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Research Article

Analytic Comparison of MHD Squeezing Flow in Porous Medium with Slip Condition

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The aim of this paper is to compare the efficiency of various techniques for squeezing flow of an incompressible viscous fluid in a porous medium under the influence of a uniform magnetic field squeezed between two large parallel plates having slip boundary. Fourth-order nonlinear ordinary differential equation is obtained by transforming the Navier-Stokes equations. Resulting boundary value problem is solved using Differential Transform Method (DTM), Daftardar Jafari Method (DJM), Adomian Decomposition Method (ADM), Homotopy Perturbation Method (HPM), and Optimal Homotopy Asymptotic Method (OHAM). The problem is also solved numerically using Mathematica solver NDSolve. The residuals of the problem are used to compare and analyze the efficiency and consistency of the abovementioned schemes.

1. Introduction

The study of squeezing flow started in 19th century and it continues to receive considerable attention due to its practical applications in physical and biophysical areas, namely, food industry, chemical engineering, polymer processing, compression, and injection modeling. Stefan [1] accomplished elementary research in this field. Analysis of Newtonian fluid squeezed between two infinite planar plates is studied by Ran et al. [2]. Thin Newtonian liquid films squeezing between two plates were studied by Grimm [3]. Squeezing flow under the influence of magnetic field is broadly applied to bearing with liquid-metal lubrication [4–7].

The study of magnetic field effects on lubrication fluid films has attracted many scientists for a number of years. The flows of electrically conducting fluid through porous medium have attained incomparable status and have been the limelight of concern of many researchers in the last few decades. The particular applications are investigated in the study of ground water flow, irrigation problems, crude petroleum recovery, heat-storage beds, thermal and insulating engineering, chromatography, chemical catalytic reactors, and many more. Hughes and Elco [8] investigated the dynamics of an electrically conducting fluid in the presence of magnetic field between two parallel disks, one rotating at a constant angular velocity, for two cases, an axial magnetic field with a radial current and a radial magnetic field with an axial current. They discovered that the magnetic field affects the load capacity of the bearing and that the frictional torque on the rotor becomes zero for both the cases by applying electrical energy through the electrodes to the fluid. Ullah et al. studied the squeezing flow, in a porous medium, of a Newtonian fluid under the influence of imposed magnetic field [9]. The velocity profile of the fluid is discussed in the last work by considering various relations between the values of Reynolds and Hartmann number.

High order nonlinear boundary value problems arise in the study of squeezing flow of Newtonian as well as non-Newtonian fluids. The exact solution of these problems is sometimes difficult to find due to the mathematical complexity of Navier-Stokes equations. In order to solve these problems, various seminumerical techniques are widely used. We discuss here one by one these techniques and apply them to obtain the velocity profile of the fluid.

Homotopy Perturbation Method (HPM) was first introduced by He [10, 11]. Marinca et al. [12, 13] introduced OHAM for approximate solution of nonlinear problems of thin film flow of a fourth-grade fluid down a vertical cylinder and for the study of the behavior of nonlinear mechanical vibration of electrical machines. It is scrutinized that HPM and HAM are the special cases of OHAM [14].

Differential Transform Method (DTM) was initially introduced by Zhou in 1986 [15]. Islam et al. [16] successfully applied this technique for squeezing flow of a Newtonian fluid in porous medium channel. Ullah et al. [17] investigated the squeezing fluid flow under the influence of magnetic field with slip boundary condition using DTM. Ayaz [18] studied the applications of two-dimensional DTM in case of partial differential equations. Hassan [19] compared DTM with ADM in solving PDEs.

Adomian [20, 21] (1923–1996), in 1980, introduced Adomian Decomposition Method for solving nonlinear functional equations. The technique is based on the decomposition of solution of nonlinear operator equation in a series of functions. Wazwaz [22] introduced the modified form of ADM and used it in many BVPs successfully. The basic idea of Daftardar Jafari Method (DJM) is introduced by Daftardar-Gejji et al. [23, 24] to solve fractional boundary value problems with Dirichlet boundary conditions. The solution of fifth- and sixth-order boundary value problem using DJM is studied by Ullah et al. and they got excellent results [25].

The goal of this research paper is to solve the model of squeezing flow of a Newtonian fluid in a porous medium with MHD effect by using HPM, OHAM, DTM, ADM, NIM, and the Mathematica solver NDSolve. Furthermore, to check the efficiency of each scheme, the residuals of the problem are used. Preparation of the model and basic ideas of the mentioned techniques along with their applications are discussed in the respective sections.

