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Research Article

On Self-Centeredness of Product of Graphs

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A graph G is said to be a self-centered graph if the eccentricity of every vertex of the graph is the same. In other words, a graph is a self-centered graph if radius and diameter of the graph are equal. In this paper, self-centeredness of strong product, co-normal product, and lexicographic product of graphs is studied in detail. The necessary and sufficient conditions for these products of graphs to be a self-centered graph are also discussed. The distance between any two vertices in the co-normal product of a finite number of graphs is also computed analytically.

1. Introduction

The concept of self-centered graphs is widely used in applications, for example, the facility location problem. The facility location problem is to locate facilities in a locality (network) so that these facilities can be used efficiently. All graphs in this paper are simple and connected graphs. The *distance* between two vertices u and v in a graph G, denoted by $d_G(u,v)$ (or simply $d(u,v)$), is the minimum length of u-v path in the graph. The *eccentricity* of a vertex v in G, denoted by $\mathrm{ecc}_G(v)$, is defined as the distance between v and a vertex farthest from v; that is, $\mathrm{ecc}_G(v) = \max\{d_G(v,u) : u \in V(G)\}$. The radius $\mathrm{rad}(G)$ and diameter $\mathrm{diam}(G)$ of the graph G are, respectively, the minimum and maximum eccentricity of the vertices of graph G; that is, $\mathrm{rad}(G) = \min\{\mathrm{ecc}(v) : v \in V(G)\}$ and $\mathrm{diam}(G) = \max\{\mathrm{ecc}(v) : v \in V(G)\}$. The center $C(G)$ of graph G is the induced subgraph of G on the set of all vertices with minimum eccentricity. A graph G is said to be a *self-centered* graph if the eccentricity of every vertex is the same; that is, $C(G) = G$ or $\mathrm{rad}(G) = \mathrm{diam}(G)$. If the eccentricity of every vertex is equal to d, then G is called *d-self-centered* graph.

For any kind of graph product G of the graphs $G_1, G_2, \ldots, G_n$, the vertex set is taken as $V(G) = \{(x_1, x_2, \ldots, x_n) : x_i \in V(G_i)\}$. Because of their adjacency rules, product names are different. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two vertices in $V(G)$. Then the product is called

(i) *Cartesian product*, denoted by $G = G_1 \square G_2 \square \cdots \square G_n$, where $x \sim y$ if and only if $x_i y_i \in E(G_i)$ for exactly one index i, $1 \le i \le n$, and $x_j = y_j$ for each index $j \ne i$,

(ii) *strong product*, denoted by $G = G_1 \boxtimes \cdots \boxtimes G_n$, where $x \sim y$ if and only if $x_i y_i \in E(G_i)$ or $x_i = y_i$, for every i, $1 \le i \le n$,

(iii) *lexicographic product*, denoted by $G = G_1 \circ \cdots \circ G_n$, where $x \sim y$ if and only if, for some $j \in \{1, 2, \ldots, n\}$, $x_j y_j \in E(G_j)$ and $x_i = y_i$ for each $1 \le i < j$,

(iv) *co-normal product*, denoted by $G = G_1 * G_2 * \cdots * G_n$, where $x \sim y$ if and only if $x_i \sim y_i$ for some $i \in \{1, 2, \ldots, n\}$.

Self-centered graphs have been broadly studied and surveyed in [1–3]. In [4], the authors described several algorithms to construct self-centered graphs. Stanic [5] proved that the Cartesian product of two self-centered graphs is a self-centered graph. Inductively, one can prove that Cartesian product of n-self-centered graphs is also a self-centered graph.

In this paper, we find conditions for self-centeredness of strong product, co-normal product, and lexicographic product of graphs.

2. Main Results

In this section, we will discuss the self-centeredness of different types of product graphs. As mentioned before, all graphs considered here are simple and connected. The following result is given by Stanic [5].

Theorem 1. *If G_1 and G_2 are m- and n-self-centered graphs, respectively, then $G_1\square G_2$ is $(m+n)$-self-centered graph. Reciprocally, if $G_1\square G_2$ is self-centered, then both graphs G_1 and G_2 are self-centered.*

By method of induction, one can extend the above theorem and get the result given below.

Theorem 2. *Let $G = G_1\square G_2\square\cdots\square G_n$ be the Cartesian product of graphs $G_1, G_2, \ldots, G_n$. If every G_i is d_i-self-centered graph, then G is m-self-centered graph, where $m = \sum_{i=1}^{n} d_i$, $1 \le i \le n$. Conversely, if G is a self-centered graph, then every G_i is a self-centered graph.*

Next we will discuss self-centeredness of strong product of graphs.

