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Research Article

Uncertainty Principles for the Dunkl-Wigner Transforms

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We prove a version of Heisenberg-type uncertainty principle for the Dunkl-Wigner transform of magnitude $s > 0$; and we deduce a local uncertainty principle for this transform.

1. Introduction

In this paper, we consider $\mathbb{R}^d$ with the Euclidean inner product $\langle \cdot, \cdot \rangle$ and norm $|y| := \sqrt{\langle y, y \rangle}$. For $\alpha \in \mathbb{R}^d \setminus \{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α:

$$\sigma_\alpha y := y - \frac{2\langle \alpha, y \rangle}{|\alpha|^2}\alpha. \tag{1}$$

A finite set $\mathfrak{R} \subset \mathbb{R}^d \setminus \{0\}$ is called a root system, if $\mathfrak{R} \cap \mathbb{R}.\alpha = \{-\alpha, \alpha\}$ and $\sigma_\alpha \mathfrak{R} = \mathfrak{R}$ for all $\alpha \in \mathfrak{R}$. We assume that it is normalized by $|\alpha|^2 = 2$ for all $\alpha \in \mathfrak{R}$. For a root system $\mathfrak{R}$, the reflections σ_α, $\alpha \in \mathfrak{R}$, generate a finite group G. The Coxeter group G is a subgroup of the orthogonal group $O(d)$. All reflections in G correspond to suitable pairs of roots. For a given $\beta \in \mathbb{R}^d \setminus \bigcup_{\alpha \in \mathfrak{R}} H_\alpha$, we fix the positive subsystem $\mathfrak{R}_+ := \{\alpha \in \mathfrak{R} : \langle \alpha, \beta \rangle > 0\}$. Then for each $\alpha \in \mathfrak{R}$ either $\alpha \in \mathfrak{R}_+$ or $-\alpha \in \mathfrak{R}_+$.

Let $k : \mathfrak{R} \to \mathbb{C}$ be a multiplicity function on $\mathfrak{R}$ (a function which is constant on the orbits under the action of G). As an abbreviation, we introduce the index $\gamma = \gamma_k := \sum_{\alpha \in \mathfrak{R}_+} k(\alpha)$.

Throughout this paper, we will assume that $k(\alpha) \geq 0$ for all $\alpha \in \mathfrak{R}$. Moreover, let w_k denote the weight function $w_k(y) := \prod_{\alpha \in \mathfrak{R}_+} |\langle \alpha, y \rangle|^{2k(\alpha)}$, for all $y \in \mathbb{R}^d$, which is G-invariant and homogeneous of degree 2γ.

Let c_k be the Mehta-type constant given by $c_k := (\int_{\mathbb{R}^d} e^{-|y|^2/2} w_k(y) dy)^{-1}$. We denote by μ_k the measure on $\mathbb{R}^d$ given by $d\mu_k(y) := c_k w_k(y) dy$, by $L^p(\mu_k)$, $1 \leq p \leq \infty$, the space of measurable functions f on $\mathbb{R}^d$, such that

$$\|f\|_{L^p(\mu_k)} := \left(\int_{\mathbb{R}^d} |f(y)|^p \, d\mu_k(y) \right)^{1/p} < \infty, \quad 1 \leq p < \infty, \tag{2}$$

$$\|f\|_{L^\infty(\mu_k)} := \operatorname*{ess\,sup}_{y \in \mathbb{R}^d} |f(y)| < \infty,$$

and by $L^p_{\text{rad}}(\mu_k)$ the subspace of $L^p(\mu_k)$ consisting of radial functions.

For $f \in L^1(\mu_k)$ the Dunkl transform of f is defined (see [1]) by

$$\mathcal{F}_k(f)(x) := \int_{\mathbb{R}^d} E_k(-ix, y) f(y) \, d\mu_k(y), \quad x \in \mathbb{R}^d, \tag{3}$$

where $E_k(-ix, y)$ denotes the Dunkl kernel. (For more details see the next section.)

Many uncertainty principles have already been proved for the Dunkl transform $\mathcal{F}_k$, namely, by Rösler [2] and Shimeno [3] who established the Heisenberg-type uncertainty inequality for this transform, by showing that for $f \in L^2(\mu_k)$,

$$\|f\|^2_{L^2(\mu_k)} \leq \frac{2}{2\gamma + d} \| |x| f \|_{L^2(\mu_k)} \| |y| \mathcal{F}_k(f) \|_{L^2(\mu_k)}. \tag{4}$$

Recently, the author [4–7] proved general forms of the Heisenberg-type inequality for the Dunkl transform $\mathscr{F}_k$.

