Hindawi Publishing Corporation
Chinese Journal of Mathematics
Volume 2016, Article ID 4361806, 8 pages
http://dx.doi.org/10.1155/2016/4361806

Research Article

Some New Generalized Integral Inequalities for GA-s-Convex Functions via Hadamard Fractional Integrals

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Received 22 April 2016; Revised 14 July 2016; Accepted 1 August 2016

Academic Editor: Chang-Jian Zhao

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We prove new generalization of Hadamard, Ostrowski, and Simpson inequalities in the framework of GA-s-convex functions and Hadamard fractional integral.

1. Introduction

Let a real function f be defined on a nonempty interval I of real line $\mathbb{R}$. The function f is said to be convex on I if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0,1]$.

In [1], Breckner introduced s-convex functions as a generalization of convex functions as follows.

Definition 1. Let $s \in (0,1]$ be a fixed real number. A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex (in the second sense), or that f belongs to the class K_s^2, if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \tag{2}$$

for all $x, y \in [0,\infty)$ and $t \in [0,1]$.

Of course, s-convexity means just convexity when $s = 1$.

The following inequalities are well known in the literature as Hermite-Hadamard inequality, Ostrowski inequality, and Simpson inequality, respectively.

Theorem 2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{3}$$

Theorem 3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping differentiable in I°, the interior of I, and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$, $x \in [a,b]$, then the following inequality holds:*

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ &\quad \leq \frac{M}{b-a}\left[\frac{(x-a)^2 + (b-x)^2}{2}\right] \end{aligned} \tag{4}$$

for all $x \in [a,b]$.

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be a four times' continuously differentiable mapping on (a,b) and $\|f^{(4)}\|_\infty = \sup_{x\in(a,b)}|f^{(4)}(x)| < \infty$. Then the following inequality holds:*

$$\begin{aligned} &\left| \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] \right. \\ &\quad \left. - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4. \end{aligned} \tag{5}$$

We will give definitions of the right and left hand side Hadamard fractional integrals which are used throughout this paper.

Definition 5. Let $f \in L[a,b]$. The right-sided and left-sided Hadamard fractional integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $b > a \geq 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} f(t) \frac{dt}{t}, \quad a < x < b, \tag{6}$$

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} f(t) \frac{dt}{t}, \quad a < x < b, \tag{7}$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$ (see [2]).

In recent years, many authors have studied errors estimations for Hermite-Hadamard, Ostrowski, and Simpson inequalities; for refinements, counterparts, and generalization see [3–10].

Definition 6 (see [11, 12]). A function $f : I \subseteq (0,\infty) \to \mathbb{R}$ is said to be GA-convex (geometric-arithmetically convex) if

$$f\left(x^t y^{1-t}\right) \leq t f(x) + (1-t) f(y) \tag{8}$$

for all $x, y \in I$ and $t \in [0,1]$.

Definition 7 (see [13]). For $s \in (0,1]$, a function $f : I \subseteq (0,\infty) \to \mathbb{R}$ is said to be GA-s-convex (geometric-arithmetically s-convex) if

$$f\left(x^t y^{1-t}\right) \leq t^s f(x) + (1-t)^s f(y) \tag{9}$$

for all $x, y \in I$ and $t \in [0,1]$.

It can be easily seen that if $s = 1$, GA-s-convexity reduces to GA-convexity.

For recent results and generalizations concerning GA-convex and GA-s-convex functions see [13–19].

Lemma 8 (see [20]). *For $\alpha > 0$ and $\mu > 0$, one has*

$$\int_0^1 t^{\alpha-1} \mu^t dt = \mu \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (\ln \mu)^{k-1}}{(\alpha)_k} < \infty, \tag{10}$$

where

$$(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1). \tag{11}$$

Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, the interior of I; in sequel of this paper we will take

$$\begin{aligned} I_f(x,\lambda,\alpha,a,b) = {} & (1-\lambda)\left[\ln^{\alpha}\frac{x}{a} + \ln^{\alpha}\frac{b}{x}\right] f(x) \\ & + \lambda\left[f(a)\ln^{\alpha}\frac{x}{a} + f(b)\ln^{\alpha}\frac{b}{x}\right] \\ & - \Gamma(\alpha+1)\left[J_{x-}^{\alpha} f(a) + J_{x+}^{\alpha} f(b)\right], \end{aligned} \tag{12}$$

where $a, b \in I$ with $a < b$, $x \in [a,b]$, $\lambda \in [0,1]$, $\alpha > 0$, and Γ is Euler Gamma function.

In [21], İşcan gave Hermite-Hadamard's inequalities for GA-convex functions in fractional integral forms as follows.

