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Research Article

On Inclusion Relations between Some Sequence Spaces

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We determine the relations between the classes $\widehat{S}_\lambda$ of almost λ-statistically convergent sequences and the relations between the classes $[\widehat{V},\lambda]$ of strongly almost (V,λ)-summable sequences for various sequences λ, μ in the class Λ. Furthermore we also give the relations between the classes $\widehat{S}_\lambda$ of almost λ-statistically convergent sequences and the classes $[\widehat{V},\lambda]$ of strongly almost (V,λ)-summable sequences for various sequences $\lambda,\mu\in\Lambda$.

1. Introduction

A sequence $x=(x_k)$ of real (or complex) numbers is said to be *statistically convergent* to the number L if for every $\varepsilon>0$

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k\le n:\left|x_k-L\right|\ge\varepsilon\right\}\right|=0. \tag{1}$$

In this case, we write $S-\lim x=L$ or $x_k\to L(S)$ and S denotes the set of all statistically convergent sequences.

A sequence $x=(x_k)$ of real (or complex) numbers is said to be *almost statistically convergent* to the number L if for every $\varepsilon>0$

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k\le n:\left|x_{k+m}-L\right|\ge\varepsilon\right\}\right|=0\quad\text{uniformly in } m. \tag{2}$$

In this case, we write $\widehat{S}-\lim x=L$ or $x_k\to L(\widehat{S})$ and $\widehat{S}$ denotes the set of all almost statistically convergent sequences [1].

Let $\lambda=(\lambda_n)$ be a nondecreasing sequence of positive real numbers tending to ∞ such that

$$\lambda_{n+1}\le\lambda_n+1,\quad \lambda_1=1. \tag{3}$$

The set of all such sequences will be denoted by Λ.

The generalized de la Vallée-Poussin mean is defined by

$$t_n(x)=\frac{1}{\lambda_n}\sum_{k\in I_n}x_k, \tag{4}$$

where $I_n=[n-\lambda_n+1,n]$.

A sequence $x=(x_k)$ is said to be (V,λ)-summable to a number L (see [2]) if

$$t_n(x)\longrightarrow L\quad\text{as } n\longrightarrow\infty. \tag{5}$$

If $\lambda_n=n$ for each $n\in\mathbb{N}$, then (V,λ)-summability reduces to $(C,1)$-summability.

We write

$$\begin{aligned}&[C,1]\\&=\left\{x=(x_k):\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left|x_k-L\right|=0\text{ for some } L\right\},\\&[V,\lambda]\\&=\left\{x=(x_k):\lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{k\in I_n}\left|x_k-L\right|=0\text{ for some } L\right\}\end{aligned} \tag{6}$$

for the sets of sequences $x=(x_k)$ which are strongly Cesàro summable and strongly (V,λ)-summable to L; that is, $x_k\to L[C,1]$ and $x_k\to L[V,\lambda]$, respectively.

Savaš [1] defined the following sequence space:

$$\left[\widehat{V},\lambda\right] = \left\{x = (x_k) : \lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{k\in I_n}\left|x_{k+m} - L\right| = 0 \text{ for some } L, \text{ uniformly in } m\right\} \tag{7}$$

for the sets of sequences $x = (x_k)$ which are strongly almost (V,λ)-summable to L; that is, $x_k \to L[V,\lambda]$. We will write $[\widehat{V},\lambda]_\infty = [\widehat{V},\lambda] \cap \ell_\infty$.

The λ-statistical convergence was introduced by Mursaleen in [3] as follows.

Let $\lambda = (\lambda_n) \in \Lambda$. A sequence $x = (x_k)$ is said to be λ-*statistically convergent* or S_λ-convergent to L if for every $\varepsilon > 0$

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\left|\{k \in I_n : \left|x_k - L\right| \geq \varepsilon\}\right| = 0, \tag{8}$$

where $I_n = [n-\lambda_n+1, n]$. In this case we write $S_\lambda - \lim x = L$ or $x_k \to L(S_\lambda)$, and $S_\lambda = \{x = (x_k) : S_\lambda - \lim x = L \text{ for some } L\}$.

The sequence $x = (x_k)$ is said to be λ-almost statistically convergent if there is a complex number L such that

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\left|\{k \in I_n : \left|x_{k+m} - L\right| \geq \varepsilon\}\right| = 0, \quad \text{uniformly in } m. \tag{9}$$

In this case, we write $\widehat{S}_\lambda - \lim x = L$ or $x_k \to L(\widehat{S}_\lambda)$ and $\widehat{S}_\lambda$ denotes the set of all λ-almost statistically convergent sequences. If we choose $\lambda_n = n$ for all n, then λ-almost statistical convergence reduces to almost statistical convergence [1].