The Dunkl translation operators τ_x, $x \in \mathbb{R}^d$, [8] are defined on $L^2(\mu_k)$ by

$$\mathscr{F}_k(\tau_x f)(y) = E_k(ix, y)\mathscr{F}_k(f)(y), \quad y \in \mathbb{R}^d. \tag{5}$$

Let $g \in L^2_{\mathrm{rad}}(\mu_k)$. The Dunkl-Wigner transform V_g is the mapping defined for $f \in L^2(\mu_k)$ by

$$V_g(f)(x, y) := \int_{\mathbb{R}^d} f(t)\overline{\tau_x g_{k,y}(-t)}\,\mathrm{d}\mu_k(t), \tag{6}$$

where

$$g_{k,y}(z) := \mathscr{F}_k\left(\sqrt{\tau_y |\mathscr{F}_k(g)|^2}\right)(z). \tag{7}$$

This transform is studied in [9, 10] where the author established some applications (Plancherel formula, inversion formula, Calderón's reproducing formula, extremal function, etc.).

In this paper we use formula (4); we prove uncertainty principle intervening $\mathscr{F}_k$ and V_g of magnitudes $a, b \geq 1$; that is, for every $f \in L^2(\mu_k)$,

$$\begin{aligned} &\left\||x|^a V_g(f)\right\|^b_{L^2(\mu_k\otimes\mu_k)} \left\||z|^b \mathscr{F}_k(f)\right\|^a_{L^2(\mu_k)} \\ &\quad \geq \left(\gamma + \frac{d}{2}\right)^{ab} \|f\|^{a+b}_{L^2(\mu_k)} \|g\|^b_{L^2_{\mathrm{rad}}(\mu_k)}. \end{aligned} \tag{8}$$

Next, we prove a Heisenberg-type uncertainty principle for the Dunkl-Wigner transform V_g of magnitude $s > 0$; that is, there exists a constant $c(k, s) > 0$ such that, for $f \in L^2(\mu_k)$,

$$\begin{aligned} &\left\||x|^s V_g(f)\right\|_{L^2(\mu_k\otimes\mu_k)} \left\||y|^s V_g(f)\right\|_{L^2(\mu_k\otimes\mu_k)} \\ &\quad \geq c(k, s)\|f\|^2_{L^2(\mu_k)} \|g\|^2_{L^2_{\mathrm{rad}}(\mu_k)}. \end{aligned} \tag{9}$$

Finally, we prove a local uncertainty principle for the Dunkl-Wigner transform V_g; that is, there exists a constant $b(k, s) > 0$ such that, for $f \in L^2(\mu_k)$ and for measurable subset E of $\mathbb{R}^d \times \mathbb{R}^d$ such that $0 < \mu_k \otimes \mu_k(E) < \infty$,

$$\begin{aligned} &\left\|\chi_E V_g(f)\right\|_{L^2(\mu_k\otimes\mu_k)} \\ &\quad \leq b(k, s)(\mu_k \otimes \mu_k(E))^{1/2} \left\||(x, y)|^s V_g(f)\right\|_{L^2(\mu_k\otimes\mu_k)}, \end{aligned} \tag{10}$$

where χ_E is the indicator function of the set E.

In the classical case, the Fourier-Wigner transforms are studied by Weyl [11] and Wong [12]. In the Bessel-Kingman hypergroups, these operators are studied by Dachraoui [13].

This paper is organized as follows. In Section 2, we recall some properties of the Dunkl-Wigner transform V_g. In Section 3, we prove a Heisenberg-type uncertainty principle for the Dunkl-Wigner transform V_g of magnitude $s > 0$; and we deduce a local uncertainty principle for this transform.

2. The Dunkl-Wigner Transform

The Dunkl operators $\mathscr{D}_j$, $j = 1, \ldots, d$, on $\mathbb{R}^d$ associated with the finite reflection group G and multiplicity function k are given, for a function f of class C^1 on $\mathbb{R}^d$, by

$$\mathscr{D}_j f(y) := \frac{\partial}{\partial y_j} f(y) + \sum_{\alpha\in\mathfrak{R}_+} k(\alpha)\alpha_j \frac{f(y) - f(\sigma_\alpha y)}{\langle \alpha, y\rangle}. \tag{11}$$

For $y \in \mathbb{R}^d$, the initial value problem $\mathscr{D}_j u(\cdot, y)(x) = y_j u(x, y)$, $j = 1, \ldots, d$, with $u(0, y) = 1$ admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $E_k(x, y)$ and called Dunkl kernel [14, 15]. This kernel has a unique analytic extension to $\mathbb{C}^d \times \mathbb{C}^d$ (see [16]). The Dunkl kernel has the Laplace-type representation [17]

$$E_k(x, y) = \int_{\mathbb{R}^d} e^{\langle y,z\rangle}\,\mathrm{d}\Gamma_x(z), \quad x \in \mathbb{R}^d, \; y \in \mathbb{C}^d, \tag{12}$$

where $\langle y, z\rangle := \sum_{i=1}^d y_i z_i$ and Γ_x is a probability measure on $\mathbb{R}^d$, such that $\mathrm{supp}(\Gamma_x) \subset \{z \in \mathbb{R}^d : |z| \leq |x|\}$. In our case,

$$\left|E_k(\pm ix, y)\right| \leq 1, \quad x, y \in \mathbb{R}^d. \tag{13}$$

The Dunkl kernel gives rise to an integral transform, which is called Dunkl transform on $\mathbb{R}^d$, and was introduced by Dunkl in [1], where already many basic properties were established. Dunkl's results were completed and extended later by de Jeu [15]. The Dunkl transform of a function f in $L^1(\mu_k)$ is defined by

$$\mathscr{F}_k(f)(x) := \int_{\mathbb{R}^d} E_k(-ix, y) f(y)\,\mathrm{d}\mu_k(y), \quad x \in \mathbb{R}^d. \tag{14}$$

