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Research Article

Convergence Results for a Common Solution of a Finite Family of Equilibrium Problems and Quasi-Bregman Nonexpansive Mappings in Banach Space

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We introduce an iterative process for finding common fixed point of finite family of quasi-Bregman nonexpansive mappings which is a unique solution of some equilibrium problem.

1. Introduction

Let E be a real reflexive Banach space and C a nonempty subset of E. Let $T : C \to C$ be a map, a point $x \in C$ is called a fixed point of T if $Tx = x$, and the set of all fixed points of T is denoted by $F(T)$. The mapping T is called L-Lipschitzian or simply Lipschitz if there exists $L > 0$, such that $\|Tx - Ty\| \le L\|x - y\|$, $\forall x, y \in C$, and if $L = 1$, then the map T is called nonexpansive.

Let $g : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem with respect to g is to find

$$z \in C \text{ such that } g(z, y) \ge 0, \quad \forall y \in C. \tag{1}$$

The set of solutions of equilibrium problem is denoted by $\mathrm{EP}(g)$. Thus

$$\mathrm{EP}(g) := \{z \in C : g(z, y) \ge 0,\ \forall y \in C\}. \tag{2}$$

Numerous problems in physics, optimization, and economics reduce to finding a solution of the equilibrium problem. Some methods have been proposed to solve equilibrium problem in Hilbert spaces; see, for example, Blum and Oettli [1], Combettes and Hirstoaga [2]. Recently, Tada and Takahashi [3, 4] and S. Takahashi and W. Takahashi [5] obtain weak and strong convergence theorems for finding a common element of the set of solutions of an equilibrium problem and set of fixed points of nonexpansive mapping in Hilbert space. In particular, Takahashi and Zembayashi [4] establish a strong convergence theorem for finding a common element of the two sets by using the hybrid method introduced in Nakajo and Takahashi [6]. They also proved such a strong convergence theorem in a uniformly convex and uniformly smooth Banach space.

In 1967, Bregman [7] discovered an elegant and effective technique for using so-called Bregman distance function D_f; see (3) in the process of designing and analyzing feasibility and optimization algorithms. This opened a growing area of research in which Bregman's technique has been applied in various ways in order to design and analyze iterative algorithms for solving feasibility and optimization problems.

Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $D_f : \operatorname{dom} f \times \operatorname{int} \operatorname{dom} f \to [0, +\infty)$ defined as

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle \tag{3}$$

is called the Bregman distance with respect to f (see [8]). It is obvious from the definition of D_f that

$$D_f(z, x) = D_f(z, y) + D_f(y, x) + \langle \nabla f(y) - \nabla f(x), z - y \rangle. \tag{4}$$

We observed from (4) that, for any $y_1, y_2, \ldots, y_N \in E$, the following holds:

$$D_f(y_1, y_N) = \sum_{k=2}^{N} D_f(y_{k-1}, y_k) + \sum_{k=3}^{N} \langle \nabla f(y_{k-1}) - \nabla f(y_k), y_{k-1} - y_1 \rangle. \tag{5}$$

Recall that the Bregman projection [7] of $x \in \operatorname{int}\operatorname{dom} f$ onto the nonempty closed and convex set $C \subset \operatorname{dom} f$ is the necessarily unique vector $P_C^f(x) \in C$ satisfying

$$D_f\left(P_C^f(x), x\right) = \inf\left\{D_f(y, x) : y \in C\right\}. \tag{6}$$

A mapping T is said to be Bregman firmly nonexpansive [9], if, for all $x, y \in C$,

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty \rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty \rangle, \tag{7}$$

or, equivalently,

$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leq D_f(Tx, y) + D_f(Ty, x). \tag{8}$$

A point $p \in C$ is said to be asymptotic fixed point of a map T, if, for any sequence $\{x_n\}$ in C which converges weakly to p, $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. We denote by $\widehat{F}(T)$ the set of asymptotic fixed points of T. Let $f : E \to \mathbb{R}$; a mapping $T : C \to C$ is said to be Bregman relatively nonexpansive [10] if $F(T) \neq \emptyset$, $\widehat{F}(T) = F(T)$, and $D_f(p, T(x)) \leq D_f(p, x)$ for all $x \in C$ and $p \in F(T)$. T is said to be quasi-Bregman relatively nonexpansive if $F(T) \neq \emptyset$, and $D_f(p, T(x)) \leq D_f(p, x)$ for all $x \in C$ and $p \in F(T)$.

Recently, by using the Bregman projection, in 2011 Reich and Sabach [9] proposed algorithms for finding common fixed points of finitely many Bregman firmly nonexpansive operators in a reflexive Banach space:

$$\begin{aligned}
& x_0 \in E \\
& Q_0^i = E, \quad i = 1, 2, \ldots, N \\
& u_n \in C \text{ such that} \\
& y_n^i = T_i\left(x_n + e_n^i\right), \\
& Q_{n+1}^i = \left\{z \in Q_n^i : \left\langle \nabla f\left(x_n + e_n^i\right) - \nabla f\left(y_n^i\right), z - y_n^i \right\rangle \leq 0\right\}, \\
& C_n = \bigcap_{i=1}^{N} C_n^i, \\
& x_{n+1} = P_{C_{n+1}}^f x_0, \quad n \geq 0.
\end{aligned} \tag{9}$$

Under some suitable conditions, they proved that the sequence generated by (9) converges strongly to $\bigcap_{i=1}^N F(T_i)$ and applied the result for the solution of convex feasibility and equilibrium problems.

In 2011, Chen et al. [11] introduced the concept of weak Bregman relatively nonexpansive mappings in a reflexive Banach space and gave an example to illustrate the existence of a weak Bregman relatively nonexpansive mapping and the difference between a weak Bregman relatively nonexpansive mapping and a Bregman relatively nonexpansive mapping. They also proved strong convergence of the sequences generated by the constructed algorithms with errors for finding a fixed point of weak Bregman relatively nonexpansive mappings and Bregman relatively nonexpansive mappings under some suitable conditions.

Recently in 2014, Alghamdi et al. [12] proved a strong convergence theorem for the common fixed point of finite family of quasi-Bregman nonexpansive mappings. Pang et al. [13] proved weak convergence theorems for Bregman relatively nonexpansive mappings, while Zegeye and Shahzad in [14, 15] proved a strong convergence theorem for the common fixed point of finite family of right Bregman strongly nonexpansive mappings and Bregman weak relatively nonexpansive mappings in reflexive Banach space, respectively.

In 2015 Kumam et al. [16] introduced the following algorithm:

$$\begin{aligned}
x_1 &= x \in C \\
z_n &= \operatorname{Res}_g^f(x_n) \\
y_n &= \nabla f^*\left(\beta_n \nabla f^*(x_n) + (1 - \beta_n) \nabla f^*(T_n(z_n))\right) \\
x_{n+1} &= \nabla f^*\left(\alpha_n \nabla f^*(x_n) + (1 - \alpha_n) \nabla f^*(T_n(y_n))\right),
\end{aligned} \tag{10}$$

where T_n, $n \in \mathbb{N}$, is a Bregman strongly nonexpansive mapping. They proved that the sequence $\{x_n\}$ which is generated by algorithm (10) converges strongly to the point $P_\Omega^f x$, where $\Omega := F(T) \cap \mathrm{EP}(g)$.

Motivated and inspired by the above works, in this paper, we prove a new strong convergence theorem for finite family of quasi-Bregman nonexpansive mapping and system of equilibrium problem in a real Banach space.

