

Research Article

Ostrowski Inequalities for Functions Whose First Derivatives Are Logarithmically Preinvex

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Some Ostrowski type inequalities for functions whose first derivatives are logarithmically preinvex are established.

1. Introduction

In 1938, A. M. Ostrowski proved the following important inequality.

Theorem 1 (see [1]). *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° (interior of I), and let $a, b \in I^\circ$ with $a < b$. If $|f'| \leq M$ for all $x \in (a, b)$, then*

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq M(b-a) \left[\frac{1}{4} + \frac{(x-(a+b)/2)^2}{(b-a)^2} \right], \quad (1) \\ & \forall x \in [a, b]. \end{aligned}$$

This is well known in the literature as Ostrowski's inequality. Due to its wide range of applications in numerical analysis and in probability, many researchers have established generalizations, extensions, and variants of inequality (1); we refer readers to [2–10] and the references cited therein.

In recent years, a lot of efforts have been made by many mathematicians to generalize classical convexity. Hanson [11] introduced a new class of generalized convexity, called invexity. In [12], the authors gave the concept of preinvex function which is a special case of invexity. Pini [13], Noor [14, 15], Yang and Li [16], and Weir and Mond [17] have studied the basic properties of the preinvex functions and their

role in optimization, variational inequalities, and equilibrium problems.

In [5], İşcan established some Ostrowski type inequalities for functions whose derivatives in absolute value are preinvex, by using the following identity.

Lemma 2 (see [5]). *Let $A \subset \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$ with $a < a + \eta(b, a)$. Suppose that $f : A \rightarrow \mathbb{R}$ is a differentiable function. If f' is integrable on $[a, a + \eta(b, a)]$, then the following equality holds:*

$$\begin{aligned} & f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(u) du = \eta(b, a) \\ & \cdot \left(\int_0^{(x-a)/\eta(b,a)} t f'(a + t\eta(b, a)) dt \right. \\ & \left. + \int_{(x-a)/\eta(b,a)}^1 (t-1) f'(a + t\eta(b, a)) dt \right), \quad (2) \end{aligned}$$

for all $x \in [a, a + \eta(b, a)]$.

Motivated by the results given in [5], in the present paper, we establish some new Ostrowski type inequalities for functions whose first derivatives in absolute value are logarithmically preinvex.

2. Preliminaries

In this section, we recall some concepts of convexity that are well known in the literature. Throughout this section, I is an interval of \mathbb{R} .

Definition 3 (see [18]). A positive function $f : I \rightarrow \mathbb{R}$ is said to be logarithmically convex, if, for all $x, y \in I$ and all $t \in [0, 1]$, we have

$$f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{(1-t)}. \quad (3)$$

Definition 4 (see [17]). A set K is said to be invex at x with respect to η , if, for all $x, y \in K$ and $t \in [0, 1]$, we have

$$x + t\eta(y, x) \in K. \quad (4)$$

K is said to be an invex set with respect to η if K is invex at each $x \in K$.

Definition 5 (see [14]). A positive function f on the invex set K is said to be log-preinvex with respect to η , if, for all $x, y \in K$ and $t \in [0, 1]$, we have

$$f(x + t\eta(y, x)) \leq [f(x)]^{(1-t)} [f(y)]^t. \quad (5)$$

Lemma 6 (see [19]). For $\alpha > 0$, $k > 0$, and $z > 0$, we have

$$\begin{aligned} J(\alpha, k) &= \int_0^1 (1-t)^{\alpha-1} k^t dt = \sum_{i=1}^{\infty} \frac{(\ln k)^{i-1}}{(\alpha)_i} < \infty \\ H(\alpha, k, z) &= \int_0^1 t^{\alpha-1} k^t dt = z^\alpha k^z \sum_{i=1}^{\infty} \frac{(-z \ln k)^{i-1}}{(\alpha)_i} < \infty, \end{aligned} \quad (6)$$

where $(\alpha)_i = \prod_{j=0}^{i-1} (\alpha + j)$.

3. Main Results

Theorem 7. Let $K \subseteq [0, \infty)$ be an invex subset with respect to $\eta : K \times K \rightarrow \mathbb{R}$ and $a, b \in K^\circ$ (K° interior of K) with $\eta(b, a) > 0$ and $[a, a + \eta(b, a)] \subset K$. Let $f : [a, a + \eta(b, a)] \rightarrow (0, \infty)$ be a differentiable function such that $f' \in L([a, a + \eta(b, a)])$ and $f'(a) \neq 0$. If $|f'|$ is logarithmically preinvex function, then the following inequality holds:

$$\begin{aligned} \left| f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(u) du \right| &\leq \frac{\eta(b, a) |f'(a)|}{2} \\ &\cdot \begin{cases} \left(\left(\frac{x-a}{\eta(b,a)} \right)^2 + \left(1 - \frac{x-a}{\eta(b,a)} \right)^2 \right) & \text{if } A = 1, \\ 2 \left[\left(2 \frac{x-a}{\eta(b,a)} - 1 \right) \frac{A^{(x-a)/\eta(b,a)}}{\ln A} + \frac{1 - 2A^{(x-a)/\eta(b,a)} + A}{\ln^2 A} \right] & \text{if } A \neq 1, \end{cases} \end{aligned} \quad (7)$$

for all $x \in [a, a + \eta(b, a)]$, where $A = |f'(b)|/|f'(a)|$.

