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Research Article

Singular Differential Equations and g-Drazin Invertible Operators

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We extend results of Favini, Nashed, and Zhao on singular differential equations using the g-Drazin inverse and the order of a quasinilpotent operator in the sense of Miekka and Nevanlinna. Two classes of singularly perturbed differential equations are studied using the continuity properties of the g-Drazin inverse obtained by Koliha and Rakočević.

1. Introduction

Let A be a bounded or closed linear operator in a Banach space X and let f be a X-valued function. The following initial value problem

$$A\frac{dx(t)}{dt} + x(t) = f(t) \quad x(0) = x_0, \; t \in [0,T] \tag{1}$$

is central to the analysis of the abstract singular equation

$$\frac{dMx(t)}{dt} + Nx(t) = f(t)$$
$$\lim_{t\to 0} Mx(t) = x_0, \; t \in (0,\infty), \tag{2}$$

where M and N are closed linear operators from a Banach space Y to X. Problem (2) and its variations were extensively studied in [1–3] and the references therein. In [4–6], Campbell studied (2) in matrix setting and applied his results in optimal control problems. More recently, related singular equations with delay are studied in [7–10]. Thus far problem (1) has been considered when A is singular (noninvertible) but Drazin invertible in the classical sense. A bounded linear operator is Drazin invertible in the classical sense if 0 is a pole of the resolvent of A. In [11], Koliha generalized the concept of Drazin invertibility to the case where 0 is only an isolated spectral point of the spectrum of A. Drazin invertibility in the generalized sense for closed linear operators was studied in [12].

In this paper we study problem (1) for the case where the bounded linear operator A is singular but g-Drazin invertible. Even though the case of A being closed can be dealt with using the g-Drazin inverse for closed linear operators in [12], we focus on the bounded case since it has been pointed out in [1–3] that it is enough to consider problem (1) when A is bounded. Following [11], a bounded linear operator A is g-Drazin invertible if 0 is not an accumulated spectral point of A. We write $\sigma(A)$ for the spectrum of A. A bounded linear operator B is called a g-Drazin inverse of A if

$$BA = AB,$$
$$BAB = B, \tag{3}$$
$$\sigma(A(I - AB)) = \{0\}.$$

Such an operator is unique, if it exists and is denoted by A^D. It follows that if A is g-Drazin invertible, then A can be decomposed to an invertible operator and a quasinilpotent operator. This fact plays a crucial role in our analysis. Recall that a bounded linear operator N is quasinilpotent if the spectrum of N is identical to 0 and N is nilpotent if there is a positive integer k such that $N^k = 0$. The smallest such k is the index of the nilpotency. The following result, which is due to Koliha [11], allows such decomposition of a g-Drazin invertible operator.

Theorem 1 (see [11, Theorem 7.1]). *If A is a g-Drazin invertible operator in a Banach space X, then $X = R(A^D A) \oplus N(A^D A)$, $A = A_1 \oplus A_2$, where A_1 is invertible, A_2 is quasinilpotent with respect to this direct sum, and*

$$A^D = A_1^{-1} \oplus 0. \tag{4}$$

Moreover, if P is the spectral projection corresponding to 0, then $P = I - AA^D$.

We will show that, under certain condition on the rate of which the powers of the quasinilpotent part decay, the solution to problem (1) exists and is given by an explicit formula. A function $u(\cdot)$ is a solution to problem (1) if it is differentiable and satisfies the differential equation in $[0,T]$ and the initial condition $x(0) = x_0$.

In Section 3 we study two classes of the so-called "singular singularly perturbed initial value problem":

$$\begin{gathered} A(\varepsilon)\frac{dx_\varepsilon(t)}{dt} + x_\varepsilon(t) = f_\varepsilon(t) \\ x_\varepsilon(0) = x_0^{(\varepsilon)},\ \varepsilon \in [0,\varepsilon_0),\ t \in [0,T], \end{gathered} \tag{5}$$

$$\begin{gathered} \varepsilon A(\varepsilon)\frac{dx_\varepsilon(t)}{dt} + x_\varepsilon(t) = f_\varepsilon(t) \\ x_\varepsilon(0) = x_0^{(\varepsilon)},\ \varepsilon \in [0,\varepsilon_0),\ t \in [0,T]. \end{gathered} \tag{6}$$

Problem (5) was extensively studied by Campbell [4, 6] in matrix setting. We will show that if the continuity of the g-Drazin inverse is assumed, then the solution to (5) converges to the solution of the reduced system when ε converges to 0^+. We will also show that the solution to (6) converges to 0 as $\varepsilon \to 0^+$, assuming the continuity of the g-Drazin inverse and the appropriate location of the spectrum of $A(0)$. The operators $A(\varepsilon)$ under consideration are a family of bounded linear operators on a Banach space X. For properties of the continuity of the classical Drazin inverse and the g-Drazin inverse, see [13–15].

