *and* $\mathbf{u}\cdot\boldsymbol{\nu} \in W^{1-1/p,p}(\partial\Omega)$, *then* $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^3)$, *and there exists a constant* $c_1(\Omega) > 0$ *such that*

$$\begin{aligned}\|\mathbf{u}\|_{W^{1,p}(\Omega)} \leq c_1(\Omega)\Big(&\|\mathbf{u}\|_{L^p(\Omega)} + \|\operatorname{curl}\mathbf{u}\|_{L^p(\Omega)} \\ &+ \|\operatorname{div}\mathbf{u}\|_{L^p(\Omega)} + \|\mathbf{u}\cdot\boldsymbol{\nu}\|_{W^{1-1/p,p}(\partial\Omega)}\Big).\end{aligned} \tag{18}$$

Here we note that if furthermore Ω *is simply connected, we can delete the first term* $\|\mathbf{u}\|_{L^p(\Omega)}$ *in the right-hand side of (18).*

(ii) *If* $\mathbf{u} \in L^p(\Omega, \mathbb{R}^3)$, $\operatorname{curl}\mathbf{u} \in L^p(\Omega, \mathbb{R}^3)$, $\operatorname{div}\mathbf{u} \in L^p(\Omega)$, *and* $\mathbf{u}_T \in W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3)$, *then* $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^3)$, *and there exists a constant* $c_2(\Omega) > 0$ *such that*

$$\begin{aligned}\|\mathbf{u}\|_{W^{1,p}(\Omega)} \leq c_2(\Omega)\Big(&\|\mathbf{u}\|_{L^p(\Omega)} + \|\operatorname{curl}\mathbf{u}\|_{L^p(\Omega)} \\ &+ \|\operatorname{div}\mathbf{u}\|_{L^p(\Omega)} + \|\mathbf{u}_T\|_{W^{1-1/p,p}(\partial\Omega)}\Big).\end{aligned} \tag{19}$$

We note that if furthermore Ω *has no holes, we can delete the first term* $\|\mathbf{u}\|_{L^p(\Omega)}$ *in the right-hand side of (19).*

For the proof of (18) and (19), see [5, Theorem 3.4 and Corollary 5.2]. If Ω is simply connected or has no holes, see Aramaki [8, Lemma 2.2].

Lemma 4. *The space* $\mathbb{K}_T^p(\Omega)$ *is a closed subspace of* $L^p(\Omega, \mathbb{R}^3)$.

Proof. Let $\mathbb{K}_T^p(\Omega) \ni \mathbf{u}_j \to \mathbf{u}$ in $L^p(\Omega, \mathbb{R}^3)$. Then from (19) we have

$$\left\|\mathbf{u}_j - \mathbf{u}_k\right\|_{W^{1,p}(\Omega)} \leq c_2(\Omega)\left\|\mathbf{u}_j - \mathbf{u}_k\right\|_{L^p(\Omega)}. \tag{20}$$

Therefore $\{\mathbf{u}_j\}$ is a Cauchy sequence in $W^{1,p}(\Omega, \mathbb{R}^3)$. Hence there exists $\mathbf{u}_0 \in W^{1,p}(\Omega, \mathbb{R}^3)$ such that $\mathbf{u}_j \to \mathbf{u}_0$ in $W^{1,p}(\Omega, \mathbb{R}^3)$, so we have $\mathbf{u} = \mathbf{u}_0$ and $\mathbf{u}_j \to \mathbf{u}$ in $W^{1,p}(\Omega, \mathbb{R}^3)$ as $j \to \infty$. It is clear that $\operatorname{curl}\mathbf{u} = \mathbf{0}$, $\operatorname{div}\mathbf{u} = 0$ in Ω, and $\mathbf{u}_T = \mathbf{0}$ on $\partial\Omega$. This implies that $\mathbf{u} \in \mathbb{K}_T^p(\Omega)$. □

3. Proof of the Main Theorem 1

In this section, we give a proof of Theorem 1. The proof consists of some lemmas and propositions. Throughout this section, we assume that $\mathscr{H}_T$ is a given tangential vector field on $\partial\Omega$.

Lemma 5. *Let* $\mathbf{A} \in H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathscr{H}_T)$. *Then the minimization problem*

$$\gamma = \inf_{\mathbf{u}\in\mathbb{K}_T^p(\Omega)} \|\mathbf{A} - \mathbf{u}\|_{L^p(\Omega)} \tag{21}$$

has a unique minimizer.

Proof. From Lemma 4, we know that $\mathbb{K}_T^p(\Omega)$ is a closed subspace of $L^p(\Omega, \mathbb{R}^3)$. Thus it is well known that (21) has a minimizer. For the uniqueness of the minimizer, it suffices to show that the unit sphere $B = \{\mathbf{u} \in L^p(\Omega, \mathbb{R}^3); \|\mathbf{u}\|_{L^p(\Omega)} = 1\}$ does not contain any line segment $[\mathbf{u}, \mathbf{v}] = \{\lambda\mathbf{u} + (1-\lambda)\mathbf{v}; 0 \leq \lambda \leq 1\}$ for $\mathbf{u}, \mathbf{v} \in B$ and $\mathbf{u} \neq \mathbf{v}$. (cf. Fujita et al. [9, p. 306 and the remark]). However, this is clear because the functional

$$f(\mathbf{u}) = \int_\Omega |\mathbf{u}|^p\, dx \tag{22}$$

is strictly convex. □

For $\mathbf{A} \in H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathscr{H}_T)$, let $\mathbf{u} \in \mathbb{K}_T^p(\Omega)$ be a unique minimizer of (21) and define $\mathbf{B} = \mathbf{A} - \mathbf{u}$. Then since for any $\mathbf{z} \in \mathbb{K}_T^p(\Omega)$ and $t \in \mathbb{R}$, $\|\mathbf{B}\|_{L^p(\Omega)}^p \leq \|\mathbf{B} + t\mathbf{z}\|_{L^p(\Omega)}^p$, we have

$$0 = \left.\frac{d}{dt}\right|_{t=0} \int_\Omega |\mathbf{B} + t\mathbf{z}|^p\, dx = p\int_\Omega |\mathbf{B}|^{p-2}\,\mathbf{B}\cdot\mathbf{z}\, dx. \tag{23}$$

If we define a space

$$\begin{aligned}B(\Omega, \mathscr{H}_T) = \Big\{&\mathbf{B} \in L^p\left(\Omega, \mathbb{R}^3\right); \operatorname{curl}\mathbf{B} \\ &\in L^p\left(\Omega, \mathbb{R}^3\right), \operatorname{div}\mathbf{B} = 0 \text{ in } \Omega, \mathbf{B}_T \\ &= \mathscr{H}_T \text{ on } \partial\Omega, \int_\Omega |\mathbf{B}|^{p-2}\,\mathbf{B}\cdot\mathbf{z}\, dx = 0\ \forall\mathbf{z} \\ &\in \mathbb{K}_T^p(\Omega)\Big\},\end{aligned} \tag{24}$$

then we see that $\mathbf{B} \in B(\Omega, \mathscr{H}_T)$. Then we have the following.