Theorem 3. *Let $G = G_1 \boxtimes \cdots \boxtimes G_n$ be the strong product of graphs $G_1, G_2, \ldots, G_n$. Then G is d-self-centered graph if and only if, for some $k \in \{1, \ldots, n\}$, G_k is d-self-centered graph and* $\operatorname{diam}(G_i) \le d$ *for every i, $1 \le i \le n$.*

Proof. For any two vertices $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, the distance between them is given in [6]:

$$d(x, y) = \max_{1\le i\le n} \left\{ d_{G_i}(x_i, y_i) \right\}. \qquad (1)$$

Now, the eccentricity of any vertex x of G is given by

$$\begin{aligned} \operatorname{ecc}(x) &= \max \{ d(x, y) : y \in V(G) \} \\ &= \max \left\{ \max_{1\le i\le n} \left\{ d_{G_i}(x_i, y_i) \right\} : y \in V(G) \right\}, \end{aligned} \qquad (2)$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

First, let G_k be d-self-centered graph for some $k \in \{1, 2, \ldots, n\}$ and $\operatorname{diam}(G_i) \le d$ for all i, $1 \le i \le n$. Since G_k is d-self-centered, $\operatorname{ecc}(x_k) = d$ and there exists some y_k in G_k such that $d(x_k, y_k) = d$. As $\operatorname{diam}(G_i) \le d$ for all i, $1 \le i \le n$, the distance between any two vertices in any G_i cannot exceed d. Hence, $\operatorname{ecc}(x) = d$ for all $x \in V(G)$ and thus G is d-self-centered graph.

Conversely, let G be a d-self-centered graph. If, for some $l \in \{1, \ldots, n\}$, $\operatorname{diam}(G_l) = d_l > d$, then there exist vertices x_l and y_l in G_l such that $d(x_l, y_l) = d_l$. Now for $x = (x_1, \ldots, x_l, \ldots, x_n)$ and $y = (y_1, \ldots, y_l, \ldots, y_n)$ in $V(G)$, $d(x, y) \ge d(x_l, y_l) = d_l > d$ and so $\operatorname{ecc}(x) \ge d_l > d$. This contradicts the fact that G is d-self-centered graph and thus it is proven that $\operatorname{diam}(G_i) \le d$ for all i. Now, our claim is that there exists $k \in \{1, \ldots, n\}$ such that G_k is d-self-centered graph. On the contrary, suppose that none of G_i is d-self-centered graph. Then there exist vertices $x_i \in V(G_i)$ for all i such that $\operatorname{ecc}(x_i) = d_i < d$. Let $x = (x_1, \ldots, x_n)$. Then $\operatorname{ecc}(x) = \max_{1\le i\le n}\{d_i\} < d$, which contradicts the fact that G is d-self-centered graph. ☐

In the following lemma, we determine the formula for the distance between two vertices in the co-normal product of a finite number of graphs.

Lemma 4. *Let $G = G_1 * G_2 * \cdots * G_n$ be the co-normal product of graphs $G_1, G_2, \ldots, G_n$. The distance between $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in G is*

$$d(x, y) = \begin{cases} 1 & \text{if } x_i \sim y_i \text{ for some } i \in \{1, 2, \ldots, n\} \\ d(x_i, y_i) & \text{if } G_j = K_1,\ \forall j \ne i \\ 2 & \text{if } x \nsim y,\ x_l \ne y_l \text{ for exactly one index } l \text{ and } G_j \ne K_1 \text{ for some } j \ne l \\ 2 & \text{if } x \nsim y \text{ and } \exists \text{ at least two indices } k, l \text{ s.t. } x_k \ne y_k \text{ and } x_l \ne y_l. \end{cases} \qquad (3)$$

Proof. Consider two vertices $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ of G. If, for some $i \in \{1, 2, \ldots, n\}$, $x_i \sim y_i$, then by the definition of co-normal product $x \sim y$ and thus $d(x, y) = 1$.

Next, let $G_j = K_1$ for all $j \ne i$. In this case, for any path P between x and y, every adjacent pair of vertices in P differ only in the ith coordinate. So $d(x, y) = d(x_i, y_i)$. For the third option of the distance formula, we have vertices x and y as $x = (x_1, x_2, \ldots, x_l, \ldots, x_j, \ldots, x_n)$ and $y = (x_1, x_2, \ldots, y_l, \ldots, x_j, \ldots, x_n)$ such that $x_l \ne y_l$ and $G_j \ne K_1$ for some $j \ne l$. Since G_j is connected graph, there exists a vertex $z_j \in V(G_j)$ such that $x_j \sim z_j$ and thus we get a vertex $z = (x_1, x_2, \ldots, x_l, \ldots, z_j, \ldots, x_n) \in G$ such that $x \sim z$ and $z \sim y$ (because $x_j = y_j$) and xzy is a path of length two and hence $d(x, y) = 2$.