2. Problem Modeling

The continuity and momentum equation for steady squeezing flow in a porous medium under the influence of magnetic field, as shown in Figure 1, are

$$\nabla \cdot W = 0,$$
$$\rho D W = \nabla \cdot T + J \times B + r. \tag{1}$$

Here W is the velocity vector, ∇ is the material time derivative, and T is the Cauchy stress tensor given by $T = -pI + \mu A$ with $A = \nabla W + (\nabla W)^t$. B is the total magnetic field given by $B = B_0 + b$. B_0 and b represent the imposed and induced magnetic fields, respectively. r is Darcy's resistance given by [26, 27]

$$r = -\frac{\mu W}{k}. \tag{2}$$

The magnetohydrodynamic force can be written as follows:

$$J \times B = -\sigma B_0^2 W. \tag{3}$$

Suppose that the magnetic field is applied along z-axis and the plates are nonconducting. For small velocity w, the gap distance $2L$ between the plates changes slowly with time t so that it can be taken constant. The flow is axisymmetric with

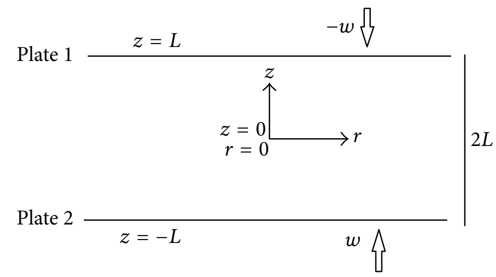

Figure 1: Geometry of the squeezing flow.

z-axis perpendicular to plates and $z = \pm L$ at the plates. The components of W for the present case are $W = (w_r, 0, w_z)$. If $P = (\rho/2)(w_r^2 + w_z^2) + p$ is the generalized pressure and the flow is steady then by comparing components the Navier-Stokes equations (1) can be written as

$$\frac{\partial P}{\partial r} - \rho\left(\frac{\partial w_z}{\partial r} - \frac{\partial w_r}{\partial z}\right) w_z = -\left(\mu \frac{\partial}{\partial z}\left(\frac{\partial w_z}{\partial r} - \frac{\partial w_r}{\partial z}\right) + \left(\frac{\mu}{k} + \sigma B_0^2\right) w_r\right), \tag{4}$$

$$\frac{\partial P}{\partial z} + \rho\left(\frac{\partial w_z}{\partial r} - \frac{\partial w_r}{\partial z}\right) w_r = \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\left(\frac{\partial w_z}{\partial r} - \frac{\partial w_r}{\partial z}\right)\right) - \left(\frac{\mu}{k}\right) w_z. \tag{5}$$

Introducing stream function $\psi(r, z)$ [9], eliminating the generalized pressure P from (4) and (5), and using the transformation $\psi(r, z) = r^2 t(z)$ and the boundary conditions

at $z = 0$

$$w_z = 0, \quad \frac{\partial w_r}{\partial z} = 0, \tag{6}$$

at $z = L$

$$w_r = \beta \frac{\partial w_r}{\partial z}, \quad w_z = -w. \tag{7}$$

We have

$$t^{(\mathrm{iv})}(z) - \left(\frac{1}{k} + \frac{\sigma B_0^2}{\mu}\right) t''(z) + 2\frac{\rho}{\mu} t(z)\, t'''(z) = 0, \tag{8}$$

subject to boundary conditions

$$t(0) = 0, \quad t''(0) = 0, \quad t(L) = \frac{w}{2}, \quad t'(L) = \beta t''(L). \tag{9}$$

Introducing nondimensional parameters,

$$T^{*}=\frac{t}{w/2},\quad z^{*}=\frac{z}{L},\quad \mathscr{R}=\frac{\rho L w}{\mu},\quad \mathscr{M}=L\sqrt{\frac{1}{k}+\frac{\sigma B_0^2}{\mu}}. \tag{10}$$

Omitting *, (8) and (9) become

$$T^{(\mathrm{iv})}(z)-\mathscr{M}^2T''(z)+\mathscr{R}T(z)T'''(z)=0, \tag{11}$$

$$T(0)=0,\quad T''(0)=0,\quad T(1)=1,\quad T'(1)=\gamma T''(1). \tag{12}$$

We solve (11) and (12) by fixing $\mathscr{M}=1$, $\mathscr{R}=1$, and $\gamma=1$ to find the particular solution in each case for comparison purpose.

3. Basic Idea and Application of DTM

For the function $t(z)$, one-dimensional differential transform is defined as follows [28, 29]:

$$T(z)=\frac{1}{k!}\left[\frac{d^k t(z)}{dz^k}\right]_{z=0}. \tag{13}$$

The inverse transform of $T(z)$ is defined as follows:

$$t(z)=\sum_{k=0}^{\infty} z^k T(z). \tag{14}$$

Combining (13) and (14), we can write

$$t(z)=\sum_{k=0}^{\infty}\frac{z^k}{k!}\left[\frac{d^k t(z)}{dz^k}\right]. \tag{15}$$

$t(z)$ in a finite series is as follows:

$$t(z)=\sum_{k=0}^{N}\frac{z^k}{k!}\left[\frac{d^k t(z)}{dz^k}\right], \tag{16}$$

which means that

$$t(z)=\sum_{k=N+1}^{\infty}\frac{z^k}{k!}\left[\frac{d^k t(z)}{dz^k}\right] \tag{17}$$

can be considered negligibly small.

Some fundamental theorems on one-dimensional differential transform are as follows.