Lemma 6. *One can see that*

$$H_t^p\left(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathscr{H}_T\right) = B\left(\Omega, \mathscr{H}_T\right) \oplus \mathbb{K}_T^p(\Omega) \quad (\textit{the direct sum}). \tag{25}$$

Proof. For any $\mathbf{A} \in H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathscr{H}_T)$, as the above we can write

$$\mathbf{A} = \mathbf{B} + \mathbf{u}, \quad \text{where } \mathbf{B} \in B\left(\Omega, \mathscr{H}_T\right),\ \mathbf{u} \in \mathbb{K}_T^p(\Omega). \tag{26}$$

We show the uniqueness of the above decomposition. If we can write

$$\mathbf{A} = \mathbf{B}_1 + \mathbf{u}_1 = \mathbf{B}_2 + \mathbf{u}_2, \tag{27}$$

where $\mathbf{B}_1, \mathbf{B}_2 \in B(\Omega, \mathscr{H}_T)$, $\mathbf{u}_1$ and $\mathbf{u}_2 \in \mathbb{K}_T^p(\Omega)$, then $\mathbf{B}_1 - \mathbf{B}_2 = \mathbf{u}_2 - \mathbf{u}_1 \in \mathbb{K}_T^p(\Omega)$. Therefore we have

$$\begin{aligned} &\int_\Omega |\mathbf{B}_1|^{p-2}\mathbf{B}_1 \cdot (\mathbf{B}_1 - \mathbf{B}_2)\, dx = 0, \\ &\int_\Omega |\mathbf{B}_2|^{p-2}\mathbf{B}_2 \cdot (\mathbf{B}_1 - \mathbf{B}_2)\, dx = 0. \end{aligned} \tag{28}$$

Hence

$$\int_\Omega \left(|\mathbf{B}_1|^{p-2}\mathbf{B}_1 - |\mathbf{B}_2|^{p-2}\mathbf{B}_2\right) \cdot (\mathbf{B}_1 - \mathbf{B}_2)\, dx = 0. \tag{29}$$

Here we use the following inequality. There exists a constant $c > 0$ such that

$$\begin{aligned} &\left(|\mathbf{a}|^{p-2}\mathbf{a} - |\mathbf{b}|^{p-2}\mathbf{b}\right) \cdot (\mathbf{a} - \mathbf{b}) \\ &\quad \geq \begin{cases} c\,|\mathbf{a} - \mathbf{b}|^p & \text{if } p \geq 2, \\ c\,(|\mathbf{a}| + |\mathbf{b}|)^{p-2}\,|\mathbf{a} - \mathbf{b}|^2 & \text{if } 1 < p < 2 \end{cases} \end{aligned} \tag{30}$$

for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. For the proof of this inequality, see DiBenedetto [10, Lemma 4.4] for $p \geq 2$, and see Miranda et al. [11, (7C')]. Applying (30) with $\mathbf{a} = \mathbf{B}_1$, $\mathbf{b} = \mathbf{B}_2$ to (29), we have

$$\begin{aligned} &\int_\Omega |\mathbf{B}_1 - \mathbf{B}_2|^p\, dx = 0 \quad \text{for } p \geq 2, \\ &\int_\Omega \left(|\mathbf{B}_1| + |\mathbf{B}_2|\right)^{p-2} |\mathbf{B}_1 - \mathbf{B}_2|^2\, dx = 0 \quad \text{for } 1 < p < 2. \end{aligned} \tag{31}$$

From these equalities, we have $\mathbf{B}_1 = \mathbf{B}_2$, so $\mathbf{u}_1 = \mathbf{u}_2$. □

Now we state a refinement of Fatou's lemma (cf. Evans [12, pp. 11-12]).

Lemma 7. *Assume that $1 < p < \infty$. Let $\mathbf{B}_j \to \mathbf{B}$ weakly in $L^p(\Omega, \mathbb{R}^3)$ and a.e. in Ω. Then one has*

$$\begin{aligned} &\lim_{j\to\infty} \int_\Omega \left(|\mathbf{B}_j|^p - \left||\mathbf{B}_j|^{p-2}\mathbf{B}_j - |\mathbf{B}|^{p-2}\mathbf{B}\right|^{p'}\right) dx \\ &\quad = \int_\Omega |\mathbf{B}|^p\, dx. \end{aligned} \tag{32}$$

If furthermore

$$\lim_{j\to\infty} \int_\Omega |\mathbf{B}_j|^p\, dx = \int_\Omega |\mathbf{B}|^p\, dx, \tag{33}$$

then

$$|\mathbf{B}_j|^{p-2}\mathbf{B}_j \longrightarrow |\mathbf{B}|^{p-2}\mathbf{B} \quad \textit{strongly in } L^{p'}\left(\Omega, \mathbb{R}^3\right), \tag{34}$$

where p' denotes the conjugate exponent of p; that is, $(1/p) + (1/p') = 1$. In particular, if $\mathbf{B}_j \to \mathbf{B}$ strongly in $L^p(\Omega, \mathbb{R}^3)$ and a.e. in Ω, then (34) holds.