We notice that $\mathscr{F}_0$ agrees with the Fourier transform $\mathscr{F}$ that is given by

$$\mathscr{F}(f)(x) := (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-i\langle x,y\rangle} f(y)\,\mathrm{d}y, \quad x \in \mathbb{R}^d. \tag{15}$$

The Dunkl transform of a function $f \in L^1_{\mathrm{rad}}(\mu_k)$ which is radial is again radial and could be computed via the associated Fourier-Bessel transform $\mathscr{F}^B_{\gamma+d/2-1}$ (see [18], Proposition 4); that is,

$$\mathscr{F}_k(f)(x) = \mathscr{F}^B_{\gamma+d/2-1}(F)(|x|), \tag{16}$$

where $f(x) = F(|x|)$ and

$$\begin{aligned} &\mathscr{F}^B_{\gamma+d/2-1}(F)(|x|) \\ &\quad := \int_0^\infty F(r)\frac{j_{\gamma+d/2-1}(|x|r)}{2^{\gamma+d/2-1}\Gamma(\gamma + d/2)} r^{2\gamma+d-1}\,\mathrm{d}r. \end{aligned} \tag{17}$$

Here j_γ is the spherical Bessel function (see [19]).

Some of the properties of Dunkl transform $\mathscr{F}_k$ are collected below (see [1, 15]).

Theorem 1. *(i) $L^1 - L^\infty$-Boundedness. For all $f \in L^1(\mu_k)$, $\mathscr{F}_k(f) \in L^\infty(\mu_k)$, and*

$$\|\mathscr{F}_k(f)\|_{L^\infty(\mu_k)} \le \|f\|_{L^1(\mu_k)}. \quad (18)$$

(ii) Inversion Theorem. Let $f \in L^1(\mu_k)$, such that $\mathscr{F}_k(f) \in L^1(\mu_k)$. Then

$$f(x) = \mathscr{F}_k(\mathscr{F}_k(f))(-x), \quad \text{a.e. } x \in \mathbb{R}^d. \quad (19)$$

(iii) Plancherel Theorem. The Dunkl transform $\mathscr{F}_k$ extends uniquely to an isometric isomorphism of $L^2(\mu_k)$ onto itself. In particular, one has

$$\|f\|_{L^2(\mu_k)} = \|\mathscr{F}_k(f)\|_{L^2(\mu_k)}. \quad (20)$$

(iv) Parseval Theorem. For $f, g \in L^2(\mu_k)$, one has

$$\langle f, g\rangle_{L^2(\mu_k)} = \langle \mathscr{F}_k(f), \mathscr{F}_k(g)\rangle_{L^2(\mu_k)}. \quad (21)$$

The Dunkl transform $\mathscr{F}_k$ allows us to define a generalized translation operators on $L^2(\mu_k)$ by setting

$$\mathscr{F}_k(\tau_x f)(y) = E_k(ix, y)\mathscr{F}_k(f)(y), \quad y \in \mathbb{R}^d. \quad (22)$$

It is the definition of Thangavelu and Xu given in [8]. It plays the role of the ordinary translation $\tau_x f = f(x+\cdot)$ in $\mathbb{R}^d$, since the Euclidean Fourier transform satisfies $\mathscr{F}(\tau_x f)(y) = e^{i\langle x,y\rangle}\mathscr{F}(f)(y)$. Note that, from (13) and Theorem 1(iii), relation (22) makes sense, and $\|\tau_x f\|_{L^2(\mu_k)} \le \|f\|_{L^2(\mu_k)}$, for all $f \in L^2(\mu_k)$.

Rösler [20] introduced the Dunkl translation operators for radial functions. If f are radial functions, $f(x) = F(|x|)$, then

$$\tau_x f(y) = \int_{\mathbb{R}^d} F\left(\sqrt{|x|^2 + |y|^2 + 2\langle y, z\rangle}\right) d\Gamma_x(z); \quad x, y \in \mathbb{R}^d, \quad (23)$$

where Γ_x is the representing measure given by (12).

This formula allows us to establish the following results [8, 21].