Proof. From Lemma 2 and properties of modulus, we have

$$\begin{aligned} \left| f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(u) du \right| &\leq \eta(b, a) \\ &\cdot \left(\int_0^{(x-a)/\eta(b,a)} t |f'(a + t\eta(b, a))| dt \right. \\ &\left. + \int_{(x-a)/\eta(b,a)}^1 (1-t) |f'(a + t\eta(b, a))| dt \right). \end{aligned} \quad (8)$$

Since $|f'|$ is a logarithmically preinvex function, we deduce

$$\begin{aligned} \left| f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(u) du \right| &\leq \eta(b, a) \\ &\cdot \left(\int_0^{(x-a)/\eta(b,a)} t |f'(a)|^{(1-t)} |f'(b)|^t dt \right. \\ &\left. + \left(\left(\frac{x-a}{\eta(b,a)} \right)^2 + \left(1 - \frac{x-a}{\eta(b,a)} \right)^2 \right) \right). \end{aligned} \quad (10)$$

$$\begin{aligned} &+ \int_{(x-a)/\eta(b,a)}^1 (1-t) |f'(a)|^{(1-t)} |f'(b)|^t dt \\ &= \eta(b, a) |f'(a)| \left(\int_0^{(x-a)/\eta(b,a)} t A^t dt \right. \\ &\left. + \int_{(x-a)/\eta(b,a)}^1 (1-t) A^t dt \right). \end{aligned} \quad (9)$$

If $A = 1$, then (9) gives

$$\begin{aligned} \left| f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(u) du \right| &\leq \frac{\eta(b, a) |f'(a)|}{2} \\ &\cdot \left(\left(\frac{x-a}{\eta(b,a)} \right)^2 + \left(1 - \frac{x-a}{\eta(b,a)} \right)^2 \right). \end{aligned} \quad (10)$$

In the case where $A \neq 1$, (9) gives

$$\begin{aligned} & \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) du \right| \leq \eta(b,a) |f'(a)| \\ & \cdot \left(\left(2 \frac{x-a}{\eta(b,a)} - 1 \right) \frac{A^{(x-a)/\eta(b,a)}}{\ln A} \right. \\ & \left. + \frac{1 - 2A^{(x-a)/\eta(b,a)} + A}{(\ln A)^2} \right), \end{aligned} \quad (11)$$

where we use the fact that

$$\begin{aligned} & \int_0^{(x-a)/\eta(b,a)} t A^t dt \\ &= \frac{x-a}{\eta(b,a)} \frac{A^{(x-a)/\eta(b,a)}}{\ln A} - \frac{A^{(x-a)/\eta(b,a)} - 1}{(\ln A)^2}, \end{aligned}$$

$$\begin{aligned} & \int_{(x-a)/\eta(b,a)}^1 (1-t) A^t dt \\ &= - \left(1 - \frac{x-a}{\eta(b,a)} \right) \frac{A^{(x-a)/\eta(b,a)}}{\ln A} \\ &+ \frac{A - A^{(x-a)/\eta(b,a)}}{(\ln A)^2}. \end{aligned} \quad (12)$$

The desired result follows from (10) and (11). \square

Corollary 8. In Theorem 7, if we choose $x = (2a + \eta(b,a))/2$, we obtain the following midpoint inequality:

$$\begin{aligned} & \left| f\left(\frac{2a + \eta(b,a)}{2}\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) du \right| \\ & \leq \begin{cases} \frac{\eta(b,a) |f'(a)|}{4} & \text{if } A = 1, \\ \eta(b,a) |f'(a)| \left[\frac{\sqrt{A} - 1}{\ln A} \right]^2 & \text{if } A \neq 1. \end{cases} \end{aligned} \quad (13)$$

Corollary 9. Let $f : [a,b] \rightarrow (0, \infty)$ be a differentiable function such that $f' \in L([a,b])$ and $f'(a) \neq 0$. If $|f'|$ is a logarithmically convex function, then the following inequality holds:

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{(b-a) |f'(a)|}{2} \begin{cases} \left(\left(\frac{x-a}{b-a} \right)^2 + \left(\frac{b-x}{b-a} \right)^2 \right) & \text{if } A = 1 \\ 2 \left[\left(2 \frac{x-a}{b-a} - 1 \right) \frac{A^{(x-a)/(b-a)}}{\ln A} + \frac{1 - 2A^{(x-a)/(b-a)} + A}{(\ln A)^2} \right] & \text{if } A \neq 1, \end{cases} \end{aligned} \quad (14)$$

for all $x \in [a,b]$, where $A = |f'(b)|/|f'(a)|$.

Example 10. In Theorem 7, if we choose $\eta(b,a) = \sqrt{ab}$, the geometric mean, we obtain the following inequality:

$$\begin{aligned} & \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\sqrt{ab}} f(u) du \right| \leq \frac{|f'(a)| \sqrt{ab}}{2} \begin{cases} \left(\left(\frac{x-a}{\sqrt{ab}} \right)^2 + \left(1 - \frac{x-a}{\sqrt{ab}} \right)^2 \right) & \text{if } A = 1, \\ 2 \left[\left(2 \frac{x-a}{\sqrt{ab}} - 1 \right) \frac{A^{(x-a)/\sqrt{ab}}}{\ln A} + \frac{1 - 2A^{(x-a)/\sqrt{ab}} + A}{(\ln A)^2} \right] & \text{if } A \neq 1. \end{cases} \end{aligned} \quad (15)$$