Proof. We use an elementary estimate. Let $1 \leq q < \infty$. Then, for any fixed $\varepsilon > 0$, there exists a constant $C = C(\varepsilon, q) > 0$ such that

$$\left||\mathbf{a} + \mathbf{b}|^q - |\mathbf{a}|^q\right| \leq \varepsilon\,|\mathbf{a}|^q + C\,|\mathbf{b}|^q \tag{35}$$

for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ (cf. [12, (1.13)]). Define

$$\begin{aligned} g_j^\varepsilon = \Bigg[&\left|\left||\mathbf{B}_j|^{p-2}\mathbf{B}_j\right|^{p'} - \left||\mathbf{B}_j|^{p-2}\mathbf{B}_j - |\mathbf{B}|^{p-2}\mathbf{B}\right|^{p'} \right. \\ &\left. - \left||\mathbf{B}|^{p-2}\mathbf{B}\right|^{p'}\right| - \varepsilon \left||\mathbf{B}_j|^{p-2}\mathbf{B}_j - |\mathbf{B}|^{p-2}\mathbf{B}\right|^{p'} \Bigg]^+, \end{aligned} \tag{36}$$

where $[a]^+ = \max\{a, 0\}$ for $a \in \mathbb{R}$. Then we have

$$\begin{aligned} g_j^\varepsilon \leq \Bigg[&\left|\left||\mathbf{B}_j|^{p-2}\mathbf{B}_j\right|^{p'} - \left||\mathbf{B}_j|^{p-2}\mathbf{B}_j - |\mathbf{B}|^{p-2}\mathbf{B}\right|^{p'}\right| \\ &+ \left||\mathbf{B}|^{p-2}\mathbf{B}\right|^{p'} - \varepsilon \left||\mathbf{B}_j|^{p-2}\mathbf{B}_j - |\mathbf{B}|^{p-2}\mathbf{B}\right|^{p'} \Bigg]^+ \\ = \Bigg[&\left|\left||\mathbf{B}_j|^{p-2}\mathbf{B}_j - |\mathbf{B}|^{p-2}\mathbf{B} + |\mathbf{B}|^{p-2}\mathbf{B}\right|^{p'} \right. \\ &- \left||\mathbf{B}_j|^{p-2}\mathbf{B}_j - |\mathbf{B}|^{p-2}\mathbf{B}\right|^{p'}\right| + \left||\mathbf{B}|^{p-2}\mathbf{B}\right|^{p'} \\ &\left. - \varepsilon \left||\mathbf{B}_j|^{p-2}\mathbf{B}_j - |\mathbf{B}|^{p-2}\mathbf{B}\right|^{p'} \Bigg]^+. \end{aligned} \tag{37}$$

If we apply (35) with $\mathbf{a} = |\mathbf{B}_j|^{p-2}\mathbf{B}_j - |\mathbf{B}|^{p-2}\mathbf{B}$, $\mathbf{b} = |\mathbf{B}|^{p-2}\mathbf{B}$ and $q = p'$, we have

$$g_j^\varepsilon \leq (C+1)\left||\mathbf{B}|^{p-2}\mathbf{B}\right|^{p'} = (C+1)\,|\mathbf{B}|^p. \tag{38}$$

We note that the right-hand side is integrable. By the hypothesis, we can see that $g_j^\varepsilon \to 0$ a.e. in Ω. Therefore by the Lebesgue dominated theorem, we have

$$\lim_{j\to\infty} \int_\Omega g_j^\varepsilon dx = 0. \tag{39}$$

Therefore we have

$$\begin{aligned} &\limsup_{j\to\infty} \int_\Omega \left|\left||\mathbf{B}_j|^{p-2}\mathbf{B}_j\right|^{p'} - \left||\mathbf{B}_j|^{p-2}\mathbf{B}_j - |\mathbf{B}|^{p-2}\mathbf{B}\right|^{p'} \right. \\ &\quad \left. - \left||\mathbf{B}|^{p-2}\mathbf{B}\right|^{p'}\right| dx \leq \varepsilon \limsup_{j\to\infty} \int_\Omega \left||\mathbf{B}_j|^{p-2}\mathbf{B}_j \right. \\ &\quad \left. - |\mathbf{B}|^{p-2}\mathbf{B}\right|^{p'} dx \\ &\quad \leq \varepsilon 2^{p'} \limsup_{j\to\infty} \int_\Omega \left(\left||\mathbf{B}_j|^{p-2}\mathbf{B}_j\right|^{p'} \right. \\ &\quad \left. + \left||\mathbf{B}|^{p-2}\mathbf{B}\right|^{p'}\right) dx = \varepsilon 2^{p'} \limsup_{j\to\infty} \int_\Omega \left(|\mathbf{B}_j|^p \right. \\ &\quad \left. + |\mathbf{B}|^p\right) dx. \end{aligned} \tag{40}$$

Since $\mathbf{B}_j \to \mathbf{B}$ weakly in $L^p(\Omega,\mathbb{R}^3)$, $\|\mathbf{B}_j\|_{L^p(\Omega)}$ is bounded. Since ε is arbitrary, we have

$$\lim_{j\to\infty}\int_\Omega\left(\left|\mathbf{B}_j\right|^p-\left|\left|\mathbf{B}_j\right|^{p-2}\mathbf{B}_j-|\mathbf{B}|^{p-2}\mathbf{B}\right|^{p'}\right)dx = \int_\Omega|\mathbf{B}|^p\,dx. \tag{41}$$

If furthermore

$$\lim_{j\to\infty}\int_\Omega\left|\mathbf{B}_j\right|^p dx=\int_\Omega|\mathbf{B}|^p\,dx, \tag{42}$$

then we have

$$\lim_{j\to\infty}\int_\Omega\left|\left|\mathbf{B}_j\right|^{p-2}\mathbf{B}_j-|\mathbf{B}|^{p-2}\mathbf{B}\right|^{p'}dx=0. \tag{43}$$

□

Lemma 8. *$B(\Omega,\mathscr{H}_T)$ is a weakly closed set in $W^{1,p}(\Omega,\mathbb{R}^3)$.*

Proof. Let $\mathbf{B}_j\in B(\Omega,\mathscr{H}_T)$, $\mathbf{B}_j\to\mathbf{B}$ weakly in $W^{1,p}(\Omega,\mathbb{R}^3)$. Then we have $\operatorname{curl}\mathbf{B}\in L^p(\Omega,\mathbb{R}^3)$, $\operatorname{div}\mathbf{B}=0$ in Ω, $\mathbf{B}_T=\mathscr{H}_T$ on $\partial\Omega$, and

$$\int_\Omega\left|\mathbf{B}_j\right|^{p-2}\mathbf{B}_j\cdot\mathbf{z}\,dx=0\quad\forall\mathbf{z}\in\mathbb{K}_T^p(\Omega). \tag{44}$$