Theorem 1. *If $t(z)=f(z)\pm h(z)$, then $T(k)=F(k)\pm H(k)$.*

Theorem 2. *If $t(z)=h^{(n)}(z)$, then $T(z)=((k+n)!/k!)H(k+n)$.*

Theorem 3. *If $t(z)=h(z)\cdot f(z)$, then $T(z)=\sum_{r=0}^{k}H(r)F(k-r)$.*

Theorem 4. *If $t(z)=z^n$, then*

$$T(z)=\delta(k-n)=\begin{cases}1 & if\ k=n\\ 0 & if\ k\neq n.\end{cases} \tag{18}$$

Keeping in view the abovementioned theorems, the differential transform of (11) is given by

$$\widetilde{T}(n+4)=\frac{n!}{(n+4)!}\left(m^2(n+1)(n+2)\widetilde{T}(n+2)-R\sum_{r=0}^{n}(r+1)(r+2)(r+3)\widetilde{T}(r+3)\widetilde{T}(n-r)\right), \tag{19}$$

with transformed boundary conditions

$$\widetilde{T}(0)=0,\quad \widetilde{T}(1)=\alpha,\quad \widetilde{T}(2)=0,\quad \widetilde{T}(3)=\beta. \tag{20}$$

Using (19) and (20), the values of $\widetilde{T}(i)$, $i=1,2,3,\ldots,15$, are

$$\widetilde{T}(2n)=0,\quad \text{for } n=0(1)7,$$

$$\widetilde{T}(5)=\frac{1}{120}(6\beta-6\alpha\beta),$$

$$\widetilde{T}(7)=\frac{1}{840\beta}\left(1-4\alpha+3\left(-2\beta+\alpha^2\right)\right),$$

$$\widetilde{T}(9)=\frac{\beta}{60480}\left(1-9\alpha+3\alpha\left(32\beta-5\alpha^2\right)+\left(-72\beta+23\alpha^2\right)\right),$$

$$\widetilde{T}(11)=\frac{\beta}{6652800}\left(1-16\alpha-44\alpha\left(-39\beta+4\alpha^2\right)+\left(-414\beta+86\alpha^2\right)+3\left(432\beta^2-482\alpha^2\beta+35\alpha^4\right)\right),$$

$$\widetilde{T}(13)=\frac{\beta}{1037836800}\left(1-25\alpha+2\alpha\left(7446\beta-475\alpha^2\right)+2\left(-948b+115\alpha^2\right)-9\alpha\left(7384\beta^2-2620\alpha^2\beta+105\alpha^4\right)+3\left(14616\beta^2-11808\alpha^2\beta+563\alpha^4\right)\right),$$

$$\widetilde{T}(15)=\frac{\beta}{217945728000}\left(1-36\alpha+24\alpha\left(4106\beta-145\alpha^{2}\right)+\left(-7974\beta+505\alpha^{2}\right)-12\alpha\left(239508\beta^{2}-62828\alpha^{2}\beta+1627\alpha^{4}\right)+\left(703296\beta^{2}-427716\alpha^{2}\beta+12139\alpha^{4}\right)+9\left(-151344\beta^{3}+282256\alpha^{2}\beta^{2}-47590\alpha^{4}\beta+1155\alpha^{6}\right)\right). \quad (21)$$

To find the values of α and β, we use the following transformed boundary conditions:

$$\sum_{n=0}^{15}\left(n\widetilde{T}[n]-n(n-1)\widetilde{T}[n]\right)=0,\qquad \sum_{n=0}^{15}\widetilde{T}[n]=1, \quad (22)$$

which leads us to the following values:

$$\alpha=0.754966,\qquad \beta=0.242565. \quad (23)$$

The approximate solution of the problem is as follows:

$$T(z)=0.754966z+0.242565z^{3}+0.00297182z^{5}-0.00050977z^{7}+3.913152541684915\times10^{-6}z^{9}+3.1529964918115006\times10^{-6}z^{11}-1.2149332568415323\times10^{-7}z^{13}-2.0892317176411676\times10^{-8}z^{15}. \quad (24)$$

4. Basic Idea and Application of DJM

Consider the nonlinear boundary value problem [9, 23, 24]:

$$\mathfrak{L}(T(z))+\mathfrak{M}(T(z))+\mathfrak{N}(T(z))=f(z), \quad (25)$$

where $\mathfrak{L}$ represents the highest order derivative with respect to z, $\mathfrak{M}$ is the linear term reminder, and $\mathfrak{N}$ represents the nonlinear term. Using the operator $\mathfrak{L}=d^{4}/dx^{4}$, (25) becomes

$$T(z)=\eta_0+\eta_1 z+\eta_2\frac{z^{2}}{2!}+\eta_3\frac{z^{3}}{3!}+\mathfrak{L}^{-1}f(z)-\mathfrak{L}^{-1}\mathfrak{M}(T(z))-\mathfrak{L}^{-1}\mathfrak{N}(T(z)). \quad (26)$$