Theorem 11. Let $K \subseteq [0, \infty)$ be an invex subset with respect to $\eta : K \times K \rightarrow \mathbb{R}$ and $a, b \in K^\circ$ (K° interior of K) with $\eta(b,a) > 0$ and $[a, a + \eta(b,a)] \subset K$. Let $f : [a, a + \eta(b,a)] \rightarrow (0, \infty)$

be a differentiable function such that $f' \in L([a, a + \eta(b,a)])$ and $f'(a) \neq 0$; let $q > 1$ with $1/p + 1/q = 1$. If $|f'|^q$ is a logarithmically preinvex function, then the following inequality holds:

$$\begin{aligned}
& \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) du \right| \\
& \leq \frac{\eta(b,a) |f'(a)|}{(p+1)^{1/p}} \\
& \cdot \begin{cases} \left(\frac{x-a}{\eta(b,a)} \right)^2 + \left(1 - \frac{x-a}{\eta(b,a)} \right)^2 & \text{if } A = 1, \\ \left(\frac{x-a}{\eta(b,a)} \right)^{1+1/p} \left(\frac{A^{q((x-a)/\eta(b,a))} - 1}{q \ln A} \right)^{1/q} + \left(1 - \frac{x-a}{\eta(b,a)} \right)^{1+1/p} \left(\frac{A^q - A^{q((x-a)/\eta(b,a))}}{q \ln A} \right)^{1/q} & \text{if } A \neq 1, \end{cases} \quad (16)
\end{aligned}$$

for all $x \in [a, a + \eta(b,a)]$, where $A = |f'(b)|/|f'(a)|$.

Proof. From Lemma 2, properties of modulus, and Hölder's inequality, we have

$$\begin{aligned}
& \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) du \right| \leq \eta(b,a) \\
& \cdot \left(\left(\int_0^{(x-a)/\eta(b,a)} t^p dt \right)^{1/p} \right. \\
& \cdot \left(\int_0^{(x-a)/\eta(b,a)} |f'(a + t\eta(b,a))|^q dt \right)^{1/q} \\
& + \left(\int_{(x-a)/\eta(b,a)}^1 (1-t)^p dt \right)^{1/p} \\
& \cdot \left. \left(\int_{(x-a)/\eta(b,a)}^1 |f'(a + t\eta(b,a))|^q dt \right)^{1/q} \right) \quad (17)
\end{aligned}$$

$$\begin{aligned}
& = \frac{\eta(b,a)}{(p+1)^{1/p}} \left(\left(\frac{x-a}{\eta(b,a)} \right)^{1+1/p} \right. \\
& \cdot \left(\int_0^{(x-a)/\eta(b,a)} |f'(a + t\eta(b,a))|^q dt \right)^{1/q} \\
& + \left(1 - \frac{x-a}{\eta(b,a)} \right)^{1+1/p} \\
& \cdot \left. \left(\int_{(x-a)/\eta(b,a)}^1 |f'(a + t\eta(b,a))|^q dt \right)^{1/q} \right).
\end{aligned}$$

Since $|f'|^q$ is a logarithmically preinvex function, we deduce

$$\begin{aligned}
& \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) du \right| \\
& \leq \frac{\eta(b,a)}{(p+1)^{1/p}} \left(\left(\frac{x-a}{\eta(b,a)} \right)^{1+1/p} \right. \\
& \cdot \left. \left(\frac{A^{q((x-a)/\eta(b,a))} - 1}{q \ln A} \right)^{1/q} + \left(1 - \frac{x-a}{\eta(b,a)} \right)^{1+1/p} \right. \\
& \cdot \left. \left(\frac{A^q - A^{q((x-a)/\eta(b,a))}}{q \ln A} \right)^{1/q} \right),
\end{aligned}$$

$$\begin{aligned}
& \cdot \left(\int_0^{(x-a)/\eta(b,a)} |f'(a)|^{q(1-t)} |f'(b)|^{qt} dt \right)^{1/q} \\
& + \left(1 - \frac{x-a}{\eta(b,a)} \right)^{1+1/p} \\
& \cdot \left(\int_{(x-a)/\eta(b,a)}^1 |f'(a)|^{q(1-t)} |f'(b)|^{qt} dt \right)^{1/q} \\
& = \frac{\eta(b,a) |f'(a)|}{(p+1)^{1/p}} \left(\left(\frac{x-a}{\eta(b,a)} \right)^{1+1/p} \right. \\
& \cdot \left(\int_0^{(x-a)/\eta(b,a)} A^{qt} dt \right)^{1/q} + \left(1 - \frac{x-a}{\eta(b,a)} \right)^{1+1/p} \\
& \cdot \left. \left(\int_{(x-a)/\eta(b,a)}^1 A^{qt} dt \right)^{1/q} \right). \quad (18)
\end{aligned}$$

If $A = 1$, then (18) gives

$$\begin{aligned}
& \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) du \right| \\
& \leq \frac{\eta(b,a) |f'(a)|}{(p+1)^{1/p}} \left(\left(\frac{x-a}{\eta(b,a)} \right)^2 + \left(1 - \frac{x-a}{\eta(b,a)} \right)^2 \right). \quad (19)
\end{aligned}$$

In the case where $A \neq 1$, (18) becomes

$$\begin{aligned}
& \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) du \right| \\
& \leq \frac{\eta(b,a) |f'(a)|}{(p+1)^{1/p}} \left(\left(\frac{x-a}{\eta(b,a)} \right)^{1+1/p} \right. \\
& \cdot \left(\frac{A^{q((x-a)/\eta(b,a))} - 1}{q \ln A} \right)^{1/q} + \left(1 - \frac{x-a}{\eta(b,a)} \right)^{1+1/p} \\
& \cdot \left. \left(\frac{A^q - A^{q((x-a)/\eta(b,a))}}{q \ln A} \right)^{1/q} \right), \quad (20)
\end{aligned}$$

where we use the fact that

$$\begin{aligned} \int_0^{(x-a)/\eta(b,a)} A^{qt} dt &= \frac{A^{q((x-a)/\eta(b,a))} - 1}{q \ln A}, \\ \int_{(x-a)/\eta(b,a)}^1 A^{qt} dt &= \frac{A^q - A^{q((x-a)/\eta(b,a))}}{q \ln A}. \end{aligned} \quad (21)$$