Finally, consider the case, where, for at least two indices k and l, $x_k \ne y_k$ and $x_l \ne y_l$; that is, for at least two indices k and l, $G_k \ne K_1$ and $G_l \ne K_1$. Since $x \nsim y$, $x_k \nsim y_k$, and $x_l \nsim y_l$, then from the connectivity of graphs G_k and G_l there exist vertices $z_k \in V(G_k)$ and $z_l \in V(G_l)$ such that $z_k \sim x_k$ in G_k and $z_l \sim y_l$ in G_l. Then we have a vertex $z = (x_1, \ldots, z_k, \ldots, z_l, \ldots, x_n) \in V(G)$ such that $x \sim z$ and $z \sim y$. Thus xzy will be an x-y path of length two and this proves that $d(x, y) = 2$. ☐

The following theorem gives necessary and sufficient conditions for a co-normal product of graphs to be a self-centered graph.

Theorem 5. *Let $G = G_1 * G_2 * \cdots * G_n$ be the co-normal product of graphs $G_1, G_2, \ldots, G_n$ with $|V(G_i)| = n_i$. Then the following hold:*

(i) *Let $G_i \neq K_1$ and $G_j = K_1$ for all $j \neq i$. Then G is d-self-centered graph if and only if G_i is d-self-centered graph.*

(ii) *Let there be at least two values of i such that $G_i \neq K_1$. Then G is 2-self-centered graph if and only if there exists an index l such that $\Delta(G_l) \neq n_l - 1$, where $\Delta(G)$ is the maximum degree of a vertex in G.*

Proof. (i) The result is true because G is isomorphic to G_i in this case through the isomorphism

$$f : V(G) \longrightarrow V(G_i) \tag{4}$$

with $f(x_1, \ldots, x_i, \ldots, x_n) = x_i$.

(ii) Let G be a 2-self-centered graph. If, for all the indices i, $\Delta(G_i) = n_i - 1$, then there are vertices $x_i \in V(G_i)$, $1 \leq i \leq n$, such that $\deg(x_i) = n_i - 1$. Now, the vertex $x = (x_1, x_2, \ldots, x_n)$, $\mathrm{ecc}(x) = 1$, which contradicts the fact that G is 2-self-centered graph. Hence there exists an index l such that $\Delta(G_l) \neq n_l - 1$.

Conversely, let there be an index l such that $\Delta(G_l) \neq n_l - 1$. Then for any vertex $x = (x_1, x_2, \ldots, x_l, x_{l+1}, \ldots, x_n)$ in G there exists another vertex $y = (x_1, x_2, \ldots, y_l, x_{l+1}, \ldots, x_n)$, where $y_l \in V(G_l)$ and $x_l \nsim y_l$. Since $x \nsim y$, from the third option of the distance formula given in Lemma 4, $\mathrm{ecc}(x) = 2$. Since x is an arbitrary vertex, G is 2-self-centered graph. □

In the following two theorems, we discuss self-centeredness of lexicographic product of graphs.

Theorem 6. *Let $G = G_1 \circ G_2 \circ \cdots \circ G_n$ be the lexicographic product of graphs $G_1, G_2, \ldots, G_n$ and let $k \geq 1$ be the smallest index for which $G_k \neq K_1$. If G_k is d-self-centered graph, where $d \geq 2$, then G is d-self-centered graph. The converse is true for $d \geq 3$.*

Proof. For vertices $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ of G, the following distance formula is due to Hammack et al. [6]:

$$d(x, y) = \begin{cases} d_{G_1}(x_1, y_1) & \text{if } x_1 \neq y_1 \\ d_{G_i}(x_i, y_i) & \text{if } d_{G_l}(x_l) = 0 \ \forall 1 \leq l < i \\ \min\{d_{G_i}(x_i, y_i), 2\} & \text{if } d_{G_l}(x_l) \neq 0 \text{ for some } 1 \leq l < i, \end{cases} \tag{5}$$

where i is the smallest index for which $x_i \neq y_i$.