η_i are constants to be determined later. The function $T(z)$ is then expressed by the infinite series as

$$T(z)=\sum_{k=0}^{\infty}T_k(z). \quad (27)$$

The nonlinear term $\mathfrak{N}(T(z))$ is written in the sum of Daftardar-Geiji et al. polynomials as

$$\mathfrak{N}(T(z))=\sum_{n=0}^{\infty}G_n. \quad (28)$$

Here G_n's are defined as

$$G_0(z)=\mathfrak{N}(T_0(z)),\qquad G_m(z)=\mathfrak{N}\left(\sum_{n=0}^{m}T_n(z)\right)-\mathfrak{N}\left(\sum_{n=0}^{m-1}T_n(z)\right). \quad (29)$$

Using these G_n's, we have the following components of $T(z)$:

$$T_0(z)=\beta_0+\beta_1 z+\beta_2\frac{z^{2}}{2!}+\beta_3\frac{z^{3}}{3!}-L^{-1}f(z),\qquad T_{k+1}(z)=-L^{-1}\mathscr{M}(T_k(z))-L^{-1}(G_k). \quad (30)$$

For the solution of (11) with the help of (12), we have $f(z)=0$, $\mathfrak{M}(T(z))=-m^{2}T''(z)$, and $\mathfrak{N}(T(z))=RT(z)T'''(z)$ the components of $T(z)$ using DJM are as follows:

$$T_0(z)=Bz+\frac{Az^{3}}{6},$$
$$T_1(z)=\frac{1}{120}(A-AB)z^{5}-\frac{A^{2}z^{7}}{5040},$$
$$T_2(z)=\frac{\left(A-4AB+3AB^{2}\right)z^{7}}{5040}+\left(-\frac{A^{2}}{30240}+\frac{A^{2}B}{22680}\right)z^{9}+\frac{\left(-7A^{2}+12A^{3}+14A^{2}B-7A^{2}B^{2}\right)z^{11}}{13305600}+\frac{\left(A^{3}-A^{3}B\right)z^{13}}{38438400}-\frac{A^{4}z^{15}}{3962649600}. \quad (31)$$

$T_3(z)$ is also obtained in the same manner. The series solution up to $T_3(z)$ is then given by

$$\begin{aligned}
T(z) &= T_0(z) + T_1(z) + T_2(z) + T_3(z) + O\left(z^{14}\right) \\
&= Bz + \frac{Az^3}{6} + \frac{1}{120}(A - AB)z^5 + \left(-\frac{A^2}{5040} + \frac{A - 4AB + 3AB^2}{5040}\right)z^7 \\
&\quad + \left(-\frac{A^2}{30240} + \frac{A^2B}{22680} + \frac{A - 9AB + 23AB^2 - 15AB^3}{362880}\right)z^9 \\
&\quad + \left(\frac{-12A^2 + 61A^2B - 55A^2B^2}{9979200} + \frac{-7A^2 + 12A^3 + 14A^2B - 7A^2B^2}{13305600}\right)z^{11} \\
&\quad + \left(\frac{A^3 - A^3B}{38438400} + \frac{-183A^2 + 1056A^3 + 1041A^2B - 1684A^3B - 1533A^2B^2 + 675A^2B^3}{6227020800}\right)z^{13}.
\end{aligned} \tag{32}$$

Use the boundary conditions at $z = 1$ to get

$$\begin{aligned}
A &= 1.45535, \\
B &= 0.754972
\end{aligned} \tag{33}$$

so that the last equation becomes

$$\begin{aligned}
T(z) &= 0.75z + 0.24z^3 + 2.97 \times 10^{-3}z^5 - 5.10 \\
&\quad \times 10^{-4}z^7 + 3.91 \times 10^{-6}z^9 + 3.30 \times 10^{-6}z^{11} \\
&\quad - 8.03 \times 10^{-8}z^{13} + O\left(z^{14}\right).
\end{aligned} \tag{34}$$

5. Basic Idea and Application of ADM

Consider the differential equation

$$\mathfrak{L}(T(z)) + \mathfrak{M}(T(z)) + \mathfrak{N}(T(z)) = f(z). \tag{35}$$

Following the basic concept of DJM, for the nonlinear term $\mathfrak{N}(T(z))$ Adomian introduced polynomials so called Adomian polynomials defined as

$$A_n = \sum_{r=1}^{n} c(r, n) T^r(z_0), \tag{36}$$

where $c(r, n)$ are products (or sum of products) of r components of $T(z)$ whose subscripts sum to n, divided by the factorial of the number of repeated subscripts. $\mathfrak{N}(T(z))$ is written in the form of infinite series of Adomian polynomials as

$$\mathfrak{N}(T(z)) = \sum_{k=0}^{\infty} A_k. \tag{37}$$

In our case $\mathfrak{N}(T(z)) = T(z)T'''(z)$. Some polynomials for this nonlinear term are

$$\begin{aligned}
A_0 &= \mathfrak{N}(T_0(z)) = T_0(z) \cdot T_0'''(z), \\
A_1 &= T_1(z)\mathfrak{N}'(T_0(z)) = T_1(z)\left(T_0(z)T_0^{(\mathrm{iv})}\right. \\
&\quad \left. + T_0'T_0'''(z)\right), \\
A_2 &= T_2(z)\mathfrak{N}'(T_0(z)) + \frac{T_1(z)}{2!}\mathfrak{N}''(T_0(z)) = T_2(z) \\
&\quad \cdot \left(T_0(z)T_0^{(\mathrm{iv})} + T_0'T_0'''(z)\right) \\
&\quad + \frac{T_1(z)}{2!}\left(T_0(z)T_0^{(\mathrm{v})}(z) + 2T_0'(z)T_0^{(\mathrm{iv})}(z)\right. \\
&\quad \left. + T_0'''(z)T_0''(z)\right).
\end{aligned} \tag{38}$$