From (19) and (20), we get the desired result. \square

Corollary 12. In Theorem 11, if we choose $x = (2a + \eta(b, a))/2$, we obtain the following midpoint inequality:

$$\left| f\left(\frac{2a + \eta(b, a)}{2}\right) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(u) du \right|$$

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a} \int_a^b f(u) du \right| \\ &\leq \frac{(b-a)|f'(a)|}{(p+1)^{1/p}} \begin{cases} \left(\frac{x-a}{b-a}\right)^2 + \left(\frac{b-x}{b-a}\right)^2 & \text{if } A = 1, \\ \left(\frac{x-a}{b-a}\right)^{1+1/p} \left(\frac{A^{q((x-a)/(b-a))} - 1}{q \ln A}\right)^{1/q} + \left(\frac{b-x}{b-a}\right)^{1+1/p} \left(\frac{A^q - A^{q((x-a)/(b-a))}}{q \ln A}\right)^{1/q} & \text{if } A \neq 1, \end{cases} \end{aligned} \quad (23)$$

for all $x \in [a, b]$, where $A = |f'(b)|/|f'(a)|$.

$$\begin{aligned} &\left| f(x) - \frac{2}{a+b} \int_a^{a+(a+b)/2} f(u) du \right| \\ &\leq \frac{(a+b)|f'(a)|}{2(p+1)^{1/p}} \\ &\cdot \begin{cases} \frac{4}{(a+b)^2} \left[(x-a)^2 + \left(\frac{b+3a-2x}{2}\right)^2 \right] & \text{if } A = 1, \\ \left(\frac{x-a}{a+b}\right)^{1+1/p} \left(\frac{A^{2q((x-a)/(a+b))} - 1}{q \ln A}\right)^{1/q} + \left(\frac{b+3a-2x}{2}\right)^{1+1/p} \left(\frac{A^q - A^{2q((x-a)/(a+b))}}{q \ln A}\right)^{1/q} & \text{if } A \neq 1. \end{cases} \end{aligned} \quad (24)$$

Theorem 15. Let $K \subseteq [0, \infty)$ be an invex subset with respect to $\eta : K \times K \rightarrow \mathbb{R}$ and $a, b \in K^\circ$ (K° interior of K) with $\eta(b, a) > 0$ and $[a, a + \eta(b, a)] \subset K$. Let $f : [a, a + \eta(b, a)] \rightarrow (0, \infty)$

Corollary 13. Let $f : [a, b] \rightarrow (0, \infty)$ be a differentiable function such that $f' \in L([a, b])$ and $f'(a) \neq 0$; let $q > 1$ with $1/p + 1/q = 1$. If $|f'|^q$ is a logarithmically convex function, then the following inequality holds:

Example 14. In Theorem 11, if we choose $\eta(b, a) = (a+b)/2$, the arithmetic mean, we obtain the following inequality:

$$\begin{aligned} &\left| f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(u) du \right| \leq \frac{\eta(b, a)}{2^{1-1/q}} |f'(a)| \\ &\cdot \begin{cases} \frac{1}{2^{1/q}} \left(\left(\frac{x-a}{\eta(b, a)}\right)^2 + \left(1 - \frac{x-a}{\eta(b, a)}\right)^2 \right) & \text{if } A = 1, \\ \left(\left(\frac{x-a}{\eta(b, a)}\right)^{2-2/q} \left(\frac{x-a}{\eta(b, a)} \frac{A^{q((x-a)/\eta(b,a))}}{\ln A} + \frac{1-A^{q((x-a)/\eta(b,a))}}{\ln^2 A}\right)^{1/q} + \left(1 - \frac{x-a}{\eta(b, a)}\right)^{2-2/q} \left(\frac{A^q - A^{q((x-a)/\eta(b,a))}}{\ln^2 A} - \left(1 - \frac{x-a}{\eta(b, a)}\right) \frac{A^{q((x-a)/\eta(b,a))}}{\ln A}\right)^{1/q} \right) & \text{if } A \neq 1, \end{cases} \end{aligned} \quad (25)$$

for all $x \in [a, a + \eta(b, a)]$, where $A = |f'(b)|/|f'(a)|$.

be a differentiable function such that $f' \in L([a, a + \eta(b, a)])$ and $f'(a) \neq 0$; let $q > 1$. If $|f'|^q$ is a logarithmically preinvex function, then the following inequality holds:

Proof. From Lemma 2, properties of modulus, and power mean inequality, we have

$$\begin{aligned}
& \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) du \right| \leq \eta(b,a) \\
& \cdot \left(\left(\int_0^{(x-a)/\eta(b,a)} t dt \right)^{1-1/q} \right. \\
& \cdot \left(\int_0^{(x-a)/\eta(b,a)} t |f'(a+t\eta(b,a))|^q dt \right)^{1/q} \\
& + \left(\int_{(x-a)/\eta(b,a)}^1 (1-t) dt \right)^{1-1/q} \\
& \cdot \left(\int_{(x-a)/\eta(b,a)}^1 (1-t) |f'(a+t\eta(b,a))|^q dt \right)^{1/q} \Big) \quad (26) \\
& = \frac{\eta(b,a)}{2^{1-1/q}} \left(\left(\frac{x-a}{\eta(b,a)} \right)^{2(1-1/q)} \right. \\
& \cdot \left(\int_0^{(x-a)/\eta(b,a)} t |f'(a+t\eta(b,a))|^q dt \right)^{1/q} \\
& + \left(1 - \frac{x-a}{\eta(b,a)} \right)^{2(1-1/q)} \\
& \cdot \left. \left(\int_{(x-a)/\eta(b,a)}^1 (1-t) |f'(a+t\eta(b,a))|^q dt \right)^{1/q} \right).
\end{aligned}$$

