

# Research Article Sectional Category of the Ganea Fibrations and Higher Relative Category

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Received 7 April 2016; Accepted 29 June 2016

Academic Editor: Dan Burghelea

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We first compute James' sectional category (secat) of the Ganea map  $g_k$  of any map  $\iota_X$  in terms of the sectional category of  $\iota_X$ : we show that secat  $g_k$  is the integer part of secat  $\iota_X/(k + 1)$ . Next we compute the relative category (relcat) of  $g_k$ . In order to do this, we introduce the relative category of order k (relcat<sub>k</sub>) of a map and show that relcat  $g_k$  is the integer part of relcat<sub>k</sub> $\iota_X/(k + 1)$ . Then we establish some inequalities linking secat and relcat of any order: we show that secat  $\iota_X \leq \text{relcat}_k \iota_X \leq \text{secat} \iota_X + k + 1$  and relcat<sub>k</sub> $\iota_X \leq \text{relcat}_{k+1}\iota_X \leq \text{relcat}_k \iota_X + 1$ . We give examples that show that these inequalities may be strict.

# 1. Introduction

The "Lusternik-Schnirelmann category" catX of a topological space *X* is the least integer  $n \ge 0$  such that *X* can be covered by n + 1 open subsets  $U_i$  ( $0 \le i \le n$ ) such that each inclusion  $U_i \hookrightarrow X$  is nullhomotopic; that is, the based path-space fibration  $PX \rightarrow X$  has a partial section on  $U_i$ . More generally, the "sectional category" secat p of a fibration  $p : E \rightarrow X$ , originally defined by Schwarz [1], is the least integer  $n \ge 0$ such that X can be covered by n + 1 open subsets with a partial section of *p* on each of these sets. This notion extends to any continuous map  $\iota_X : A \to X$  by taking the standard homotopy replacement of  $\iota_X$  by a fibration  $p : E \to X$  and setting secat  $\iota_X$  = secat p. So cat X = secat(\*  $\rightarrow$  X). Sectional category earned its renown recently as Farber's notion of "topological complexity" [2] of a space A, which measures the difficulty of solving the motion planning problem: the topological complexity of A is the sectional category of the diagonal  $\Delta : A \rightarrow A \times A$  or equivalently of the (unbased) fibration  $\pi : A^I \to A \times A : \alpha \mapsto (\alpha(0), \alpha(1)).$ 

For a given space *X*, Ganea [3] defined a sequence of fibrations  $p_k : E_k \to X$  for  $k \ge 0$ , starting with  $p_0 : PX \to X$ . The fundamental property of the sequence is that it gives another criterion for detecting the category: cat*X* is the least *n* such that  $p_n$  has a section (at least for a sufficiently nice space:

normal, well pointed). This construction can be generalized for any map  $\iota_X : A \to X$ ; that is, there is a sequence of maps  $g_k(\iota_X) : G_k(\iota_X) \to X$ , starting with  $g_0(\iota_X) = \iota_X$ , and secat $(\iota_X)$ is the least *n* such that  $g_n(\iota_X)$  has a homotopy section; see Definition 3. We recover the Ganea construction when A = \*; in this case we write  $g_k(X)$  instead of  $g_k(\iota_X)$ .

In this paper, we first show that the sectional category of kth Ganea map  $g_k(X)$  of X is the integer part of  $\operatorname{cat} X/(k + 1)$ . More generally, the sectional category of the Ganea map  $g_k(\iota_X)$  associated with any map  $\iota_X$  is the integer part of secat  $\iota_X/(k + 1)$ .

As we may "think of" the sectional category as the degree of obstruction for a map to have a homotopy section, this shows us how this degree of obstruction decreases when we consider the successive Ganea maps. For instance, for a space X with catX = 7, the successive values of secat( $g_k(X)$ ) for  $0 \le k \le 7$  are

$$7 \ 3 \ 2 \ 1 \ 1 \ 1 \ 0. \tag{1}$$

In [4], we used the same Ganea-type construction to define the "relative category" of a map (relcat for short). As a particular case, the relative category of the diagonal map  $\Delta : X \to X \times X$  is the "monoidal topological complexity" of *X* defined in [5]. It turns out that the relative category can differ

from the sectional category by at most one. More precisely, we have

secat 
$$\iota_X \leq \operatorname{relcat} \iota_X \leq \operatorname{secat} \iota_X + 1.$$
 (2)

This establishes a dichotomy between maps: those for which the sectional category equals the relative category and those for which they differ by 1.

In this paper we introduce the "relative category of order k" (relcat<sub>k</sub>) and show that the relative category of kth Ganea map  $g_k(\iota_X)$  associated with a map  $\iota_X$  is the integer part of relcat<sub>k</sub> $\iota_X/(k + 1)$ . When  $\iota_X : * \to X$ , we write relcat<sub>k</sub> $\iota_X = \operatorname{cat}_k X$ .

*Warning*. Despite  $cat_k$  is sometimes used in the literature for Fox's *k*-dimensional category, this is *not* the meaning of this notation in this paper.