The recursive process to find the components of $T(z)$ is

$$\begin{aligned}
T_0(z) &= \beta_0 + \beta_1 z + \beta_2\frac{z^2}{2!} + \beta_3\frac{z^3}{3!} - L^{-1}f(z), \\
T_{k+1}(z) &= -L^{-1}M(T_k(z)) - L^{-1}(A_k), \\
&\qquad k = 0, 1, 2, \ldots.
\end{aligned} \tag{39}$$

By means of boundary conditions at $z = 0$ the components of $T(z)$ are obtained as follows:

$$\begin{aligned}
T_0(z) &= bz + \frac{az^3}{6}, \\
T_1(z) &= \frac{1}{120}(a - ab)z^5 - \frac{a^2z^7}{5040}, \\
T_2(z) &= \frac{(a - ab)z^7}{5040} + \frac{\left(-a^2 - a^2b + a^2b^2\right)z^9}{362880} \\
&\quad + \left(-\frac{a^3}{1900800} + \frac{a^3b}{1814400}\right)z^{11} \\
&\quad + \frac{a^4z^{13}}{172972800}.
\end{aligned} \tag{40}$$

Considering the components up to $T_4(z)$, we have the following solution:

$$\begin{aligned}
T(z) &= bz + \frac{az^3}{6} + \frac{1}{120}(a-ab)z^5 + \left(-\frac{a^2}{5040} + \frac{a-ab}{5040}\right) \\
&\cdot z^7 + \left(\frac{a-ab}{362880} + \frac{-a^2-a^2b+a^2b^2}{362880}\right)z^9 + \left(-\frac{a^3}{1900800}\right. \\
&+ \left.\frac{a^3b}{1814400} + \frac{a-ab}{39916800} + \frac{-a^2-2a^2b+2a^2b^2}{39916800}\right)z^{11} \\
&+ \left(\frac{a^4}{172972800} + \frac{-a^2-3a^2b+3a^2b^2}{6227020800}\right. \\
&+ \left.\frac{-57a^3+59a^3b+a^3b^2-a^3b^3}{6227020800}\right)z^{13} \\
&+ \left(\frac{-1295a^4+2848a^4b-1463a^4b^2}{1307674368000}\right. \\
&+ \left.\frac{-112a^3+115a^3b+3a^3b^2-3a^3b^3}{1307674368000}\right)z^{15} \\
&+ \left(\frac{a^5}{29804544000} - \frac{251a^5b}{7410154752000}\right. \\
&+ \left.\frac{-11513a^4+23653a^4b-11974a^4b^2-a^4b^3+a^4b^4}{355687428096000}\right) \\
&\cdot z^{17} + \left(-\frac{41a^6}{168951528345600}\right. \\
&+ \left.\frac{-26697a^5+309950a^5b-441391a^5b^2+157724a^5b^3}{121645100408832000}\right) \\
&\cdot z^{19}.
\end{aligned} \tag{41}$$

Use the conditions at $z = 1$ to get two equations which, on solving, give

$$\begin{aligned}
a &= 1.44861, \\
b &= 0.755971.
\end{aligned} \tag{42}$$

The approximate solution thus obtained is

$$\begin{aligned}
T(z) &= 0.755971z + 0.241436z^3 + 0.002946z^5 \\
&- 0.000346z^7 - 5.875514 \times 10^{-6}z^9 \\
&- 3.958095 \times 10^{-7}z^{11} + 1.895084 \times 10^{-8}z^{13} \\
&+ 1.649921 \times 10^{-11}z^{15} + 4.480188 \\
&\times 10^{-11}z^{17} - 1.009835 \times 10^{-12}z^{19}.
\end{aligned} \tag{43}$$

6. Basic Idea and Application of HPM

Let us consider a nonlinear differential equation as follows [10, 11]:

$$\mathfrak{M}(T) = g(z). \tag{44}$$

The operator $\mathfrak{M}$ is usually divided into two parts, namely, linear ($\mathfrak{L}$) and nonlinear ($\mathfrak{N}$); that is,

$$\mathfrak{M} = \mathfrak{L} + \mathfrak{N}, \tag{45}$$

and $g(z)$ is a known analytic function. Equation (44) can be written as

$$\mathfrak{L}(T) + \mathfrak{N}(T) - g(z) = 0, \quad r \in \Omega, \tag{46}$$

with boundary conditions

$$\mathfrak{B}\left(T, \frac{\partial T}{\partial z}\right) = 0, \quad z \in \Gamma, \tag{47}$$

where Γ is the boundary of the domain Ω. A homotopy $\mathfrak{H}(T(z,p),p) : \mathbf{R} \times [0,1] \to \mathbf{R}$ is constructed which satisfies

$$\begin{aligned}
\mathfrak{H}(v,p) &= (1-p)\left[\mathfrak{L}(v) - \mathfrak{L}(T_0)\right] \\
&+ p\left[\mathfrak{M}v - g(z)\right].
\end{aligned} \tag{48}$$