Since $|f'|^q$ is a logarithmically preinvex function, we deduce

$$\begin{aligned}
& \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) du \right| \leq \frac{\eta(b,a)}{2^{1-1/q}} |f'(a)| \\
& \cdot \left(\left(\frac{x-a}{\eta(b,a)} \right)^{2-2/q} \left(\int_0^{(x-a)/\eta(b,a)} t A^{qt} dt \right)^{1/q} \right. \\
& + \left(1 - \frac{x-a}{\eta(b,a)} \right)^{2-2/q} \\
& \cdot \left. \left(\int_{(x-a)/\eta(b,a)}^1 (1-t) A^{qt} dt \right)^{1/q} \right). \quad (27)
\end{aligned}$$

In the case where $A = 1$, (27) gives

$$\begin{aligned}
& \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) du \right| \\
& \leq \frac{\eta(b,a)}{2} |f'(a)| \\
& \cdot \left(\left(\frac{x-a}{\eta(b,a)} \right)^2 + \left(1 - \frac{x-a}{\eta(b,a)} \right)^2 \right). \quad (28)
\end{aligned}$$

For $A \neq 1$, (27) gives

$$\begin{aligned}
& \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) du \right| \leq \frac{\eta(b,a)}{2^{1-1/q}} |f'(a)| \\
& \cdot \left(\left(\frac{x-a}{\eta(b,a)} \right)^{2-2/q} \left(\frac{x-a}{\eta(b,a)} \frac{A^{q((x-a)/\eta(b,a))}}{\ln A} \right. \right. \\
& + \frac{1 - A^{q((x-a)/\eta(b,a))}}{\ln^2 A} \Big)^{1/q} + \left(1 - \frac{x-a}{\eta(b,a)} \right)^{2-2/q} \\
& \cdot \left(\frac{A^q - A^{q((x-a)/\eta(b,a))}}{\ln^2 A} \right. \\
& \left. \left. - \left(1 - \frac{x-a}{\eta(b,a)} \right) \frac{A^{q((x-a)/\eta(b,a))}}{\ln A} \right)^{1/q} \right), \quad (29)
\end{aligned}$$

where we use the fact that

$$\begin{aligned}
& \int_0^{(x-a)/\eta(b,a)} t A^{qt} dt \\
& = \frac{x-a}{\eta(b,a)} \frac{A^{q(x-a)/\eta(b,a)}}{\ln A} + \frac{1 - A^{q(x-a)/\eta(b,a)}}{\ln^2 A}, \\
& \int_{(x-a)/\eta(b,a)}^1 (1-t) A^{qt} dt \quad (30) \\
& = \frac{A^q - A^{q((x-a)/\eta(b,a))}}{\ln^2 A} \\
& - \left(1 - \frac{x-a}{\eta(b,a)} \right) \frac{A^{q((x-a)/\eta(b,a))}}{\ln A}.
\end{aligned}$$

From (28) and (29), we obtain the desired result. \square

Corollary 16. In Theorem 15, if we choose $x = (2a + \eta(b,a))/2$, we obtain the following midpoint inequality:

$$\begin{aligned}
& \left| f\left(\frac{2a + \eta(b,a)}{2}\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) du \right| \\
& \leq \frac{\eta(b,a)}{4} |f'(a)| \begin{cases} 1 & \text{if } A = 1, \\ \frac{1}{2^{1-3/q}} \left(\left(\frac{1 - A^{q/2}}{\ln^2 A} + \frac{A^{q/2}}{2 \ln A} \right)^{1/q} + A^{1/2} \left(\frac{A^{q/2} - 1}{\ln^2 A} - \frac{1}{2 \ln A} \right)^{1/q} \right) & \text{if } A \neq 1. \end{cases} \quad (31)
\end{aligned}$$

Corollary 17. Let $f : [a, a + \eta(b, a)] \rightarrow (0, \infty)$ be a differentiable function such that $f' \in L([a, a + \eta(b, a)])$ and $f'(a) \neq 0$; let $q > 1$. If $|f'|^q$ is a logarithmically convex function, then the following inequality holds:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{(b-a)}{2^{1-1/q}} |f'(a)|$$

$$\cdot \begin{cases} \frac{1}{2^{1/q}} \left(\left(\frac{x-a}{b-a} \right)^2 + \left(\frac{b-x}{b-a} \right)^2 \right) & \text{if } A = 1, \\ \left(\left(\frac{x-a}{b-a} \right)^{2-2/q} \left(\frac{x-a}{b-a} \frac{A^{q((x-a)/(b-a))}}{\ln A} + \frac{1-A^{q((x-a)/(b-a))}}{\ln^2 A} \right)^{1/q} + \left(\frac{b-x}{b-a} \right)^{2-2/q} \left(\frac{A^q - A^{q((x-a)/(b-a))}}{\ln^2 A} - \left(\frac{b-x}{b-a} \right) \frac{A^{q((x-a)/(b-a))}}{\ln A} \right)^{1/q} \right) & \text{if } A \neq 1, \end{cases} \quad (32)$$

for all $x \in [a, b]$, where $A = |f'(b)|/|f'(a)|$.