Theorem 9. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a function such that $f \in L[a,b]$, where $a, b \in I$ with $a < b$. If f is a GA-convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$\begin{aligned} f\left(\sqrt{ab}\right) & \leq \frac{\Gamma(\alpha+1)}{2(\ln(b/a))^{\alpha}} \left\{J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right\} \\ & \leq \frac{f(a)+f(b)}{2} \end{aligned} \tag{13}$$

with $\alpha > 0$.

In [21], İşcan obtained some new inequalities for quasi-geometrically convex functions via fractional integrals by using the following lemma.

Lemma 10. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. Then for all $x \in [a,b]$, $\lambda \in [0,1]$, and $\alpha > 0$ one has*

$$\begin{aligned} & I_f(x,\lambda,\alpha,a,b) \\ & = a\left(\ln\frac{x}{a}\right)^{\alpha+1} \int_0^1 \left(t^{\alpha}-\lambda\right)\left(\frac{x}{a}\right)^t f'\left(x^t a^{1-t}\right) dt \\ & \quad - b\left(\ln\frac{b}{x}\right)^{\alpha+1} \int_0^1 \left(t^{\alpha}-\lambda\right)\left(\frac{x}{b}\right)^t f'\left(x^t b^{1-t}\right) dt. \end{aligned} \tag{14}$$

In this paper, we will use Lemma 10 to obtain some new inequalities on generalization of Hadamard, Ostrowski, and Simpson type inequalities for GA-s-convex functions via Hadamard fractional integral.

2. Generalized Integral Inequalities for Some GA-s-Convex Functions via Fractional Integrals

Theorem 11. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I^{\circ}$ with $a < b$. If $|f'|^q$ is GA-s-convex on $[a,b]$ in the second sense for some fixed $q \geq 1$, $x \in [a,b]$, $\lambda \in [0,1]$, and $\alpha > 0$ then the following inequality for fractional integrals holds:*

$$\begin{aligned} \left|I_f(x,\lambda,\alpha,a,b)\right| \leq {} & A_1^{1-1/q}(\alpha,\lambda)\left\{a\left(\ln\frac{x}{a}\right)^{\alpha+1}\right. \\ & \cdot\left(\left|f'(x)\right|^q A_2\left(\left(\frac{x}{a}\right)^q,\alpha,\lambda,s\right)\right. \\ & \left.+\left|f'(a)\right|^q A_3\left(\left(\frac{x}{a}\right)^q,\alpha,\lambda,s\right)\right)^{1/q} + b\left(\ln\frac{b}{x}\right)^{\alpha+1} \end{aligned}$$

$$\cdot \left(\left| f'(x) \right|^q A_2 \left(\left(\frac{x}{b} \right)^q, \alpha, \lambda, s \right) + \left| f'(b) \right|^q A_3 \left(\left(\frac{x}{b} \right)^q, \alpha, \lambda, s \right) \right)^{1/q} \Bigg\}, \quad (15)$$

where

$$A_1(\alpha, \lambda) = \frac{2\alpha\lambda^{1+1/\alpha} + 1}{\alpha + 1} - \lambda,$$
$$A_2\left(\left(\frac{x}{u} \right)^q, \alpha, \lambda, s \right) = \int_0^1 \left| t^\alpha - \lambda \right| \left(\frac{x}{u} \right)^{qt} t^s dt, \quad (16)$$
$$A_3\left(\left(\frac{x}{u} \right)^q, \alpha, \lambda, s \right) = \int_0^1 \left| t^\alpha - \lambda \right| \left(\frac{x}{u} \right)^{qt} (1-t)^s dt, \quad u = a, b.$$

Proof. Using Lemma 10, property of the modulus, and the power-mean inequality, we have

$$\begin{aligned} \left| I_f(x, \lambda, \alpha, a, b) \right| &\le a \left(\ln \frac{x}{a} \right)^{\alpha+1} \cdot \int_0^1 \left| t^\alpha - \lambda \right| \left(\frac{x}{a} \right)^t \left| f'\left(x^t a^{1-t} \right) \right| dt + b \left(\ln \frac{b}{x} \right)^{\alpha+1} \\ &\cdot \int_0^1 \left| t^\alpha - \lambda \right| \left(\frac{x}{b} \right)^t \left| f'\left(x^t b^{1-t} \right) \right| dt \le a \left(\ln \frac{x}{a} \right)^{\alpha+1} \\ &\cdot \left(\int_0^1 \left| t^\alpha - \lambda \right| dt \right)^{1-1/q} \\ &\cdot \left(\int_0^1 \left| t^\alpha - \lambda \right| \left(\frac{x}{a} \right)^{qt} \left| f'\left(x^t a^{1-t} \right) \right|^q dt \right)^{1/q} \\ &+ b \left(\ln \frac{b}{x} \right)^{\alpha+1} \left(\int_0^1 \left| t^\alpha - \lambda \right| dt \right)^{1-1/q} \\ &\cdot \left(\int_0^1 \left| t^\alpha - \lambda \right| \left(\frac{x}{b} \right)^{qt} \left| f'\left(x^t b^{1-t} \right) \right|^q dt \right)^{1/q}. \end{aligned} \quad (17)$$