2. Preliminaries

Let E be a real reflexive Banach space with the norm $\|\cdot\|$ and E^* the dual space of E. Throughout this paper, we will assume $f : E \to (-\infty, +\infty]$ is a proper, lower semicontinuous, and convex function. We denote by $\operatorname{dom} f := \{x \in E : f(x) < +\infty\}$ the domain of f.

Let $x \in \operatorname{int}\operatorname{dom} f$; the subdifferential of f at x is the convex set defined by

$$\partial f(x) = \left\{x^* \in E^* : f(x) + \langle x^*, y - x \rangle \leq f(y),\ \forall y \in E\right\}, \tag{11}$$

where the Fenchel conjugate of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}. \tag{12}$$

We know that the Young-Fenchel inequality holds:

$$\langle x^*, x\rangle \le f(x) + f^*(x^*), \quad \forall x \in E,\ x^* \in E^*. \tag{13}$$

A function f on E is coercive [17] if the sublevel set of f is bounded; equivalently,

$$\lim_{\|x\|\to+\infty} f(x) = +\infty. \tag{14}$$

A function f on E is said be strongly coercive [18] if

$$\lim_{\|x\|\to+\infty} \frac{f(x)}{\|x\|} = +\infty. \tag{15}$$

For any $x \in \operatorname{int}\operatorname{dom} f$ and $y \in E$, the right-hand derivative of f at x in the direction y is defined by

$$f^\circ(x, y) \coloneqq \lim_{t\to 0^+} \frac{f(x+ty) - f(x)}{t}. \tag{16}$$

The function f is said to be Gâteaux differentiable at x if $\lim_{t\to 0^+}((f(x+ty) - f(x))/t)$ exists for any y. In this case, $f^\circ(x, y)$ coincides with $\nabla f(x)$, the value of the gradient ∇f of f at x. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable for any $x \in \operatorname{int}\operatorname{dom} f$. The function f is said to be Fréchet differentiable at x if this limit is attained uniformly in $\|y\| = 1$. Finally, f is said to be uniformly Fréchet differentiable on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$. It is known that if f is Gâteaux differentiable (resp., Fréchet differentiable) on $\operatorname{int}\operatorname{dom} f$, then f is continuous and its Gâteaux derivative ∇f is norm-to-weak* continuous (resp., continuous) on int dom f (see also [19, 20]). We will need the following results.

Lemma 1 (see [21]). *If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Definition 2 (see [22]). The function f is said to be

(i) essentially smooth, if ∂f is both locally bounded and single-valued on its domain,

(ii) essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of dom ∂f,

(iii) Legendre, if it is both essentially smooth and essentially strictly convex.

Remark 3. Let E be a reflexive Banach space. Then we have the following:

(i) f is essentially smooth if and only if f^* is essentially strictly convex (see [22], Theorem 5.4).

(ii) $(\partial f)^{-1} = \partial f^*$ (see [20]).

(iii) f is Legendre if and only if f^* is Legendre (see [22], Corollary 5.5).

(iv) If f is Legendre, then ∇f is a bijection satisfying $\nabla f = (\nabla f^*)^{-1}$, $\operatorname{ran}\nabla f = \operatorname{dom}\nabla f^* = \operatorname{int}\operatorname{dom} f^*$, and $\operatorname{ran}\nabla f^* = \operatorname{dom} f = \operatorname{int}\operatorname{dom} f$ (see [22], Theorem 5.10).

The following result was proved in [23] (see also [24]).

Lemma 4. *Let E be a Banach space, let $r > 0$ be a constant, let ρ_r be the gauge of uniform convexity of g, and let $g : E \to \mathbb{R}$ be a convex function which is uniformly convex on bounded subsets of E. Then,*

(i) *for any $x, y \in B_r$ and $\alpha \in (0, 1)$,*

$$\begin{aligned} g(\alpha x + (1-\alpha) y) \le{}& \alpha g(x) + (1-\alpha) g(y) \\ &- \alpha(1-\alpha)\rho_r(\|x - y\|), \end{aligned} \tag{17}$$

(ii) *for any $x, y \in B_r$,*

$$\rho_r(\|x - y\|) \le D_g(x, y), \tag{18}$$

(iii) *if, in addition, g is bounded on bounded subsets and uniformly convex on bounded subsets of E then, for any $x \in E$, $y^*, z^* \in B_r$, and $\alpha \in (0, 1)$,*

$$\begin{aligned} &V_g(x, \alpha y^* + (1-\alpha) z^*) \\ &\quad\le \alpha V_g(x, y^*) + (1-\alpha) V_g(x, z^*) \\ &\qquad - \alpha(1-\alpha)\rho_r^*(\|y^* - x^*\|). \end{aligned} \tag{19}$$

Lemma 5 (see [25]). *Let E be a Banach space, let $r > 0$ be a constant, and let $f : E \to \mathbb{R}$ be a continuous and convex function which is uniformly convex on bounded subsets of E. Then*

$$f\left(\sum_{k=0}^{\infty} \alpha_k x_k\right) \le \sum_{k=0}^{\infty} \alpha_k f(x_k) - \alpha_i \alpha_j \rho_r(\|x_i - x_j\|), \tag{20}$$

for all $i, j \in \mathbb{N} \cup \{0\}$, $x_k \in B_r$, $\alpha_k \in (0, 1)$, and $k \in \mathbb{N} \cup \{0\}$ with $\sum_{k=0}^{\infty} \alpha_k = 1$, where ρ_r is the gauge of uniform convexity of f.

We know the following two results; see [18].

Theorem 6. *Let E be a reflexive Banach space and let $f : E \to \mathbb{R}$ be a convex function which is bounded on bounded subsets of E. Then the following assertions are equivalent:*

(1) *f is strongly coercive and uniformly convex on bounded subsets of E.*

(2) dom $f^* = E^*$, *f^* is bounded on bounded subsets and uniformly smooth on bounded subsets of E^*.*

(3) dom $f^* = E^*$, *f^* is Fréchet differentiable and ∇f is uniformly norm-to-norm continuous on bounded subsets of E^*.*

Theorem 7. *Let E be a reflexive Banach space and let $f : E \to \mathbb{R}$ be a continuous convex function which is strongly coercive. Then the following assertions are equivalent:*

(1) *f is bounded on bounded subsets and uniformly smooth on bounded subsets of E.*

(2) *f^* is Fréchet differentiable and f^* is uniformly norm-to-norm continuous on bounded subsets of E^*.*

(3) *$\operatorname{dom} f^* = E^*$, f^* is strongly coercive and uniformly convex on bounded subsets of E^*.*

The following result was first proved in [26] (see also [27]).

Lemma 8. *Let E be a reflexive Banach space, let $f : E \to \mathbb{R}$ be a strongly coercive Bregman function, and let V be the function defined by*

$$V(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*), \quad x \in E,\ x^* \in E^*. \tag{21}$$

Then the following assertions hold:

(1) *$D_f(x, \nabla f(x^*)) = V(x, x^*)$ for all $x \in E$ and $x^* \in E^*$.*

(2) *$V(x, x^*) + \langle \nabla f^*(x^*) - x, y^* \rangle \leq V(x, x^* + y^*)$ for all $x \in E$ and $x^*, y^* \in E^*$.*

Examples of Legendre functions were given in [22, 28]. One important and interesting Legendre function is $(1/p)\|\cdot\|^p$ $(1 < p < \infty)$ when E is a smooth and strictly convex Banach space. In this case the gradient ∇f of f is coincident with the generalized duality mapping of E; that is, $\nabla f = J_p$ $(1 < p < \infty)$. In particular, $\nabla f = I$, the identity mapping in Hilbert spaces. In the rest of this paper, we always assume that $f : E \to (-\infty, +\infty]$ is Legendre.

