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Research Article

Analysis of Subgrid Stabilization Method for Stokes-Darcy Problems

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A number of techniques, used as remedy to the instability of the Galerkin finite element formulation for Stokes like problems, are found in the literature. In this work we consider a coupled Stokes-Darcy problem, where in one part of the domain the fluid motion is described by Stokes equations and for the other part the fluid is in a porous medium and described by Darcy law and the conservation of mass. Such systems can be discretized by heterogeneous mixed finite elements in the two parts. A better method, from a computational point of view, consists in using a unified approach on both subdomains. Here, the coupled Stokes-Darcy problem is analyzed using equal-order velocity and pressure approximation combined with subgrid stabilization. We prove that the obtained finite element solution is stable and converges to the classical solution with optimal rates for both velocity and pressure.

1. Introduction

The transport of substances between surface water and groundwater has attracted a lot of interest into the coupling of viscous flows and porous media flows [1–5]. In this work we consider coupled problems in fluid dynamics where the fluid in one part of the domain is described by the Stokes equations and in the other, porous media part, by the Darcy equation and mass conservation. Velocity and pressure on these two parts are mutually coupled by interface conditions derived in [6]. Such systems can be discretized by heterogeneous finite elements as analyzed by Layton et al. [1]. In more recent works, unified approaches become more popular. For instance, discontinuous Galerkin methods were analyzed by Girault and Rivière [3], mixed methods by Karper et al. [4], and local pressure gradient stabilized methods by Braack and Nafa [7].

In this work, we consider the L^2-formulation of the coupled Stokes-Darcy problem as in [4], but we discretize by equal-order finite elements and use subgrid method and grad-div term to stabilize the pressure and control the natural $H^1(\mathrm{div})$ velocity norm on the Darcy subdomain.

2. Formulations of the Stokes-Darcy Coupled Equations

2.1. Model Equations. Let $\Omega \subset R^d$, $d = 2$ or 3, be a bounded region with Lipschitz boundary $\partial\Omega$. Ω_S and Ω_D are, respectively, the fluid and porous media subdomains of Ω such that $\Omega_S \cap \Omega_D = \emptyset$. The subdomains have a common interface $\Gamma = \overline{\Omega}_S \cap \overline{\Omega}_D$. We denote by $\mathbf{v} = (\mathbf{v}_S, \mathbf{v}_D)$ the fluid velocity and by $p = (p_S, p_D)$ the fluid pressure, where $\mathbf{v}_i = \mathbf{v}|_{\Omega_i}$, $p_i = p|_{\Omega_i}$, $i = S, D$. The flow in the domain Ω_S is assumed to be of Stokes type and governed by the equations

$$\begin{aligned} -2\nu \operatorname{div}\left(D\left(\mathbf{v}_S\right)\right) + \nabla p_S &= \mathbf{f}, \quad \text{in } \Omega_S \\ \operatorname{div} \mathbf{v}_S &= 0, \quad \text{in } \Omega_S \end{aligned} \tag{1}$$

with symmetric strain tensor $D(\mathbf{v}_S) = (1/2)(\nabla \mathbf{v}_S + \nabla \mathbf{v}_S^T)$, external force $\mathbf{f}$, and constant viscosity $\nu > 0$. In the porous region Ω_D the filtration of an incompressible flow through porous media is described by Darcy equations

$$\begin{aligned} K^{-1}\mathbf{v}_D + \nabla p_D &= \mathbf{f}, \quad \text{in } \Omega_D \\ \operatorname{div} \mathbf{v}_D &= g, \quad \text{in } \Omega_D, \end{aligned} \tag{2}$$

where the permeability $K = K(x)$ is a positive definite symmetric tensor and g denotes an external Darcy force.

2.2. Boundary Conditions. On $\Gamma_S = \partial\Omega_S \setminus \Gamma$, we prescribe homogeneous Dirichlet conditions for the velocity $\mathbf{v}_S$.

$$\mathbf{v}_S = \mathbf{0}, \quad \text{on } \Gamma_S. \tag{3}$$

The boundary of Ω_D is split into three parts $\partial\Omega_D = \Gamma \cup \Gamma_{D,1} \cup \Gamma_{D,2}$. We prescribe zero flux on $\Gamma_{D,1}$ and a homogeneous Dirichlet condition for the pressure on $\Gamma_{D,2}$.

$$\begin{aligned} \mathbf{v}_D \cdot \mathbf{n}_D &= 0, \quad \text{on } \Gamma_{D,1} \\ p_D &= 0, \quad \text{on } \Gamma_{D,2}, \end{aligned} \tag{4}$$

where $\mathbf{n}_D$ denotes the outer normal vector on the boundary pointing from Ω_D into Ω_S. This boundary condition ensures a zero mass flux.

2.3. The Beavers-Joseph-Saffman Condition. The flows in Ω_S and Ω_D are coupled across the interface Γ. Conditions describing the interaction of the flows are as follows [6, 8]:

(i) The continuity of the normal velocity:

$$\mathbf{v}_S \cdot \mathbf{n}_S = -\mathbf{v}_D \cdot \mathbf{n}_D, \quad \text{on } \Gamma \tag{5}$$

(ii) The balance of normal forces:

$$-\left(-p_S I + 2\nu D\left(\mathbf{v}_S\right)\right)\mathbf{n}_S \cdot \mathbf{n}_S = p_D, \quad \text{on } \Gamma \tag{6}$$

(iii) The Beavers-Joseph-Saffman condition written in terms of the strain tensor:

$$\mathbf{v}_S \cdot \tau = -\frac{2\sqrt{\tilde{k}}}{\alpha}\left(D\left(\mathbf{v}_S\right) \cdot \mathbf{n}_S\right) \cdot \tau, \tag{7}$$

where $\tilde{k} = \nu K \tau \cdot \tau$ and α is a dimensionless parameter to be determined experimentally, this condition relating the tangential slip velocity $\mathbf{v}_S \cdot \tau$ to the normal derivative of the tangential velocity component in the Stokes region

3. Variational Formulation

As variational formulation we consider the so-called L^2-formulation used by Karper et al. [4] and recently by [9, 10]. We denote

$$\begin{aligned} (\mathbf{v}, \mathbf{w})_\Omega &= \int_\Omega \mathbf{v}\mathbf{w}\, dx, \quad \mathbf{v}, \mathbf{w} \in L^2(\Omega)^d, \\ \langle v, w\rangle_\Gamma &= \int_\Gamma v w\, ds, \quad v, w \in L^2(\Gamma), \end{aligned} \tag{8}$$

where $L^2(\Omega)$ and $H^1(\Omega)$ denote the usual Sobolev spaces.

Next, we define the spaces

$$\begin{aligned} \mathbf{H}^1_{\Gamma_S}(\Omega_S) &= \left\{\mathbf{w} \in \left(H^1(\Omega_S)\right)^d \mid \mathbf{w} = \mathbf{0} \text{ on } \Gamma_S\right\} \\ \mathbf{H}^1(\operatorname{div}, \Omega_D) &= \left\{\mathbf{w} \in L^2(\Omega_D)^d \mid \operatorname{div}\mathbf{w} \in L^2(\Omega_D)\right\}, \\ \mathbf{H}^1_{\Gamma_{D,1}}(\Omega_D) &= \left\{\mathbf{w} \in \mathbf{H}^1(\operatorname{div}, \Omega_D) \mid \mathbf{w} \cdot \mathbf{n}_D = 0 \text{ on } \Gamma_D\right\}. \end{aligned} \tag{9}$$

Then, multiplying the Stokes equations (1) by the test functions $\mathbf{w}_S \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)$, $q_S \in L^2(\Omega_S)$, respectively, and integrating by part on the domain Ω_S, we obtain

$$\begin{aligned} &\left(2\nu D\left(\mathbf{v}_S\right), D\left(\mathbf{w}_S\right)\right)_{\Omega_S} - \left\langle 2\nu D\left(\mathbf{v}_S\right)\mathbf{n}_S, \mathbf{w}_S\right\rangle \\ &\quad - \left(p_S, \operatorname{div}\mathbf{w}_S\right)_{\Omega_S} + \left\langle p_S, \mathbf{w}_S \cdot \mathbf{n}_S\right\rangle_\Gamma = \left(\mathbf{f}, \mathbf{w}_S\right)_{\Omega_S}, \\ &\left(\operatorname{div}\mathbf{v}_S, q_S\right)_{\Omega_S} = 0. \end{aligned} \tag{10}$$