Since $|f'|^q$ is GA-s-convex on $[a, b]$, we get

$$\begin{aligned} \int_0^1 \left| t^\alpha - \lambda \right| \left(\frac{x}{a} \right)^{qt} \left| f'\left(x^t a^{1-t} \right) \right|^q dt &\le \int_0^1 \left| t^\alpha - \lambda \right| \\ &\cdot \left(\frac{x}{a} \right)^{qt} \left(t^s \left| f'(x) \right|^q + (1-t)^s \left| f'(a) \right|^q \right) dt \\ &= \left| f'(x) \right|^q A_2\left(\frac{x}{a}, \alpha, \lambda, s, q \right) + \left| f'(a) \right|^q \\ &\cdot A_3\left(\frac{x}{a}, \alpha, \lambda, s, q \right), \end{aligned} \quad (18)$$

$$\begin{aligned} \int_0^1 \left| t^\alpha - \lambda \right| \left(\frac{x}{b} \right)^{qt} \left| f'\left(x^t b^{1-t} \right) \right|^q dt &\le \int_0^1 \left| t^\alpha - \lambda \right| \\ &\cdot \left(\frac{x}{b} \right)^{qt} \left(t^s \left| f'(x) \right|^q + (1-t)^s \left| f'(b) \right|^q \right) dt \\ &= \left| f'(x) \right|^q A_2\left(\frac{x}{b}, \alpha, \lambda, s, q \right) + \left| f'(b) \right|^q \\ &\cdot A_3\left(\frac{x}{b}, \alpha, \lambda, s, q \right), \end{aligned} \quad (19)$$

and by a simple computation, we have

$$\begin{aligned} \int_0^1 \left| t^\alpha - \lambda \right| dt &= \int_0^{\lambda^{1/\alpha}} \left(\lambda - t^\alpha \right) dt + \int_{\lambda^{1/\alpha}}^1 \left(t^\alpha - \lambda \right) dt \\ &= \frac{2\alpha\lambda^{1+1/\alpha} + 1}{\alpha + 1} - \lambda. \end{aligned} \quad (20)$$

Hence, If we use (18), (19), and (20) in (17), we obtain the desired result. This completes the proof. □

Corollary 12. *Under the assumptions of Theorem 11 with $s = 1$, inequality (15) reduces to the following inequality:*

$$\begin{aligned} \left| I_f(x, \lambda, \alpha, a, b) \right| &\le A_1^{1-1/q}(\alpha, \lambda) \left\{ a \left(\ln \frac{x}{a} \right)^{\alpha+1} \right. \\ &\cdot \left(\left| f'(x) \right|^q A_2\left(\left(\frac{x}{a} \right)^q, \alpha, \lambda, 1 \right) \right. \\ &+ \left. \left| f'(a) \right|^q A_3\left(\left(\frac{x}{a} \right)^q, \alpha, \lambda, 1 \right) \right)^{1/q} + b \left(\ln \frac{b}{x} \right)^{\alpha+1} \\ &\cdot \left(\left| f'(x) \right|^q A_2\left(\left(\frac{x}{b} \right)^q, \alpha, \lambda, 1 \right) \right. \\ &+ \left. \left. \left| f'(b) \right|^q A_3\left(\left(\frac{x}{b} \right)^q, \alpha, \lambda, 1 \right) \right)^{1/q} \right\}. \end{aligned} \quad (21)$$

Corollary 13. *Under the assumptions of Theorem 11 with $s = 1$ and $\alpha = 1$, inequality (15) reduces to the following inequality:*

$$\begin{aligned} \left(\ln \frac{b}{a} \right)^{-1} \left| I_f(x, \lambda, 1, a, b) \right| &= \left| (1-\lambda) f(x) \right. \\ &+ \lambda \left[\frac{f(a)\ln(x/a) + f(b)\ln(b/x)}{\ln(b/a)} \right] - \frac{1}{\ln(b/a)} \\ &\cdot \left. \int_a^b \frac{f(u)}{u} du \right| \le \left(\ln \frac{b}{a} \right)^{-1} A_1^{1-1/q}(1, \lambda) \left\{ a \left(\ln \frac{x}{a} \right)^2 \right. \\ &\cdot \left(\left| f'(x) \right|^q A_2(\mu_a, 1, \lambda, 1) \right. \\ &+ \left. \left| f'(a) \right|^q A_3(\mu_a, 1, \lambda, 1) \right)^{1/q} + b \left(\ln \frac{b}{x} \right)^2 \\ &\cdot \left(\left| f'(x) \right|^q A_2(\mu_b, 1, \lambda, 1) \right. \\ &+ \left. \left. \left| f'(b) \right|^q A_3(\mu_b, 1, \lambda, 1) \right)^{1/q} \right\}, \end{aligned} \quad (22)$$