Passing to a subsequence, we may assume that $\mathbf{B}_j\to\mathbf{B}$ strongly in $L^p(\Omega,\mathbb{R}^3)$ and a.e. in Ω. Thus from Lemma 7, we have $|\mathbf{B}_j|^{p-2}\mathbf{B}_j\to|\mathbf{B}|^{p-2}\mathbf{B}$ in $L^{p'}(\Omega,\mathbb{R}^3)$. Therefore we have

$$\int_\Omega|\mathbf{B}|^{p-2}\mathbf{B}\cdot\mathbf{z}\,dx=0\quad\forall\mathbf{z}\in\mathbb{K}_T^p(\Omega). \tag{45}$$

This implies that $\mathbf{B}\in B(\Omega,\mathscr{H}_T)$. □

Lemma 9. *There exists a constant $c(\Omega)>0$ such that for all $\mathbf{B}\in W^{1,p}(\Omega,\mathbb{R}^3)$ satisfying $\operatorname{div}\mathbf{B}=0$ in Ω and*

$$\int_\Omega|\mathbf{B}|^{p-2}\mathbf{B}\cdot\mathbf{z}\,dx=0\quad\forall\mathbf{z}\in\mathbb{K}_T^p(\Omega), \tag{46}$$

one has

$$\|\mathbf{B}\|_{W^{1,p}(\Omega)}\le c(\Omega)\left(\|\operatorname{curl}\mathbf{B}\|_{L^p(\Omega)}+\|\mathbf{B}_T\|_{W^{1-1/p,p}(\partial\Omega)}\right). \tag{47}$$

Proof. If the conclusion (47) is false, there exists a sequence $\{\mathbf{B}_j\}\subset W^{1,p}(\Omega,\mathbb{R}^3)$ satisfying $\operatorname{div}\mathbf{B}_j=0$ in Ω and

$$\int_\Omega\left|\mathbf{B}_j\right|^{p-2}\mathbf{B}_j\cdot\mathbf{z}\,dx=0\quad\forall\mathbf{z}\in\mathbb{K}_T^p(\Omega), \tag{48}$$

such that $\|\mathbf{B}_j\|_{W^{1,p}(\Omega)}=1$, $\|\operatorname{curl}\mathbf{B}_j\|_{L^p(\Omega)}\to0$, $\|\mathbf{B}_{j,T}\|_{W^{1-1/p,p}(\partial\Omega)}\to0$ as $j\to\infty$. After passing to a subsequence, we may assume that $\mathbf{B}_j\to\mathbf{B}_0$ weakly in $W^{1,p}(\Omega,\mathbb{R}^3)$, strongly in $L^p(\Omega,\mathbb{R}^3)$, and a.e. in Ω. Therefore we have $\operatorname{div}\mathbf{B}_0=0$, $\operatorname{curl}\mathbf{B}_0=\mathbf{0}$ in Ω and $\mathbf{B}_{0,T}=\mathbf{0}$ on $\partial\Omega$, so $\mathbf{B}_0\in\mathbb{K}_T^p(\Omega)$. From Lemma 7,

$$\begin{aligned}\int_\Omega\left|\mathbf{B}_0\right|^p dx&=\int_\Omega\left|\mathbf{B}_0\right|^{p-2}\mathbf{B}_0\cdot\mathbf{B}_0dx\\&=\lim_{j\to\infty}\int_\Omega\left|\mathbf{B}_j\right|^{p-2}\mathbf{B}_j\cdot\mathbf{B}_0dx=0.\end{aligned} \tag{49}$$

Thus we have $\mathbf{B}_0=\mathbf{0}$. Hence $\mathbf{B}_j\to\mathbf{0}$ strongly in $L^p(\Omega,\mathbb{R}^3)$. From (19), we see that

$$\begin{aligned}\|\mathbf{B}_j\|_{W^{1,p}(\Omega)}&\le c_2(\Omega)\\&\quad\cdot\left(\|\mathbf{B}_j\|_{L^p(\Omega)}+\|\operatorname{curl}\mathbf{B}_j\|_{L^p(\Omega)}+\|\mathbf{B}_{j,T}\|_{W^{1-1/p,p}(\partial\Omega)}\right)\longrightarrow0\end{aligned} \tag{50}$$

as $j\to\infty$. This contradicts $\|\mathbf{B}_j\|_{W^{1,p}(\Omega)}=1$. □

Proposition 10. *Let $\mathscr{H}_T\in W^{1-1/p,p}(\partial\Omega,\mathbb{R}^3)$. Then the minimization problem*

$$\inf_{\mathbf{B}\in B(\Omega,\mathscr{H}_T)}\int_\Omega S\left(x,|\operatorname{curl}\mathbf{B}|^2\right)dx \tag{51}$$

is achieved and

$$R_t^p(\mathscr{H}_T)=\inf_{\mathbf{B}\in B(\Omega,\mathscr{H}_T)}\int_\Omega S\left(x,|\operatorname{curl}\mathbf{B}|^2\right)dx \tag{52}$$

Proof. By Lemma 2, we can see that

$$R_t^p(\mathscr{H}_T)=\inf_{\mathbf{A}\in H_t^p(\Omega,\operatorname{curl},\operatorname{div}0,\mathscr{H}_T)}\int_\Omega S\left(x,|\operatorname{curl}\mathbf{A}|^2\right)dx. \tag{53}$$

Since $B(\Omega,\mathscr{H}_T)\subset H_t^p(\Omega,\operatorname{curl},\operatorname{div}0,\mathscr{H}_T)$, it is clear that

$$R_t^p(\mathscr{H}_T)\le\inf_{\mathbf{B}\in B(\Omega,\mathscr{H}_T)}\int_\Omega S\left(x,|\operatorname{curl}\mathbf{B}|^2\right)dx. \tag{54}$$

On the other hand, for any $\mathbf{A}\in H_t^p(\Omega,\operatorname{curl},\operatorname{div}0,\mathscr{H}_T)$, we can write $\mathbf{A}=\mathbf{B}+\mathbf{u}$, where $\mathbf{B}\in B(\Omega,\mathscr{H}_T)$, and $\mathbf{u}\in\mathbb{K}_T^p(\Omega)$. Hence we have

$$\begin{aligned}\int_\Omega S\left(x,|\operatorname{curl}\mathbf{A}|^2\right)dx&=\int_\Omega S\left(x,|\operatorname{curl}\mathbf{B}|^2\right)dx\\&\ge\inf_{\mathbf{B}\in B(\Omega,\mathscr{H}_T)}\int_\Omega S\left(x,|\operatorname{curl}\mathbf{B}|^2\right)dx.\end{aligned} \tag{55}$$