2. Main Results

Throughout the paper, unless stated otherwise, by "for all $n \in \mathbb{N}_{n_o}$" we mean "for all $n \in \mathbb{N}$ except finite numbers of positive integers" where $\mathbb{N}_{n_o} = \{n_o, n_o+1, n_o+2, \ldots\}$ for some $n_o \in \mathbb{N} = \{1,2,3,\ldots\}$.

Theorem 1. *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}_{n_o}$. Consider the following:*

(i) *If*

$$\liminf_{n\to\infty}\frac{\lambda_n}{\mu_n} > 0 \tag{10}$$

then $\widehat{S}_\mu \subseteq \widehat{S}_\lambda$.

(ii) *If*

$$\lim_{n\to\infty}\frac{\lambda_n}{\mu_n} = 1 \tag{11}$$

then $\widehat{S}_\lambda \subseteq \widehat{S}_\mu$.

Proof. (i) Suppose that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}_{n_o}$ and let (10) be satisfied. Then $I_n \subset J_n$ so that for $\varepsilon > 0$ we may write

$$\begin{aligned}&\left|\{k \in J_n : \left|x_{k+m} - L\right| \geq \varepsilon\}\right| \\ &\quad \geq \left|\{k \in I_n : \left|x_{k+m} - L\right| \geq \varepsilon\}\right|\end{aligned} \tag{12}$$

and therefore we have

$$\begin{aligned}&\frac{1}{\mu_n}\left|\{k \in J_n : \left|x_{k+m} - L\right| \geq \varepsilon\}\right| \\ &\quad \geq \frac{\lambda_n}{\mu_n}\frac{1}{\lambda_n}\left|\{k \in I_n : \left|x_{k+m} - L\right| \geq \varepsilon\}\right|\end{aligned} \tag{13}$$

for all $n \in \mathbb{N}_{n_o}$, where $J_n = [n-\mu_n+1, n]$. Now taking the limit as $n \to \infty$ uniformly in m in the last inequality and using (10) we get $x_k \to L(\widehat{S}_\mu) \Rightarrow x_k \to L(\widehat{S}_\lambda)$ so that $\widehat{S}_\mu \subseteq \widehat{S}_\lambda$.

(ii) Let $(x_k) \in \widehat{S}_\lambda$ and (11) be satisfied. Since $I_n \subset J_n$, for $\varepsilon > 0$, we may write

$$\begin{aligned}&\frac{1}{\mu_n}\left|\{k \in J_n : \left|x_{k+m} - L\right| \geq \varepsilon\}\right| \\ &\quad = \frac{1}{\mu_n}\left|\{n-\mu_n+1 \leq k \leq n-\lambda_n : \left|x_{k+m} - L\right| \geq \varepsilon\}\right| \\ &\qquad + \frac{1}{\mu_n}\left|\{k \in I_n : \left|x_{k+m} - L\right| \geq \varepsilon\}\right| \\ &\quad \leq \frac{\mu_n - \lambda_n}{\mu_n} + \frac{1}{\lambda_n}\left|\{k \in I_n : \left|x_{k+m} - L\right| \geq \varepsilon\}\right| \\ &\quad \leq \left(1 - \frac{\lambda_n}{\mu_n}\right) + \frac{1}{\lambda_n}\left|\{k \in I_n : \left|x_{k+m} - L\right| \geq \varepsilon\}\right|\end{aligned} \tag{14}$$

for all $n \in \mathbb{N}_{n_o}$. Since $\lim_n(\lambda_n/\mu_n) = 1$ by (11) and since $x = (x_k) \in \widehat{S}_\lambda$ the first term and second term of right hand side of above inequality tend to 0 as $n \to \infty$ uniformly in m. This implies that $(1/\mu_n)|\{k \in J_n : |x_{k+m} - L| \geq \varepsilon\}| \to 0$ as $n \to \infty$ uniformly in m. Therefore $\widehat{S}_\lambda \subseteq \widehat{S}_\mu$. □

From Theorem 1 we have the following result.

Corollary 2. *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}_{n_o}$. If (11) holds then $\widehat{S}_\lambda = \widehat{S}_\mu$.*

If we take $\mu = (\mu_n) = (n)$ in Corollary 2 we have the following result.