where

$$
\begin{aligned}
A_1(1,\lambda) &= \frac{(2\lambda^2-2\lambda+1)}{2},\\
A_2(\mu_u,1,\lambda,1) &= \frac{\left\{\left(\mu_u-2\lambda^2\mu_u^\lambda\right)\ln^2\mu_u+\left(\lambda\mu_u^\lambda\ln\mu_u-\mu_u^\lambda+1\right)(\lambda\ln\mu_u+\lambda+4)-(\lambda+2)(\mu_u\ln\mu_u-\mu_u+1)\right\}}{(\ln\mu_u)^3},\\
A_3(\mu_u,1,\lambda,1) &= \frac{\left[2\mu_u^\lambda+\mu_u\ln\mu_u-\lambda(1+\mu_u)\ln\mu_u-\mu_u-1\right]}{(\ln\mu_u)^2}-A_2(\mu_u,1,\lambda,1),\\
\mu_u &= \left(\frac{x}{u}\right)^q,\quad u=a,b.
\end{aligned} \tag{23}
$$

Corollary 14. *Under the assumptions of Theorem 11 with $q=1$, inequality (15) reduces to the following inequality:*

$$
\begin{aligned}
\left|I_f(x,\lambda,\alpha,a,b)\right| &\le \Bigg\{a\left(\ln\frac{x}{a}\right)^{\alpha+1}\\
&\cdot\left(\left|f'(x)\right|A_2\left(\frac{x}{a},\alpha,\lambda,s\right)\right.\\
&\left.+\left|f'(a)\right|A_3\left(\frac{x}{a},\alpha,\lambda,s\right)\right)+b\left(\ln\frac{b}{x}\right)^{\alpha+1}\\
&\cdot\left(\left|f'(x)\right|A_2\left(\frac{x}{b},\alpha,\lambda,s\right)\right.\\
&\left.+\left|f'(b)\right|A_3\left(\frac{x}{b},\alpha,\lambda,s\right)\right)\Bigg\}.
\end{aligned} \tag{24}
$$

Corollary 15. *Under the assumptions of Theorem 11 with $x=\sqrt{ab}$, $\lambda=1/3$, from inequality (15), one gets the following Simpson type inequality for fractional integrals:*

$$
\begin{aligned}
&\left|2^{\alpha-1}\left(\ln\frac{b}{a}\right)^{-\alpha}I_f\left(\sqrt{ab},\frac{1}{3},\alpha,a,b\right)\right| = \left|\frac{1}{6}\left[f(a)\right.\right.\\
&\left.+4f\left(\sqrt{ab}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ln(b/a))^\alpha}\left[J^\alpha_{\sqrt{ab}-}f(a)\right.\\
&\left.\left.+J^\alpha_{\sqrt{ab}+}f(b)\right]\right| \le \frac{\ln(b/a)}{4}A_1^{1-1/q}\left(\alpha,\frac{1}{3}\right)\\
&\cdot\Bigg\{a\left(\left|f'\left(\sqrt{ab}\right)\right|^q A_2\left(\left(\frac{b}{a}\right)^{q/2},\alpha,\frac{1}{3},s\right)\right.\\
&\left.+\left|f'(a)\right|^q A_3\left(\left(\frac{b}{a}\right)^{q/2},\alpha,\frac{1}{3},s\right)\right)^{1/q}\\
&+b\left(\left|f'\left(\sqrt{ab}\right)\right|^q A_2\left(\left(\frac{a}{b}\right)^{q/2},\alpha,\frac{1}{3},s\right)\right.\\
&\left.+\left|f'(b)\right|^q A_3\left(\left(\frac{a}{b}\right)^{q/2},\alpha,\frac{1}{3},s\right)\right)^{1/q}\Bigg\}.
\end{aligned} \tag{25}
$$