Concerning the Bregman projection, the following are well known.

Lemma 9 (see [26]). *Let C be a nonempty, closed, and convex subset of a reflexive Banach space E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. Then*

(a) *$z = P_C^f(x)$ if and only if $\langle \nabla f(x) - \nabla f(z), y - z \rangle \leq 0$, $\forall y \in C$.*

(b) *$D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x)$, $\forall x \in E$, $y \in C$.*

Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The modulus of total convexity of f at $x \in \operatorname{int}\operatorname{dom} f$ is the function $\nu_f(x, \cdot) : [0, +\infty) \to [0, +\infty]$ defined by

$$\nu_f(x, t) := \inf \left\{ D_f(y, x) : y \in \operatorname{dom} f,\ \|y - x\| = t \right\}, \tag{22}$$

The function f is called totally convex at x if $\nu_f(x, t) > 0$ whenever $t > 0$. The function f is called totally convex if it is totally convex at any point $x \in \operatorname{int}\operatorname{dom} f$ and is said to be totally convex on bounded sets if $\nu_f(B, t) > 0$ for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $\nu_f : \operatorname{int}\operatorname{dom} f \times [0, +\infty) \to [0, +\infty]$ defined by

$$\nu_f(B, t) := \inf \left\{ \nu_f(x, t) : x \in B \cap \operatorname{dom} f \right\}. \tag{23}$$

Lemma 10 (see [29]). *If $x \in \operatorname{dom} f$, then the following statements are equivalent:*

(i) *The function f is totally convex at x.*

(ii) *For any sequence $\{y_n\} \subset \operatorname{dom} f$,*

$$\lim_{n \to +\infty} D_f(y_n, x) = 0 \Longrightarrow \lim_{n \to +\infty} \|y_n - x\| = 0. \tag{24}$$

Recall that the function f is called sequentially consistent [26] if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first one is bounded

$$\lim_{n \to +\infty} D_f(y_n, x_n) = 0 \Longrightarrow \lim_{n \to +\infty} \|y_n - x_n\| = 0. \tag{25}$$

Lemma 11 (see [30]). *The function f is totally convex on bounded sets if and only if the function f is sequentially consistent.*

Lemma 12 (see [31]). *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is bounded too.*

Lemma 13 (see [31]). *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function, $x_0 \in E$, and let C be a nonempty, closed, and convex subset of E. Suppose that the sequence $\{x_n\}$ is bounded and any weak subsequential limit of $\{x_n\}$ belongs to C. If $D_f(x_n, x_0) \leq D_f(P_C^f(x_0), x_0)$ for any $n \in \mathbb{R}$, then $\{x_n\}$ converges strongly to $P_C^f(x_0)$.*

Lemma 14 (see [32]). *Let E be a real reflexive Banach space, let $f : E \to (-\infty, +\infty]$ be a proper lower semicontinuous function, and then $f^* : E^* \to (-\infty, +\infty]$ is a proper weak* lower semicontinuous and convex function. Thus, for all $z \in E$, one has*

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^{N} t_i \nabla f(x_i)\right)\right) \leq \sum_{i=1}^{N} t_i D_f(z, x_i). \tag{26}$$

In order to solve the equilibrium problem, let us assume that a bifunction $g : C \times C \to \mathbb{R}$ satisfies the following conditions [1]:

(A1) $g(x, x) = 0$, $\forall x \in C$.

(A2) g is monotone; that is, $g(x, y) + g(y, x) \leq 0$, $\forall x, y \in C$.

(A3) $\limsup_{t \downarrow 0} g(x + t(z - x), y) \leq g(x, y)$ $\forall x, z, y \in C$.

(A4) The function $y \mapsto g(x, y)$ is convex and lower semicontinuous.

The resolvent of a bifunction g [2] is the operator $\mathrm{Res}_g^f : E \to 2^C$ defined by

$$\mathrm{Res}_g^f(x) = \{z \in C : g(z, y) + \langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0,\ \forall y \in C\}. \tag{27}$$

From Lemma 1, in [33], if $f : (-\infty, +\infty]$ is a strongly coercive and Gâteaux differentiable function and g satisfies conditions (A1)–(A4), then $\mathrm{dom}(\mathrm{Res}_g^f) = E$. The following lemma gives some characterization of the resolvent Res_g^f.

Lemma 15 (see [33]). *Let E be a real reflexive Banach space and let C be a nonempty closed convex subset of E. Let $f : E \to (-\infty, +\infty]$ be a Legendre function. If the bifunction $g : C \times C \to \mathbb{R}$ satisfies the conditions (A1)–(A4), then, the following hold:*

(i) *Res_g^f is single-valued.*

(ii) *Res_g^f is a Bregman firmly nonexpansive operator.*

(iii) *$F(Res_g^f) = \mathrm{EP}(g)$.*

(iv) *$\mathrm{EP}(g)$ is closed and convex subset of C.*

(v) *For all $x \in E$ and for all $q \in F(Res_g^f)$, one has*

$$D_f\left(q, Res_g^f(x)\right) + D_f\left(Res_g^f(x), x\right) \leq D_f(q, x). \tag{28}$$

Lemma 16 (see [34]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n) a_n + \alpha_n \delta_n, \quad n \geq n_0, \tag{29}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\delta_n\}$ is a real sequence satisfying the following conditions:

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} = \infty, \quad \text{as } \limsup_{n\to\infty} \delta_n \leq 0. \tag{30}$$

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 17 (see [35]). *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1}, \quad a_k \leq a_{m_k+1}. \tag{31}$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

3. Main Results

We now prove the following theorem.

Theorem 18. *Let C be a nonempty, closed, and convex subset of a real reflexive Banach space E and $f : E \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subset of E. For each $j = 1, 2, \ldots, m$, let g_j be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) and let $\{T_{i=1}^N\}$ be a finite family of quasi-Bregman nonexpansive self-mapping of C such that $F := \bigcap_{i=1}^N F(T_i) \neq \emptyset$, where $F = F(T_N T_{N-1} T_{N-2} \cdots T_2 T_1) = F(T_1 T_N T_{N-1} T_{N-2} \cdots T_2) = \cdots = F(T_{N-1} T_{N-2} \cdots T_2 T_1 T_N) \neq \emptyset$ and $\Omega := (\bigcap_{j=1}^m \mathrm{EP}(g_j)) \cap F \neq \emptyset$. Let $\{x_n\}_{n=1}^\infty$ be a sequence generated by $x_1 = x \in C$, $C_1 = C$, and*

$$\begin{aligned} & x_1 \in C \\ & u_{j,n} = Res_{g_j}^f x_n, \quad j = 1, 2, 3, \ldots, m \\ & y_n = P_C\left(\nabla f^*\left((1 - \alpha_n) \nabla f\left(u_{j,n}\right)\right)\right) \\ & x_{n+1} = P_C\left(\nabla f^*\left(\beta_n \nabla f(y_n) + (1 - \beta_n) \nabla f\left(T_{[n]} y_n\right)\right)\right), \end{aligned} \tag{32}$$

where $T_{[n]} = T_{n (\mathrm{mod}\, N)}$ and $\{\alpha_n\}_{n=1}^\infty \subset (0,1)$ and $\{\beta_n\}_{n=1}^\infty \subset [c,d] \subset (0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$. Then $\{x_n\}_{n=1}^\infty$ converges strongly to $P_\Omega^f(x)$, where P_Ω^f is the Bregman projection of C onto Ω.