Example 18. In Theorem 15, if we choose $\eta(b, a) = (b-a)/(\ln b - \ln a)$ with $a \neq b$, the logarithmic mean, we obtain the following inequality:

$$\left| f(x) - \frac{\ln b - \ln a}{b-a} \int_a^{a+(b-a)/(\ln b - \ln a)} f(u) du \right| \leq \frac{b-a}{2^{1-1/q} (\ln b - \ln a)} |f'(a)|$$

$$\cdot \begin{cases} \frac{1}{2^{1/q}} (\theta^2 + (1-\theta)^2) & \text{if } A = 1, \\ \left(\theta^{2-2/q} \left(\theta \frac{A^{q\theta}}{\ln A} + \frac{1-A^{q\theta}}{\ln^2 A} \right)^{1/q} + (1-\theta)^{2-2/q} \left(\frac{A^q - A^{q\theta}}{\ln^2 A} - (1-\theta) \frac{A^{q\theta}}{\ln A} \right)^{1/q} \right) & \text{if } A \neq 1, \end{cases} \quad (33)$$

where $\theta = (x-a)(\ln b - \ln a)/(b-a)$.

Theorem 19. Suppose that all the assumptions of Theorem 15 are satisfied, then the following inequality holds:

$$\left| f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(u) du \right| \leq \eta(b, a) |f'(a)|$$

$$\cdot \begin{cases} \frac{1}{(q+1)^{1/q}} \left(\left(\frac{x-a}{\eta(b, a)} \right)^2 + \left(1 - \frac{x-a}{\eta(b, a)} \right)^2 \right) & \text{if } A = 1, \\ A^{(x-a)/\eta(b, a)} \left(\left(1 - \frac{x-a}{\eta(b, a)} \right) \frac{x-a}{\eta(b, a)} \left(\sum_{i=1}^{\infty} \frac{(\ln A)^{q(1-(x-a)/\eta(b, a))}}{(\alpha)_i} \right)^{i-1} \right)^{1/q} + \left(\frac{x-a}{\eta(b, a)} \right)^2 \left(\sum_{i=1}^{\infty} \frac{(-\ln A)^{q((x-a)/\eta(b, a))}}{(\alpha)_i} \right)^{i-1} & \text{if } A \neq 1, \end{cases} \quad (34)$$

for all $x \in [a, a + \eta(b, a)]$, where $A = |f'(b)|/|f'(a)|$ and $(q+1)_i = \prod_{j=0}^{i-1} (q+1+j)$.

Proof. From Lemma 2, properties of modulus, and power mean inequality, we have

$$\left| f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(u) du \right| \leq \eta(b, a)$$

$$\cdot \left(\left(\int_0^{(x-a)/\eta(b, a)} dt \right)^{1-1/q} \cdot \left(\left| f'(a + t\eta(b, a)) \right|^q dt \right)^{1/q} \right) = \eta(b, a)$$

$$\cdot \left(\left(\frac{x-a}{\eta(b, a)} \right)^{1-1/q} \cdot \left(\left| f'(a + t\eta(b, a)) \right|^q dt \right)^{1/q} \right)$$

$$\cdot \left(\int_0^{(x-a)/\eta(b, a)} t^q |f'(a + t\eta(b, a))|^q dt \right)^{1/q}$$

$$+ \left(\int_{(x-a)/\eta(b, a)}^1 dt \right)^{1-1/q} \left(\int_{(x-a)/\eta(b, a)}^1 (1-t)^q \right)$$

$$\cdot \left(\left| f'(a + t\eta(b, a)) \right|^q dt \right)^{1/q} \right)$$

$$\cdot \left(\left(\frac{x-a}{\eta(b, a)} \right)^{1-1/q} \cdot \left(\left| f'(a + t\eta(b, a)) \right|^q dt \right)^{1/q} \right)$$

$$\begin{aligned}
& \cdot \left(\left(\int_0^{(x-a)/\eta(b,a)} t^q |f'(a + t\eta(b,a))|^q dt \right)^{1/q} + \left(1 - \frac{x-a}{\eta(b,a)} \right) \frac{x-a}{\eta(b,a)} A^{(x-a)/\eta(b,a)} \right. \\
& \quad \cdot \left(\left(\int_{(x-a)/\eta(b,a)}^1 (1-t)^q |f'(a + t\eta(b,a))|^q dt \right)^{1/q} \right. \\
& \quad \cdot \left. \left. \left(\int_0^1 (t-1)^q (A^{q(1-(x-a)/\eta(b,a))})^t dt \right)^{1/q} \right) \right). \tag{35}
\end{aligned}$$

Applying Lemma 6 with $z = 1$, we get

Since $|f'|^q$ is a logarithmically preinvex function, we deduce

$$\begin{aligned}
& \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) du \right| \leq \eta(b,a) |f'(a)| \\
& \cdot \left(\left(\frac{x-a}{\eta(b,a)} \right)^{1-1/q} \left(\int_0^{(x-a)/\eta(b,a)} t^q A^{qt} dt \right)^{1/q} \right. \\
& \quad \left. + \left(1 - \frac{x-a}{\eta(b,a)} \right)^{1-1/q} \right. \\
& \quad \cdot \left. \left(\int_{(x-a)/\eta(b,a)}^1 (1-t)^q A^{qt} dt \right)^{1/q} \right). \tag{36}
\end{aligned}$$

$$\begin{aligned}
& \int_0^1 t^q (A^{q((x-a)/\eta(b,a))})^t dt \\
& = A^{q((x-a)/\eta(b,a))} \sum_{i=1}^{\infty} \frac{(-\ln A^{q((x-a)/\eta(b,a))})^{i-1}}{(\alpha)_i} \\
& \int_0^1 (t-1)^q (A^{q(1-(x-a)/\eta(b,a))})^t dt \\
& = \sum_{i=1}^{\infty} \frac{(\ln A^{q(1-(x-a)/\eta(b,a))})^{i-1}}{(\alpha)_i}. \tag{39}
\end{aligned}$$