Corollary 3. *Let $\lambda = (\lambda_n) \in \Lambda$. If $\lim_n(\lambda_n/n) = 1$ then we have $\widehat{S}_\lambda = \widehat{S}$.*

Theorem 4. *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n) \in \Lambda$ and suppose that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}_{n_o}$. Consider the following:*

(i) *If (10) holds then $[\widehat{V},\mu] \subseteq [\widehat{V},\lambda]$.*

(ii) *If (11) holds then $[\widehat{V},\lambda]_\infty \subseteq [\widehat{V},\mu]$.*

Proof. (i) Suppose that $\lambda_n \le \mu_n$ for all $n \in \mathbb{N}_{n_o}$. Then $I_n \subseteq J_n$ so that we may write

$$\frac{1}{\mu_n}\sum_{k\in J_n}\left|x_{k+m}-L\right| \ge \frac{1}{\mu_n}\sum_{k\in I_n}\left|x_{k+m}-L\right| \tag{15}$$

for all $n \in \mathbb{N}_{n_o}$. This gives that

$$\frac{1}{\mu_n}\sum_{k\in J_n}\left|x_{k+m}-L\right| \ge \frac{\lambda_n}{\mu_n}\frac{1}{\lambda_n}\sum_{k\in I_n}\left|x_{k+m}-L\right|. \tag{16}$$

Then taking limit as $n \to \infty$, uniformly in m in the last inequality, and using (10) we obtain $x_k \to L[\widehat{V},\mu] \Rightarrow x_k \to L[\widehat{V},\lambda]$.

(ii) Let $x = (x_k) \in [\widehat{V},\lambda]_\infty$ be any sequence. Suppose that $x_k \to L[\widehat{V},\mu]$ and that (11) holds. Since $x = (x_k) \in l_\infty$ then there exists some $M > 0$ such that $|x_{k+m} - L| \le M$ for all k and m. Since $\lambda_n \le \mu_n$ so that $1/\mu_n \le 1/\lambda_n$, and $I_n \subset J_n$ for all $n \in \mathbb{N}_{n_o}$, we may write

$$\begin{aligned}\frac{1}{\mu_n}\sum_{k\in J_n}\left|x_{k+m}-L\right| &= \frac{1}{\mu_n}\sum_{k\in J_n-I_n}\left|x_{k+m}-L\right| \\ &\quad+\frac{1}{\mu_n}\sum_{k\in I_n}\left|x_{k+m}-L\right| \\ &\le \frac{\mu_n-\lambda_n}{\mu_n}M+\frac{1}{\mu_n}\sum_{k\in I_n}\left|x_{k+m}-L\right| \\ &\le \left(1-\frac{\lambda_n}{\mu_n}\right)M+\frac{1}{\lambda_n}\sum_{k\in I_n}\left|x_{k+m}-L\right|\end{aligned} \tag{17}$$

for every $n \in \mathbb{N}_{n_o}$. Since $\lim_n(\lambda_n/\mu_n) = 1$ by (11) and since $x_k \to L[\widehat{V},\lambda]$ the first term and the second term of right hand side of above inequality tend to 0 as $n \to \infty$, uniformly in m (note that $1 - \lambda_n/\mu_n \ge 0$ for all $n \in \mathbb{N}_{n_o}$). Then we get $x_k \to L[\widehat{V},\lambda] \Rightarrow x_k \to L[\widehat{V},\mu]$. Since $x = (x_k) \in [\widehat{V},\lambda]_\infty$ is an arbitrary sequence we obtain $[\widehat{V},\lambda]_\infty \subseteq [\widehat{V},\mu]$. □

Since clearly (11) implies (10) from Theorem 4 we have the following result.

Corollary 5. *Let $\lambda, \mu \in \Lambda$ such that $\lambda_n \le \mu_n$ for all $n \in \mathbb{N}_{n_o}$. If (11) holds then $[\widehat{V},\lambda]_\infty = [\widehat{V},\mu]_\infty$.*

Theorem 6. *Let $\lambda, \mu \in \Lambda$ such that $\lambda_n \le \mu_n$ for all $n \in \mathbb{N}_{n_o}$. Consider the following:*

(i) *If (10) holds then*

$$x_k \longrightarrow L\left[\widehat{V},\mu\right] \Longrightarrow x_k \longrightarrow L\left(\widehat{S}_\lambda\right) \tag{18}$$

and the inclusion $[\widehat{V},\mu] \subset \widehat{S}_\lambda$ holds for some $\lambda, \mu \in \Lambda$.