In the sequel we will use the following definition, which is attributed to to Miekkala and Nevanlinna [16].

Definition 2. A quasinilpotent operator A is of finite order ω if the resolvent of A is of finite order ω as an entire function in $1/\lambda$. The value of ω is a nonnegative number for which

$$\|R(\lambda, A)\| \leq e^{1/|\lambda|^{\omega+\varepsilon}} \tag{7}$$

holds for $\varepsilon > 0$ with small enough $|\lambda|$ but fails for $\varepsilon < 0$.

Nilpotent operators are quasinilpotent of order zero but the converse is not true since a quasinilpotent is nilpotent of order n if and only if the resolvent is a polynomial in $1/\lambda$ of order n. The following result in [16] is important for our analysis.

Theorem 3 (see [16, Proposition 3.5]). *A quasinilpotent operator A is of finite order if*

$$\mu := \limsup_{n\to\infty} \frac{n\log n}{\log(1/\|A^n\|)} \tag{8}$$

is finite, and then the order ω is equal to μ.

Using Theorem 1, we say that a g-Drazin invertible operator A is of order ω if the quasinilpotent part of A is not 0 and of order ω.

2. Singular Initial Value Problem

In this section we extend the results on singular differential equations in [1, Theorem 3.1] and [3, Theorem 4.1] for the case where A is Drazin invertible in the classical sense to the case where A is g-Drazin invertible. The next theorem shows that when the function f is analytic in $[0,T]$, problem (1) can be solved when A is quasinilpotent of order $\omega < 1$. This result extends [1, Lemma 3.1].

Theorem 4. *If the operator A is quasinilpotent of order $\omega < 1$ and f is analytic in $[0,T]$, then problem (1) has a unique solution if and only if $x_0 = \sum_{n=0}^{\infty}(-1)^n A^n f^{(n)}(0)$, and the solution is given by*

$$x(t) = \sum_{n=0}^{\infty} (-1)^n A^n f^{(n)}(t). \tag{9}$$

Proof. By direct verification it is clear that if $x(t) = \sum_{n=0}^{\infty}(-1)^n A^n f^{(n)}(t)$ converges uniformly on $[0,T]$, then $x(t)$ is a solution of (1) if and only if $x_0 = \sum_{n=0}^{\infty}(-1)^n A^n f^{(n)}(0)$. Our proof of the existence of the solution is therefore reduced to showing that the infinite series converges uniformly on $[0,T]$. Observe that $\sum_{n=0}^{\infty}(-1)^n A^n f^{(n)}(t)$ converges pointwise in X if it converges absolutely. Since each $f^{(n)}$ is continuous on $[0,T]$, there exists $\tau_n \in [0,T]$ such that $\|f^{(n)}(t)\| \leq \|f^{(n)}(\tau_n)\|$ for all $t \in [0,T]$. Hence by the Weierstrass M-test, for uniform convergence it is sufficient to show that $\sum_{n=0}^{\infty} \|A^n f^{(n)}(\tau_n)\|$ converges.

For each $\tau_n \in [0,T]$,

$$\begin{aligned} &\limsup_{n\to\infty} \left\|A^n f^{(n)}(\tau_n)\right\|^{1/n} \\ &\quad \leq \limsup_{n\to\infty} \left\|n! A^n\right\|^{1/n} \limsup_{n\to\infty} \left\|\frac{f^{(n)}(\tau_n)}{n!}\right\|^{1/n}. \end{aligned} \tag{10}$$

Since the quasinilpotent A is of order $\omega < 1$, by Theorem 3 there exist $\varepsilon > 0$ and $n > N(\varepsilon)$ such that

$$\frac{n\log n}{\log(1/\|A^n\|)} < \omega + \varepsilon < 1, \tag{11}$$

which implies that $n\log n < \log(1/\|A^n\|)^{\omega+\varepsilon}$. Since $\log n! = O(n\log n)$, for sufficiently large n,

$$\log n! < \log\left(\frac{1}{\|A^n\|}\right)^{\omega+\varepsilon} \quad \text{so} \quad (n!)^{1/n} < \left\|A^n\right\|^{-(\omega+\varepsilon)/n}, \tag{12}$$

which implies $\|n!A^n\|^{1/n} < (\|A^n\|^{1/n})^{1-(\varepsilon+\omega)}$. Since $\omega + \varepsilon < 1$,

$$\limsup_{n\to\infty} \left\|n!A^n\right\|^{1/n} \le \lim_{n\to\infty} \left(\left\|A^n\right\|^{1/n}\right)^{1-(\varepsilon+\omega)} = \left(\lim_{n\to\infty} \left\|A^n\right\|^{1/n}\right)^{1-(\omega+\varepsilon)} = 0. \quad (13)$$