Let $|V(G_i)| = 1$ for $i = 1, 2, \ldots, k-1$ and let G_k be d-self-centered graph, where $d \geq 2$. First let $k = 1$. Since $|V(G_1)| > 1$, G_1 is connected and degree of no vertex in G_1 is zero; then the second option in the distance formula will not arise. Then the above formula to calculate the distance reduces to

$$d(x, y) = \begin{cases} d_{G_1}(x_1, y_1) & \text{if } i = 1 \\ \min\{d_{G_i}(x_i, y_i), 2\} & \text{if } i \geq 2, \end{cases} \tag{6}$$

where i is the smallest index for which $x_i \neq y_i$. For $i \geq 2$, let $r = \min\{d_{G_i}(x_i, y_i), 2\}$. Then $r \leq 2$. Since $d \geq 2$, we get $r \leq d$. Now, for $x \in V(G)$,

$$\begin{aligned} \mathrm{ecc}(x) &= \max\{d(x, y) : y \in V(G)\} \\ &= \max\{d_{G_1}(x_1, y_1), r : y_1 \in V(G_1)\} \\ &= d, \end{aligned} \tag{7}$$

because $\mathrm{ecc}(x_1) = d$ and there exists $y_1 \in G_1$ such that $d(x_1, y_1) = d$. This proves that $\mathrm{ecc}(x) = d$ for all $x \in V(G)$ and hence G is a d-self-centered graph.

Next, let $k > 1$. Since $|V(G_1)| = 1$, there is no $y_1 \in G_1$ such that $x_1 \neq y_1$. So, first option in the distance formula will not arise. Since the degree of the vertex in G_j for $j = 1, 2, \ldots, k-1$ is zero, if $i = k$ in the above distance formula then $d(x, y) = d_{G_k}(x_k, y_k)$. Since $G_k \neq K_1$ and is connected $\deg(x_k) \neq 0$. So if $i \geq k+1$ in the above formula, $d(x, y) = \min\{d_{G_i}(x_i, y_i), 2\}$ and thus the above formula to calculate the distance reduces to

$$d(x, y) = \begin{cases} d_{G_k}(x_k, y_k) & \text{if } i = k \\ \min\{d_{G_i}(x_i, y_i), 2\} & \text{if } i \geq k+1, \end{cases} \tag{8}$$

where i is the smallest index for which $x_i \neq y_i$. For $i \geq k+1$ let $r_1 = \min\{d_{G_i}(x_i, y_i), 2\}$. Then $r_1 \leq 2$. Since $d \geq 2$, we get $r_1 \leq d$. Thus, for any vertex $x \in V(G)$, we have

$$\begin{aligned} \mathrm{ecc}(x) &= \max\{d(x, y) : y \in V(G)\} \\ &= \max\{d_{G_k}(x_k, y_k), r_1 : y_k \in V(G_k)\} \\ &= d. \end{aligned} \tag{9}$$

This proves that $\mathrm{ecc}(x) = d$ for all $x \in V(G)$ and hence G is a d-self-centered graph.

Conversely, let G be a d-self-centered graph, where $d \geq 3$. Then $\mathrm{ecc}(x) = d$ for all $x \in V(G)$. Notice that, for any vertex $x = (x_1, x_2, \ldots, x_n)$ in G,

$$\begin{aligned} \mathrm{ecc}(x) &= \max\{d(x, y) : y \in V(G)\} \\ &= \begin{cases} \max\{\mathrm{ecc}(x_1), r : x_1 \in V(G_1)\} & \text{if } k = 1 \\ \max\{\mathrm{ecc}(x_k), r_1 : x_k \in V(G_k)\} & \text{if } k > 1, \end{cases} \end{aligned} \tag{10}$$

where r and r_1 are as defined above. Since $\mathrm{ecc}(x)$ (which is the maximum of $\mathrm{ecc}(x_k)$ and r or r_1) is equal to d, $d \geq 3$ and $r, r_1 \leq 2$, we get $\mathrm{ecc}(x_k) = d$ for all $x_k \in V(G_k)$. So G_k is d-self-centered graph. □

If we take $d = 2$, then $\mathrm{ecc}(x) = 2$ may not imply that $\mathrm{ecc}(x_k) = 2$ (there may be $\mathrm{ecc}(x_k) < 2$ and r or r_1 is equal to 2; see example below).