Using the decomposition $\mathbf{w}_S = (\mathbf{w}_S \cdot \mathbf{n}_S)\mathbf{n}_S + (\mathbf{w}_S \cdot \tau)\tau$, the fluid normal stress condition (6), and the BJS interface condition (7) in (10), we obtain the weak formulation of the Stokes equations: find $\mathbf{v}_S \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)$, $p_S \in L^2(\Omega_S)$ such that

$$\begin{aligned} &\left(2\nu D\left(\mathbf{v}_S\right), D\left(\mathbf{w}_S\right)\right)_{\Omega_S} + \frac{\nu\alpha}{\sqrt{\tilde{k}}}\left\langle \mathbf{v}_S \cdot \tau, \mathbf{w}_S \cdot \tau\right\rangle_\Gamma \\ &\quad - \left(p_S, \operatorname{div}\mathbf{w}_S\right)_{\Omega_S} + \left\langle p_D, \mathbf{w}_S \cdot \mathbf{n}_S\right\rangle_\Gamma = \left(\mathbf{f}, \mathbf{w}_S\right)_{\Omega_S}, \\ &\left(\operatorname{div}\mathbf{v}_S, q_S\right)_{\Omega_S} = 0, \end{aligned} \tag{11}$$

$\forall \mathbf{w}_S \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)$, $q_S \in L^2(\Omega_S)$.

Similarly, taking $\delta > 0$ and testing the Darcy equations (2) by $\mathbf{w}_D \in \mathbf{H}^1_{\Gamma_{D,1}}(\Omega_D)$, $q_D \in L^2(\Omega_D)$, respectively, together with the weighted grad-div term we obtain the weak formulation of Darcy equations: find $\mathbf{v}_D \in \mathbf{H}^1_{\Gamma_D}(\Omega_D)$, $p_D \in H^1_{D,2}(\Omega_D)$ such that

$$\begin{aligned} &\left(K^{-1}\mathbf{v}_D, \mathbf{w}_D\right)_{\Omega_D} + \left(\nabla p_D, \mathbf{w}_D\right)_{\Omega_D} \\ &\quad + \delta\left(\operatorname{div}\mathbf{v}_D, \operatorname{div}\mathbf{w}_D\right)_{\Omega_D} = \delta\left(g, \operatorname{div}\mathbf{w}_D\right)_{\Omega_D}, \\ &- \left(\mathbf{v}_D, \nabla q_D\right)_{\Omega_D} + \left\langle \mathbf{v}_D \cdot \mathbf{n}_D, q_D\right\rangle_\Gamma = \left(g, q_D\right)_{\Omega_D}. \end{aligned} \tag{12}$$

Summing up (11) and (12) the weak form of the coupled problem is given by the following: find $\mathbf{v}_S \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)$, $p_S \in L^2(\Omega_S)$, $\mathbf{v}_D \in \mathbf{H}^1_{\Gamma_D}(\Omega_D)$, and $p_D \in L^2(\Omega_D)$ such that

$$\begin{aligned} &\left(2\nu D\left(\mathbf{v}_S\right), D\left(\mathbf{w}_S\right)\right)_{\Omega_S} - \left(p_S, \operatorname{div}\mathbf{w}_S\right)_{\Omega_S} \\ &\quad + \left(K^{-1}\mathbf{v}_D, \mathbf{w}_D\right)_{\Omega_D} + \left(\nabla p_D, \mathbf{w}_D\right)_{\Omega_D} \\ &\quad + \delta\left(\operatorname{div}\mathbf{v}_D, \operatorname{div}\mathbf{w}_D\right)_{\Omega_D} + \frac{\nu\alpha}{\sqrt{\tilde{k}}}\left\langle \mathbf{v}_S \cdot \tau, \mathbf{w}_S \cdot \tau\right\rangle_\Gamma \\ &\quad + \left\langle p_D, \mathbf{w}_S \cdot \mathbf{n}_S\right\rangle_\Gamma = \left(\mathbf{f}, \mathbf{w}_S\right)_{\Omega_S} + \delta\left(g, \operatorname{div}\mathbf{w}_D\right)_{\Omega_D}, \\ &\left(\operatorname{div}\mathbf{v}_S, q_S\right)_{\Omega_S} - \left(\mathbf{v}_D, \nabla q_D\right)_{\Omega_D} \\ &\quad - \left\langle \mathbf{v}_S \cdot \mathbf{n}_S, q_D\right\rangle_\Gamma = \left(g, q_D\right)_{\Omega_D}. \end{aligned} \tag{13}$$

To analyze the weak formulation of the coupled problem we introduce the following spaces

$$\begin{aligned}\mathbf{V} = \Big\{ & \mathbf{v} \in \mathbf{H}(\operatorname{div}, \Omega) \mid \mathbf{v}_S \in \left(H^1(\Omega_S)\right)^d,\ \mathbf{v}_S \\ & = 0 \text{ on } \Gamma_S,\ \mathbf{v}\cdot\mathbf{n}_D = 0 \text{ on } \Gamma_{D,1} \Big\}, \\ Q = \Big\{ & q \in L^2(\Omega) \mid p_D \in H^1(\Omega_D),\ p = 0 \in \Gamma_{D,2} \Big\}, \\ X = {} & \mathbf{V} \times Q.\end{aligned} \tag{14}$$

The velocity and pressure spaces $\mathbf{V}$ and Q are equipped with the natural norms

$$\begin{aligned}\|\mathbf{v}\|_{\mathbf{V}} &= \left(\|\nabla \mathbf{v}\|_{\Omega_S}^2 + \|\mathbf{v}\|_{\Omega_D}^2 + \|\operatorname{div}\mathbf{v}\|_{\Omega_D}^2\right)^{1/2}, \\ \|p\|_Q &= \left(\|p\|_{\Omega_S}^2 + \|\nabla p\|_{\Omega_D}^2\right)^{1/2}.\end{aligned} \tag{15}$$

Further, due to the positive definiteness of K with respect to the $L^2(\Omega_D)$ norm $\|\cdot\|_{\Omega_D}$, there exist positive real numbers k_1 and k_2 such that

$$k_1 \|\mathbf{v}\|_{\Omega_D}^2 \le \left(K^{-1}\mathbf{v}, \mathbf{v}\right)_{\Omega_D} \le k_2 \|\mathbf{v}\|_{\Omega_D}^2, \quad \forall \mathbf{v} \in \mathbf{V}. \tag{16}$$

Next, we define the bilinear forms for $\mathbf{v} = (\mathbf{v}_S, \mathbf{v}_D)$, $\mathbf{w} = (\mathbf{w}_S, \mathbf{w}_D)$ in $\mathbf{V}$ and $p = (p_S, p_D)$, $q = (q_S, q_D)$ in Q on the two parts of the domain by

$$\begin{aligned}\mathscr{A}_S(\mathbf{v}, p; \mathbf{w}, q) &= \left(2\nu D(\mathbf{v}_S), D(\mathbf{w}_S)\right)_{\Omega_S} \\ &\quad + \frac{\nu\alpha}{\sqrt{k}} \langle \mathbf{v}_S\cdot\tau, \mathbf{w}_S\cdot\tau\rangle_\Gamma \\ &\quad - (p_S, \operatorname{div}\mathbf{w}_S)_{\Omega_S} \\ &\quad + (\operatorname{div}\mathbf{v}_S, q_S)_{\Omega_S}, \\ \mathscr{A}_D(\mathbf{v}, p; \mathbf{w}, q) &= \left(K^{-1}\mathbf{v}_D, \mathbf{w}_D\right)_{\Omega_D} \\ &\quad + \delta(\operatorname{div}\mathbf{v}_D, \operatorname{div}\mathbf{w}_D)_{\Omega_D} \\ &\quad + (\nabla p_D, \mathbf{w}_D)_{\Omega_D} - (\mathbf{v}_D, \nabla q_D)_{\Omega_D}.\end{aligned} \tag{17}$$