Thus (52) holds. We show that the right-hand side of (52) has a minimizer. Let $\{\mathbf{B}_j\}\subset B(\Omega,\mathscr{H}_T)$ be a minimizing sequence. Then

$$\int_\Omega S\left(x,\left|\operatorname{curl}\mathbf{B}_j\right|^2\right)dx=R_t^p(\mathscr{H}_T)+o(1)\quad\text{as } j\longrightarrow\infty. \tag{56}$$

By (1), we have

$$\begin{aligned}\frac{2}{p}\lambda\int_\Omega\left|\operatorname{curl}\mathbf{B}_j\right|^p dx&\le\int_\Omega S\left(x,\left|\operatorname{curl}\mathbf{B}_j\right|^2\right)dx\\&=R_t^p(\mathscr{H}_T)+o(1).\end{aligned} \tag{57}$$

Thus, by Lemma 9, $\{\mathbf{B}_j\}$ is bounded in $W^{1,p}(\Omega,\mathbb{R}^3)$. Passing to a subsequence, we may assume that $\mathbf{B}_j \to \mathbf{B}_0$ weakly in $W^{1,p}(\Omega,\mathbb{R}^3)$, strongly in $L^p(\Omega,\mathbb{R}^3)$, and a.e. in Ω. Therefore we have $\operatorname{div}\mathbf{B}_0 = 0$, $\mathbf{B}_{0,T} = \mathscr{H}_T$ on $\partial\Omega$. Since

$$\int_\Omega \left|\mathbf{B}_j\right|^{p-2}\mathbf{B}_j\cdot\mathbf{z}\,dx = 0 \quad \forall \mathbf{z}\in\mathbb{K}_T^p(\Omega), \tag{58}$$

it follows from Lemma 7 that

$$\int_\Omega \left|\mathbf{B}_0\right|^{p-2}\mathbf{B}_0\cdot\mathbf{z}\,dx = 0 \quad \forall \mathbf{z}\in\mathbb{K}_T^p(\Omega). \tag{59}$$

Therefore $\mathbf{B}_0 \in B(\Omega,\mathscr{H}_T)$. It suffices to prove that

$$\begin{aligned}&\int_\Omega S\left(x,\left|\operatorname{curl}\mathbf{B}_0\right|^2\right)dx\\ &\quad\le \liminf_{j\to\infty}\int_\Omega S\left(x,\left|\operatorname{curl}\mathbf{B}_j\right|^2\right)dx.\end{aligned} \tag{60}$$

In fact, we can choose a subsequence $\{\operatorname{curl}\mathbf{B}_{j_k}\}$ of $\{\operatorname{curl}\mathbf{B}_j\}$ so that

$$\begin{aligned}&\lim_{k\to\infty}\int_\Omega S\left(x,\left|\operatorname{curl}\mathbf{B}_{j_k}\right|^2\right)dx\\ &\quad= \liminf_{j\to\infty}\int_\Omega S\left(x,\left|\operatorname{curl}\mathbf{B}_j\right|^2\right)dx.\end{aligned} \tag{61}$$

Since $\operatorname{curl}\mathbf{B}_{j_k}\to\operatorname{curl}\mathbf{B}_0$ weakly in $L^p(\Omega,\mathbb{R}^3)$, it follows from the Mazur theorem that there exist $\mathbf{g}_l\in L^p(\Omega,\mathbb{R}^3)$ such that $\mathbf{g}_l\in$ convex hull of $\{\operatorname{curl}\mathbf{B}_{j_k}; k\ge l\}$ and $\mathbf{g}_l\to\operatorname{curl}\mathbf{B}_0$ strongly in $L^p(\Omega,\mathbb{R}^3)$. Hence we can choose a subsequence $\{\mathbf{g}_{l_m}\}$ of $\{\mathbf{g}_l\}$ so that $\mathbf{g}_{l_m}\to\operatorname{curl}\mathbf{B}_0$ strongly in $L^p(\Omega,\mathbb{R}^3)$ and a.e. in Ω. By the Fatou lemma, we have

$$\int_\Omega S\left(x,\left|\operatorname{curl}\mathbf{B}_0\right|^2\right)dx \le \liminf_{m\to\infty}\int_\Omega S\left(x,\left|\mathbf{g}_{l_m}\right|^2\right)dx. \tag{62}$$

Since $S(x,t^2)$ is a convex function with respect to t, we have

$$\begin{aligned}&\int_\Omega S\left(x,\left|\mathbf{g}_{l_m}\right|^2\right)dx\\ &\quad\le \sup\left\{\int_\Omega S\left(x,\left|\operatorname{curl}\mathbf{B}_{j_k}\right|^2\right)dx; k\ge l_m\right\}.\end{aligned} \tag{63}$$

Therefore we have

$$\begin{aligned}&\int_\Omega S\left(x,\left|\operatorname{curl}\mathbf{B}_0\right|^2\right)dx \le \liminf_{m\to\infty}\int_\Omega S\left(x,\left|\mathbf{g}_{l_m}\right|^2\right)dx\\ &\quad\le \lim_{m\to\infty}\sup\left\{\int_\Omega S\left(x,\left|\operatorname{curl}\mathbf{B}_{j_k}\right|^2\right)dx; k\ge l_m\right\}\\ &\quad= \lim_{k\to\infty}\int_\Omega S\left(x,\left|\operatorname{curl}\mathbf{B}_{j_k}\right|^2\right)dx\\ &\quad= \liminf_{j\to\infty}\int_\Omega S\left(x,\left|\operatorname{curl}\mathbf{B}_j\right|^2\right)dx.\end{aligned} \tag{64}$$

This completes the proof. □

Lemma 11. *Let $\mathbf{A}\in H_t^p(\Omega,\operatorname{curl},\operatorname{div}0,\mathscr{H}_T)$ be a minimizer of $R_t^p(\mathscr{H}_T)$. Then $\mathbf{A}$ is a weak solution of the following system:*

$$\begin{aligned}&\operatorname{curl}\left[S_t\left(x,\left|\operatorname{curl}\mathbf{A}\right|^2\right)\operatorname{curl}\mathbf{A}\right] = \mathbf{0},\quad \operatorname{div}\mathbf{A} = 0 \text{ in } \Omega,\\ &\qquad\qquad \mathbf{A}_T = \mathscr{H}_T \quad \text{on } \partial\Omega.\end{aligned} \tag{65}$$