$p \in [0,1]$ is an embedding parameter and T_0 is the first approximation satisfying the boundary conditions. Taylor's series expansion of $T(z,p)$ about p is used for the approximate solution of the differential equation as follows:

$$T(z) = v_0(z) + \sum_{r=1}^{\infty} v_r(z)\,p^r. \tag{49}$$

Putting (49) in (48) and equating the coefficients of like powers of p for the present problem, we get the following:

Zeroth-order problem:

$$\begin{aligned}
T^{(\mathrm{iv})}(z) &= 0, \\
T_0(0) &= 0, \\
T_0''(0) &= 0, \\
T_0(1) &= 1, \\
T_0'(1) &= T_0''(1).
\end{aligned} \tag{50}$$

First-order problem:

$$\begin{aligned}
T_1^{(\mathrm{iv})}(z) &= T_0''(z) - T_0(z)\,T_0'''(z), \\
T_1(0) &= 0, \\
T_1''(0) &= 0, \\
T_1(1) &= 0, \\
T_1'(1) &= T_1''(1).
\end{aligned} \tag{51}$$

Second-order problem:

$$\begin{aligned}
T_2^{(\mathrm{iv})}(z) &= T_1''(z) - T_1(z)\,T_0'''(z) - T_0(z)\,T_1'''(z), \\
T_2(0) &= 0, \\
T_2''(0) &= 0, \\
T_2(1) &= 0, \\
T_2'(1) &= T_2''(1).
\end{aligned} \tag{52}$$

Third-order problem:

$$\begin{aligned} T_3^{(\mathrm{iv})}(z) &= T_2''(z) - T_2(z)T_0'''(z) - T_1(z)T_1'''(z) \\ &\quad - T_0(z)T_2'''(z), \\ T_3(0) &= 0, \\ T_3''(0) &= 0, \\ T_3(1) &= 0, \\ T_3'(1) &= T_3''(1). \end{aligned} \tag{53}$$

Fourth-order problem:

$$\begin{aligned} T_4^{(\mathrm{iv})}(z) &= T_3''(z) - T_3(z)T_0'''(z) - T_2(z)T_1'''(z) \\ &\quad - T_1(z)T_2'''(z) - T_0(z)T_3'''(z), \\ T_4(0) &= 0, \\ T_4''(0) &= 0, \\ T_4(1) &= 0, \\ T_4'(1) &= T_4''(1). \end{aligned} \tag{54}$$

By considering the fourth-order solution, we have

$$\begin{aligned} T(z) &= T_0(z) + T_1(z) + T_2(z) + T_3(z) + T_4(z) \\ &= 0.755z + 0.243z^3 + 0.003z^5 - 5.097\times 10^{-4}z^7 \\ &\quad + 3.877\times 10^{-6}z^9 + 3.150\times 10^{-6}z^{11} - 1.293 \\ &\quad \times 10^{-7}z^{13} + O\left(z^{14}\right). \end{aligned} \tag{55}$$

7. Basic Idea and Application of OHAM

If $t(z)$ is an unknown function, $f(z)$ is known function, and $\mathfrak{L}, \mathfrak{N}, \mathfrak{B}$ are linear, nonlinear, and boundary operator, respectively, then for boundary value problem [12–14]

$$\mathfrak{L}[t(z)] + f(z) + \mathfrak{N}[t(z)] = 0, \tag{56}$$

with boundary conditions

$$\mathfrak{B}\left(T(z), \frac{dT(z)}{dz}\right) = 0, \tag{57}$$

a homotopy $\mathfrak{H}(T(z,p),p) : \mathbf{R}\times[0,1] \to \mathbf{R}$ is constructed which satisfies the following:

$$\begin{aligned} &(1-p)\left[\mathfrak{L}(T(z,p)) + f(z)\right] \\ &\quad = \mathfrak{H}(p)\left[\mathfrak{L}(T(z,p)) + f(z) + \mathfrak{N}(T(z,p))\right], \\ &\mathfrak{B}\left(T(z,p), \frac{\partial T(z,p)}{\partial z}\right) = 0, \end{aligned} \tag{58}$$

where $p \in [0,1]$ and $\mathfrak{H}(p)$ is a nonzero auxiliary function. If $p = 0$, then $T(z,0) = t_0(z)$ and if $p = 1$, then $T(z,1) = t(z)$ hold. It means that the solution $T(z,p)$ approaches from $t_0(z)$ to $t(z)$ as p varies from 0 to 1.