Hence, the bilinear form for the coupled problem is the sum of $\mathscr{A}_S(\mathbf{v}, p; \mathbf{w}, q)$, $\mathscr{A}_D(\mathbf{v}, p; \mathbf{w}, q)$, and terms to enforce the continuity of the normal part of the velocities across the interface.

$$\begin{aligned}\mathscr{A}(\mathbf{v}, p; \mathbf{w}, q) &= \mathscr{A}_S(\mathbf{v}, p; \mathbf{w}, q) + \mathscr{A}_D(\mathbf{v}, p; \mathbf{w}, q) \\ &\quad + \langle p_D, \mathbf{w}_S\cdot\mathbf{n}_S\rangle_\Gamma - \langle q_D, \mathbf{v}_S\cdot\mathbf{n}_S\rangle_\Gamma.\end{aligned} \tag{18}$$

Assuming, for simplicity, that $\mathbf{f}$ and g are extended by zero to the whole domain, the variational formulation of the coupled Stokes-Darcy system in compact form reads as follows: find $(\mathbf{v}, p) \in \mathbf{V} \times Q$ solution of

$$\mathscr{A}(\mathbf{v}, p; \mathbf{w}, q) = \mathscr{F}(\mathbf{w}, q), \quad \forall (\mathbf{w}, q) \in \mathbf{V} \times Q, \tag{19}$$

with

$$\mathscr{F}(\mathbf{w}, q) = (\mathbf{f}, \mathbf{w}_S)_\Omega + (g, q_D)_\Omega + \delta(g, \operatorname{div}\mathbf{w}_D)_\Omega. \tag{20}$$

It can easily be shown that a sufficiently regular solution $(\mathbf{v}, p) \in \mathbf{V} \times Q$ of (19) such that $\mathbf{v}_S \in H^2(\Omega_S)^d$, $\mathbf{v}_D \in H^1(\Omega_D)^d$, $p \in H^1(\Omega_S \cup \Omega_D)$ is also a classical solution of (1) and (2). We note that there is an alternative variational formulation to the one given here called $H(\text{div})$-formulation. The latter uses the term $-(p, \operatorname{div}\mathbf{w})_{\Omega_D} + (\operatorname{div}\mathbf{v}, q)_{\Omega_D}$ instead of $(\mathbf{w}, \nabla p)_{\Omega_D} - (\mathbf{v}, \nabla q)_{\Omega_D}$ [4].

The existence and uniqueness of the solution of problem (19) follows from Brezzi's conditions for saddle point problems [11]; namely,

$$\begin{aligned}A(\mathbf{v}, p; \mathbf{v}, p) &\ge \tilde{\gamma}\|\mathbf{v}\|_{\mathbf{V}}^2, \\ &\quad \forall v \in \mathbf{V},\ \tilde{\gamma} > 0,\end{aligned} \tag{21}$$

$$\inf_{q\in L^2(\Omega_S)} \sup_{\mathbf{v}\in H^1(\Omega_S)^d} \frac{(\operatorname{div}\mathbf{v}, q)_{\Omega_S}}{\|\nabla\mathbf{v}\|_{\Omega_S}\|q\|_{\Omega_S}} \ge \beta_S, \tag{22}$$

$$\inf_{q\in H^1(\Omega_D)} \sup_{\mathbf{v}\in L^2(\Omega_D)^d} \frac{-(\mathbf{v}, \nabla q)_{\Omega_D}}{\|\mathbf{v}\|_{\Omega_D}\|\nabla q\|_{\Omega_D}} \ge \beta_D. \tag{23}$$

with positive constants β_S and β_D [7].

The following lemma is needed in the analysis below and is a consequence of the continuous inf-sup conditions (23) [10].

Lemma 1. *For every $(\mathbf{v}, p) \in X$ there is $\mathbf{w} \in \mathbf{V}$ such that $\mathbf{w}_S \cdot \mathbf{n}_S = 0$ on Γ, satisfying*

$$\begin{aligned}\mathscr{A}(\mathbf{v}, p; \mathbf{w}, 0) &\ge c_2\|p\|_Q^2 - c_1\|\mathbf{v}\|_{\mathbf{V}}^2, \\ \|\mathbf{w}\|_{\mathbf{V}} &\le c_3\|p\|_Q,\end{aligned} \tag{24}$$

with positive constants c_1, c_2, and c_3.

Proof. Let $(\mathbf{v}, p) \in X$. Then, due to Stokes inf-sup condition there exists $\mathbf{w}_S \in H^1(\Omega_S)^d$ with $\mathbf{w}_S = \mathbf{0}$ on Γ_S and $\mathbf{w}_S\cdot\mathbf{n} = 0$ on Γ such that

$$\begin{aligned}-(\operatorname{div}\mathbf{w}_S, p)_{\Omega_S} &= \|p\|_{\Omega_S}^2, \\ \|\nabla\mathbf{w}_S\|_{\Omega_S} &\le c_S\|p\|_{\Omega_S}.\end{aligned} \tag{25}$$

For the Darcy equation, due to the condition $p = 0$ on $\Gamma_{D,2}$, there exists $\mathbf{w}_D \in H^1(\Omega_D)^d$ with $\mathbf{w}_D\cdot\mathbf{n} = 0$ on $\Gamma_{D,2}$ and Γ, such that

$$\begin{aligned}-(\operatorname{div}\mathbf{w}_D, p)_{\Omega_D} &= \|\nabla p\|_{\Omega_D}^2, \\ \|\nabla\mathbf{w}_D\|_{\Omega_D} &\le c_D\|\nabla p\|_{\Omega_D}.\end{aligned} \tag{26}$$

Define

$$\mathbf{w} = \begin{cases}\mathbf{w}_S & \text{in } \Omega_S \\ \mathbf{w}_D & \text{in } \Omega_D,\end{cases} \tag{27}$$

and then

$$\begin{aligned}
\mathscr{A}(\mathbf{v},p;\mathbf{w},0) &= (2\nu D(\mathbf{v}), D(\mathbf{w}))_{\Omega_S} - (p, \operatorname{div}\mathbf{w})_{\Omega_S} \\
&\quad + \left(K^{-1}\mathbf{v},\mathbf{w}\right)_{\Omega_D} + (\nabla p,\mathbf{w})_{\Omega_D} \\
&\quad + \delta(\operatorname{div}\mathbf{v}, \operatorname{div}\mathbf{w})_{\Omega_D} \\
&\geq -2\nu\|D(\mathbf{v})\|_{\Omega_S}\|D(\mathbf{w})\|_{\Omega_S} + \|p\|^2_{\Omega_S} \\
&\quad - k_2\|\mathbf{v}\|_{\Omega_D}\|\mathbf{w}_D\|_{\Omega_D} + \|\nabla p\|^2_{\Omega_D} \\
&\quad - \delta\|\operatorname{div}\mathbf{v}\|_{\Omega_D}\|\operatorname{div}\mathbf{w}\|_{\Omega_D} \\
&\geq -2\nu\|\nabla\mathbf{v}\|_{\Omega_S}\|\nabla\mathbf{w}\|_{\Omega_S} + \|p\|^2_{\Omega_S} \\
&\quad - k_2\|\mathbf{v}\|_{\Omega_D}\|\mathbf{w}_D\|_{\Omega_D} + \|\nabla p\|^2_{\Omega_D} \\
&\quad - \delta\|\operatorname{div}\mathbf{v}\|_{\Omega_D}\|\nabla\mathbf{w}\|_{\Omega_D} \\
&\geq -2\nu c_S\|\nabla\mathbf{v}\|_{\Omega_S}\|p\|_{\Omega_S} + \|p\|^2_{\Omega_S} \\
&\quad - c_p c_D k_2\|\mathbf{v}\|_{\Omega_D}\|\nabla p\|_{\Omega_D} + \|\nabla p\|^2_{\Omega_D} \\
&\quad - \delta c_D\|\operatorname{div}\mathbf{v}\|_{\Omega_D}\|\nabla p\|_{\Omega_D},
\end{aligned} \tag{28}$$

where c_p denote the Poincaré constant.