(ii) *If $(x_k) \in \ell_\infty$ and $x_k \to L(\widehat{S}_\lambda)$ then $x_k \to L[\widehat{V},\mu]$, whenever (11) holds.*

(iii) *If (11) holds then $\ell_\infty \cap \widehat{S}_\lambda = [\widehat{V},\mu]_\infty$.*

Proof. (i) Let $\varepsilon > 0$ be given and let $x_k \to L[\widehat{V},\mu]$. Now for every $\varepsilon > 0$ we may write

$$\begin{aligned}\sum_{k\in J_n}\left|x_{k+m}-L\right| &\ge \sum_{k\in I_n}\left|x_{k+m}-L\right| \\ &\ge \sum_{\substack{k\in I_n\\ |x_{k+m}-L|\ge\varepsilon}}\left|x_{k+m}-L\right| \\ &\ge \varepsilon\left|\left\{k\in I_n : \left|x_{k+m}-L\right|\ge\varepsilon\right\}\right|\end{aligned} \tag{19}$$

so that

$$\begin{aligned}&\frac{1}{\mu_n}\sum_{k\in J_n}\left|x_{k+m}-L\right| \\ &\quad\ge \frac{\lambda_n}{\mu_n}\frac{1}{\lambda_n}\left|\left\{k\in I_n : \left|x_{k+m}-L\right|\ge\varepsilon\right\}\right|\varepsilon\end{aligned} \tag{20}$$

for all $n \in \mathbb{N}_{n_o}$. Then taking limit as $n \to \infty$, uniformly in m in the last inequality, and using (10) we obtain that $x_k \to L(\widehat{S}_\lambda)$ whenever $x_k \to L[\widehat{V},\mu]$. Since $x = (x_k) \in [\widehat{V},\mu]$ is an arbitrary sequence we obtain that $[\widehat{V},\mu] \subset \widehat{S}_\lambda$.

(ii) Suppose that $x_k \to L(\widehat{S}_\lambda)$ and $x = (x_k) \in \ell_\infty$. Then there exists some $M > 0$ such that $|x_{k+m} - L| \le M$ for all k and m. Since $1/\mu_n \le 1/\lambda_n$, then for every $\varepsilon > 0$ we may write

$$\begin{aligned}\frac{1}{\mu_n}\sum_{k\in J_n}\left|x_{k+m}-L\right| &= \frac{1}{\mu_n}\sum_{k\in J_n-I_n}\left|x_{k+m}-L\right| \\ &\quad+\frac{1}{\mu_n}\sum_{k\in I_n}\left|x_{k+m}-L\right| \\ &\le \frac{\mu_n-\lambda_n}{\mu_n}M+\frac{1}{\mu_n}\sum_{k\in I_n}\left|x_{k+m}-L\right| \\ &\le \left(1-\frac{\lambda_n}{\mu_n}\right)M+\frac{1}{\lambda_n}\sum_{k\in I_n}\left|x_{k+m}-L\right| \\ &\le \left(1-\frac{\lambda_n}{\mu_n}\right)M \\ &\quad+\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ |x_{k+m}-L|\ge\varepsilon}}\left|x_{k+m}-L\right| \\ &\quad+\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ |x_{k+m}-L|<\varepsilon}}\left|x_{k+m}-L\right| \\ &\le \left(1-\frac{\lambda_n}{\mu_n}\right)M \\ &\quad+\frac{M}{\lambda_n}\left|\left\{k\in I_n : \left|x_{k+m}-L\right|\ge\varepsilon\right\}\right| \\ &\quad+\varepsilon\end{aligned} \tag{21}$$

for all $n \in \mathbb{N}_{n_o}$. Using (11) we obtain that $x_k \to L[\widehat{V}, \mu]$ whenever $x_k \to L(\widehat{S}_\lambda)$. Since $x = (x_k) \in \ell_\infty \cap \widehat{S}_\lambda$ is an arbitrary sequence we obtain $\ell_\infty \cap \widehat{S}_\lambda \subseteq [\widehat{V}, \mu]$.

(iii) The proof follows from (i) and (ii), so we omit it. □

From Theorem 1(i) and Theorem 6(i) we obtain the following result.

Corollary 7. *If* $\liminf_{n\to\infty}(\lambda_n/\mu_n) > 0$ *then* $\widehat{S}_\mu \cap [\widehat{V}, \mu] \subset \widehat{S}_\lambda$.

If we take $\mu_n = n$ for all n in Theorem 6 then we have the following results, because $\lim_{n\to\infty}(\lambda_n/\mu_n) = 1$ implies that $\liminf_{n\to\infty}(\lambda_n/\mu_n) = 1 > 0$; that is, (11) $\Rightarrow$ (10).

Corollary 8. *If* $\lim_{n\to\infty}(\lambda_n/n) = 1$ *then*

(i) *if* $(x_k) \in \ell_\infty$ *and* $x_k \to L(\widehat{S}_\lambda)$ *then* $x_k \to L[C, 1]$,

(ii) *if* $x_k \to L[C, 1]$ *then* $x_k \to L(\widehat{S}_\lambda)$.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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