Proof. Let $p = P_\Omega^f \in \Omega$ from Lemma 15; we obtain

$$D_f\left(p, u_{j,n}\right) = D_f\left(p, \mathrm{Res}_{g_j}^f x_n\right) \leq D_f(p, x_n). \tag{33}$$

Now from (32), we obtain

$$\begin{aligned} D_f(p, y_n) &\leq D_f\left(p, \nabla f^*\left((1 - \alpha_n) \nabla f\left(u_{j,n}\right)\right)\right) \\ &= D_f\left(p, \nabla f^*\left(\alpha_n \nabla f(0) + (1 - \alpha_n) \nabla f\left(u_{j,n}\right)\right)\right) \\ &\leq \alpha_n D_f(p, 0) + (1 - \alpha_n) D_f\left(p, u_{j,n}\right) \\ &\leq \alpha_n D_f(p, 0) + (1 - \alpha_n) D_f(p, x_n). \end{aligned} \tag{34}$$

Also from (32), (26), and (34), we have

$$\begin{aligned} & D_f(p, x_{n+1}) \\ &\quad \leq D_f\left(p, \nabla f^*\left((1 - \beta_n) \nabla f(y_n) + \beta_n \nabla f\left(T_{[n]} y_n\right)\right)\right) \\ &\quad \leq (1 - \beta_n) D_f(p, y_n) + \beta_n D_f\left(p, T_{[n]} y_n\right) \\ &\quad \leq (1 - \beta_n) D_f(p, y_n) + \beta_n D_f(p, y_n) \\ &\quad = D_f(p, y_n) \leq \alpha_n D_f(p, 0) + (1 - \alpha_n) D_f(p, x_n) \\ &\quad \leq \max\left\{D_f(p, 0), D_f(p, x_n)\right\}. \end{aligned} \tag{35}$$

Thus, by induction we obtain

$$D_f(p, x_{n+1}) \leq \max\left\{D_f(p, 0), D_f(p, x_n)\right\}, \quad \forall n \geq 0, \tag{36}$$

which implies that $\{x_n\}$ is bounded and hence $\{y_n\}$, $\{T_{[n]}y_n\}$, $\{T_{[n]}x_n\}$, and $\{u_{j,n}\}$ are all bounded for each $j = 1, 2, \ldots, m$. Now from (32) let $z_n := \nabla f^*((1-\alpha_n)\nabla f(u_{j,n}))$. Furthermore since $\alpha_n \to 0$ as $n \to \infty$, we obtain

$$\left\|\nabla f(z_n) - \nabla f(u_{j,n})\right\| = \alpha_n \left\|(-\nabla f(u_{j,n}))\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{37}$$

Since f is strongly coercive and uniformly convex on bounded subsets of E, f^* is uniformly Fréchet differentiable on bounded sets. Moreover, f^* is bounded on bounded sets; from (37), we obtain

$$\lim_{n\to\infty} \left\|z_n - u_{j,n}\right\| = 0. \tag{38}$$

On the other hand, in view of (3) in Theorem 6, we know that $\operatorname{dom} f^* = E^*$ and f^* is strongly coercive and uniformly convex on bounded subsets. Let $s = \sup\{\|\nabla f(y_n)\|, \|\nabla f(T_{[n]}y_n)\|\}$ and $\rho_s^* : E^* \to \mathbb{R}$ be the gauge of uniform convexity of the conjugate function f^*. Now from (32) and Lemmas 4 and 8, we obtain

$$\begin{aligned}
D_f(p, y_n) &\le D_f(p, z_n) = V(p, \nabla f(z_n)) \\
&\le V(p, \nabla f(z_n) + \alpha_n \nabla f(p)) \\
&\quad + \alpha_n \langle -\nabla f(p), z_n - p\rangle \\
&= D_f\left(p, \nabla f^*\left((1-\alpha_n)\nabla f\left(u_{j,n}\right) + \alpha_n \nabla f(p)\right)\right) \\
&\quad + \alpha_n \nabla f(p) + \alpha_n \langle -\nabla f(p), z_n - p\rangle \\
&\le \alpha_n D_f(p, p) + (1-\alpha_n) D_f\left(p, u_{j,n}\right) \\
&\quad + \alpha_n \langle -\nabla f(p), z_n - p\rangle \\
&\le (1-\alpha_n) D_f(p, x_n) + \alpha_n \langle -\nabla f(p), z_n - p\rangle,
\end{aligned} \tag{39}$$

$$\begin{aligned}
&D_f(p, x_{n+1}) \\
&\quad \le D_f(p, \nabla f^*((1-\beta_n)\nabla f(y_n) + \beta_n \nabla f(T_{[n]}y_n))) \\
&\quad = V(p, (1-\beta_n)\nabla f(y_n) + \beta_n \nabla f(T_{[n]}y_n)) \\
&\quad = f(p) - \langle p, (1-\beta_n)\nabla f(y_n) + \beta_n \nabla f(T_{[n]}y_n)\rangle \\
&\qquad + f^*((1-\beta_n)\nabla f(y_n) + \beta_n \nabla f(T_{[n]}y_n)) \\
&\quad \le (1-\beta_n) f(p) + \beta_n f(p) \\
&\qquad - (1-\beta_n)\langle p, \nabla f(y_n)\rangle - \beta_n \langle p, \nabla f(T_{[n]}y_n)\rangle \\
&\qquad + (1-\beta_n) f^*(\nabla f(T_{[n]}y_n)) \\
&\qquad + \beta_n f^*(\nabla f(T_{[n]}y_n)) \\
&\qquad - \beta_n(1-\beta_n)\rho_s^*(\|\nabla f(y_n) - \nabla f(T_{[n]}y_n)\|) \\
&\quad = (1-\beta_n) V(p, \nabla f(y_n)) + \beta_n V(p, \nabla f(T_{[n]}y_n)) \\
&\qquad - \beta_n(1-\beta_n)\rho_s^*(\|\nabla f(y_n) - \nabla f(T_{[n]}y_n)\|) \\
&\quad = (1-\beta_n) D_f(p, y_n) + \beta_n D_f(p, T_{[n]}y_n) \\
&\qquad - \beta_n(1-\beta_n)\rho_s^*(\|\nabla f(y_n) - \nabla f(T_{[n]}y_n)\|) \\
&\quad \le (1-\beta_n) D_f(p, y_n) + \beta_n D_f(p, y_n) \\
&\qquad - \beta_n(1-\beta_n)\rho_s^*(\|\nabla f(y_n) - \nabla f(T_{[n]}y_n)\|) \\
&\quad = D_f(p, y_n) \\
&\qquad - \beta_n(1-\beta_n)\rho_s^*(\|\nabla f(y_n) - \nabla f(T_{[n]}y_n)\|) \\
&\quad \le (1-\alpha_n) D_f(p, x_n) + \alpha_n \langle -\nabla f(p), z_n - p\rangle \\
&\qquad - \beta_n(1-\beta_n)\rho_s^*(\|\nabla f(y_n) - \nabla f(T_{[n]}y_n)\|)
\end{aligned} \tag{40}$$

$$\le (1-\alpha_n) D_f(p, x_n) + \alpha_n \langle -\nabla f(p), z_n - p\rangle. \tag{41}$$