Then, using Young's inequality we obtain

$$\begin{aligned}
\mathscr{A}(\mathbf{v},p;\mathbf{w},0) &\geq -\frac{\nu c_S}{\varepsilon_1}\|\nabla\mathbf{v}\|^2_{\Omega_S} + (1-\nu c_S\varepsilon_1)\|p\|^2_{\Omega_S} \\
&\quad - \frac{c_p c_D k_2}{2\varepsilon_2}\|\mathbf{v}\|^2_{\Omega_D} \\
&\quad + \left(1 - \frac{c_p c_D k_2\varepsilon_2}{2} - \frac{\delta c_D\varepsilon_3}{2}\right)\|\nabla p\|^2_{\Omega_D} \\
&\quad - \frac{\delta c_D}{2\varepsilon_3}\|\operatorname{div}\mathbf{v}\|^2_{\Omega_D}.
\end{aligned} \tag{29}$$

Choosing ε_1, ε_2, ε_3 positive constants such that

$$\begin{aligned}
\varepsilon_1 &< \frac{1}{\nu c_S}, \\
\varepsilon_2 &< \frac{2}{c_p c_D k_2}, \\
\varepsilon_3 &< \frac{2 - c_p c_D k_2\varepsilon_2}{\delta c_D},
\end{aligned} \tag{30}$$

we obtain the required result

$$\mathscr{A}(\mathbf{v},p;\mathbf{w},0) \geq c_2\|p\|^2_Q - c_1\|\mathbf{v}\|^2_{\mathbf{V}}, \tag{31}$$

where

$$\begin{aligned}
c_1 &= \max\left\{\frac{\nu c_S}{\varepsilon_1}, \frac{c_p c_D k_2}{2\varepsilon_2}, \frac{\delta c_D}{2\varepsilon_3}\right\}, \\
c_2 &= \min\left\{1 - \nu c_S\varepsilon_1, 1 - \frac{c_p c_D k_2\varepsilon_2}{2} - \frac{\delta c_D\varepsilon_3}{2}\right\}.
\end{aligned} \tag{32}$$

In addition, we also have

$$\begin{aligned}
\|\mathbf{w}\|^2_{\mathbf{V}} &= \|\nabla\mathbf{w}\|^2_{\Omega_S} + \|\mathbf{w}\|^2_{\Omega_D} + \|\operatorname{div}\mathbf{w}\|^2_{\Omega_D} \\
&\leq c_S^2\|p\|^2_{\Omega_S} + c_D^2\left(c_p^2+1\right)\|\nabla p\|^2_{\Omega_D} \leq c_3\|p\|_Q,
\end{aligned} \tag{33}$$

where $c_3^2 = \max\{c_S^2, c_D^2(c_p^2+1)\}$. □

4. Finite Element Discretization

Let $\mathscr{T}_h$ be a shape-regular partition of quadrilaterals for $d = 2$ or hexahedra for $d = 3$ [12, 13]. The diameter of element $T \in \mathscr{T}_h$ will be denoted by h_T and the global mesh size is defined by $h := \max\{h_T : T \text{ in } \mathscr{T}_h\}$. Let $\widehat{T} := (-1;1)^d$ be the reference element, F_T the mapping from $\widehat{T}$ to element T, and $Q^r(\widehat{T})$ the space of all polynomials on $\widehat{T}$ with maximal degree $r \geq 0$ in each coordinate. We assume that the mesh $\mathscr{T}_h$ is obtained from a coarser mesh $\mathscr{T}_{2h}$ by global refinement. Hence, $\mathscr{T}_{2h}$ consists of patches of elements of $\mathscr{T}_h$. We define the finite element space

$$\begin{aligned}
X_h^r &:= \left\{v \in C(\Omega_S) \cup C(\Omega_D) : v|_T \right. \\
&\qquad \left. \circ F_T \text{ in } Q_r\left(\widehat{T}\right), \ \forall T \in \mathscr{T}_h\right\}.
\end{aligned} \tag{34}$$

For the discrete spaces $\mathbf{V}_h$ and Q_h we use the equal-order finite element functions that are continuous in Ω_S and Ω_D and piecewise polynomials of degree $r \geq 1$.

$$\begin{aligned}
\mathbf{V}_h &= \left(X_h^r\right)^d \cap \mathbf{V}, \\
Q_h &= X_h^r \cap Q \cap H^1(\Omega).
\end{aligned} \tag{35}$$

We define the *Scott-Zhang* interpolation operator which preserves the boundary condition [13], as $j_r^h : H^1(\Omega) \to X_h^r$ with stability and interpolation properties, respectively, as

$$\left\|\nabla j_r^h\phi\right\|_\Omega \leq c_s|\phi|_{1,\Omega}, \quad \phi \in H^1(\Omega). \tag{36}$$

$$\begin{aligned}
\left\|\phi - j_r^h\phi\right\|_{m,\Omega} &\leq c_i h^{r+1-m}|\phi|_{r+1,\Omega}, \\
&\qquad \phi \in H^{r+1}(\Omega), \ m = 0 \text{ or } 1,
\end{aligned} \tag{37}$$

where c_i, c_s are positive constants.

We will also use the inverse inequality

$$\left(\sum_{T\in\mathscr{T}_h} h_T^2\|\nabla\phi\|_T\right)^{1/2} \leq c_I\|\phi\|_\Omega, \quad \forall\phi \in H^1(\Omega). \tag{38}$$

Similarly, for vector functions we define the interpolation operator

$$\mathbf{j}_r^h : H^1(\Omega)^d \longrightarrow \left(X_h^r\right)^d, \tag{39}$$

with interpolation and stability properties as above.

It is known that the standard Galerkin discretizations of the Darcy system are not stable for equal-order elements. This instability stems from the violation of the discrete analogue on to the inf-sup condition. One possibility to circumvent this condition is to work with a modified bilinear form $\mathscr{A}_h(\cdot;\cdot)$ by adding a stabilization term $\mathscr{S}_h(\cdot;\cdot)$; that is,

$$\mathscr{A}_h(\mathbf{v}_h, p_h; \mathbf{w}, q) = \mathscr{A}(\mathbf{v}_h, p_h; \mathbf{w}, q) + \mathscr{S}_h(p_h; q), \quad (40)$$

such that the stabilized discrete problem reads

$$\mathscr{A}_h(\mathbf{v}_h, p_h; \mathbf{w}, q) = \mathscr{F}(\mathbf{w}, q) \quad \forall (\mathbf{w}, q) \in \mathbf{V}_h \times Q_h. \quad (41)$$

Unlike in [10] where a combination of a generalized mini element and local projection (LPS) is analyzed and in [14] where a method based on two local Gauss integrals for the Stokes equations is used, here we will analyze the problem using a subgrid method [12, 15, 16].

For this method the filter, with respect to the global Lagrange interpolant I_{2h}, onto a coarser mesh $\mathscr{T}_{2h}$ is used. Defining $\kappa_{2h} = I - I_{2h}$ the subgrid stabilization term reads

$$\mathscr{S}_h(p_h; q) = \sum_{M \in \mathscr{T}_{2h}} h_M (\gamma \nabla \kappa_{2h} p_h, \nabla \kappa_{2h} q)_M, \quad r \geq 1, \quad (42)$$

where γ is patchwise constant.