Substituting (39) into (38), we obtain

If $A = 1$, (36) gives

$$\begin{aligned}
& \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) du \right| \\
& \leq \frac{\eta(b,a) |f'(a)|}{(q+1)^{1/q}} \\
& \cdot \left(\left(\frac{x-a}{\eta(b,a)} \right)^2 + \left(1 - \frac{x-a}{\eta(b,a)} \right)^2 \right). \tag{37}
\end{aligned}$$

In the case where $A \neq 1$, we can restate (36) as

$$\begin{aligned}
& \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) du \right| \leq \eta(b,a) |f'(a)| \\
& \cdot \left(\left(\frac{x-a}{\eta(b,a)} \right)^2 \left(\int_0^1 t^q (A^{q((x-a)/\eta(b,a))})^t dt \right)^{1/q} \right)
\end{aligned}$$

$$\begin{aligned}
& \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) du \right| \leq \eta(b,a) |f'(a)| \\
& \cdot \left(\left(\frac{x-a}{\eta(b,a)} \right)^2 \right. \\
& \quad \cdot A^{(x-a)/\eta(b,a)} \left(\sum_{i=1}^{\infty} \frac{(-\ln A^{q((x-a)/\eta(b,a))})^{i-1}}{(\alpha)_i} \right)^{1/q} \\
& \quad + \left(1 - \frac{x-a}{\eta(b,a)} \right) \frac{x-a}{\eta(b,a)} A^{(x-a)/\eta(b,a)} \\
& \quad \cdot \left. \left(\sum_{i=1}^{\infty} \frac{(\ln A^{q(1-(x-a)/\eta(b,a))})^{i-1}}{(\alpha)_i} \right)^{1/q} \right). \tag{40}
\end{aligned}$$

The desired result follows from (37) and (40). \square

Corollary 20. In Theorem 19, if we choose $x = (2a + \eta(b,a))/2$, we obtain the following midpoint inequality:

$$\begin{aligned}
& \left| f\left(\frac{2a + \eta(b, a)}{2}\right) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(u) du \right| \\
& \leq \frac{\eta(b, a) |f'(a)|}{2} \begin{cases} \frac{1}{(q+1)^{1/q}} & \text{if } A = 1, \\ \frac{\sqrt{A}}{2} \left(\left(\sum_{i=1}^{\infty} \frac{(\ln A^{q/2})^{i-1}}{(\alpha)_i} \right)^{1/q} + \left(\sum_{i=1}^{\infty} \frac{(-\ln A^{q/2})^{i-1}}{(\alpha)_i} \right)^{1/q} \right) & \text{if } A \neq 1. \end{cases} \quad (41)
\end{aligned}$$

Corollary 21. In Theorem 19, if we choose $\eta(b, a) = b - a$, we obtain the following inequality:

$$\begin{aligned}
& \left| f(x) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq (b-a) |f'(a)| \\
& \cdot \begin{cases} \frac{1}{(q+1)^{1/q}} \left(\left(\frac{x-a}{b-a} \right)^2 + \left(\frac{b-x}{b-a} \right)^2 \right) & \text{if } A = 1, \\ A^{(x-a)/(b-a)} \left(\frac{(b-x)(x-a)}{(b-a)^2} \left(\sum_{i=1}^{\infty} \frac{(\ln A^{q((b-x)/(b-a))})^{i-1}}{(\alpha)_i} \right)^{1/q} + \left(\frac{x-a}{b-a} \right)^2 \left(\sum_{i=1}^{\infty} \frac{(-\ln A^{q((x-a)/(b-a))})^{i-1}}{(\alpha)_i} \right)^{1/q} \right) & \text{if } A \neq 1. \end{cases} \quad (42)
\end{aligned}$$

Remark 22. In all the above theorems, inequalities for non-convex functions could be drawn by just replacing $\eta(b, a)$ by other means than those in the previously mentioned examples.

Competing Interests

The author declares that they have no competing interests.