Now, we consider two cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{D_f(p, x_n)\}$ is nonincreasing. In this situation $\{D_f(p, x_n)\}$ is convergent. Then from (40) we obtain

$$\beta_n(1-\beta_n)\rho_s^*(\|\nabla f(y_n) - \nabla f(T_{[n]}y_n)\|) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{42}$$

which implies, by the property of ρ_s and since $\beta_n \in [c, d] \subset (0, 1)$,

$$\lim_{n\to\infty} \|\nabla f(y_n) - \nabla f(T_{[n]}y_n)\| = 0. \tag{43}$$

Since f is strongly coercive and uniformly convex on bounded subsets of E, f^* is uniformly Fréchet differentiable on bounded sets. Moreover, f^* is bounded on bounded sets; from (43), we obtain

$$\lim_{n\to\infty} \|y_n - T_{[n]}y_n\| = 0. \tag{44}$$

Now from (4), we obtain

$$\begin{aligned}
&D_f(y_n, T_{[n]}y_n) \\
&\quad = D_f(p, T_{[n]}y_n) - D_f(p, y_n) \\
&\qquad + \langle \nabla f(T_{[n]}y_n) - \nabla f(y_n), p - y_n\rangle \\
&\quad \le D_f(p, y_n) - D_f(p, y_n) \\
&\qquad + \langle \nabla f(T_{[n]}y_n) - \nabla f(y_n), p - y_n\rangle,
\end{aligned} \tag{45}$$

and therefore

$$\begin{aligned}
&D_f(y_n, T_{[n]}y_n) \\
&\quad \le \|\nabla f(y_n) - \nabla f(T_{[n]}y_n)\| \|p - y_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{46}$$

Also, from (28) in Lemma 15, we have

$$\begin{aligned}D_f\left(x_n,u_{j,n}\right)&=D_f\left(x_n,\operatorname{Res}_{g_j}^f x_n\right)\\&\leq D_f\left(p,\operatorname{Res}_{g_j}^f x_n\right)-D_f\left(p,x_n\right)\\&\leq D_f\left(p,x_n\right)-D_f\left(p,x_n\right)\longrightarrow 0\\&\qquad\text{as } n\longrightarrow\infty.\end{aligned}\tag{47}$$

Then, we have from Lemma 10 that

$$\lim_{n\to\infty}\left\|x_n-u_{j,n}\right\|=0.\tag{48}$$

Also, from (b) of Lemma 9, we have

$$\begin{aligned}D_f\left(y_n,P_C z_n\right)&=D_f\left(y_n,z_n\right)\\&=D_f\left(y_n,\nabla f^*\left(\nabla f(0)+\left(1-\alpha_n\right)\nabla f\left(u_{j,n}\right)\right)\right)\\&\leq\alpha_n D_f\left(y_n,0\right)+\left(1-\alpha_n\right)D_f\left(y_n,u_{j,n}\right)\\&\leq\alpha_n D_f\left(y_n,0\right)+\left(1-\alpha_n\right)D_f\left(u_{j,n},u_{j,n}\right)\longrightarrow 0\\&\qquad\text{as } n\longrightarrow\infty.\end{aligned}\tag{49}$$

Then, we have from Lemma 10 that

$$\lim_{n\to\infty}\left\|y_n-z_n\right\|=0.\tag{50}$$

From (38) and (48), we obtain

$$\lim_{n\to\infty}\left\|x_n-z_n\right\|=0.\tag{51}$$

From (50) and (51), we obtain

$$\lim_{n\to\infty}\left\|x_n-y_n\right\|=0.\tag{52}$$

Since f is strongly coercive and uniformly convex on bounded subsets of E, f^* is uniformly Fréchet differentiable on bounded sets. Moreover, f^* is bounded on bounded sets; from (52), we obtain

$$\lim_{n\to\infty}\left\|\nabla f\left(x_n\right)-\nabla f\left(z_n\right)\right\|=0.\tag{53}$$

Also from (44) and (52)

$$\lim_{n\to\infty}\left\|x_n-T_{[n]}y_n\right\|=0.\tag{54}$$

Now from (4) and (34), we obtain

$$\begin{aligned}D_f\left(x_n,y_n\right)&=D_f\left(p,y_n\right)-D_f\left(p,x_n\right)\\&\quad+\left\langle\nabla f\left(x_n\right)-\nabla f\left(y_n\right),p-x_n\right\rangle\\&\leq\alpha_n D_f\left(p,0\right)+\left(1-\alpha_n\right)D_f\left(p,x_n\right)\\&\quad-D_f\left(p,x_n\right)\\&\quad+\left\langle\nabla f\left(x_n\right)-\nabla f\left(y_n\right),p-x_n\right\rangle\\&=\alpha_n\left(D_f\left(p,0\right)-D_f\left(p,x_n\right)\right)\\&\quad+\left\langle\nabla f\left(x_n\right)-\nabla f\left(y_n\right),p-x_n\right\rangle;\end{aligned}\tag{55}$$

therefore, from (53), we obtain

$$\begin{aligned}D_f\left(x_n,y_n\right)&\leq\alpha_n\left(D_f\left(p,0\right)-D_f\left(p,x_n\right)\right)\\&\quad+\left\|\nabla f\left(x_n\right)-\nabla f\left(y_n\right)\right\|\left\|p-x_n\right\|\longrightarrow 0\\&\qquad\text{as } n\longrightarrow\infty.\end{aligned}\tag{56}$$

Also

$$\begin{aligned}&D_f\left(x_n,T_{[n]}y_n\right)\\&\quad=D_f\left(p,T_{[n]}y_n\right)-D_f\left(p,x_n\right)\\&\qquad+\left\langle\nabla f\left(x_n\right)-\nabla f\left(T_{[n]}y_n\right),p-x_n\right\rangle\\&\quad\leq D_f\left(p,y_n\right)-D_f\left(p,x_n\right)\\&\qquad+\left\langle\nabla f\left(x_n\right)-\nabla f\left(T_{[n]}y_n\right),p-x_n\right\rangle\\&\quad\leq\alpha_n D_f\left(p,0\right)+\left(1-\alpha_n\right)-D_f\left(p,x_n\right)\\&\qquad+\left\langle\nabla f\left(x_n\right)-\nabla f\left(T_{[n]}y_n\right),p-x_n\right\rangle\\&\quad=\alpha_n\left(D_f\left(p,0\right)-D_f\left(p,x_n\right)\right)\\&\qquad+\left\langle\nabla f\left(x_n\right)-\nabla f\left(T_{[n]}y_n\right),p-x_n\right\rangle;\end{aligned}\tag{57}$$

thus

$$\begin{aligned}&D_f\left(x_n,T_{[n]}y_n\right)\\&\quad\leq\alpha_n\left|D_f\left(p,0\right)-D_f\left(p,x_n\right)\right|\\&\qquad+\left\|\nabla f\left(T_{[n]}y_n\right)-\nabla f\left(x_n\right)\right\|\left\|p-x_n\right\|\longrightarrow 0\\&\qquad\text{as } n\longrightarrow\infty.\end{aligned}\tag{58}$$

Also, from (56)

$$\begin{aligned}D_f\left(T_{[n]}x_n,T_{[n]}y_n\right)&\leq D_f\left(x_n,y_n\right)\longrightarrow 0\\&\qquad\text{as } n\longrightarrow\infty.\end{aligned}\tag{59}$$