Example 7. Here we consider the lexicographic product of three graphs, G_1, G_2, and G_3, where $G_1 = K_2$, $G_2 = P_4$, and $G_3 = K_2$. Let $V(G_1) = \{x, y\}$, $V(G_2) = \{a, b, c, d\}$, and $V(G_3) = \{1, 2\}$. The lexicographic product $G = K_2 \circ P_4 \circ K_2$

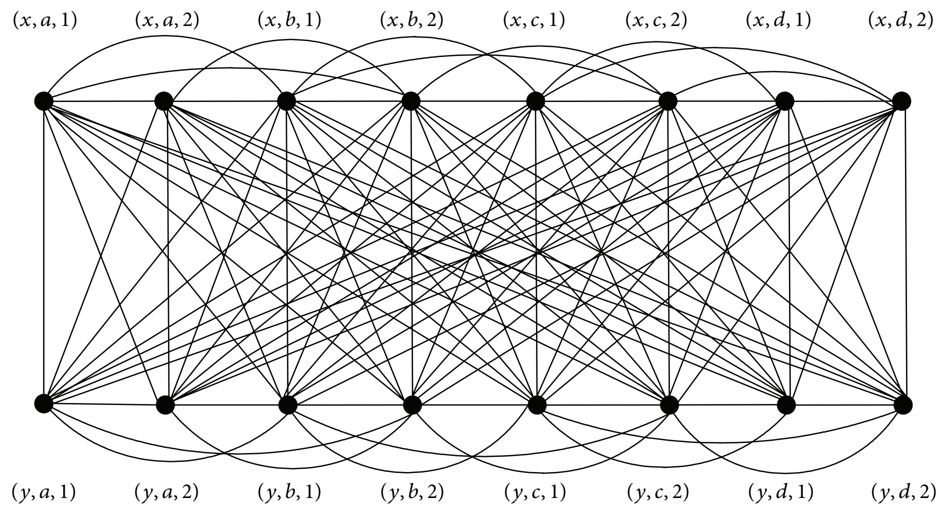

Figure 1

of graphs K_2, P_4, and K_2 is shown in Figure 1. One can check that the eccentricity of every vertex of G is two and hence G is a 2-self-centered graph. However, G_1 is not a 2-self-centered graph.

In the theorem below, we present the general version of the 2-self-centered product graphs included in the previous example.

Theorem 8. *Let $G = G_1 \circ G_2 \circ \cdots \circ G_n$ be the lexicographic product of graphs $G_1, G_2, \ldots, G_n$ with $|V(G_i)| = n_i$, let G_k be 1-self-centered graph for some $k \in \{1, \ldots, n-1\}$, and let G_i (if it exists) be K_1 for all $i < k$. Then G is a 2-self-centered graph if and only if $\Delta(G_j) \neq n_j - 1$ for some $j \geq k+1$.*

Proof. First let G be a 2-self-centered graph. It is given that, for some $k \in \{1, \ldots, n-1\}$, G_k is 1-self-centered graph and let G_i be K_1 for all $i < k$. Our claim is that $\Delta(G_j) \neq n_j - 1$ for some $j \geq k+1$. On the contrary, let $\Delta(G_j) = n_j - 1$ for all $j \geq k+1$. Then there are vertices $g_i \in G_i$ such that $\mathrm{ecc}(g_i) = 1$ for every $i, k \leq i \leq n$. Now, by using above distance formula, for every $x = (x_1, \ldots, x_{k-1}, g_k, \ldots, g_n)$ in G, one gets $\mathrm{ecc}(x) = 1$. This contradicts the fact that G is a 2-self-centered graph.

Conversely, let $\Delta(G_l) \neq n_l - 1$ for some $l \geq k+1$. Then for any vertex $x_l \in G_l$ there exists $y_l \in G_l$ such that $x_l \nsim y_l$. For any vertex $x = (x_1, \ldots, x_k, \ldots, x_l, \ldots, x_n)$ there exists a vertex $y = (x_1, \ldots, x_k, \ldots, y_l, \ldots, x_n)$ such that $x \nsim y$. So, $\mathrm{ecc}(x) \geq 2$. Since $G_i = K_1$ for all $i < k$ (if any), the distance formula will be

$$d(x, y) = \begin{cases} d_{G_i}(x_i, y_i) & \text{if } i = k \\ \min\left\{d_{G_i}(x_i, y_i), 2\right\} & \text{if } i \geq k+1, \end{cases} \tag{11}$$

where i is the smallest index for which $x_i \neq y_i$. Since G_k is a 1-self-centered graph, $d_{G_i}(x_i, y_i) = 1$ if $i = k$. Also, for $i \geq k+1$, $\min\{d_{G_i}(x_i, y_i), 2\} \leq 2$. Thus eccentricity of no vertex is more than two and we get $\mathrm{ecc}(x) = 2$ for every $x \in G$. Hence G is a 2-self-centered graph. □

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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