A more attractive method from the computational point is obtained using only the fine mesh with smaller stencil. Defining $\kappa_h = I - I_h$ the subgrid stabilization term reads

$$\mathscr{S}_h(p_h; q) = \sum_{K \in \mathscr{T}_h} h_K (\gamma \nabla \kappa_h p_h, \nabla \kappa_h q)_K, \quad r \geq 2. \quad (43)$$

Next, we prove the stability of the discrete coupled Stokes-Darcy problem with respect to the norm

$$|||(\mathbf{v}, p)|||_h = \left(\|\mathbf{v}\|_{\mathbf{V}}^2 + \|p\|_Q^2 + \mathscr{S}_h(p; p)\right)^{1/2}. \quad (44)$$

5. Stability

Theorem 2. *Let $\mathscr{T}_h$ be a quasi-regular partition [13]. Then, the following discrete inf-sup condition holds for some positive constant $\widetilde{\beta}$ independent of the mesh size h.*

$$\inf_{(\mathbf{v}_h, p_h) \in \mathbf{V}_h \times Q_h \setminus \{\mathbf{0},0\}} \sup_{(\mathbf{w}_h, q_h) \in \mathbf{V}_h \times Q_h \setminus \{\mathbf{0},0\}} \frac{\mathscr{A}(\mathbf{v}_h, p_h; \mathbf{w}_h, q_h)}{|||(\mathbf{v}_h, p_h)|||_h |||(\mathbf{w}_h, q_h)|||_h} \geq \widetilde{\beta}. \quad (45)$$

Proof. First, let $(\mathbf{v}_h, p_h) \in \mathbf{V}_h \times Q_h$, and then the diagonal testing combined with Korn's inequality and the positivity of K^{-1} give

$$\begin{aligned}\mathscr{A}_h(\mathbf{v}_h, p_h; \mathbf{v}_h, p_h) &= \mathscr{A}(\mathbf{v}_h, p_h; \mathbf{v}_h, p_h) + \mathscr{S}_h(p_h; p_h) \\ &\geq \widetilde{\alpha} \|\mathbf{v}\|_{\mathbf{V}}^2 + \mathscr{S}_h(p_h; p_h).\end{aligned} \quad (46)$$

In addition, let $\mathbf{w}$ be as in Lemma 1, corresponding to $(\mathbf{v}_h, p_h) \in \mathbf{V}_h \times Q_h$, and set $\mathbf{z} = \mathbf{j}_r^h \mathbf{w} - \mathbf{w}$. Then,

$$\begin{aligned}\mathscr{A}(\mathbf{v}_h, p_h; \mathbf{j}_r^h \mathbf{w}, 0) &= \mathscr{A}(\mathbf{v}_h, p_h; \mathbf{w}, 0) + \mathscr{A}(\mathbf{v}_h, p_h; \mathbf{z}, 0) \\ &\geq c_2 \|p_h\|_Q^2 - c_1 \|\mathbf{v}_h\|_{\mathbf{V}}^2 \\ &\quad + \mathscr{A}_S(\mathbf{v}_h, p_h; \mathbf{z}, 0) \\ &\quad + \mathscr{A}_D(\mathbf{v}_h, p_h; \mathbf{z}, 0).\end{aligned} \quad (47)$$

Next, we estimate $\mathscr{A}_S(\mathbf{v}_h, p_h; \mathbf{z}, 0)$ and $\mathscr{A}_D(\mathbf{v}_h, p_h; \mathbf{z}, 0)$ as follows:

$$\begin{aligned}\mathscr{A}_S(\mathbf{v}_h, p_h; \mathbf{z}, 0) &= (2\nu D(\mathbf{v}_h), D(\mathbf{z}))_{\Omega_S} + (\nabla p_h, \mathbf{z})_{\Omega_S} \\ &\quad + \frac{\nu\alpha}{\sqrt{\tilde{k}}} \langle \mathbf{v}_{hS} \cdot \tau, \mathbf{z}_S \cdot \tau \rangle_\Gamma,\end{aligned} \quad (48)$$

where the first two terms are bounded using Cauchy inequality together with the interpolation, stability, and inverse inequalities

$$\begin{aligned}\left|(\nu D(\mathbf{v}_h), D(\mathbf{z}))_{\Omega_S}\right| &\leq \nu \|D(\mathbf{v}_h)\|_{\Omega_S} \|D(\mathbf{z})\|_{\Omega_S} \\ &\leq \nu \|\mathbf{v}_h\|_{\mathbf{V}} \|\nabla \mathbf{z}\|_{\Omega_S} \leq \nu c_i \|\mathbf{v}_h\|_{\mathbf{V}} \|\nabla \mathbf{w}\|_{\Omega_S} \\ &\leq \nu c_3 c_i \|\mathbf{v}_h\|_{\mathbf{V}} \|p_h\|_Q, \\ (\nabla p_h, \mathbf{z})_{\Omega_S} &\leq \left(\sum_{T \in \mathscr{T}_h, T \subset \Omega_S} h_T^{-2} \|\mathbf{z}\|_T^2\right)^{1/2} \\ &\quad \cdot \left(\sum_{T \in \mathscr{T}_h, T \subset \Omega_S} h_T^2 \|\nabla p_h\|_T^2\right)^{1/2} \\ &\leq \left(\sum_{T \in \mathscr{T}_h, T \subset \Omega_S} h_T^{-2} h_T^{2r} \|\nabla \mathbf{w}\|_T^2\right)^{1/2} c_I \|p_h\|_{\Omega S} \\ &\leq c c_i c_I \|\nabla \mathbf{w}\|_{\Omega_S} \|p_h\|_{\Omega_S} \leq c c_i c_I c_3 \|p_h\|_Q^2.\end{aligned} \quad (49)$$

The boundary term is bounded using the trace theorem and the H^1- stability by

$$\begin{aligned}\left|\frac{\nu\alpha}{\sqrt{\tilde{k}}} \langle \mathbf{v}_{h_S} \cdot \tau, \mathbf{z}_S \cdot \tau \rangle_\Gamma\right| &\leq c_\Gamma^2 \frac{\nu\alpha}{\sqrt{\tilde{k}}} \|\mathbf{v}_h\|_{\mathbf{V}} \|\nabla \mathbf{z}\|_{\Omega_S} \\ &\leq c_\Gamma^2 c_s c_3 \frac{\nu\alpha}{\sqrt{\tilde{k}}} \|\mathbf{v}_h\|_{\mathbf{V}} \|p_h\|_Q.\end{aligned} \quad (50)$$

Hence, by Young inequality with

$$\begin{aligned}\epsilon_1 &= \frac{c_2}{8\nu c_i c_3}, \\ \epsilon_2 &= \frac{c_2 \sqrt{\tilde{k}}}{8\nu\alpha c_\Gamma^2 c_s c_3}\end{aligned} \quad (51)$$

we obtain

$$\mathscr{A}_S(\mathbf{v}_h, p_h; \mathbf{z}, 0) \leq \frac{c_2}{8} c_4 \|p_h\|_Q^2 + c_4 \|\mathbf{v}_h\|_{\mathbf{V}}^2, \quad (52)$$

where $c_4 = (4(\nu c_3 c_i)^2 + 0.25(c_\Gamma^2 c_s c_3)^2)/c_2$.

For the Darcy bilinear form we have

$$\begin{aligned}\mathscr{A}_D(\mathbf{v}_h, p_h; \mathbf{z}, 0) &= \left(K^{-1}\mathbf{v}_h, \mathbf{z}\right)_{\Omega_D} + \delta\left(\operatorname{div}\mathbf{v}_h, \operatorname{div}\mathbf{z}\right)_{\Omega_D} \\ &\quad + \left(\nabla p_h, \mathbf{z}\right)_{\Omega_D} \\ &= \left(K^{-1}\mathbf{v}_h, \mathbf{z}\right)_{\Omega_D} + \delta\left(\operatorname{div}\mathbf{v}_h, \operatorname{div}\mathbf{z}\right)_{\Omega_D} \\ &\quad + \left(\nabla\left(p_h - \kappa_{2h}p_h\right), \mathbf{z}\right)_{\Omega_D} \\ &\quad + \left(\nabla\kappa_{2h}p_h, \mathbf{z}\right)_{\Omega_D} \\ &\le \left\|K^{-1}\mathbf{v}_h\right\|_{\Omega_D}\|\mathbf{z}\|_{\Omega_D} \\ &\quad + \delta\left\|\operatorname{div}\mathbf{v}_h\right\|_{\Omega_D}\|\mathbf{z}\|_{\Omega_D} \\ &\quad + \left\|\nabla\left(p_h - \kappa_{2h}p_h\right)\right\|_{\Omega_D}\|\mathbf{z}\|_{\Omega_D} \\ &\quad + \left\|\nabla\kappa_{2h}p_h\right\|_{\Omega_D}\|\mathbf{z}\|_{\Omega_D} \\ &\le k_2\left\|\mathbf{v}_h\right\|_{\Omega_D} c_i\|\mathbf{w}\|_{\Omega_D} \\ &\quad + \delta\left\|\operatorname{div}\mathbf{v}_h\right\|_{\Omega_D}\left(1 + c_s\right)\|\mathbf{w}\|_{\Omega_D} \\ &\quad + \left\|\nabla\left(p_h - \kappa_{2h}p_h\right)\right\|_{\Omega_D} c_i\|\mathbf{w}\|_{\Omega_D} \\ &\quad + c_s\left\|\nabla p_h\right\|_{\Omega_D} \\ &\le k_2c_ic_3\left\|p_h\right\|_Q \\ &\quad + \delta c_3\left(1 + c_s\right)\left\|\operatorname{div}\mathbf{v}_h\right\|_{\Omega_D}\left\|p_h\right\|_Q \\ &\quad + c_ic_3\left\|\nabla\left(p_h - \kappa_{2h}p_h\right)\right\|_{\Omega_D} \\ &\quad + c_s\left\|p_h\right\|_Q.\end{aligned} \tag{53}$$