Then, we have from Lemma 10 that

$$\lim_{n\to\infty}\left\|T_{[n]}x_n-T_{[n]}y_n\right\|=0.\tag{60}$$

Then from (32) and (44), we have

$$\begin{aligned}&\left\|\nabla f\left(x_{n+1}\right)-\nabla f\left(y_n\right)\right\|\\&\quad=\beta_n\left\|\nabla f\left(T_{[n]}y_n\right)-\nabla f\left(y_n\right)\right\|\longrightarrow 0\quad\text{as } n\longrightarrow\infty.\end{aligned}\tag{61}$$

This implies

$$\left\|x_{n+1}-y_n\right\|\longrightarrow 0\quad\text{as } n\longrightarrow\infty,\tag{62}$$

$$\begin{aligned}\left\|x_n-T_{[n]}x_n\right\|&\leq\left\|x_n-y_n\right\|+\left\|y_n-T_{[n]}y_n\right\|\\&\quad+\left\|T_{[n]}y_n-T_{[n]}x_n\right\|;\end{aligned}\tag{63}$$

from (44), (52), and (60), we obtain

$$\lim_{n\to\infty}\left\|x_n-T_{[n]}x_n\right\|=0.\tag{64}$$

This implies that

$$\lim_{n\to\infty} \left\|\nabla f\left(x_n\right) - \nabla f\left(T_{[n]}x_n\right)\right\| = 0. \tag{65}$$

Also from (52) and (62), we obtain

$$\left\|x_{n+1} - x_n\right\| \le \left\|x_{n+1} - y_n\right\| + \left\|y_n - x_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{66}$$

But

$$\begin{aligned}\left\|x_{n+N} - x_n\right\| &\le \left\|x_{n+N} - x_{n+N-1}\right\| + \left\|x_{n+N-1} - x_{n+N-2}\right\| \\ &\quad + \cdots + \left\|x_{n+1} - x_n\right\| \longrightarrow 0,\end{aligned} \tag{67}$$

as $n \to \infty$. Hence

$$\lim_{n\to\infty} \left\|x_{n+N} - x_n\right\| = 0. \tag{68}$$

From the uniformly continuous ∇f, we have from (66) that

$$\lim_{n\to\infty} \left\|\nabla f\left(x_{n+1}\right) - \nabla f\left(x_n\right)\right\| = 0. \tag{69}$$

From (4), (35), and (69), we obtain

$$\begin{aligned}D_f\left(x_n, x_{n+1}\right) &= D_f\left(p, x_{n+1}\right) - D_f\left(p, x_n\right) \\ &\quad + \left\langle \nabla f\left(x_n\right) - \nabla f\left(x_{n+}\right), p - x_n\right\rangle \\ &\le \alpha_n D_f\left(p, 0\right) + \left(1 - \alpha_n\right) D_f\left(p, x_n\right) \\ &\quad - D_f\left(p, x_n\right) \\ &\quad + \left\langle \nabla f\left(x_n\right) - \nabla f\left(x_{n+1}\right), p - x_n\right\rangle,\end{aligned} \tag{70}$$

which implies

$$D_f\left(x_n, x_{n+1}\right) \le \alpha_n \left|D_f\left(p, 0\right) - D_f\left(p, x_n\right)\right| + \left\|\nabla f\left(x_{n+1}\right) - \nabla f\left(x_n\right)\right\| \left\|p - x_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{71}$$

Also from quasi-Bregman nonexpansivity of $T_{[n]}$, we have

$$D_f\left(T_{[n]}x_n, T_{[n]}x_{n+1}\right) \le D_f\left(x_n, x_{n+1}\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{72}$$

which implies

$$\lim_{n\to\infty} \left\|T_{[n]}x_n - T_{[n]}x_{n+1}\right\| = 0, \tag{73}$$

and from the uniform continuous ∇f, we obtain

$$\lim_{n\to\infty} \left\|\nabla f\left(T_{[n]}x_n\right) - \nabla f\left(T_{[n]}x_{n+1}\right)\right\| = 0. \tag{74}$$

Also from (4) and (64), we obtain

$$\begin{aligned}&D_f\left(x_n, T_{[n]}x_n\right) \\ &\quad = D_f\left(p, T_{[n]}x_n\right) - D_f\left(p, x_n\right) \\ &\qquad + \left\langle \nabla f\left(x_n\right) - \nabla f\left(T_{[n]}x_n\right), p - x_n\right\rangle \\ &\quad \le D_f\left(p, x_n\right) - D_f\left(p, x_n\right) \\ &\qquad + \left\langle \nabla f\left(x_n\right) - \nabla f\left(T_{[n]}x_n\right), p - x_n\right\rangle \\ &\quad \le \left\|\nabla f\left(T_{[n]}x_n\right) - \nabla f\left(x_n\right)\right\| \left\|p - x_n\right\| \longrightarrow 0 \\ &\qquad \text{as } n \longrightarrow \infty.\end{aligned} \tag{75}$$

From (64), (66), and (73), we obtain

$$\begin{aligned}\left\|x_n - T_{[n+1]}x_n\right\| &\le \left\|x_n - x_{n+1}\right\| + \left\|x_{n+1} - T_{[n+1]}x_{n+1}\right\| \\ &\quad + \left\|T_{[n+1]}x_{n+1} - T_{[n+1]}x_n\right\| \longrightarrow 0 \\ &\quad \text{as } n \longrightarrow \infty,\end{aligned} \tag{76}$$

which from uniform continuous ∇f implies

$$\lim_{n\to\infty} \left\|\nabla f\left(T_{[n]}x_n\right) - \nabla f\left(T_{[n+1]}x_n\right)\right\| = 0, \tag{77}$$

and from (4) and (77), we obtain

$$\begin{aligned}&D_f\left(x_n, T_{[n+1]}x_n\right) \\ &\quad \le D_f\left(p, T_{[n+1]}x_n\right) - D_f\left(p, x_n\right) \\ &\qquad + \left\langle \nabla f\left(x_n\right) - \nabla f\left(T_{[n+1]}x_n\right), p - x_n\right\rangle \\ &\quad \le D_f\left(p, x_n\right) - D_f\left(p, x_n\right) \\ &\qquad + \left\|\nabla f\left(T_{[n+1]}x_n\right) - \nabla f\left(x_n\right)\right\| \left\|p - x_n\right\| \longrightarrow 0 \\ &\qquad \text{as } n \longrightarrow \infty.\end{aligned} \tag{78}$$

From (4), (71), (77), and (78)

$$\begin{aligned}&D_f\left(x_{n+1}, T_{[n+1]}x_n\right) \\ &\quad = D_f\left(x_{n+1}, x_n\right) + D_f\left(x_n, T_{[n+1]}x_n\right) \\ &\qquad + \left\langle \nabla f\left(T_{[n+1]}x_n\right) - \nabla f\left(x_n\right), x_n - x_{n+1}\right\rangle \\ &\quad \le D_f\left(x_{n+1}, x_n\right) + D_f\left(x_n, T_{[n+1]}x_n\right) \\ &\qquad + \left\|\nabla f\left(T_{[n+1]}x_n\right) - \nabla f\left(x_n\right)\right\| \left\|x_n - x_{n+1}\right\| \longrightarrow 0 \\ &\qquad \text{as } n \longrightarrow \infty.\end{aligned} \tag{79}$$