Corollary 16. *Under the assumptions of Theorem 11 with $x=\sqrt{ab}$, $\lambda=0$, from inequality (15), one gets*

$$
\begin{aligned}
&\left|2^{\alpha-1}\left(\ln\frac{b}{a}\right)^{-\alpha}I_f\left(\sqrt{ab},0,\alpha,a,b\right)\right| = \left|f\left(\sqrt{ab}\right)\right.\\
&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ln(b/a))^\alpha}\left[J^\alpha_{\sqrt{ab}-}f(a)+J^\alpha_{\sqrt{ab}+}f(b)\right]\right|\\
&\le \frac{\ln(b/a)}{4}\left(\frac{1}{\alpha+1}\right)^{1-1/q}\\
&\cdot\Bigg\{a\left(\left|f'\left(\sqrt{ab}\right)\right|^q A_2\left(\left(\frac{b}{a}\right)^{q/2},\alpha,0,s\right)\right.\\
&\left.+\left|f'(a)\right|^q A_3\left(\left(\frac{b}{a}\right)^{q/2},\alpha,0,s\right)\right)^{1/q}\\
&+b\left(\left|f'\left(\sqrt{ab}\right)\right|^q A_2\left(\left(\frac{a}{b}\right)^{q/2},\alpha,0,s\right)\right.\\
&\left.+\left|f'(b)\right|^q A_3\left(\left(\frac{a}{b}\right)^{q/2},\alpha,0,s\right)\right)^{1/q}\Bigg\}.
\end{aligned} \tag{26}
$$

Corollary 17. *Under the assumptions of Theorem 11 with $x=\sqrt{ab}$ and $\lambda=1$, from inequality (15) one gets*

$$
\begin{aligned}
&\left|2^{\alpha-1}\left(\ln\frac{b}{a}\right)^{-\alpha}I_f\left(\sqrt{ab},1,\alpha,a,b\right)\right| = \left|\frac{f(a)+f(b)}{2}\right.\\
&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ln(b/a))^\alpha}\left[J^\alpha_{\sqrt{ab}-}f(a)+J^\alpha_{\sqrt{ab}+}f(b)\right]\right|\\
&\le \frac{\ln(b/a)}{4}\left(\frac{\alpha}{\alpha+1}\right)^{1-1/q}\\
&\cdot\Bigg\{a\left[\left|f'\left(\sqrt{ab}\right)\right|^q A_2\left(\left(\frac{b}{a}\right)^{q/2},\alpha,1,s\right)\right.\\
&\left.+\left|f'(a)\right|^q A_3\left(\left(\frac{b}{a}\right)^{q/2},\alpha,1,s\right)\right]^{1/q}
\end{aligned}
$$

$$+b\left[\left|f'\left(\sqrt{ab}\right)\right|^{q}A_2\left(\left(\frac{a}{b}\right)^{q/2},\alpha,1,s\right)\right.$$

$$\left.\left.+\left|f'(b)\right|^{q}A_3\left(\left(\frac{a}{b}\right)^{q/2},\alpha,1,s\right)\right]^{1/q}\right\}. \tag{27}$$

Corollary 18. *Let the assumptions of Theorem 11 hold. If $|f'(x)|\le M$ for all $x\in[a,b]$ and $\lambda=0$, then from inequality (15), one gets the following Ostrowski type inequality for fractional integrals:*

$$\left|\left[\left(\ln\frac{x}{a}\right)^{\alpha}+\left(\ln\frac{b}{x}\right)^{\alpha}\right]f(x)-\Gamma(\alpha+1)\left[J_{x-}^{\alpha}f(a)\right.\right.$$

$$\left.\left.+J_{x+}^{\alpha}f(b)\right]\right|\le M\left(\frac{1}{\alpha+1}\right)^{1-1/q}\left\{a\left(\ln\frac{x}{a}\right)^{\alpha}\right.$$

$$\cdot\left[A_2\left(\left(\frac{b}{a}\right)^{q/2},\alpha,0,s\right)\right.$$

$$\left.+A_3\left(\left(\frac{b}{a}\right)^{q/2},\alpha,0,s\right)\right]^{1/q}+b\left(\ln\frac{b}{x}\right)^{\alpha}$$

$$\cdot\left[A_2\left(\left(\frac{a}{b}\right)^{q/2},\alpha,0,s\right)\right.$$

$$\left.\left.+A_3\left(\left(\frac{a}{b}\right)^{q/2},\alpha,0,s\right)\right]^{1/q}\right\} \tag{28}$$

for all $x\in[a,b]$.