Proposition 2. *(i) For all $p \in [1,2]$ and for all $x \in \mathbb{R}^d$, the Dunkl translation $\tau_x : L^p_{rad}(\mu_k) \to L^p(\mu_k)$ is a bounded operator, and for $f \in L^p_{rad}(\mu_k)$, one has*

$$\|\tau_x f\|_{L^p(\mu_k)} \le \|f\|_{L^p_{rad}(\mu_k)}. \quad (24)$$

(ii) Let $f \in L^1_{rad}(\mu_k)$. Then, for all $x \in \mathbb{R}^d$, one has

$$\int_{\mathbb{R}^d} \tau_x f(y)\, d\mu_k(y) = \int_{\mathbb{R}^d} f(y)\, d\mu_k(y). \quad (25)$$

The Dunkl convolution product $*_k$ of two functions f and g in $L^2(\mu_k)$ is defined by

$$f *_k g(x) := \int_{\mathbb{R}^d} \tau_x f(-y)\, g(y)\, d\mu_k(y), \quad x \in \mathbb{R}^d. \quad (26)$$

We notice that $*_k$ generalizes the convolution $*$ that is given by

$$f * g(x) := (2\pi)^{-d/2}\int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy, \quad x \in \mathbb{R}^d. \quad (27)$$

Proposition 2 allows us to establish the following properties for the Dunkl convolution on $\mathbb{R}^d$ (see [8]).

Proposition 3. *(i) Assume that $p \in [1,2]$ and $q, r \in [1,\infty]$ such that $1/p + 1/q = 1 + 1/r$. Then the map $(f, g) \to f *_k g$ extends to a continuous map from $L^p_{rad}(\mu_k) \times L^q(\mu_k)$ to $L^r(\mu_k)$, and*

$$\|f *_k g\|_{L^r(\mu_k)} \le \|f\|_{L^p_{rad}(\mu_k)} \|g\|_{L^q(\mu_k)}. \quad (28)$$

(ii) For all $f \in L^1_{rad}(\mu_k)$ and $g \in L^2(\mu_k)$, one has

$$\mathscr{F}_k(f *_k g) = \mathscr{F}_k(f)\mathscr{F}_k(g). \quad (29)$$

(iii) Let $f \in L^2_{rad}(\mu_k)$ and $g \in L^2(\mu_k)$. Then $f *_k g$ belongs to $L^2(\mu_k)$ if and only if $\mathscr{F}_k(f)\mathscr{F}_k(g)$ belongs to $L^2(\mu_k)$, and

$$\mathscr{F}_k(f *_k g) = \mathscr{F}_k(f)\mathscr{F}_k(g), \quad \text{in the } L^2(\mu_k)\text{—case}. \quad (30)$$

(iv) Let $f \in L^2_{rad}(\mu_k)$ and $g \in L^2(\mu_k)$. Then

$$\int_{\mathbb{R}^d} |f * g(x)|^2\, d\mu_k(x) = \int_{\mathbb{R}^d} |\mathscr{F}_k(f)(z)|^2 |\mathscr{F}_k(g)(z)|^2\, d\mu_k(z), \quad (31)$$

where both sides are finite or infinite.

Let $g \in L^2_{\mathrm{rad}}(\mu_k)$ and $y \in \mathbb{R}^d$. The modulation of g by y is the function $g_{k,y}$ defined by

$$g_{k,y}(z) := \mathscr{F}_k\left(\sqrt{\tau_y |\mathscr{F}_k(g)|^2}\right)(z), \quad z \in \mathbb{R}^d. \quad (32)$$

Thus,

$$\|g_{k,y}\|_{L^2(\mu_k)} = \|g\|_{L^2_{\mathrm{rad}}(\mu_k)}. \quad (33)$$

Let $g \in L^2_{\mathrm{rad}}(\mu_k)$. The Fourier-Wigner transform associated with the Dunkl operators is the mapping V_g defined for $f \in L^2(\mu_k)$ by

$$V_g(f)(x, y) := \int_{\mathbb{R}^d} f(t)\, \overline{\tau_x g_{k,y}(-t)}\, d\mu_k(t), \quad x, y \in \mathbb{R}^d. \quad (34)$$

In the following we recall some properties of the Dunkl-Wigner transform (Plancherel formula, inversion formula, reproducing inversion formula of Calderón's type, etc.).

Proposition 4 (see [10]). *Let $(f, g) \in L^2(\mu_k) \times L^2_{rad}(\mu_k)$. Then*

(i) $V_g(f)(x, y) = \overline{g_{k,y}} *_k f(x)$.

(ii) $V_g(f)(x, y) = \int_{\mathbb{R}^d} E_k(ix, z)\mathscr{F}_k(f)(z)\sqrt{\tau_y|\mathscr{F}_k(g)|^2(z)}\, d\mu_k(z)$.