Then, by Young inequality and (52) we obtain

$$\begin{aligned}\mathscr{A}_h\left(\mathbf{v}_h, p_h; \mathbf{j}_r^h\mathbf{w}, 0\right) &\ge \frac{5c_2}{8}\left\|p_h\right\|_Q^2 \\ &\quad - C\left(\left\|\mathbf{v}_h\right\|_{\mathbf{V}}^2 + \mathcal{S}_h\left(p_h; p_h\right)\right).\end{aligned} \tag{54}$$

Scaling $\mathbf{j}_r^h\mathbf{w}$ we obtain

$$\begin{aligned}\mathscr{A}_h\left(\mathbf{v}_h, p_h; \mathbf{j}_r^h\mathbf{w}, 0\right) &\ge \left\|p_h\right\|_Q^2 \\ &\quad - C_1\left(\left\|\mathbf{v}_h\right\|_{\mathbf{V}}^2 + \mathcal{S}_h\left(p_h; p_h\right)\right).\end{aligned} \tag{55}$$

Choosing $(\mathbf{w}_h, q_h) = (\mathbf{v}_h, p_h) + (1/(1 + C_1))(\mathbf{j}_r^h\mathbf{w}, 0)$ we obtain

$$\begin{aligned}\mathscr{A}_h\left(\mathbf{v}_h, p_h; \mathbf{w}_h, q_h\right) &\ge \left\|\mathbf{v}_h\right\|_{\mathbf{V}}^2 + \frac{1}{1 + C_1}\left\|p_h\right\|_Q^2 \\ &\quad - \frac{C_1}{1 + C_1}\left\|\mathbf{v}_h\right\|_{\mathbf{V}}^2 \\ &= \frac{1}{1 + C_1}\left(\left\|\mathbf{v}_h\right\|_{\mathbf{V}}^2 + \left\|p_h\right\|_Q^2\right) \\ &= \frac{1}{1 + C_1}|||\left(\mathbf{v}_h, p_h\right)|||_h^2,\end{aligned}$$

$$\begin{aligned}|||\mathbf{w}_h, q_h|||_h &\le |||\left(\mathbf{v}_h, p_h\right)|||_h \\ &\quad + \frac{1}{1 + C_1}|||\left(\mathbf{j}_r^h\mathbf{w}, 0\right)|||_h \\ &\le |||\left(\mathbf{v}_h, p_h\right)|||_h + C_2\left\|\nabla\mathbf{j}_r^h\mathbf{w}\right\|_\Omega \\ &\le C_3|||\left(\mathbf{v}_h, p_h\right)|||_h\end{aligned} \tag{56}$$

which implies the required result

$$\inf_{(\mathbf{v}_h, p_h)\in\mathbf{V}_h\times Q_h\setminus\{\mathbf{0}\}}\ \sup_{(\mathbf{w}_h, q_h)\in\mathbf{V}_h\times Q_h\setminus\{\mathbf{0}\}}\frac{\mathscr{A}_h\left(\mathbf{v}_h, p_h; \mathbf{w}_h, q_h\right)}{|||\left(\mathbf{v}_h, p_h\right)|||_h|||\left(\mathbf{w}_h, q_h\right)|||_h} \ge \widetilde{\beta}, \tag{57}$$

with $\widetilde{\beta} = C_3^{-1}/(1 + C_1)$. □

6. Error Analysis

Theorem 3. *Assume that the solution $(\mathbf{v}, p)$ of the Stokes-Darcy problem (19) is such that $(\mathbf{v}_S, p_S) \in \mathbf{V}_S \cap H^{r+1}(\Omega_S)^d \times Q \cap H^{l+1}(\Omega_S)$, $(\mathbf{v}_D, p_D) \in \mathbf{V}_D \cap H^{r+1}(\Omega_D)^d \times Q \cap H^{l+1}(\Omega_D)$, and $(\mathbf{v}_h, p_h)$ is the solution of the stabilized problem (41). Then, the following error estimate holds with constants $c_1, c_2, \ldots, c_7$ independent of h:*

$$\begin{aligned}|||\left(\mathbf{v} - \mathbf{v}_h, p - p_h\right)|||_h &\le \Big\{\left(c_1\nu + c_2\right)^2 h^{2r}\|\mathbf{v}\|_{r+1,\Omega_S}^2 \\ &\quad + \left(c_3h + c_4\delta\right)^2 h^{2r}\|\mathbf{v}\|_{r+1,\Omega_D}^2 \\ &\quad + \left(c_5 + c_6\gamma^{1/2}h^{1/2} + c_7h\right)^2 h^{2l}\|p\|_{l+1,\Omega_S}^2 \\ &\quad + \left(c_5 + c_6\gamma^{1/2}h^{1/2} + c_7h\right)^2 h^{2l}\|p\|_{l+1,\Omega_D}^2\Big\}^{1/2}.\end{aligned} \tag{58}$$

Proof. Using the stability estimate of Theorem 3, there exists $(\mathbf{w}_h, q_h) \in \mathbf{V}_h \times Q_h$, with $|||(\mathbf{w}_h, q_h)|||_h \le \widetilde{C}$ satisfying

$$\begin{aligned}|||\left(\mathbf{j}_r^h\mathbf{v} - \mathbf{v}_h, j_l^h p - p_h\right)|||_h &\le \frac{1}{\widetilde{\beta}}\frac{\mathscr{A}_h\left(\mathbf{j}_r^h\mathbf{v} - \mathbf{v}_h, j_l^h p - p; \mathbf{w}_h, q_h\right)}{|||\left(\mathbf{w}_h, q_h\right)|||_h} \\ &\le \frac{1}{\widetilde{\beta}}\frac{\mathscr{A}_h\left(\mathbf{v} - \mathbf{v}_h, p - p_h; \mathbf{w}_h, q_h\right)}{|||\left(\mathbf{w}_h, q_h\right)|||_h} \\ &\quad + \frac{1}{\widetilde{\beta}}\frac{\mathscr{A}_h\left(\mathbf{j}_r^h\mathbf{v} - \mathbf{v}, j_l^h p - p; \mathbf{w}_h, q_h\right)}{|||\left(\mathbf{w}_h, q_h\right)|||_h}.\end{aligned} \tag{59}$$

Then, by Galerkin orthogonality property, the first term of (59) is bounded by

$$\begin{aligned}\frac{\mathscr{A}_h\left(\mathbf{v} - \mathbf{v}_h, p - p_h; \mathbf{w}_h, q_h\right)}{|||\left(\mathbf{w}_h, q_h\right)|||_h} &= \frac{\mathcal{S}_h\left(p; q_h\right)}{|||\left(\mathbf{w}_h, q_h\right)|||_h} \\ &\le \frac{\mathcal{S}_h\left(p; p\right)^{1/2}\mathcal{S}_h\left(q_h; q_h\right)^{1/2}}{|||\left(\mathbf{w}_h, q_h\right)|||_h} \le \mathcal{S}_h\left(p; p\right)^{1/2}.\end{aligned} \tag{60}$$

Hence, the approximation properties of κ_{2h} and κ_h imply

$$\begin{aligned}\frac{1}{\widetilde{\beta}}&\frac{\mathscr{A}_h(\mathbf{v}-\mathbf{v}_h, p-p_h;\mathbf{w}_h,q_h)}{|||(\mathbf{w}_h,q_h)|||_h}\\ &\le \frac{1}{\widetilde{\beta}}\left\|\gamma\nabla\kappa_{2h}p\right\|_{\Omega}\left\|\nabla\kappa_{2h}p\right\|_{\Omega}\\ &\le c_1\widetilde{\beta}^{-1}\gamma^{1/2}h^{l+1/2}\left\|p\right\|_{l+1,\Omega}.\end{aligned}\tag{61}$$

To estimate the second term of (59) we consider separately each individual term of the bilinear form $(1/\widetilde{\beta})\mathscr{A}_h(\mathbf{j}_r^h\mathbf{v}-\mathbf{v}, j_l^h p-p;\mathbf{w}_h,q_h)$.