Theorem 19. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I^{\circ}$ with $a<b$. If $|f'|^{q}$ is GA-s-convex on $[a,b]$ for some fixed $q>1$, $x\in[a,b]$, $\lambda\in[0,1]$, and $\alpha>0$ then the following inequality for fractional integrals holds:*

$$\begin{aligned}\left|I_f(x,\lambda,\alpha,a,b)\right|&\le C_1^{1/p}(\alpha,\lambda)\left\{a\left(\ln\frac{x}{a}\right)^{\alpha+1}\right.\\&\cdot\left(\left|f'(x)\right|^{q}C_2\left(\left(\frac{x}{a}\right)^{q},s\right)\right.\\&\left.+\left|f'(a)\right|^{q}C_3\left(\left(\frac{x}{a}\right)^{q},s\right)\right)^{1/q}+b\left(\ln\frac{b}{x}\right)^{\alpha+1}\\&\cdot\left(\left|f'(x)\right|^{q}C_2\left(\left(\frac{x}{b}\right)^{q},s\right)\right.\\&\left.\left.+\left|f'(b)\right|^{q}C_3\left(\left(\frac{x}{b}\right)^{q},s\right)\right)^{1/q}\right\},\end{aligned} \tag{29}$$

where $1/p+1/q=1$ and

$$\begin{aligned}C_1(\alpha,\lambda)&=\begin{cases}\dfrac{1}{(\alpha p+1)}, & \lambda=0\\[2ex] \dfrac{\lambda^{1+p+1/\alpha}}{\alpha}\beta\left(\dfrac{1}{\alpha},p+1\right)+\dfrac{(1-\lambda)^{p+1}}{\alpha(p+1)}\cdot{}_2F_1\left(1-\dfrac{1}{\alpha},1;p+2;1-\lambda\right), & 0<\lambda\le1,\end{cases}\\ C_2\left(\left(\frac{x}{u}\right)^{q},s\right)&=\left(\frac{x}{u}\right)^{q}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\left(\ln(x/u)^{q}\right)^{k-1}}{(s+1)_k},\\ C_3\left(\left(\frac{x}{u}\right)^{q},s\right)&=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\left(-\ln(x/u)^{q}\right)^{k-1}}{(s+1)_k},\quad u=a,b.\end{aligned} \tag{30}$$

Proof. Using Lemma 10, property of the modulus, the Hölder inequality, and GA-s-convexity of $|f'|^{q}$, we have

$$\begin{aligned}\left|I_f(x,\lambda,\alpha,a,b)\right|&\le a\left(\ln\frac{x}{a}\right)^{\alpha+1}\int_0^1\left|t^{\alpha}-\lambda\right|\left(\frac{x}{a}\right)^{t}\\&\cdot\left|f'\left(x^{t}a^{1-t}\right)\right|dt+b\left(\ln\frac{b}{x}\right)^{\alpha+1}\int_0^1\left|t^{\alpha}-\lambda\right|\\&\cdot\left(\frac{x}{b}\right)^{t}\left|f'\left(x^{t}b^{1-t}\right)\right|dt\le a\left(\ln\frac{x}{a}\right)^{\alpha+1}\\&\cdot\left(\int_0^1\left|t^{\alpha}-\lambda\right|^{p}dt\right)^{1/p}\\&\cdot\left(\int_0^1\left(\frac{x}{a}\right)^{qt}\left|f'\left(x^{t}a^{1-t}\right)\right|^{q}dt\right)^{1/q}+b\left(\ln\frac{b}{x}\right)^{\alpha+1}\\&\cdot\left(\int_0^1\left|t^{\alpha}-\lambda\right|^{p}dt\right)^{1/p}\\&\cdot\left(\int_0^1\left(\frac{x}{b}\right)^{qt}\left|f'\left(x^{t}b^{1-t}\right)\right|^{q}dt\right)^{1/q}\\&\le\left(\int_0^1\left|t^{\alpha}-\lambda\right|^{p}dt\right)^{1/p}\left\{a\left(\ln\frac{x}{a}\right)^{\alpha+1}\right.\\&\cdot\left(\left|f'(x)\right|^{q}\int_0^1\mu_a^{t}t^{s}+\left|f'(a)\right|^{q}\int_0^1\mu_a^{t}(1-t)^{s}dt\right)^{1/q}\end{aligned}$$

$$+b\left(\ln\frac{b}{x}\right)^{\alpha+1}\left(\left|f'(x)\right|^{q}\int_{0}^{1}\mu_{b}^{t}t^{s}+\left|f'(b)\right|^{q}\int_{0}^{1}\mu_{b}^{t}(1-t)^{s}dt\right)^{1/q}\Bigg\}, \quad (31)$$

where $\mu_a=(x/a)^q$, $\mu_b=(x/b)^q$ and

$$\int_0^1\left|t^\alpha-\lambda\right|^p dt=\int_0^{\lambda^{1/\alpha}}(\lambda-t^\alpha)^p dt+\int_{\lambda^{1/\alpha}}^1(t^\alpha-\lambda)^p dt = \begin{cases}\dfrac{1}{(\alpha p+1)}, & \lambda=0\\ \dfrac{\lambda^{(\alpha p+1)/\alpha}}{\alpha}\beta\left(\dfrac{1}{\alpha},p+1\right)+\dfrac{(1-\lambda)^{p+1}}{\alpha(p+1)}\cdot{}_2F_1\left(1-\dfrac{1}{\alpha},1;p+2;1-\lambda\right), & 0<\lambda\le 1.\end{cases} \quad (32)$$