For $p = 0$

$$\begin{aligned} \mathfrak{L}(T_0(z)) + f(z) &= 0, \\ \mathfrak{B}\left(T_0, \frac{dT_0}{dz}\right) &= 0. \end{aligned} \tag{59}$$

The auxiliary function $\mathfrak{H}(p)$ is selected such that

$$\mathfrak{H}(p) = \sum_{k=0}^{n} p^k C_k, \tag{60}$$

where C_k are the convergence controlling constants to be determined. Expanding $T(z,p)$ in Taylor's series about p to get

$$T(z,p,C_k) = T_0(z) + \sum_{j=1}^{n} T_j\left(z, C_1, C_2, \ldots, C_j\right)p^j, \tag{61}$$

substituting (61) into (58) and comparing the coefficients of the same powers of p, the general nth order problem is

$$\begin{aligned} &\mathfrak{L}(T_n(z)) - \mathfrak{L}(T_{n-1}(z)) = C_n \mathfrak{N}_0(T_0(z)) \\ &\quad + \left(\sum_{j=1}^{n-1} C_j \left[\mathfrak{L}\left(T_{n-j}(z)\right)\right.\right. \\ &\quad \left.\left. + \mathfrak{N}_{n-j}\left(T_0(z), T_1(z), \ldots, T_{n-1}(z)\right)\right]\right), \end{aligned} \tag{62}$$

with boundary conditions

$$\mathfrak{B}\left(T_n, \frac{dT_n}{dz}\right) = 0, \quad n = 1, 2, 3, 4, \ldots, \tag{63}$$

where $\mathfrak{N}_m(T_0(z), T_1(z), \ldots, T_{m-1}(z))$ is the coefficient of p^m in the expansion of $\mathfrak{N}(T(z,p))$ about p. Consider

$$\begin{aligned} \mathfrak{N}(T(z,p,C_k)) &= \mathfrak{N}_0(T_0(z)) \\ &\quad + \sum_{m=1}^{\infty} \mathfrak{N}_m(T_0, T_1, T_2, \ldots, T_m)p^m. \end{aligned} \tag{64}$$

The kth-order approximation $\widetilde{T}$ is

$$\begin{aligned} \widetilde{T}(z, C_1, C_2, \ldots, C_k) &= T_0(z) \\ &\quad + \sum_{j=1}^{k} T_j\left(z, C_1, C_2, \ldots, C_j\right). \end{aligned} \tag{65}$$

The expression for the residual is

$$\begin{aligned} \mathfrak{R}(z, C_1, C_2, \ldots, C_k) &= \mathfrak{L}\left(\widetilde{T}(z, C_1, C_2, \ldots, C_k)\right) \\ &\quad + f(z) \\ &\quad + \mathfrak{N}\left(\widetilde{T}(z, C_1, C_2, \ldots, C_k)\right). \end{aligned} \tag{66}$$

If $\mathfrak{R} = 0$, then we say that $\widetilde{T}$ is the exact solution, but, in case of nonlinearity, it does not happen generally. To search the constants C_k, different methods can be applied. One of these methods is the method of least square as follows:

$$I = \int_{x_0}^{x_1} \mathfrak{R}^2 (z, C_1, C_2, \ldots, C_k)\, dz. \quad (67)$$

Minimizing this function, we have

$$\frac{\partial I}{\partial C_i} (z, C_1, C_2, \ldots, C_k) = 0, \quad i = 1, 2, 3, \ldots, k. \quad (68)$$

x_0 and x_1 are within the domain of the problem for locating suitable C_r's $(r = 1, 2, \ldots, k)$. Now we solve (11) with boundary conditions (12); we find the following different order problems:

Zeroth-order problem:

$$\begin{aligned} &T_0^{(\mathrm{iv})}(z) = 0, \\ &T_0(0) = 0, \\ &T_0''(0) = 0, \\ &T_0(1) = 1, \\ &T_0'(1) = T_0''(1). \end{aligned} \quad (69)$$

First-order problem:

$$\begin{aligned} T_1^{(\mathrm{iv})}(z) &= C_1 T_0(z) T_0'''(z) + T_0^{(\mathrm{iv})}(z) + C_1 T_0^{(\mathrm{iv})}(z) \\ &\quad - C_1 T_0''(z), \\ T_1(0) &= 0, \\ T_1''(0) &= 0, \\ T_1(1) &= 0, \\ T_1'(1) &= T_1''(1). \end{aligned} \quad (70)$$

Second-order problem:

$$\begin{aligned} T_2^{(\mathrm{iv})}(z) &= C_2 T_0(z) T_0'''(z) + C_1 T_1(z) T_0'''(z) \\ &\quad + C_1 T_0(z) T_1'''(z) + C_2 T_0^{(\mathrm{iv})} + T_1^{(\mathrm{iv})}(z) \\ &\quad + C_1 T_1^{(\mathrm{iv})}(z) - C_2 T_0''(z) - C_1 T_1''(z), \\ T_2(0) &= 0, \\ T_2''(0) &= 0, \\ T_2(1) &= 0, \\ T_2'(1) &= T_2''(1). \end{aligned} \quad (71)$$