Proof. If $\mathbf{A}\in H_t^p(\Omega,\operatorname{curl},\operatorname{div}0,\mathscr{H}_T)$ is a minimizer of $R_t^p(\mathscr{H}_T)$, then we can see that, for any $\mathbf{w}\in H_t^p(\Omega,\operatorname{curl},\operatorname{div}0,\mathbf{0})$, we have

$$\left.\frac{d}{dt}\right|_{t=0}\int_\Omega S\left(x,\left|\operatorname{curl}\mathbf{A}+t\operatorname{curl}\mathbf{w}\right|^2\right)dx = 0. \tag{66}$$

Thus we have

$$\int_\Omega S_t\left(x,\left|\operatorname{curl}\mathbf{A}\right|^2\right)\operatorname{curl}\mathbf{A}\cdot\operatorname{curl}\mathbf{w}\,dx = 0 \tag{67}$$

for all $\mathbf{w}\in H_t^p(\Omega,\operatorname{curl},\operatorname{div}0,\mathbf{0})$. We claim that

$$\begin{aligned}&\operatorname{curl}\left[H_t^p(\Omega,\operatorname{curl},\operatorname{div}0,\mathbf{0})\right]\\ &\quad= \operatorname{curl}\left[W_t^{1,p}\left(\Omega,\mathbb{R}^3,\mathbf{0}\right)\right].\end{aligned} \tag{68}$$

In fact, since it is clear that $H_t^p(\Omega,\operatorname{curl},\operatorname{div}0,\mathbf{0})\subset W_t^{1,p}(\Omega,\mathbb{R}^3,\mathbf{0})$, we have

$$\begin{aligned}&\operatorname{curl}\left[H_t^p(\Omega,\operatorname{curl},\operatorname{div}0,\mathbf{0})\right]\\ &\quad\subset \operatorname{curl}\left[W_t^{1,p}\left(\Omega,\mathbb{R}^3,\mathbf{0}\right)\right].\end{aligned} \tag{69}$$

Conversely let $\mathbf{u}\in W_t^{1,p}(\Omega,\mathbb{R}^3,\mathbf{0})$. Choose ϕ to be a solution of

$$\begin{aligned}&\Delta\phi = \operatorname{div}\mathbf{u} \quad \text{in } \Omega,\\ &\phi = 0 \quad \text{on } \partial\Omega.\end{aligned} \tag{70}$$

By the elliptic regularity theorem, we see that $\phi\in W^{2,p}(\Omega)$. Define $\mathbf{v} = \mathbf{u}-\nabla\phi$. Then $\operatorname{curl}\mathbf{v} = \operatorname{curl}\mathbf{u}\in L^p(\Omega,\mathbb{R}^3)$, $\operatorname{div}\mathbf{v} = \operatorname{div}\mathbf{u}-\Delta\phi = 0$ in Ω, and $\mathbf{v}_T = \mathbf{u}_T-(\nabla\phi)_T = \mathbf{u}_T = \mathbf{0}$ on $\partial\Omega$. Therefore $\mathbf{v}\in H_t^p(\Omega,\operatorname{curl},\operatorname{div}0,\mathbf{0})$ and $\operatorname{curl}\mathbf{u} = \operatorname{curl}\mathbf{v}\in\operatorname{curl}[H_t^p(\Omega,\operatorname{curl},\operatorname{div}0,\mathbf{0})]$.

Hence (67) holds for any $\mathbf{w}\in W_t^{1,p}(\Omega,\mathbb{R}^3,\mathbf{0})$. Since $\mathscr{D}(\Omega,\mathbb{R}^3)\subset W_t^{1,p}(\Omega,\mathbb{R}^3,\mathbf{0})$, it follows from (67) that $\mathbf{A}$ is a weak solution of (65). □

Remark 12. The system (65) is so called the p-curl system. When Ω is a bounded, simply connected domain in $\mathbb{R}^3$ without holes and with $C^{2+\alpha}$ boundary for some $\alpha\in(0,1)$. If $\mathscr{H}_T = \mathbf{0}$, then [8] showed that the weak solution $\mathbf{A}$ of system (65) satisfies the fact that $\mathbf{A}\in C^{1+\beta}(\overline{\Omega},\mathbb{R}^3)$ for some $\beta\in(0,1)$ and there exists a constant C depending only on p,Ω such that $\|\mathbf{A}\|_{C^{1+\beta}(\overline{\Omega})}\le C$.

Lemma 13. *Let $\mathbf{B}_0\in B(\Omega,\mathscr{H}_T)$ be a minimizer of (52). Then any minimizer $\mathbf{A}\in H_t^p(\Omega,\operatorname{curl},\operatorname{div}0,\mathscr{H}_T)$ of (17) must have the form $\mathbf{A} = \mathbf{B}_0+\mathbf{u}$ where $\mathbf{u}\in\mathbb{K}_T^p(\Omega)$. In particular, the minimizer of (52) is unique.*

Proof. Since for any $\mathbf{u} \in \mathbb{K}_T^p(\Omega)$, we see that

$$\begin{aligned} &\mathbf{B}_0 + \mathbf{u} \in H_t^p\left(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathscr{H}_T\right), \\ &\int_\Omega \left|\operatorname{curl}\left(\mathbf{B}_0 + \mathbf{u}\right)^p\right| dx = \int_\Omega \left|\operatorname{curl} \mathbf{B}_0\right|^p dx \\ &\quad = R_t^p\left(\mathscr{H}_T\right). \end{aligned} \tag{71}$$

Thus $\mathbf{B}_0 + \mathbf{u}$ is a minimizer of (17). On the other hand, for any minimizer $\mathbf{A} \in H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathscr{H}_T)$ of (17), define $\mathbf{w} = \mathbf{A} - \mathbf{B}_0$. Then $\mathbf{w} \in H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathbf{0})$. From (67), we have

$$\begin{aligned} &\int_\Omega S_t\left(x, |\operatorname{curl} \mathbf{A}|^2\right) \operatorname{curl} \mathbf{A} \cdot \operatorname{curl} \mathbf{w}\, dx \\ &\quad = \int_\Omega S_t\left(x, \left|\operatorname{curl} \mathbf{B}_0\right|^2\right) \operatorname{curl} \mathbf{B}_0 \cdot \operatorname{curl} \mathbf{w}\, dx = 0. \end{aligned} \tag{72}$$

Therefore,

$$\begin{aligned} &\int_\Omega \left(S_t\left(x, |\operatorname{curl} \mathbf{A}|^2\right) \operatorname{curl} \mathbf{A}\right. \\ &\quad \left. - S_t\left(x, \left|\operatorname{curl} \mathbf{B}_0\right|^2\right) \operatorname{curl} \mathbf{B}_0\right) \cdot \left(\operatorname{curl} \mathbf{A}\right. \\ &\quad \left. - \operatorname{curl} \mathbf{B}_0\right) dx = 0. \end{aligned} \tag{73}$$

By the structure condition (2), we have $\operatorname{curl}(\mathbf{A} - \mathbf{B}_0) = \mathbf{0}$ in Ω, so $\mathbf{A} - \mathbf{B}_0 \in \mathbb{K}_T^p(\Omega)$.