Also from (4), (71), and (79)

$$\begin{aligned} D_f\left(x_n, T_{[n+1]}x_n\right) &= D_f\left(x_n, x_{n+1}\right) + D_f\left(x_{n+1}, T_{[n+1]}x_n\right) + \left\langle \nabla f\left(x_{n+1}\right) - \nabla f\left(T_{[n+1]}x_{n+1}\right), x_{n+1} - x_n\right\rangle \\ &= D_f\left(x_n, x_{n+1}\right) + D_f\left(x_{n+1}, T_{[n+1]}x_n\right) + \left\|\nabla f\left(T_{[n+1]}x_n\right) - \nabla f\left(x_{n+1}\right)\right\| \left\|x_{n+1} - x_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{80}$$

Using the quasi-Bregman nonexpansivity of $T_{(i)}$ for each i, we obtain the following finite table:

$$\begin{gathered} D_f\left(x_{n+N}, T_{(n+N)}x_{n+N-1}\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \\ D_f\left(T_{(n+N)}x_{n+N-1}, T_{(n+N)}T_{(n+N-1)}x_{n+N-2}\right) \longrightarrow 0 \\ \text{as } n \longrightarrow \infty \\ \vdots \\ D_f\left(T_{(n+N)} \cdots T_{(n+2)}x_{n+1}, T_{(n+N)} \cdots T_{(n+1)}x_n\right) \longrightarrow 0 \\ \text{as } n \longrightarrow \infty. \end{gathered} \tag{81}$$

Then, applying Lemma 10 on each line above, we obtain

$$\begin{gathered} x_{n+N} - T_{(n+N)}x_{n+N-1} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \\ T_{(n+N)}x_{n+N-1} - T_{(n+N)}T_{(n+N-1)}x_{n+N-2} \longrightarrow 0 \\ \text{as } n \longrightarrow \infty \\ \vdots \\ T_{(n+N)} \cdots T_{(n+2)}x_{n+1} - T_{(n+N)} \cdots T_{(n+1)}x_n \longrightarrow 0 \\ \text{as } n \longrightarrow \infty, \end{gathered} \tag{82}$$

and adding up this table, we obtain

$$x_{n+N} - T_{(n+N)}T_{(n+N-1)} \cdots T_{(n+1)}x_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{83}$$

Using this and (68), we obtain

$$\lim_{n\to\infty} \left\|x_n - T_{(n+N)}T_{(n+N-1)} \cdots T_{(n+1)}x_n\right\| = 0. \tag{84}$$

Also from quasi-Bregman nonexpansivity of $T_{(i)}$, for each i, we have

$$\begin{aligned} &D_f\left(T_{(n+N)}T_{(n+N-1)} \cdots T_{(n+1)}x_n, T_{(n+N)}T_{(n+N-1)}\right. \\ &\quad \left.\cdots T_{(n+1)}y_n\right) \le D_f\left(x_n, y_n\right) \longrightarrow 0, \end{aligned} \tag{85}$$

as $n \to \infty$. Then, we have from Lemma 10 that

$$\begin{aligned} &T_{(n+N)}T_{(n+N-1)} \cdots T_{(n+1)}x_n \\ &\quad - T_{(n+N)}T_{(n+N-1)} \cdots T_{(n+1)}y_n \longrightarrow 0 \\ &\quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{86}$$

Since

$$\begin{aligned} \left\|y_n - T_{(n+N)}T_{(n+N-1)} \cdots T_{(n+1)}y_n\right\| &\le \left\|y_n - x_n\right\| + \left\|x_n\right. \\ &\quad \left.- T_{(n+N)}T_{(n+N-1)} \cdots + T_{(n+1)}x_n\right\| \\ &\quad + \left\|T_{(n+N)}T_{(n+N-1)} \cdots T_{(n+1)}x_n\right. \\ &\quad \left.- T_{(n+N)}T_{(n+N-1)} \cdots T_{(n+1)}y_n\right\|, \end{aligned} \tag{87}$$

then, from (52), (84), and (86), we obtain

$$\lim_{n\to\infty} \left\|y_n - T_{(n+N)}T_{(n+N-1)} \cdots T_{(n+1)}y_n\right\| = 0. \tag{88}$$

Following the argument from (85), (86), and (88) by replacing y_n with z_n and using (51), we obtain

$$\lim_{n\to\infty} \left\|z_n - T_{(n+N)}T_{(n+N-1)} \cdots T_{(n+1)}z_n\right\| = 0. \tag{89}$$

Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$. Since $\{x_n\}$ is bounded and E is reflexive, without loss of generality, we may assume that $x_{n_i} \rightharpoonup q$ for some $q \in F$ and since $x_n - z_n \to 0$ as $n \to \infty$, then $z_{n_i} \rightharpoonup q$. Since the pool of mappings of $T_{[n]}$ is finite, passing to a further subsequence if necessary, we may further assume that, for some $i \in \{1, 2, \ldots, N\}$, from (89), we get

$$z_{n_i} - T_{(i+N)} \cdots T_{(i+1)}z_{n_i} \longrightarrow 0 \quad \text{as } i \longrightarrow \infty, \tag{90}$$

and also

$$\begin{aligned} &\limsup_{n\to\infty} \left\langle -\nabla f\left(p\right), z_n - p\right\rangle \\ &\quad = \lim_{i\to\infty} \left\langle -\nabla f\left(p\right), z_{n_i} - p\right\rangle. \end{aligned} \tag{91}$$

Noticing that $u_{j,n} = \operatorname{Res}^f_{g_j}(x_n)$ for each $j = 1, 2, \ldots, m$, we obtain

$$g_j\left(u_{j,n}, y\right) + \left\langle y - u_{j,n}, \nabla f\left(u_{j,n}\right) - \nabla f\left(x_n\right)\right\rangle \ge 0, \quad \forall y \in C. \tag{92}$$

Hence

$$g_j\left(u_{j,n_i}, y\right) + \left\langle y - u_{j,n_i}, \nabla f\left(u_{j,n_i}\right) - \nabla f\left(x_{n_i}\right)\right\rangle \ge 0, \quad \forall y \in C. \tag{93}$$

From (A2), we note that, for each $j = 1, 2, \ldots, m$,

$$\begin{aligned} &\left\|y - u_{j,n}\right\| \frac{\left\|\nabla f\left(u_{j,n_i}\right) - \nabla f\left(x_{n_i}\right)\right\|}{r_{n_i}} \\ &\quad \ge \left\langle y - u_{j,n}, \nabla f\left(u_{j,n}\right) - \nabla f\left(x_n\right)\right\rangle \ge -g_j\left(u_{j,n_i}\right) \\ &\quad \ge g_j\left(y, u_{j,n_i}\right), \quad \forall y \in C. \end{aligned} \tag{94}$$

Taking the limit as $i \to \infty$ in above inequality and from (A4) and $u_{j,n_i} \rightharpoonup q$, we have $g_j(y,q) \leq 0$ for each $j = 1,2,\ldots,m$. For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)q$. Noticing that $y,q \in C$, we obtain $y_t \in C$, which yield that $g_j(y_t,q) \leq 0$. It follows from (A1) that

$$\begin{aligned} 0 &= g_j(y_t,y_t) \leq tg_j(y_t,y) + (1-t)\,g_j(y_t,q) \\ &\leq tg_j(y_t,y). \end{aligned} \tag{95}$$

That is, for each $j = 1,2,\ldots,m$, we have $g_j(y_t,y) \geq 0$.