(iii) *The function $V_g(f)$ belongs to $L^\infty(\mu_k \otimes \mu_k)$, and*

$$\left\|V_g(f)\right\|_{L^\infty(\mu_k\otimes\mu_k)} \le \|f\|_{L^2(\mu_k)} \|g\|_{L^2_{rad}(\mu_k)}. \tag{35}$$

Theorem 5 (see [10]). *Let $g \in L^2_{rad}(\mu_k)$ be a nonzero function. Then one has the following.*

(i) *Plancherel formula: for every $f \in L^2(\mu_k)$, one has*

$$\left\|V_g(f)\right\|_{L^2(\mu_k\otimes\mu_k)} = \|f\|_{L^2(\mu_k)} \|g\|_{L^2_{rad}(\mu_k)}. \tag{36}$$

(ii) *Parseval formula: for every $f, h \in L^2(\mu_k)$, one has*

$$\left\langle V_g(f), V_g(h)\right\rangle_{L^2(\mu_k\otimes\mu_k)} = \|g\|^2_{L^2_{rad}(\mu_k)} \langle f, h\rangle_{L^2(\mu_k)}. \tag{37}$$

(iii) *Inversion formula: for all $f \in L^1 \cap L^2(\mu_k)$ such that $\mathscr{F}_k(f) \in L^1(\mu_k)$, one has*

$$f(z) = \frac{1}{\|g\|^2_{L^2_{rad}(\mu_k)}} \cdot \iint_{\mathbb{R}^d} V_g(f)(x, y)\overline{\tau_z g_{k,y}(-x)}\, d\mu_k(x)\, d\mu_k(y). \tag{38}$$

Theorem 6 (Calderón's reproducing inversion formula; see [10]). *Let $\Delta = \prod_{j=1}^d [a_j, b_j]$, $-\infty < a_j < b_j < \infty$, and let $g \in L^2_{rad}(\mu_k)$ be a nonzero function, such that $\mathscr{F}_k(g) \in L^\infty(\mu_k)$. Then, for $f \in L^2(\mu_k)$, the function f_Δ given by*

$$f_\Delta(z) = \frac{1}{\|g\|_{L^2_{rad}(\mu_k)}} \cdot \int_\Delta \int_{\mathbb{R}^d} V_g(f)(x, y)\overline{\tau_z g_{k,y}(-x)}\, d\mu_k(x)\, d\mu_k(y) \tag{39}$$

belongs to $L^2(\mu_k)$ and satisfies

$$\lim_{\substack{a_j\to-\infty\\ b_j\to+\infty}} \|f_\Delta - f\|_{L^2(\mu_k)} = 0. \tag{40}$$

3. Uncertainty Principles for the Mapping V_g

In this section we establish Heisenberg-type uncertainty principle for the Dunkl-Wigner transform V_g. We begin by the following theorem.

Theorem 7. *Let $g \in L^2_{rad}(\mu_k)$ be a nonzero function. Then, for $f \in L^2(\mu_k)$, one has*

$$\left\||x| V_g(f)\right\|_{L^2(\mu_k\otimes\mu_k)} \left\||z| \mathscr{F}_k(f)\right\|_{L^2(\mu_k)} \ge \left(\gamma + \frac{d}{2}\right) \|f\|^2_{L^2(\mu_k)} \|g\|_{L^2_{rad}(\mu_k)}. \tag{41}$$

Proof. Let $f \in L^2(\mu_k)$. Assume that $\||x|V_g(f)\|_{L^2(\mu_k\otimes\mu_k)} + \||z|\mathscr{F}_k(f)\|_{L^2(\mu_k)} < \infty$. Inequality (4) leads to

$$\begin{aligned}&\int_{\mathbb{R}^d} \left|V_g(f)(x, y)\right|^2 d\mu_k(x) \\ &\le \frac{2}{2\gamma + d}\left(\int_{\mathbb{R}^d} |x|^2 \left|V_g(f)(x, y)\right|^2 d\mu_k(x)\right)^{1/2} \\ &\left(\int_{\mathbb{R}^d} |z|^2 \left|\mathscr{F}_k\left(V_g(f)(\cdot, y)\right)(z)\right|^2 d\mu_k(z)\right)^{1/2}.\end{aligned} \tag{42}$$

Integrating with respect to $d\mu_k(y)$ and using the Schwarz inequality, we get

$$\begin{aligned}\left\|V_g(f)\right\|^2_{L^2(\mu_k\otimes\mu_k)} &\le \frac{2}{2\gamma + d}\left(\iint_{\mathbb{R}^d} |x|^2 \left|V_g(f)\right.\right. \\ &\left.\left.\cdot (x, y)\right|^2 d\mu_k(x)\, d\mu_k(y)\right)^{1/2} \\ &\cdot \left(\iint_{\mathbb{R}^d} |z|^2 \left|\mathscr{F}_k\left(V_g(f)(\cdot, y)\right)\right.\right. \\ &\left.\left.\cdot (z)\right|^2 d\mu_k(z)\, d\mu_k(y)\right)^{1/2}.\end{aligned} \tag{43}$$

But by Proposition 4(ii), Fubini-Tonelli's theorem, (16), Proposition 2(ii), and Theorem 1(iii), we have

$$\begin{aligned}&\iint_{\mathbb{R}^d} |z|^2 \left|\mathscr{F}_k\left(V_g(f)(\cdot, y)\right)(z)\right|^2 d\mu_k(z)\, d\mu_k(y) \\ &= \iint_{\mathbb{R}^d} |z|^2 \tau_y \left|\mathscr{F}_k(g)\right|^2(z) \left|\mathscr{F}_k(f)\right. \\ &\left.\cdot (z)\right|^2 d\mu_k(z)\, d\mu_k(y) \\ &= \|g\|^2_{L^2_{rad}(\mu_k)} \int_{\mathbb{R}^d} |z|^2 \left|\mathscr{F}_k(f)(z)\right|^2 d\mu_k(z).\end{aligned} \tag{44}$$