Using Lemma 8, we have

$$\begin{aligned}\int_0^1\mu_u^t t^s dt&=\mu_u\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(\ln\mu_u)^{k-1}}{(s+1)_k},\\ \int_0^1\mu_u^t(1-t)^s dt&=\int_0^1\mu_u^{1-t}t^s dt\\ &=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(-\ln\mu_u)^{k-1}}{(s+1)_k},\\ &\qquad u=a,b.\end{aligned} \quad (33)$$

Hence, if we use (32)-(33) in (31) and replacing $\mu_a=(x/a)^q$, $\mu_b=(x/b)^q$, we obtain the desired result. This completes the proof. □

Corollary 20. *Under the assumptions of Theorem 19 with $s=1$, inequality (29) reduces to the following inequality:*

$$\begin{aligned}&\left|I_f(x,\lambda,\alpha,a,b)\right|\le C_1^{1/p}(\alpha,\lambda)\Bigg\{a\left(\ln\frac{x}{a}\right)^{\alpha+1}\\ &\cdot\left(\left|f'(x)\right|^q C_2\left(\left(\frac{x}{a}\right)^q,1\right)\right.\\ &\left.+\left|f'(a)\right|^q C_3\left(\left(\frac{x}{a}\right)^q,1\right)\right)^{1/q}+b\left(\ln\frac{b}{x}\right)^{\alpha+1}\\ &\cdot\left(\left|f'(x)\right|^q C_2\left(\left(\frac{x}{b}\right)^q,1\right)\right.\\ &\left.+\left|f'(b)\right|^q C_3\left(\left(\frac{x}{b}\right)^q,1\right)\right)^{1/q}\Bigg\}.\end{aligned} \quad (34)$$

Corollary 21. *Under the assumptions of Theorem 19 with $s=1$ and $\alpha=1$, inequality (29) reduces to the following inequality:*

$$\begin{aligned}&\left|I_f(x,\lambda,1,a,b)\right|=\left|\ln\frac{b}{a}(1-\lambda)f(x)+\lambda\left[f(a)\ln\frac{x}{a}\right.\right.\\ &\left.\left.+f(b)\ln\frac{b}{x}\right]-\int_a^b\frac{f(u)}{u}du\right|\\ &\le\left(\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{p+1}\right)^{1/p}\Bigg\{a\left(\ln\frac{x}{a}\right)^2\\ &\cdot\left(\left|f'(x)\right|^q C_2\left(\left(\frac{x}{a}\right)^q,1\right)\right.\\ &\left.+\left|f'(a)\right|^q C_3\left(\left(\frac{x}{a}\right)^q,1\right)\right)^{1/q}+b\left(\ln\frac{b}{x}\right)^2\\ &\cdot\left(\left|f'(x)\right|^q C_2\left(\left(\frac{x}{b}\right)^q,1\right)\right.\\ &\left.+\left|f'(b)\right|^q C_3\left(\left(\frac{x}{b}\right)^q,1\right)\right)^{1/q}\Bigg\}.\end{aligned} \quad (35)$$

Corollary 22. *Under the assumptions of Theorem 19 with $x=\sqrt{ab}$, $\lambda=1/3$, from inequality (29), one gets the following Simpson type inequality for fractional integrals:*

$$\begin{aligned}&\left|2^{\alpha-1}\left(\ln\frac{b}{a}\right)^{-\alpha}I_f\left(\sqrt{ab},\frac{1}{3},\alpha,a,b\right)\right|=\left|\frac{1}{6}\left[f(a)\right.\right.\\ &\left.+4f\left(\sqrt{ab}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ln(b/a))^\alpha}\left[J^\alpha_{\sqrt{ab}-}f(a)\right.\\ &\left.\left.+J^\alpha_{\sqrt{ab}+}f(b)\right]\right|\le\frac{\ln(b/a)}{4}C_1^{1/p}\left(\alpha,\frac{1}{3}\right)\\ &\cdot\Bigg\{a\left(\left|f'\left(\sqrt{ab}\right)\right|^q C_2\left(\left(\frac{b}{a}\right)^{q/2},s\right)\right.\end{aligned}$$

$$+\left|f'(a)\right|^{q} C_{3}\left(\left(\frac{b}{a}\right)^{q/2}, s\right)\right)^{1/q}$$
$$+ b\left(\left|f'\left(\sqrt{ab}\right)\right|^{q} C_{2}\left(\left(\frac{a}{b}\right)^{q/2}, s\right)\right.$$
$$\left.\left.+\left|f'(b)\right|^{q} C_{3}\left(\left(\frac{a}{b}\right)^{q/2}, s\right)\right)^{1/q}\right\}. \quad (36)$$