On the other hand, if a function g is analytic in an open set D (in the set of real numbers), for every compact set $K \subset D$, there exists a constant C such that for every $t \in K$ and every nonnegative integer k the following bound holds (see [17]):

$$\left\|g^{(k)}(t)\right\| \le C^{k+1}k!. \quad (14)$$

Using the above result and the condition that f is analytic in $[0,T]$, there exists a constant C independent of $\tau_n \in [0,T]$ and n such that

$$\left\|f^{(n)}(\tau_n)\right\| \le C^{n+1}n!. \quad (15)$$

Dividing both sides of the above inequality by $n!$ and raising it to the power of $1/n$, we get $\limsup_{n\to\infty}\|(n!)^{-1}f^{(n)}(\tau_n)\|^{1/n} \le C$. Therefore $\limsup_{n\to\infty}\|A^n f^{(n)}(\tau_n)\|^{1/n} = 0$, which concludes that $\sum_{n=0}^{\infty}\|A^n f^{(n)}(\tau_n)\|$ converges.

For the uniqueness of the solution, it is enough to show that 0 is the only solution of (1) with $x_0 = 0$ and $f(t) = 0$. Taking the Laplace transform of (1) with $x(0) = 0$ and $f(t) = 0$, we obtain

$$(sA + I)X(s) = 0, \quad (16)$$

where $X(s)$ denotes the Laplace transform of $x(t)$. Since the operator $sA + I$ is invertible for every complex number s, we can conclude that $X(s) = 0$, which implies that $x(t) = 0$ is the only solution. □

We are now in a position to show our main result.

Theorem 5. *If A is g-Drazin invertible operator of order $\omega < 1$, then problem (1) has a unique solution if and only if $Px_0 = \sum_{n=0}^{\infty}(-1)^n A^n P f^{(n)}(0)$, and the solution is given by*

$$x(t) = A^D \exp\left(-A^D t\right)(I-P)x_0 + \int_0^t A^D \exp\left(-A^D(t-s)\right)(I-P)f(s)\,ds + \sum_{n=0}^{\infty}(-1)^n A^n P f^{(n)}(t), \quad (17)$$

where $P = I - AA^D$.

Proof. Since A is g-Drazin invertible of order $\omega < 1$, by Theorem 1, $X = R(I-P) \oplus N(I-P)$, $A = A_1 \oplus A_2$, where A_1 is invertible and A_2 is quasinilpotent of order $\omega < 1$ with respect to the direct sum. Therefore problem (1) has a unique solution if and only if each of the following two initial value problems has a unique solution on $(I-P)X$ and PX, respectively:

$$A_1\frac{dx_1(t)}{dt} + x_1(t) = f_1(t), \quad x_1(0) = (I-P)x_0, \quad (18)$$

$$A_2\frac{dx_2(t)}{dt} + x_2(t) = f_2(t), \quad x_2(0) = Px_0, \quad (19)$$

where $f_1(t) = (I-P)f(t)$ and $f_2(t) = Pf(t)$. Applying Theorem 4 to (19),

$$x_2(t) = \sum_{n=0}^{\infty}(-1)^n A_2^n f_2^{(n)}(t) \quad (20)$$

is the unique solution of (19) if and only if $Px_0 = \sum_{n=0}^{\infty}(-1)^n A_2^n f_2^{(n)}(0)$.

Since A_1 is invertible, (18) has a unique solution given by

$$x_1(t) = A_1^{-1}\exp\left(-A_1^{-1}t\right)(I-P)x_0 + \int_0^t A_1^{-1}\exp\left(-A_1^{-1}(t-s)\right)f_1(s)\,ds. \quad (21)$$

Since $A^D(I-P) = A_1^{-1}(I-P)$ and $A^nP = A_2^nP$, we obtain

$$x(t) = A^D \exp\left(-A^D t\right)(I-P)x_0 + \int_0^t A^D \exp\left(-A^D(t-s)\right)(I-P)f(s)\,ds + \sum_{n=0}^{\infty}(-1)^n A^n P f^{(n)}(t). \quad (22)$$

□

On modifying the proof of Theorem 5, we can extend [3, Theorem 4.1] for the case where A is a closed linear operator. This can be done by replacing the g-Drazin inverse for bounded linear operators by that of closed linear operators using Definition 2.1 in [12] and by replacing Theorem 1 by Theorem 2.3 in [12].