This yields the result and completes the proof of the theorem. □

Theorem 8. *Let $g \in L^2_{rad}(\mu_k)$ be a nonzero function and $s \ge 1$. Then, for $f \in L^2(\mu_k)$, one has*

$$\begin{aligned}&\left\||x|^s V_g(f)\right\|_{L^2(\mu_k\otimes\mu_k)} \left\||z|^s \mathscr{F}_k(f)\right\|_{L^2(\mu_k)} \\ &\ge \left(\gamma + \frac{d}{2}\right)^s \|f\|^2_{L^2(\mu_k)} \|g\|_{L^2_{rad}(\mu_k)}.\end{aligned} \tag{45}$$

Proof. Let $s \ge 1$ and let $f \in L^2(\mu_k)$, $f \ne 0$, such that $\||x|^s V_g(f)\|_{L^2(\mu_k\otimes\mu_k)} + \||z|^s\mathscr{F}_k(f)\|_{L^2(\mu_k)} < \infty$. Then, for $s > 1$, we have

$$\begin{aligned}&\left\||x|^s V_g(f)\right\|^{2/s}_{L^2(\mu_k\otimes\mu_k)} \left\|V_g(f)\right\|^{2/s'}_{L^2(\mu_k\otimes\mu_k)} \\ &= \left\||x|^2 \left|V_g(f)\right|^{2/s}\right\|_{L^s(\mu_k\otimes\mu_k)} \left\|\left|V_g(f)\right|^{2/s'}\right\|_{L^{s'}(\mu_k\otimes\mu_k)},\end{aligned} \tag{46}$$

where s' is defined as usual by $1/s + 1/s' = 1$. By Hölder's inequality we get

$$\begin{aligned} &\left\| |x|^s V_g(f) \right\|^{1/s}_{L^2(\mu_k\otimes\mu_k)} \left\| V_g(f) \right\|^{1/s'}_{L^2(\mu_k\otimes\mu_k)} \\ &\quad \geq \left\| |x| V_g(f) \right\|_{L^2(\mu_k\otimes\mu_k)}. \end{aligned} \tag{47}$$

Thus, for all $s \geq 1$, we have

$$\left\| |x|^s V_g(f) \right\|^{1/s}_{L^2(\mu_k\otimes\mu_k)} \geq \frac{\left\| |x| V_g(f) \right\|_{L^2(\mu_k\otimes\mu_k)}}{\left\| V_g(f) \right\|^{1-1/s}_{L^2(\mu_k\otimes\mu_k)}}, \tag{48}$$

with equality if $s = 1$. In the same manner and using Theorem 1(iii), we have, for $s \geq 1$,

$$\left\| |z|^s \mathscr{F}_k(f) \right\|^{1/s}_{L^2(\mu_k)} \geq \frac{\left\| |z| \mathscr{F}_k(f) \right\|_{L^2(\mu_k)}}{\left\| \mathscr{F}_k(f) \right\|^{1-1/s}_{L^2(\mu_k)}}, \tag{49}$$

with equality if $s = 1$. By (48) and (49), for all $s \geq 1$, we have

$$\begin{aligned} &\left\| |x|^s V_g(f) \right\|^{1/s}_{L^2(\mu_k\otimes\mu_k)} \left\| |z|^s \mathscr{F}_k(f) \right\|^{1/s}_{L^2(\mu_k)} \\ &\quad \geq \frac{\left\| |x| V_g(f) \right\|_{L^2(\mu_k\otimes\mu_k)} \left\| |z| \mathscr{F}_k(f) \right\|_{L^2(\mu_k)}}{\left(\|f\|^2_{L^2(\mu_k)} \|g\|_{L^2_{\mathrm{rad}}(\mu_k)} \right)^{1-1/s}}, \end{aligned} \tag{50}$$

with equality if $s = 1$. Applying Theorem 7, we obtain

$$\begin{aligned} &\left\| |x|^s V_g(f) \right\|_{L^2(\mu_k\otimes\mu_k)} \left\| |z|^s \mathscr{F}_k(f) \right\|_{L^2(\mu_k)} \\ &\quad \geq \left(\gamma + \frac{d}{2} \right)^s \|f\|^2_{L^2(\mu_k)} \|g\|_{L^2_{\mathrm{rad}}(\mu_k)}, \end{aligned} \tag{51}$$

which completes the proof of the theorem. □

From (48) and (49) we deduce the following remark.