If $\mathbf{B} \in B(\Omega, \mathscr{H}_T) \subset H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathscr{H}_T)$ is a minimizer of (52), we can write $\mathbf{B} = \mathbf{B}_0 + \mathbf{u}$, where $\mathbf{u} \in \mathbb{K}_T^p(\Omega)$. If follows from Lemma 6 that we see that $\mathbf{u} = \mathbf{0}$. Thus the minimizer of (52) in $B(\Omega, \mathscr{H}_T)$ is unique. □

For $\mathscr{H}_T \in W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3)$, let $\mathbf{A} = \mathbf{A}(\mathscr{H}_T) \in H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathscr{H}_T)$ be a minimizer of (17). Then there exist uniquely $\mathbf{B}_0 = \mathbf{B}_0(\mathscr{H}_T) \in B(\Omega, \mathscr{H}_T)$ which is a minimizer of (52) and $\mathbf{u} = \mathbf{u}(\mathscr{H}_T) \in \mathbb{K}_T^p(\Omega)$ such that

$$\mathbf{A}\left(\mathscr{H}_T\right) = \mathbf{B}_0\left(\mathscr{H}_T\right) + \mathbf{u}\left(\mathscr{H}_T\right). \tag{74}$$

We note that $P\mathbf{A}(\mathscr{H}_T) = \mathbf{B}_0(\mathscr{H}_T)$.

In order to show the estimate in Theorem 1, it suffices to prove the following proposition.

Proposition 14. *There exists a constant $c = c(\Omega)$ independent of $\mathscr{H}_T$ such that*

$$\left\|\mathbf{B}_0\left(\mathscr{H}_T\right)\right\|_{W^{1,p}(\Omega)} \le c \left\|\mathscr{H}_T\right\|_{W^{1-1/p,p}(\partial\Omega)}. \tag{75}$$

Proof. Assume that the conclusion is false. Then there exists a sequence $\{\mathscr{H}_{j,T}\} \subset W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3)$ such that $\|\mathbf{B}_0(\mathscr{H}_{j,T})\|_{W^{1,p}(\Omega)} = 1$ and

$$\left\|\mathscr{H}_{j,T}\right\|_{W^{1-1/p,p}(\partial\Omega)} \longrightarrow 0 \quad \text{as } j \longrightarrow \infty. \tag{76}$$

For brevity of notation, we write $\mathbf{B}_j = \mathbf{B}_0(\mathscr{H}_{j,T})$. Passing to a subsequence, we may assume that $\mathbf{B}_j \to \mathbf{B}$ weakly in $W^{1,p}(\Omega, \mathbb{R}^3)$, strongly in $L^p(\Omega, \mathbb{R}^3)$, and a.e. in Ω. Thus $\operatorname{curl} \mathbf{B} \in L^p(\Omega, \mathbb{R}^3)$, $\operatorname{div} \mathbf{B} = 0$ in Ω, and $\mathbf{B}_T = \mathbf{0}$ on $\partial\Omega$. Since $\mathbf{B}_j$ satisfies

$$\int_\Omega \left|\mathbf{B}_j\right|^{p-2} \mathbf{B}_j \cdot \mathbf{z}\, dx = 0 \quad \forall \mathbf{z} \in \mathbb{K}_T^p(\Omega) \tag{77}$$

and $\mathbf{B}_j \to \mathbf{B}$ strongly in $L^p(\Omega, \mathbb{R}^3)$ and a.e. in Ω, it follows from Lemma 7 that

$$\int_\Omega |\mathbf{B}|^{p-2} \mathbf{B} \cdot \mathbf{z}\, dx = 0 \quad \forall \mathbf{z} \in \mathbb{K}_T^p(\Omega). \tag{78}$$

Hence we have $\mathbf{B} \in B(\Omega, \mathbf{0})$. On the other hand, $\mathbf{B}_j$ is a weak solution of

$$\begin{aligned} \operatorname{curl}\left[S_t\left(x, \left|\operatorname{curl} \mathbf{B}_j\right|^2\right) \operatorname{curl} \mathbf{B}_j\right] &= \mathbf{0} \quad \text{in } \Omega, \\ \mathbf{B}_{j,T} &= \mathscr{H}_{j,T} \quad \text{on } \partial\Omega. \end{aligned} \tag{79}$$

Since $S_t(x, |\operatorname{curl} \mathbf{B}_j|^2) \operatorname{curl} \mathbf{B}_j \in L^{p'}(\Omega, \mathbb{R}^3)$ and $\operatorname{curl}[S_t(x, |\operatorname{curl} \mathbf{B}_j|^2) \operatorname{curl} \mathbf{B}_j] = \mathbf{0}$, we see that $S_t(x, |\operatorname{curl} \mathbf{B}_j|^2) \operatorname{curl} \mathbf{B}_j|_{\partial\Omega} \in W^{-1/p',p'}(\partial\Omega, \mathbb{R}^3)$. Since $\boldsymbol{\nu} \times \mathscr{H}_{j,T} \in W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3) = W^{1/p',p}(\partial\Omega, \mathbb{R}^3)$, it follows from the Green formula that

$$\begin{aligned} 0 &= \int_\Omega \operatorname{curl}\left[S_t\left(x, \left|\operatorname{curl} \mathbf{B}_j\right|^2\right) \operatorname{curl} \mathbf{B}_j\right] \cdot \mathbf{B}_j dx \\ &= \int_\Omega S_t\left(x, \left|\operatorname{curl} \mathbf{B}_j\right|^2\right) \operatorname{curl} \mathbf{B}_j \cdot \operatorname{curl} \mathbf{B}_j dx \\ &\quad + \int_{\partial\Omega} \left\langle \mathbf{B}_{j,T}, \boldsymbol{\nu} \times S_t\left(x, \left|\operatorname{curl} \mathbf{B}_j\right|^2\right) \operatorname{curl} \mathbf{B}_j \right\rangle dS, \end{aligned} \tag{80}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality bracket of the spaces $W^{1/p',p}(\partial\Omega, \mathbb{R}^3)$ and $W^{-1/p',p'}(\partial\Omega, \mathbb{R}^3)$. Here we have