Let $t \downarrow 0$; from (A3), we obtain $g_j(q,y) \geq 0$ for any $y \in C$, for each $j = 1,2,\ldots,m$. This implies that $q \in \bigcap_{j=1}^{m} \mathrm{EP}(g_j)$. Hence $q \in \Omega$. It follows from the definition of the Bregman projection that

$$\begin{aligned} \limsup_{n\to\infty} \langle -\nabla f(p), z_n - p\rangle &= \lim_{i\to\infty} \langle -\nabla f(p), z_{n_i} - p\rangle \\ &\leq \langle -\nabla f(p), q - p\rangle \leq 0. \end{aligned} \tag{96}$$

It follows from Lemma 16 and (41) that $D_f(p,x_n) \to 0$ as $n \to \infty$. Consequently, from Lemma 10, we obtain $x_n \to p$ as $n \to \infty$.

Case 2. Suppose $D_f(p,x_n)$ is not monotone decreasing sequences; then set $\Phi_n := D_f(p,x_n)$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \geq N_0$ for some sufficiently large N_0 by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Phi_k \leq \Phi_{k+1}\}. \tag{97}$$

Then by Lemma 17 $\tau(n)$ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $\Phi_{\tau(n)} \leq \Phi_{\tau(n)+1}$, for $n \geq N_0$. Then from (40) and the fact that $\alpha_{\tau(n)} \to 0$, we obtain that

$$\rho_s^*\left(\left\|\nabla f(y_{\tau(n)}) - \nabla f(T_{[\tau(n)]})\, y_{\tau(n)}\right\|\right) \longrightarrow 0 \quad \text{as } \tau(n) \longrightarrow \infty. \tag{98}$$

Following the same argument as in Case 1, we obtain

$$y_{\tau(n)} - T_{(i+N)} \cdots T_{(i+1)} y_{\tau(n)} \longrightarrow 0 \quad \text{as } \tau(n) \longrightarrow \infty, \tag{99}$$

and also we obtain

$$\limsup_{\tau(n)\to\infty} \langle -\nabla f(p), y_{\tau(n)} - p\rangle \leq 0. \tag{100}$$

Then from (41), we obtain that

$$\begin{aligned} 0 &\leq D_f(p, x_{\tau(n)+1}) - D_f(p, x_{\tau(n)}) \\ &\leq \alpha_{\tau(n)}\left(\langle -\nabla f(p), y_{\tau(n)} - p\rangle - D_f(p, x_{\tau(n)})\right). \end{aligned} \tag{101}$$

It follows from (101) and $\Phi_n \leq \Phi_{\tau(n)+1}, \alpha_{\tau(n)} > 0$ that

$$D_f(p, x_{\tau(n)}) \leq \langle -\nabla f(p), y_{\tau(n)} - p\rangle \longrightarrow 0, \tag{102}$$

as $\tau(n) \to \infty$. Thus

$$\lim_{\tau(n)\to\infty} \Phi_{\tau(n)} = \lim_{\tau(n)\to\infty} \Phi_{\tau(n)+1} = 0. \tag{103}$$

Furthermore, for $n \geq N_0$, if $n \neq \tau(n)$ (i.e., $\tau(n) < n$), because $\Phi_j > \Phi_{j+1}$ for $\tau(n)+1 \leq j \leq n$. it then follows that for all $n \geq N_0$ we have

$$0 \leq \Phi_n \leq \max\{\Phi_{\tau(n)}, \Phi_{\tau(n)+1}\} = \Phi_{\tau(n)+1}. \tag{104}$$

This implies that $\lim_{n\to\infty}\Phi_n = 0$, and hence $D_f(p,x_n) \to 0$ as $n \to \infty$. Consequently, from Lemma 10, we obtain $x_n \to p$ as $n \to \infty$. Therefore from the above two cases, we conclude that $\{x_n\}$ converges strongly to $p \in \Omega$ and this completes the proof. □

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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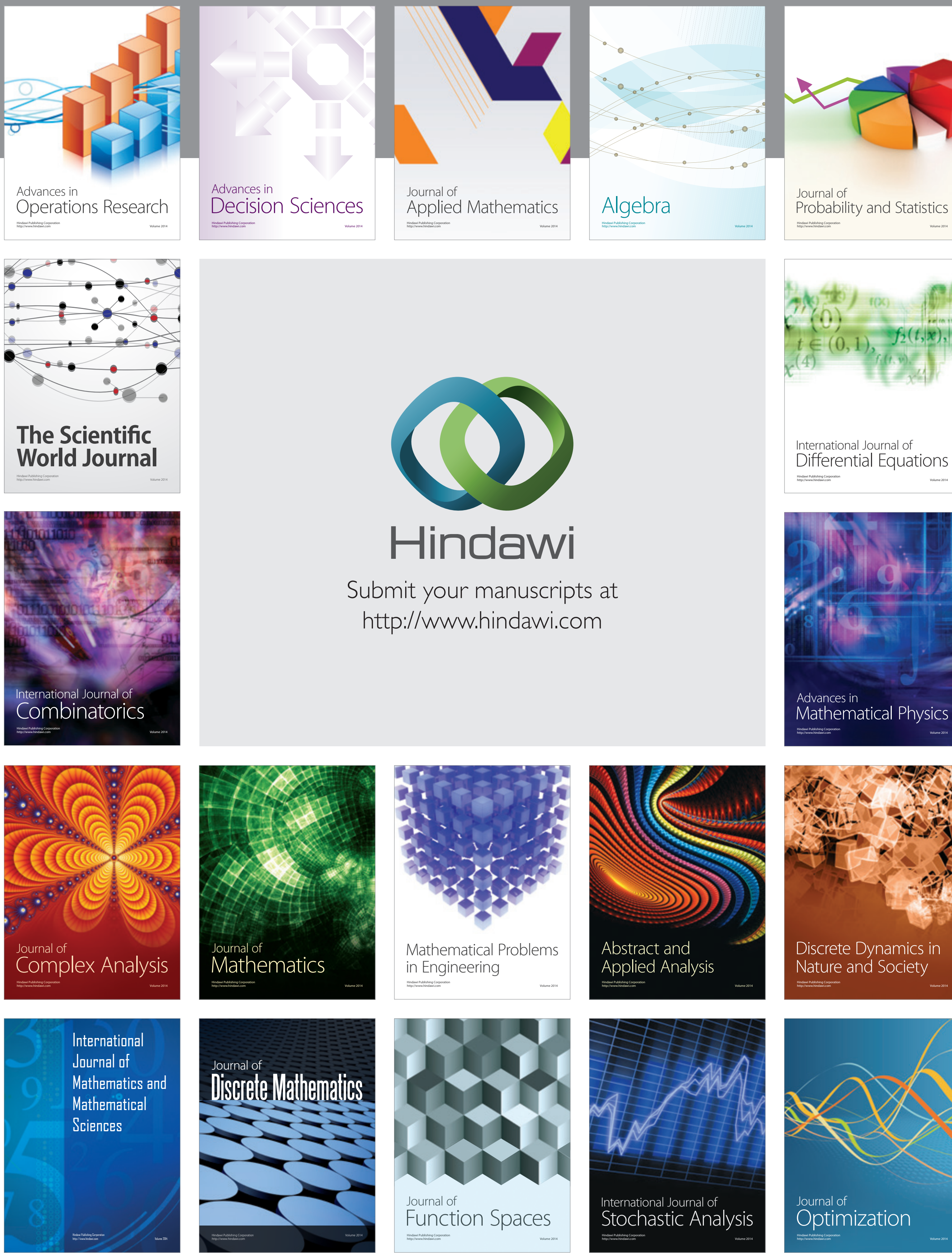
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