Remark 9. Let $g \in L^2_{\mathrm{rad}}(\mu_k)$ be a nonzero function and $a, b \geq 1$. Then, for $f \in L^2(\mu_k)$, we have

$$\begin{aligned} &\left\| |x|^a V_g(f) \right\|^b_{L^2(\mu_k\otimes\mu_k)} \left\| |z|^b \mathscr{F}_k(f) \right\|^a_{L^2(\mu_k)} \\ &\quad \geq \left(\gamma + \frac{d}{2} \right)^{ab} \|f\|^{a+b}_{L^2(\mu_k)} \|g\|^b_{L^2_{\mathrm{rad}}(\mu_k)}. \end{aligned} \tag{52}$$

For $\lambda > 0$, we define the dilation of $f \in L^2(\mu_k)$ by

$$f_\lambda(x) := \frac{1}{\lambda^{\gamma+d/2}} f\left(\frac{x}{\lambda}\right), \quad x \in \mathbb{R}^d. \tag{53}$$

Then

$$\|f_\lambda\|_{L^2(\mu_k)} = \|f\|_{L^2(\mu_k)}, \tag{54}$$

$$\begin{aligned} &\mathscr{F}_k(f_\lambda)(z) = \lambda^{\gamma+d/2} \mathscr{F}_k(f)(\lambda z), \\ &\tau_x(f_\lambda)(y) = \frac{1}{\lambda^{\gamma+d/2}} \tau_{x/\lambda} f\left(\frac{y}{\lambda}\right). \end{aligned} \tag{55}$$

Let us now turn to establishing Heisenberg-type uncertainty principle for the Dunkl-Wigner transform V_g of magnitude $s > 0$. Thus, we consider the following lemma.

Lemma 10. *Let $\lambda > 0$ and let $g \in L^2_{rad}(\mu_k)$ be a nonzero function. Then, for $f \in L^2(\mu_k)$, one has*

$$V_{g_\lambda}(f_\lambda)(x, y) = V_g(f)\left(\frac{x}{\lambda}, \lambda y\right), \quad x, y \in \mathbb{R}^d. \tag{56}$$

Proof. From Proposition 4(ii), we have

$$\begin{aligned} V_{g_\lambda}(f_\lambda)(x, y) &= \int_{\mathbb{R}^d} E_k(ix, z) \mathscr{F}_k(f_\lambda)(z) \\ &\quad \cdot \sqrt{\tau_y \left|\mathscr{F}_k(g_\lambda)\right|^2(z)} d\mu_k(z). \end{aligned} \tag{57}$$

But by (55) we have

$$\tau_y \left|\mathscr{F}_k(g_\lambda)\right|^2(z) = \lambda^{2\gamma+d} \tau_{\lambda y} \left|\mathscr{F}_k(g)\right|^2(\lambda z). \tag{58}$$

Thus,

$$\begin{aligned} V_{g_\lambda}(f_\lambda)(x, y) &= \lambda^{2\gamma+d} \int_{\mathbb{R}^d} E_k(ix, z) \mathscr{F}_k(f)(\lambda z) \\ &\quad \cdot \sqrt{\tau_{\lambda y} \left|\mathscr{F}_k(g)\right|^2(\lambda z)} d\mu_k(z) = \int_{\mathbb{R}^d} E_k\left(ix, \frac{z}{\lambda}\right) \\ &\quad \cdot \mathscr{F}_k(f)(z) \sqrt{\tau_{\lambda y} \left|\mathscr{F}_k(g)\right|^2(z)} d\mu_k(z) = V_g(f) \\ &\quad \cdot \left(\frac{x}{\lambda}, \lambda y\right), \end{aligned} \tag{59}$$

which gives the result. □

Theorem 11 (Heisenberg-type uncertainty principle for V_g). *Let $s > 0$. Then there exists a constant $c(k, s) > 0$ such that, for all $f \in L^2(\mu_k)$ and $g \in L^2_{rad}(\mu_k)$, one has*

$$\begin{aligned} &\left\| |x|^s V_g(f) \right\|_{L^2(\mu_k\otimes\mu_k)} \left\| |y|^s V_g(f) \right\|_{L^2(\mu_k\otimes\mu_k)} \\ &\quad \geq c(k, s) \|f\|^2_{L^2(\mu_k)} \|g\|^2_{L^2_{rad}(\mu_k)}. \end{aligned} \tag{60}$$

Proof. Let $s, r_0 > 0$ and $B_{r_0} := \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |(x, y)| < r_0\}$, where $|(x, y)| = (|x|^2 + |y|^2)^{1/2}$. Fix r_0 such that $\mu_k \otimes \mu_k(B_{r_0}) < 1$. We write

$$\begin{aligned} \left\| V_g(f) \right\|^2_{L^2(\mu_k\otimes\mu_k)} &= \left\| \chi_{B_{r_0}} V_g(f) \right\|^2_{L^2(\mu_k\otimes\mu_k)} \\ &\quad + \left\| \chi_{\mathbb{R}^d \setminus B_{r_0}} V_g(f) \right\|^2_{L^2(\mu_k\otimes\mu_k)} \\ &\leq \left\| \chi_{B_{r_0}} V_g(f) \right\|^2_{L^2(\mu_k\otimes\mu_k)} \\ &\quad + r_0^{-2s} \left\| |(x, y)|^s V_g(f) \right\|^2_{L^2(\mu_k\otimes\mu_k)}. \end{aligned} \tag{61}$$