$$\begin{aligned} &\left|\int_{\partial\Omega} \left\langle \mathscr{H}_{j,T}, \boldsymbol{\nu} \times S_t\left(x, \left|\operatorname{curl} \mathbf{B}_j\right|^2\right) \operatorname{curl} \mathbf{B}_j \right\rangle dS\right| \\ &\quad \le \left\|\mathscr{H}_{j,T}\right\|_{W^{1-1/p,p}(\partial\Omega)} \left\|S_t\left(x, \left|\operatorname{curl} \mathbf{B}_j\right|^2\right) \operatorname{curl} \mathbf{B}_j\right\|_{L^{p'}(\Omega)} \\ &\quad \le \left\|\mathscr{H}_{j,T}\right\|_{W^{1-1/p,p}(\partial\Omega)} \left(\int_\Omega \left(\Lambda \left|\operatorname{curl} \mathbf{B}_j\right|^{p-1}\right)^{p'} dx\right)^{1/p'} \\ &\quad \le \Lambda \left\|\mathscr{H}_{j,T}\right\|_{W^{1-1/p,p}(\partial\Omega)} \left\|\operatorname{curl} \mathbf{B}_j\right\|_{L^p(\Omega)}^{p/p'}. \end{aligned} \tag{81}$$

Since $\operatorname{curl} \mathbf{B}_j \to \operatorname{curl} \mathbf{B}$ weakly in $L^p(\Omega, \mathbb{R}^3)$, we see that $\|\operatorname{curl} \mathbf{B}_j\|_{L^p(\Omega)}$ is bounded. Since $\|\mathscr{H}_{j,T}\|_{W^{1-1/p,p}(\partial\Omega)} \to 0$, we have

$$\int_{\partial\Omega} \left\langle \boldsymbol{\nu} \times \mathscr{H}_{j,T}, S_t\left(x, \left|\operatorname{curl} \mathbf{B}_j\right|^2\right) \operatorname{curl} \mathbf{B}_j \right\rangle dS \longrightarrow 0 \tag{82}$$

as $j \to \infty$. Since $S(x,t^2)t^2$ is equivalent to $S(x,t)$, using (80), we have

$$\begin{aligned}
&\int_\Omega S_t\left(x, |\operatorname{curl}\mathbf{B}|^2\right)|\operatorname{curl}\mathbf{B}|^2\,dx \\
&\quad\le \liminf_{j\to\infty}\int_\Omega S_t\left(x, \left|\operatorname{curl}\mathbf{B}_j\right|^2\right)\left|\operatorname{curl}\mathbf{B}_j\right|^2 dx \\
&\quad= \liminf_{j\to\infty}\left[\int_\Omega S_t\left(x, \left|\operatorname{curl}\mathbf{B}_j\right|^2\right)\left|\operatorname{curl}\mathbf{B}_j\right|^2 dx\right. \\
&\quad\left.+\int_{\partial\Omega}\left\langle \boldsymbol{\nu}\times\mathscr{H}_{j,T}, S_t\left(x, \left|\operatorname{curl}\mathbf{B}_j\right|^2\right)\operatorname{curl}\mathbf{B}_j\right\rangle dS\right] \\
&\quad= \limsup_{j\to\infty}\left[\int_\Omega S_t\left(x, \left|\operatorname{curl}\mathbf{B}_j\right|^2\right)\left|\operatorname{curl}\mathbf{B}_j\right|^2 dx\right. \\
&\quad\left.+\int_{\partial\Omega}\left\langle \boldsymbol{\nu}\times\mathscr{H}_{j,T}, S_t\left(x, \left|\operatorname{curl}\mathbf{B}_j\right|^2\right)\operatorname{curl}\mathbf{B}_j\right\rangle dS\right] \\
&\quad= 0.
\end{aligned} \tag{83}$$

Since $S_t(x,|\operatorname{curl}\mathbf{B}|^2)|\operatorname{curl}\mathbf{B}|^2 \ge \lambda|\operatorname{curl}\mathbf{B}|^p$, we see that $\operatorname{curl}\mathbf{B} = \mathbf{0}$, so $\mathbf{B} \in \mathbb{K}_T^p(\Omega)$. From (78) with $\mathbf{z} = \mathbf{B}$, we have

$$0 = \int_\Omega |\mathbf{B}|^{p-2}\,\mathbf{B}\cdot\mathbf{B}\,dx = \int_\Omega |\mathbf{B}|^p\,dx. \tag{84}$$

Therefore $\mathbf{B} = \mathbf{0}$ in Ω, so $\mathbf{B}_j \to \mathbf{0}$ weakly in $W^{1,p}(\Omega,\mathbb{R}^3)$ and strongly in $L^p(\Omega,\mathbb{R}^3)$. From (80), we can see that $\|\operatorname{curl}\mathbf{B}_j\|_{L^p(\Omega)} \to 0$. By (19),

$$\begin{aligned}
&\left\|\mathbf{B}_j\right\|_{W^{1,p}(\Omega)} \le c_2(\Omega) \\
&\quad\cdot\left(\left\|\mathbf{B}_j\right\|_{L^p(\Omega)} + \left\|\operatorname{curl}\mathbf{B}_j\right\|_{L^p(\Omega)} + \left\|\mathscr{H}_{j,T}\right\|_{W^{1-1/p,p}(\partial\Omega)}\right) \longrightarrow 0
\end{aligned} \tag{85}$$

as $j \to \infty$. This contradicts $\|\mathbf{B}_j\|_{W^{1,p}(\Omega)} = 1$. □

Proof of Theorem 1. The proof of Theorem 1 follows from Lemma 2 and Propositions 10 and 14. □

Remark 15. Instead of minimizing $S(t,|\operatorname{curl}\mathbf{u}|^2)$, it is also interesting to minimize $S(x,|\operatorname{div}\mathbf{u}|^2)$. This problem is related to the mathematical theory of liquid crystals. For $p = 2$ and $S(x,t) = t$, see Aramaki [13].

Competing Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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