Next, Cauchy schwarz and Poincaré inequality for the boundary terms imply

$$\begin{aligned}
&\frac{1}{\widetilde{\beta}}\mathscr{A}_S\left(\mathbf{j}_r^h\mathbf{v}-\mathbf{v}, j_l^h p-p;\mathbf{w}_h,q_h\right)\\
&\quad\le \widetilde{\beta}^{-1}\Big[\nu\left\|\nabla\left(\mathbf{j}_r^h\mathbf{v}-\mathbf{v}\right)\right\|_{\Omega_S}\left\|\nabla\mathbf{w}_h\right\|_{\Omega_S}\\
&\quad+\left\|j_l^h p-p\right\|_{\Omega_S}\left\|\nabla\mathbf{w}_h\right\|_{\Omega_S}\left\|\nabla\left(\mathbf{j}_r^h\mathbf{v}-\mathbf{v}\right)\right\|_{\Omega_S}\left\|q_h\right\|_{\Omega_S}\\
&\quad+\frac{\nu\alpha c_\Gamma^2}{\sqrt{\widetilde{k}}}\left\|\nabla\left(\mathbf{j}_r^h\mathbf{v}-\mathbf{v}\right)\right\|_{\Omega_S}\left\|\nabla\mathbf{w}_h\right\|_{\Omega_S}\Big]\\
&\quad\le \widetilde{\beta}^{-1}c_i\widetilde{C}\Big[\nu h^r\left\|\mathbf{v}\right\|_{r+1,\Omega_S}+h^{l+1}\left\|p\right\|_{l,\Omega_S}\\
&\quad+h^r\left\|\mathbf{v}\right\|_{r+1,\Omega_S}+\frac{\nu\alpha c_\Gamma^2}{\sqrt{\widetilde{k}}}h^r\left\|\mathbf{v}\right\|_{r+1,\Omega_S}\Big],\\
&\frac{1}{\widetilde{\beta}}\mathscr{A}_D\left(\mathbf{j}_r^h\mathbf{v}-\mathbf{v}, j_l^h p-p;\mathbf{w}_h,q_h\right)\\
&\quad\le \widetilde{\beta}^{-1}\Big[k_2\left\|\mathbf{j}_r^h\mathbf{v}-\mathbf{v}\right\|_{\Omega_D}\left\|\mathbf{w}_h\right\|_{\Omega_D}\\
&\quad+\delta\left\|\nabla\left(\mathbf{j}_r^h\mathbf{v}-\mathbf{v}\right)\right\|_{\Omega_D}\left\|\operatorname{div}\mathbf{w}_h\right\|_{\Omega_D}\\
&\quad+\left\|\nabla\left(j_l^h p-p\right)\right\|_{\Omega_D}\left\|\mathbf{w}_h\right\|_{\Omega_D}\\
&\quad+\left\|\nabla q_h\right\|_{\Omega_D}\left\|\mathbf{j}_r^h\mathbf{v}-\mathbf{v}\right\|_{\Omega_D}\Big]\\
&\quad\le \widetilde{\beta}^{-1}c_i\widetilde{C}\Big[k_2h^{r+1}\left\|\mathbf{v}\right\|_{r+1,\Omega_D}+\delta h^r\left\|\mathbf{v}\right\|_{r+1,\Omega_D}\\
&\quad+h^l\left\|p\right\|_{l+1,\Omega_D}+h^{r+1}\left\|\mathbf{v}\right\|_{r+1,\Omega_D}\Big].
\end{aligned}\tag{62}$$

Thus,

$$\begin{aligned}
&|||\left(\mathbf{j}_r^h\mathbf{v}-\mathbf{v}_h, j_l^h p-p_h\right)|||_h\\
&\quad\le\left(\widetilde{c}_1\nu+\widetilde{c}_2\right)h^r\left\|\mathbf{v}\right\|_{r+1,\Omega_S}+\left(\widetilde{c}_3h+\widetilde{c}_4\delta\right)h^r\left\|\mathbf{v}\right\|_{r+1,\Omega_D}\\
&\quad+\left(\widetilde{c}_5+\widetilde{c}_6\gamma^{1/2}h^{1/2}+\widetilde{c}_7h\right)h^l\left\|p\right\|_{l+1,\Omega_S}\\
&\quad+\left(\widetilde{c}_5+\widetilde{c}_6\gamma^{1/2}h^{1/2}+\widetilde{c}_7h\right)h^l\left\|p\right\|_{l+1,\Omega_D}.
\end{aligned}\tag{63}$$

Squaring the norm and applying Young inequality we obtain

$$\begin{aligned}
&|||\left(\mathbf{j}_r^h\mathbf{v}-\mathbf{v}_h, j_l^h p-p_h\right)|||_h^2\\
&\quad\le 4\left(\widetilde{c}_1\nu+\widetilde{c}_2\right)^2h^{2r}\left\|\mathbf{v}\right\|_{r+1,\Omega_S}^2\\
&\qquad+4\left(\widetilde{c}_3h+\widetilde{c}_4\delta\right)^2h^{2r}\left\|\mathbf{v}\right\|_{r+1,\Omega_D}^2\\
&\qquad+4\left(\widetilde{c}_5+\widetilde{c}_6\gamma^{1/2}h^{1/2}+\widetilde{c}_7h\right)^2h^{2l}\left\|p\right\|_{l+1,\Omega_S}^2\\
&\qquad+4\left(\widetilde{c}_5+\widetilde{c}_6\gamma^{1/2}h^{1/2}+\widetilde{c}_7h\right)^2h^{2l}\left\|p\right\|_{l+1,\Omega_D}^2.
\end{aligned}\tag{64}$$

Next, we estimate the interpolation error by

$$\begin{aligned}
&|||\left(\mathbf{v}-\mathbf{j}_r^h\mathbf{v}, p-j_l^h p\right)|||_h^2\\
&\quad=\left\|\nabla\left(\mathbf{v}-\mathbf{j}_r^h\mathbf{v}\right)\right\|_{\Omega_S}^2+\left\|\left(\mathbf{v}-\mathbf{j}_r^h\mathbf{v}\right)\right\|_{\Omega_D}^2\\
&\qquad+\left\|\operatorname{div}\left(\mathbf{v}-\mathbf{j}_r^h\mathbf{v}\right)\right\|_{\Omega_D}^2+\left\|p-j_l^h p\right\|_{\Omega_S}^2\\
&\qquad+\left\|\nabla\left(p-j_l^h p\right)\right\|_{\Omega_D}^2+\mathscr{S}_h\left(\kappa_{2h}p,\kappa_{2h}p\right)\\
&\quad\le c_i^2h^{2r}\left\|\mathbf{v}\right\|_{r+1,\Omega_S}^2+c_i^2h^{2r}\left(h^2+1\right)h^{2r}\left\|\mathbf{v}\right\|_{r+1,\Omega_D}^2\\
&\qquad+\left(\widetilde{c}_i^2h^2+\gamma h\right)h^{2l}\left\|p\right\|_{l+1,\Omega_S}^2\\
&\qquad+\left(\widetilde{c}_i^2+\gamma h\right)h^{2l}\left\|p\right\|_{l+1,\Omega_D}^2.
\end{aligned}\tag{65}$$

Adding the interpolation error (64) to the projection error (65) we obtain the required result

$$\begin{aligned}
|||\left(\mathbf{v}-\mathbf{v}_h, p-p_h\right)|||_h\le&\Big\{\left(c_1\nu+c_2\right)^2h^{2r}\left\|\mathbf{v}\right\|_{r+1,\Omega_S}^2\\
&+\left(c_3h+c_4\delta\right)^2h^{2r}\left\|\mathbf{v}\right\|_{r+1,\Omega_D}^2\\
&+\left(c_5+c_6\gamma^{1/2}h^{1/2}+c_7h\right)^2h^{2l}\left\|p\right\|_{l+1,\Omega_S}^2\\
&+\left(c_5+c_6\gamma^{1/2}h^{1/2}+c_7h\right)^2h^{2l}\left\|p\right\|_{l+1,\Omega_D}^2\Big\}^{1/2}.
\end{aligned}\tag{66}$$

□

Remark 4. We note that the analysis above holds true for the triangular subgrid interpolation $P_r-P_r-P_r$.