But from Hölder's inequality and Proposition 4(iii) we have

$$\begin{aligned} &\left\| \chi_{B_{r_0}} V_g(f) \right\|^2_{L^2(\mu_k\otimes\mu_k)} \\ &\quad \leq \mu_k \otimes \mu_k\left(B_{r_0}\right) \left\| V_g(f) \right\|^2_{L^\infty(\mu_k\otimes\mu_k)} \\ &\quad \leq \mu_k \otimes \mu_k\left(B_{r_0}\right) \|f\|^2_{L^2(\mu_k)} \|g\|^2_{L^2_{\mathrm{rad}}(\mu_k)}. \end{aligned} \tag{62}$$

Therefore, by Theorem 5(i),

$$r_0^{2s}\left(1-\mu_k\otimes\mu_k\left(B_{r_0}\right)\right)\|f\|_{L^2(\mu_k)}^2\|g\|_{L^2_{\mathrm{rad}}(\mu_k)}^2 \leq \left\||(x,y)|^s V_g(f)\right\|_{L^2(\mu_k\otimes\mu_k)}^2. \quad (63)$$

Using the fact that $|(x,y)|^s=(|x|^2+|y|^2)^{s/2}\leq 2^{s/2}(|x|^s+|y|^s)$ we deduce that

$$2c(k,s)\|f\|_{L^2(\mu_k)}^2\|g\|_{L^2_{\mathrm{rad}}(\mu_k)}^2 \leq \left\||x|^s V_g(f)\right\|_{L^2(\mu_k\otimes\mu_k)}^2+\left\||y|^s V_g(f)\right\|_{L^2(\mu_k\otimes\mu_k)}^2, \quad (64)$$

where

$$c(k,s)=\frac{r_0^{2s}}{2^{s+1}}\left(1-\mu_k\otimes\mu_k\left(B_{r_0}\right)\right). \quad (65)$$

Replacing f and g by f_λ and g_λ, respectively, in the previous inequality, we obtain by Lemma 10 and by a suitable change of variables

$$2c(k,s)\|f\|_{L^2(\mu_k)}^2\|g\|_{L^2_{\mathrm{rad}}(\mu_k)}^2 \leq \lambda^{2s}\left\||x|^s V_g(f)\right\|_{L^2(\mu_k\otimes\mu_k)}^2 + \lambda^{-2s}\left\||y|^s V_g(f)\right\|_{L^2(\mu_k\otimes\mu_k)}^2. \quad (66)$$

By setting $\lambda=(\||y|^sV_g(f)\|_{L^2(\mu_k\otimes\mu_k)}/\||x|^sV_g(f)\|_{L^2(\mu_k\otimes\mu_k)})^{1/2s}$ in the right-hand side of the previous inequality we obtain the desired result. □

We will now prove a local uncertainty principle for the Dunkl-Wigner transform V_g, which extends the result of Faris [22].

Theorem 12 (local uncertainty principle for V_g). *Let $s>0$. Then there exists a constant $b(k,s)>0$ such that, for all $f\in L^2(\mu_k)$ and $g\in L^2_{rad}(\mu_k)$ and for all measurable subset E of $\mathbb{R}^d\times\mathbb{R}^d$ such that $0<\mu_k\otimes\mu_k(E)<\infty$, one has*

$$\left\|\chi_E V_g(f)\right\|_{L^2(\mu_k\otimes\mu_k)} \leq b(k,s)\left(\mu_k\otimes\mu_k(E)\right)^{1/2}\left\||(x,y)|^s V_g(f)\right\|_{L^2(\mu_k\otimes\mu_k)}. \quad (67)$$

Proof. Let $s>0$ and let E be a measurable subset of $\mathbb{R}^d\times\mathbb{R}^d$ such that $0<\mu_k\otimes\mu_k(E)<\infty$. From Hölder's inequality and Proposition 4(iii) we have

$$\begin{aligned}\left\|\chi_E V_g(f)\right\|_{L^2(\mu_k\otimes\mu_k)} &\leq \left(\mu_k\otimes\mu_k(E)\right)^{1/2}\left\|V_g(f)\right\|_{L^\infty(\mu_k\otimes\mu_k)} \\ &\leq \left(\mu_k\otimes\mu_k(E)\right)^{1/2}\|f\|_{L^2(\mu_k)}\|g\|_{L^2_{\mathrm{rad}}(\mu_k)}.\end{aligned} \quad (68)$$

From (63) there exists $b(k,s)>0$ such that

$$\|f\|_{L^2(\mu_k)}\|g\|_{L^2_{\mathrm{rad}}(\mu_k)} \leq b(k,s)\left\||(x,y)|^s V_g(f)\right\|_{L^2(\mu_k\otimes\mu_k)}. \quad (69)$$

Therefore we obtain the desired result. □

Competing Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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