References

- [1] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, vol. 61 of *Mathematics and Its Applications (East European Series)*, Kluwer Academic Publishers Group, Dordrecht, The Netherlands, 1993.
- [2] M. W. Alomari, M. E. Özdemir, and H. Kavurmac, “On companion of Ostrowski inequality for mappings whose first derivatives absolute value are convex with applications,” *Miskolc Mathematical Notes*, vol. 13, no. 2, pp. 233–248, 2012.
- [3] N. S. Barnett, P. Cerone, S. S. Dragomir, M. R. Pinheiro, and A. Sofo, “Ostrowski type inequalities for functions whose modulus of derivatives are convex and applications,” *RGMIA Research Report Collection*, vol. 5, no. 2, article 1, 2002.
- [4] P. Cerone and S. S. Dragomir, “Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions,” *Demonstratio Mathematica*, vol. 37, no. 2, pp. 299–308, 2004.
- [5] I. Işcan, “Ostrowski type inequalities for functions whose derivatives are preinvex,” *Bulletin of the Iranian Mathematical Society*, vol. 40, no. 2, pp. 373–386, 2014.
- [6] M. E. Kiris and M. Z. Sarikaya, “On Ostrowski type inequalities and Čebyšev type inequalities with applications,” *Filomat*, vol. 29, no. 8, pp. 1695–1713, 2015.
- [7] M. A. Noor, K. I. Noor, and M. U. Awan, “Some quantum integral inequalities via preinvex functions,” *Applied Mathematics and Computation*, vol. 269, pp. 242–251, 2015.
- [8] M. A. Noor, K. I. Noor, and M. U. Awan, “Fractional Ostrowski inequalities for (s, m) -Godunova-Levin functions,” *Facta Universitatis, Series: Mathematics and Informatics*, vol. 30, no. 4, pp. 489–499, 2015.
- [9] E. Set, M. E. Özdemir, M. Z. Sarikaya, and M. Z. Sarikaya, “New inequalities of Ostrowski’s type for s -convex functions in the second sense with applications,” *Facta Universitatis, Series: Mathematics and Informatics*, vol. 27, no. 1, pp. 67–82, 2012.
- [10] E. Set, “New inequalities of Ostrowski type for mappings whose derivatives are s -convex in the second sense via fractional integrals,” *Computers & Mathematics with Applications*, vol. 63, no. 7, pp. 1147–1154, 2012.
- [11] M. A. Hanson, “On sufficiency of the Kuhn-Tucker conditions,” *Journal of Mathematical Analysis and Applications*, vol. 80, no. 2, pp. 545–550, 1981.
- [12] A. Ben-Israel and B. Mond, “What is invexity?” *The Journal of the Australian Mathematical Society—Series B: Applied Mathematics*, vol. 28, no. 1, pp. 1–9, 1986.
- [13] R. Pini, “Invexity and generalized convexity,” *Optimization*, vol. 22, no. 4, pp. 513–525, 1991.
- [14] M. A. Noor, “Variational-like inequalities,” *Optimization*, vol. 30, no. 4, pp. 323–330, 1994.
- [15] M. A. Noor, “Invex equilibrium problems,” *Journal of Mathematical Analysis and Applications*, vol. 302, no. 2, pp. 463–475, 2005.

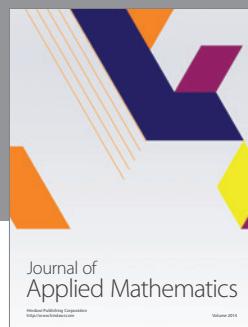
- [16] X. M. Yang and D. Li, "On properties of preinvex functions," *Journal of Mathematical Analysis and Applications*, vol. 256, no. 1, pp. 229–241, 2001.
- [17] T. Weir and B. Mond, "Pre-invex functions in multiple objective optimization," *Journal of Mathematical Analysis and Applications*, vol. 136, no. 1, pp. 29–38, 1988.
- [18] J. E. Pečarić, F. Proschan, and Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, vol. 187 of *Mathematics in Science and Engineering*, Academic Press, Boston, Mass, USA, 1992.
- [19] J. Wang, J. Deng, and M. Fečkan, "Hermite-Hadamard-type inequalities for r -convex functions based on the use of Riemann-Liouville fractional integrals," *Ukrainian Mathematical Journal*, vol. 65, no. 2, pp. 193–211, 2013.



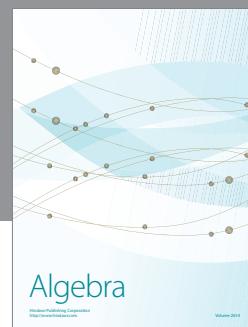
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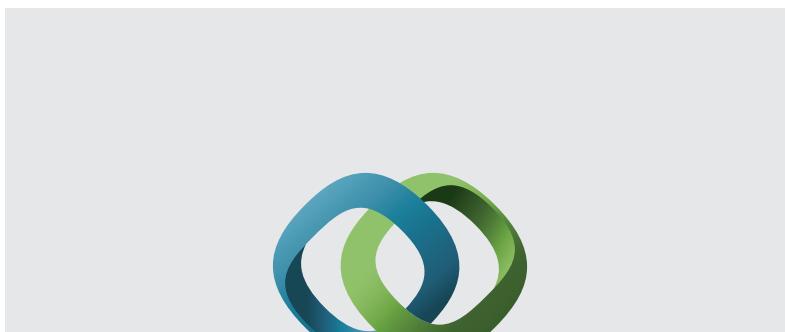
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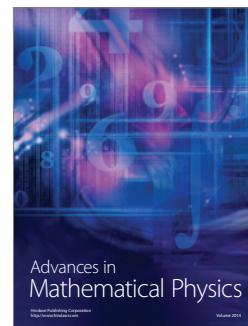


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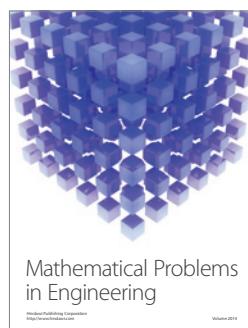
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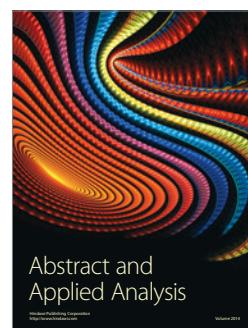
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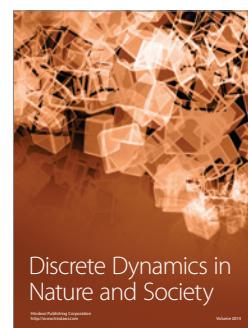
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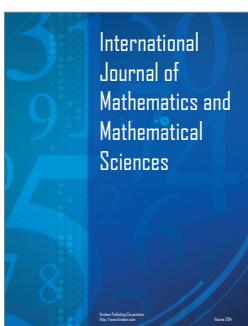
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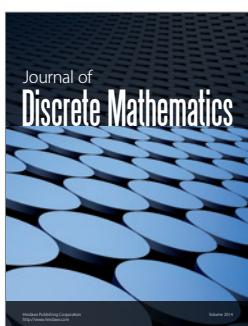
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