Remark 5. Because of the presence of divergence of the velocity and the gradient of the pressure in the discrete norm, the velocity and pressure solutions are $O(h^r)$ and $O(h^l)$, respectively. So, we expect the L_2-asymptotic rates to be $O(h^{r+1})$ and $O(h^{l+1})$.

7. Numerical Results

As a test model problem we take $\Omega=(0,1)\times(0,1)$ and split it into $\Omega_S=(0,1/2)\times(0,1)$ and $\Omega_D=(1/2,1)\times(0,1)$. The interface boundary is $\Gamma=\{(0.5,y)\mid 0<y<1\}$. We take

Table 1: Rates of convergence for velocity and pressure solution in the Stokes subdomain.

	$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega_S}$	$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega_S}$	$\|p-p_h\|_{0,\Omega_S}$
$h=\frac{1}{8}$	—	—	—
$h=\frac{1}{16}$	1.9303	1.0284	0.8480
$h=\frac{1}{32}$	1.9735	1.0208	0.9149
$h=\frac{1}{64}$	1.9890	1.0119	0.9511
$h=\frac{1}{128}$	1.9951	1.0055	0.9725

Table 2: Rates of convergence for velocity and pressure solution in the Darcy subdomain.

	$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega_D}$	$\|\mathrm{div}(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega_D}$	$\|p-p_h\|_{0,\Omega_D}$
$h=\frac{1}{8}$	—	—	—
$h=\frac{1}{16}$	0.8813	0.8412	1.0416
$h=\frac{1}{32}$	0.9534	0.9235	1.0318
$h=\frac{1}{64}$	0.9642	0.9514	1.0167
$h=\frac{1}{128}$	0.9857	0.9657	1.0085

$\nu = 1$, $\alpha = 1$, $\widetilde{k} = 1$, and $K = I$ and the right hand sides $\mathbf{f}$, g such that the velocity and pressure solution in the two subdomains are given by

$$\begin{aligned} \mathbf{u}_S &= \left(y^4e^x, e^y\cos(2x)\right), \quad (x,y)\in\Omega_S \\ \mathbf{u}_D &= \left(y^4e^x, 4y^3e^x\right), \quad (x,y)\in\Omega_D \\ p &= y^4e^x, \quad (x,y)\in\Omega. \end{aligned} \tag{67}$$

Note that for this problem forcing terms are needed to balance the equations; notably additional terms are added to the interface conditions in (6) and (7) as follows:

$$-\left(-p_SI+2\nu D(\mathbf{v}_S)\right)\mathbf{n}_S\cdot\mathbf{n}_S = p_D + g_1, \quad \text{on } \Gamma,$$
$$\mathbf{v}_S\cdot\tau = -\frac{2\sqrt{\widetilde{k}}}{\alpha}\left(D(\mathbf{v}_S)\cdot\mathbf{n}_S\right)\cdot\tau \quad \text{on } \Gamma, \tag{68}$$

where $g_1 = -2y^4e^x$, and $g_2 = e^y\cos(2x)+4y^3e^x-2e^y\sin(2x)$.

The problem is solved using a Q_1-Q_1 velocity-pressure approximation with a two-level subgrid stabilization on a uniform mesh with $\delta = 0.4$. Rates of convergence for the velocity and pressure errors for $h = 1/8, 1/16, 1/32, 1/64$, and $1/128$ are displayed in Tables 1 and 2.

In Table 1, we see clearly that the velocity field in the Stokes subdomain is of second-order accuracy with respect to the L_2-norm and first-order accuracy with respect to H^1-seminorm, and the pressure is of first-order accuracy. In addition, In Table 2, we observe that the velocity field and its divergence are of first-order accuracy in the Darcy subdomain, and the pressure is of first-order accuracy with respect to the L_2-norm. So, clearly these results are in agreement with the theoretical results of the previous section and are comparable to the ones found in [2, 5].

Competing Interests

The author declares that they have no competing interests.

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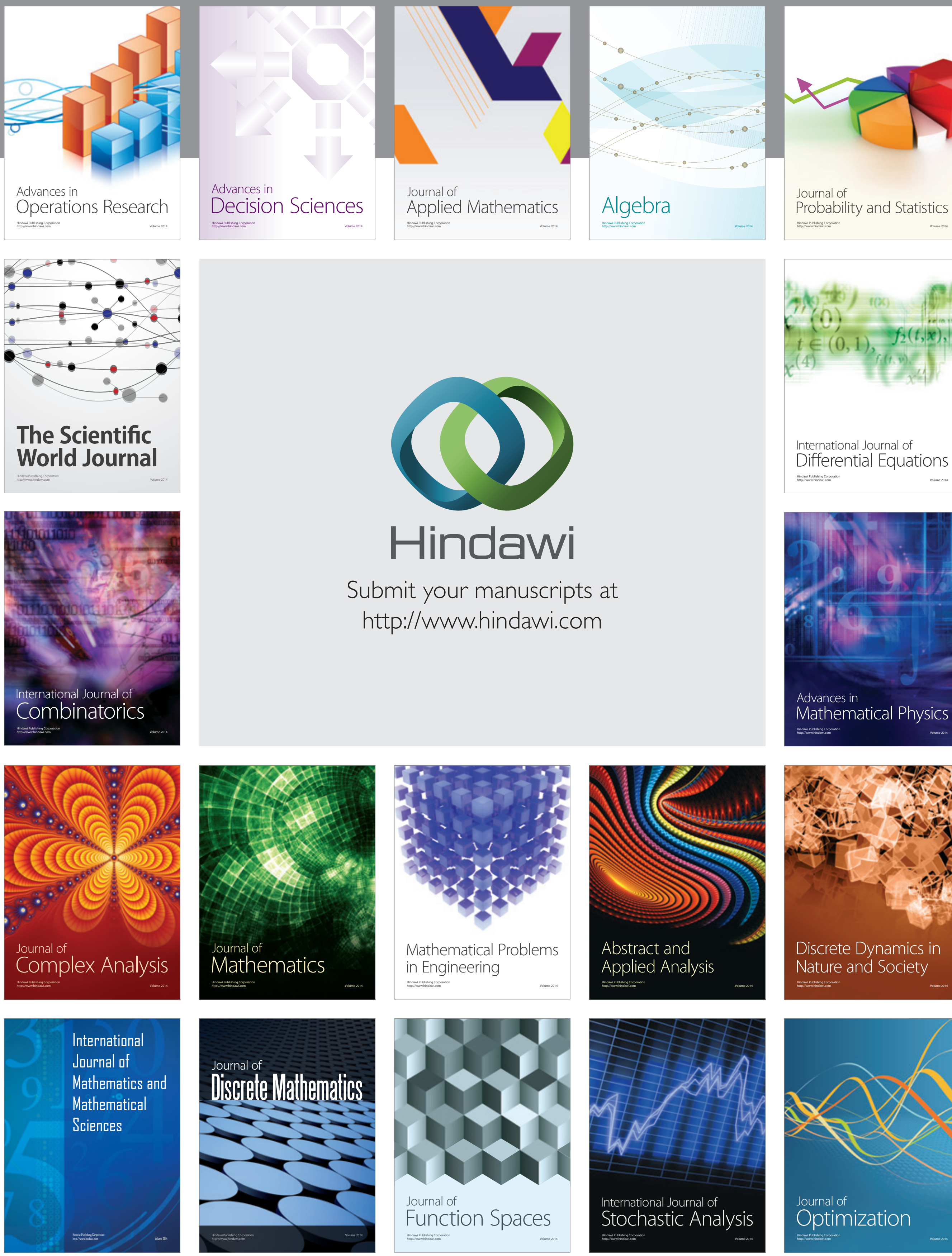
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