Third-order problem:

$$\begin{aligned} T_3^{(\mathrm{iv})}(z) &= C_2 T_1(z) T_0'''(z) + C_1 T_2(z) T_0'''(z) \\ &\quad + C_2 T_0(z) T_1'''(z) + C_1 T_1(z) T_1'''(z) \\ &\quad + C_1 T_0(z) T_2'''(z) + C_2 T_1^{(\mathrm{iv})}(z) \\ &\quad + T_2^{(\mathrm{iv})}(z) + C_1 T_2^{(\mathrm{iv})}(z) - C_2 T_1''(z) \\ &\quad + C_1 T_2''(z), \\ T_3(0) &= 0, \\ T_3''(0) &= 0, \\ T_3(1) &= 0, \\ T_3'(1) &= T_3''(1). \end{aligned} \quad (72)$$

Similarly fourth-order problem can also be found easily. By considering the fourth-order solution, we have

$$\widetilde{T}(z) = \sum_{i=0}^{4} T_i (z, C_1, C_2). \quad (73)$$

The residual of the problem is

$$\mathfrak{Res} = \widetilde{T}^{(\mathrm{iv})}(z) - \mathcal{M}^2 \widetilde{T}''(z) + \mathcal{R} \widetilde{T}(z) \widetilde{T}'''(z). \quad (74)$$

In order to find C_1 and C_2, we apply the method of least square as follows:

$$\begin{aligned} J(C_1, C_2) &= \int_0^1 \mathfrak{Res}^2 (z, C_1, C_2)\, dz, \\ \frac{\partial J}{\partial C_i} (z, C_1, C_2) &= 0, \quad i = 1, 2. \end{aligned} \quad (75)$$

Solving (75) for C_1, C_2, we get

$$\begin{aligned} C_1 &= -0.89521, \\ C_2 &= -0.00031. \end{aligned} \quad (76)$$

Using these values of C_1, C_2, the approximate solution is given by

$$\begin{aligned} T(z) &= 0.754966z + 0.242565z^3 + 0.0029716z^5 \\ &\quad - 0.000509457z^7 + 3.73464 \times 10^{-6} z^9 \\ &\quad + 3.17424 \times 10^{-6} z^{11} - 1.10266 \times 10^{-7} z^{13} \\ &\quad - 2.19862 \times 10^{-8} z^{15} + 8.28889 \times 10^{-9} z^{17} \\ &\quad + 1.26544 \times 10^{-10} z^{19}. \end{aligned} \quad (77)$$

8. Conclusion

In the present paper various analytical techniques are used along with one numerical scheme to find the approximate

Table 1: Absolute residuals for numerical and various analytical schemes.

z	ADM	DJM	DTM	HPM	OHAM	NDSolve
0.0	0.0	0.0	0.0	0.0	0.0	1.20×10^{-2}
0.1	1.34×10^{-4}	1.07×10^{-10}	1.40×10^{-17}	8.84×10^{-6}	7.53×10^{-9}	1.55×10^{-3}
0.2	1.06×10^{-3}	1.40×10^{-8}	7.00×10^{-14}	1.76×10^{-5}	6.05×10^{-9}	1.50×10^{-4}
0.3	3.58×10^{-3}	2.50×10^{-7}	1.32×10^{-11}	2.63×10^{-5}	1.66×10^{-8}	2.00×10^{-5}
0.4	8.40×10^{-3}	2.00×10^{-6}	5.61×10^{-10}	3.54×10^{-5}	6.85×10^{-8}	8.02×10^{-6}
0.5	1.62×10^{-2}	1.02×10^{-5}	1.04×10^{-8}	4.56×10^{-5}	1.30×10^{-7}	2.90×10^{-5}
0.6	2.74×10^{-2}	4.10×10^{-5}	1.14×10^{-7}	6.00×10^{-5}	1.24×10^{-7}	6.20×10^{-5}
0.7	4.21×10^{-2}	1.40×10^{-4}	8.63×10^{-7}	7.10×10^{-5}	7.81×10^{-8}	1.60×10^{-5}
0.8	5.99×10^{-2}	4.00×10^{-4}	5.03×10^{-6}	7.10×10^{-5}	5.34×10^{-7}	2.90×10^{-4}
0.9	7.96×10^{-2}	1.05×10^{-3}	2.40×10^{-5}	9.00×10^{-6}	8.58×10^{-7}	1.10×10^{-3}
1.0	9.89×10^{-2}	2.60×10^{-3}	9.74×10^{-5}	2.60×10^{-4}	4.11×10^{-7}	1.20×10^{-2}

solution for axisymmetric squeezing flow of incompressible Newtonian fluid having MHD effect and passing through porous medium channel with slip boundary. Absolute residuals of the modeled problem are obtained using these schemes, that is, ADM, DJM, DTM, HPM, and OHAM. Numerical solution is obtained using Mathematica solver NDSolve. The residuals are given in Table 1 which shows the efficiency of all the schemes used in the given scenario as compared with the numerical scheme NDSolve. In comparison with other techniques it is clear from Table 1 that OHAM is more efficient and consistent.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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