Corollary 23. *Under the assumptions of Theorem 19 with $x = \sqrt{ab}$, $\lambda = 0$, from inequality (29), one gets*

$$\left|2^{\alpha-1}\left(\ln\frac{b}{a}\right)^{-\alpha} I_{f}\left(\sqrt{ab}, 0, \alpha, a, b\right)\right| = \left|f\left(\sqrt{ab}\right) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ln(b/a))^{\alpha}}\left[J^{\alpha}_{\sqrt{ab}-} f(a) + J^{\alpha}_{\sqrt{ab}+} f(b)\right]\right|$$
$$\le \frac{\ln(b/a)}{4}\left(\frac{1}{\alpha p+1}\right)^{1/p} \cdot \left\{a\left(\left|f'\left(\sqrt{ab}\right)\right|^{q} C_{2}\left(\left(\frac{b}{a}\right)^{q/2}, s\right) + \left|f'(a)\right|^{q} C_{3}\left(\left(\frac{b}{a}\right)^{q/2}, s\right)\right)^{1/q}\right. \quad (37)$$
$$\left.+ b\left(\left|f'\left(\sqrt{ab}\right)\right|^{q} C_{2}\left(\left(\frac{a}{b}\right)^{q/2}, s\right) + \left|f'(b)\right|^{q} C_{3}\left(\left(\frac{a}{b}\right)^{q/2}, s\right)\right)^{1/q}\right\}.$$

Corollary 24. *Under the assumptions of Theorem 19 with $x = \sqrt{ab}$ and $\lambda = 1$, from inequality (29) one gets*

$$\left|2^{\alpha-1}\left(\ln\frac{b}{a}\right)^{-\alpha} I_{f}\left(\sqrt{ab}, 1, \alpha, a, b\right)\right| = \left|\frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ln(b/a))^{\alpha}}\left[J^{\alpha}_{\sqrt{ab}-} f(a) + J^{\alpha}_{\sqrt{ab}+} f(b)\right]\right|$$
$$\le \frac{\ln(b/a)}{4}\left(\frac{1}{\alpha}\beta\left(\frac{1}{\alpha}, p+1\right)\right)^{1/p} \cdot \left\{a\left[\left|f'\left(\sqrt{ab}\right)\right|^{q} C_{2}\left(\left(\frac{b}{a}\right)^{q/2}, s\right) + \left|f'(a)\right|^{q} C_{3}\left(\left(\frac{b}{a}\right)^{q/2}, s\right)\right]^{1/q}\right. \quad (38)$$
$$\left.+ b\left[\left|f'\left(\sqrt{ab}\right)\right|^{q} C_{2}\left(\left(\frac{a}{b}\right)^{q/2}, s\right) + \left|f'(b)\right|^{q} C_{3}\left(\left(\frac{a}{b}\right)^{q/2}, s\right)\right]^{1/q}\right\}.$$

Corollary 25. *Let the assumptions of Theorem 19 hold. If $|f'(x)| \le M$ for all $x \in [a,b]$ and $\lambda = 0$, then from inequality (29), one gets the following Ostrowski type inequality for fractional integrals:*

$$\left|\left[\left(\ln\frac{x}{a}\right)^{\alpha} + \left(\ln\frac{b}{x}\right)^{\alpha}\right] f(x) - \Gamma(\alpha+1) \cdot \left[J^{\alpha}_{x-} f(a) + J^{\alpha}_{x+} f(b)\right]\right| \le M\left(\frac{1}{\alpha p+1}\right)^{1/p}$$
$$\cdot \left\{a\left(\ln\frac{x}{a}\right)^{\alpha}\left[C_{2}\left(\left(\frac{x}{a}\right)^{q}, s\right) + C_{3}\left(\left(\frac{x}{a}\right)^{q}, s\right)\right]^{1/q}\right. \quad (39)$$
$$\left.+ b\left(\ln\frac{b}{x}\right)^{\alpha} \cdot \left[C_{2}\left(\left(\frac{x}{b}\right)^{q}, s\right) + C_{3}\left(\left(\frac{x}{b}\right)^{q}, s\right)\right]^{1/q}\right\}$$

for each $x \in [a,b]$.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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