3. Singularly Perturbed Differential Equations

In this section we use the results in previous sections and the continuity of the g-Drazin inverse to study two classes of singularly perturbed differential equations in the forms of (5) and (6).

We first show the stability of (6) under some Lyapunov-type conditions. Let H^- and H^+ denote the open left half- and right half-plane of the complex plane, respectively. In the next two results, we write A and P for $A(0)$ and $P(0)$, respectively.

Theorem 6. *Let $A(\varepsilon)$ be a g-Drazin invertible operator of order $\omega < 1$, let $A^D(\varepsilon)$ be the corresponding Drazin inverse, and let $f_\varepsilon(t)$ be analytic function in $[0,T]$ for each $\varepsilon \in [0,\varepsilon_0)$. Equation (6) has a unique solution for each $\varepsilon \in [0,\varepsilon_0)$ if* *and only if* $P(\varepsilon)x_0^{(\varepsilon)} = \sum_{n=0}^{\infty}(-1)^n\varepsilon^n A^n(\varepsilon)P(\varepsilon)f_\varepsilon^{(n)}(0)$*, and the solution is given by*

$$\begin{aligned} x_\varepsilon(t) &= \varepsilon^{-1}A^D(\varepsilon)\exp\left(-\varepsilon^{-1}A^D(\varepsilon)t\right)(I-P(\varepsilon))x_0^{(\varepsilon)} \\ &+ \int_0^t \varepsilon^{-1}A^D(\varepsilon)\exp\left(-\varepsilon^{-1}A^D(\varepsilon)(t-s)\right)(I-P(\varepsilon)) \\ &\cdot f_\varepsilon(s)\,ds + \sum_{n=0}^{\infty}(-1)^n\varepsilon^n A^n(\varepsilon)P(\varepsilon)f_\varepsilon^{(n)}(t), \end{aligned} \tag{23}$$

where $P(\varepsilon)$ *is the spectral projection of* $A(\varepsilon)$ *corresponding to* 0*. Furthermore, if* $\sigma(A) \subset H^+ \cup 0$, $A(\varepsilon) \to A$, $f_\varepsilon(t) \to f_0(t)$, $x_0^{(\varepsilon)} \to x_0$*, and* $P(\varepsilon) \to P$ *as* $\varepsilon \to 0^+$*, then*

$$\lim_{\varepsilon\to 0^+} x_\varepsilon(t) = 0. \tag{24}$$

Proof. The fact that the solution of (6) exists and that $x_\varepsilon(t)$ is given by (23) follows from Theorem 5. Since $P(\varepsilon) \to P$, $A(\varepsilon) \to A$, and $f_\varepsilon(t) \to f_0(t)$ as $\varepsilon \to 0^+$, we can show that $\sum_{n=0}^{\infty}(-1)^n\varepsilon^n A^n(\varepsilon)P(\varepsilon)f_\varepsilon^{(n)}(t)$ converges uniformly in a compact set of $[0,\varepsilon_0)$; hence it converges to zero as $\varepsilon \to 0^+$. Since $A(\varepsilon) \to A$ and $P(\varepsilon) \to P$, it follows that $A^D(\varepsilon) \to A^D$ as $\varepsilon \to 0^+$ (see [15, Theorem 2.4]). By [11, Theorem 4.4], $\sigma(A^D)\setminus\{0\} = \{\lambda^{-1};\ \lambda \in \sigma(A)\setminus\{0\}\} \subset H^+$, and $\sigma(-A^D)\setminus\{0\} \subset H^-$. Since 0 is an isolated spectral point, there are disjoint open sets U_1 and U_0 such that

$$\sigma\left(-A^D\right)\setminus\{0\} \subset U_1 \subset H^-, \quad 0 \in U_0. \tag{25}$$

By the upper semicontinuity of the spectrum there is $0 < \varepsilon_1 < \varepsilon_0$ such that

$$\sigma\left(-A^D(\varepsilon)\right)\setminus\{0\} \subset U_1 \quad \text{if } \varepsilon < \varepsilon_1 < \varepsilon_0, \tag{26}$$

and there are a bounded open set H_1 such that $\sigma(-A^D)\setminus\{0\} \subset H_1 \subset \overline{H_1} \subset U_1$ and a Cauchy cycle γ with respect to $(U_1, \overline{H_1})$. Since $A^D(\varepsilon) \to A^D$ as $\varepsilon \to 0^+$, it follows that

$$\lim_{\varepsilon\to 0^+}\left(\lambda I + A^D(\varepsilon)\right)^{-1} = \left(\lambda I + A^D\right)^{-1} \tag{27}$$

uniformly for $\lambda \in \gamma$. Therefore there exists a constant c such that

$$\sup_{\lambda\in\gamma}\left\|\left(\lambda I + A^D(\varepsilon)\right)^{-1}\right\| = c \tag{28}$$

for all $\varepsilon < \varepsilon_1$. Let $\nu = -\sup\{\mathrm{Re}\lambda : \lambda \in \gamma\}$. Then $\nu > 0$ as $\gamma \in H^-$ and

$$\begin{aligned} &\left\|\exp\left(-\frac{A^D(\varepsilon)}{\varepsilon}\right)(I-P(\varepsilon))\right\| \\ &\quad = \left\|\frac{1}{2\pi i}\int_\gamma e^{\lambda/\varepsilon}\left(\lambda I + A^D(\varepsilon)\right)^{-1} ds\right\| \\ &\quad \le \frac{c\cdot l(\gamma)}{2\pi}e^{-\nu/\varepsilon}, \end{aligned} \tag{29}$$

which implies $\|\exp(-A^D(\varepsilon)/\varepsilon)(I-P(\varepsilon))\| = O(e^{-\nu/\varepsilon})$. On the other hand, $A^D(\varepsilon)/\varepsilon = O(\varepsilon^{-1})$ since $A^D(\varepsilon) \to A^D$ as $\varepsilon \to 0^+$. Therefore the first two terms of (23) converge to zero as $\varepsilon \to 0^+$. We conclude $x_\varepsilon(t) \to 0$ as $\varepsilon \to 0^+$. □

We can now easily show that the solution of (5) converges to the solution of the associated reduced equation as $\varepsilon \to 0^+$ if the continuity of the g-Drazin inverse is assumed.

Theorem 7. *Let* $A(\varepsilon)$ *be* g*-Drazin invertible operator of order* $\omega < 1$ *and let* $A^D(\varepsilon)$ *be the corresponding Drazin inverse for each* $\varepsilon \in [0,\varepsilon_0)$*. Equation (5) has a unique solution for each* $\varepsilon \in [0,\varepsilon_0)$ *if and only if* $P(\varepsilon)x_0^{(\varepsilon)} = \sum_{n=0}^{\infty}(-1)^n A^n(\varepsilon)P(\varepsilon)f_\varepsilon^{(n)}(0)$*, and the solution is given by*

$$\begin{aligned} x_\varepsilon(t) &= A^D(\varepsilon)\exp\left(-A^D(\varepsilon)t\right)(I-P(\varepsilon))x_0^{(\varepsilon)} \\ &+ \int_0^t A^D(\varepsilon)\exp\left(-A^D(\varepsilon)(t-s)\right)(I-P(\varepsilon)) \\ &\cdot f_\varepsilon(s)\,ds + \sum_{n=0}^{\infty}(-1)^n A^n(\varepsilon)P(\varepsilon)f_\varepsilon^{(n)}(t), \end{aligned} \tag{30}$$

where $P(\varepsilon)$ *is the spectral projection of* $A(\varepsilon)$ *corresponding to* 0*. Furthermore, if* $A(\varepsilon) \to A$, $f_\varepsilon(t) \to f_0(t)$, $x_0^{(\varepsilon)} \to x_0$*, and* $P(\varepsilon) \to P$ *as* $\varepsilon \to 0^+$*, then*

$$\lim_{\varepsilon\to 0^+} x_\varepsilon(t) = x(t), \tag{31}$$

where $x(t)$ *is the solution of the associated reduced equation*

$$A\frac{dx(t)}{dt} + x(t) = f_0(t) \quad x(0) = x_0,\ t \in [0,T]. \tag{32}$$

Proof. The proof follows from Theorem 5 and [15, Theorem 2.4]. □

4. Conclusions

We have obtained some results on abstract singular differential equations on a Banach space using the generalized Drazin inverse. In particular, the associated singular operator is assumed to have a generalized Drazin inverse instead of a classical one. Furthermore, two classes of singularly perturbed system have been studied. Under the continuity conditions of the generalized Drazin inverses, we have shown that the solution to the singularly perturbed differential equation converges to the solution of the reduced equation.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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