We link all these invariants together by several inequalities:

secat 
$$\iota_X \leq \operatorname{relcat}_k \iota_X \leq \operatorname{secat} \iota_X + k + 1$$
,  
relcat<sub>k</sub> $\iota_X \leq \operatorname{relcat}_{k+1} \iota_X \leq \operatorname{relcat}_k \iota_X + 1$ . (3)

Finally, we show that, with some hypothesis on the connectivity of  $\iota_X$  and the homotopical dimension of the source of  $g_k(\iota_X)$ , relcat<sub>i</sub>  $\iota_X$  = secat  $\iota_X$  for all  $j \leq k$ .

For a given space X (resp.: map  $\iota_X$ ), the set of integers k for which the equality  $\operatorname{cat}_{k+1}X = \operatorname{cat}_k X$  (resp.,  $\operatorname{relcat}_{k+1}\iota_X = \operatorname{relcat}_k\iota_X$ ) holds is an interesting datum about this space (resp., map). The maximum number of such integers is  $\operatorname{cat} X$  (resp.,  $\operatorname{relcat}\iota_X$ ). For instance, for  $X = K(\mathbb{Q}, 1)$ , there is just one such k, which is 0: namely,

$$cat_0 X = cat_1 X = 2,$$

$$cat_k X = k + 1 \quad \text{for } k > 1.$$
(4)

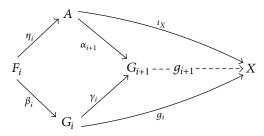
### 2. Sectional Category of the Ganea Maps

We use the symbol  $\approx$  both to mean that maps are homotopic and to mean that spaces are of the same homotopy type. We denote the integer part of a rational number *q* by  $\lfloor q \rfloor$ .

We build all our spaces and maps with "homotopy commutative diagrams," especially "homotopy pullbacks" and "homotopy pushouts," in the spirit of [6].

Recall the following construction.

Definition 1. For any map  $\iota_X : A \to X$ , the Ganea construction of  $\iota_X$  is the following sequence of homotopy commutative diagrams ( $i \ge 0$ ):



where the outside square is a homotopy pullback, the inside square is a homotopy pushout, and the map  $g_{i+1} = (g_i, \iota_X) :$  $G_{i+1} \to X$  is the whisker map induced by this homotopy pushout. The iteration starts with  $g_0 = \iota_X : A \to X$ .

In other words, the map  $g_{i+1}$  is the join of  $g_i$  and  $\iota_X$  over X; namely,  $g_{i+1} \simeq g_i \bowtie_X \iota_X$ . When we need to be precise, we denote  $G_i$  by  $G_i(\iota_X)$  and  $g_i$  by  $g_i(\iota_X)$ . If  $A \simeq *$ , we also write  $G_i(X)$  and  $g_i(X)$ , respectively.

Notice that, as the outside square is a homotopy pullback,  $g_i$  and  $\eta_i$  have a common homotopy fiber, so their connectivity is equal.

For coherence, let  $\alpha_0 = id_A$ . For any  $i \ge 0$ , there is a whisker map  $\theta_i = (id_A, \alpha_i) : A \to F_i$  induced by the homotopy pullback. Thus,  $\theta_i$  is a homotopy section of  $\eta_i$ . Moreover, we have  $\gamma_i \circ \alpha_i \simeq \alpha_{i+1}$ .

**Proposition 2.** For any map  $\iota_X : A \to X$ , we have

$$g_{j}\left(g_{i}\left(\iota_{X}\right)\right) \simeq g_{ij+i+j}\left(\iota_{X}\right).$$
(5)

*Proof.* This is just an application of the "associativity of the join" (see [7, Theorem 4.8], for instance):

$$g_{j}(g_{i}(\iota_{X})) \simeq g_{i}(\iota_{X}) \bowtie_{X} \cdots \bowtie_{X} g_{i}(\iota_{X}) \quad (j+1 \text{ times})$$

$$\simeq (\iota_{X} \bowtie_{X} \cdots \bowtie_{X} \iota_{X}) \cdots (\iota_{X} \bowtie_{X} \cdots \bowtie_{X} \iota_{X})$$

$$\simeq \iota_{X} \bowtie_{X} \cdots \bowtie_{X} \iota_{X} \qquad (6)$$

$$((j+1)(i+1) \text{ times})$$

$$\simeq g_{(j+1)(i+1)-1}(\iota_{X}).$$

*Definition 3.* Let  $\iota_X : A \to X$  be any map.

- (1) The *sectional category* of  $\iota_X$  is the least integer *n* such that the map  $g_n : G_n(\iota_X) \to X$  has a homotopy section: that is, there exists a map  $\sigma : X \to G_n(\iota_X)$  such that  $g_n \circ \sigma \simeq id_X$ .
- (2) The *relative category* of  $\iota_X$  is the least integer *n* such that the map  $g_n : G_n(\iota_X) \to X$  has a homotopy section  $\sigma$  and  $\sigma \circ \iota_X \simeq \alpha_n$ .

We denote the sectional category by  $secat(\iota_X)$  and the relative category by  $relcat(\iota_X)$ . If  $A \simeq *$ ,  $secat(\iota_X) = relcat(\iota_X)$  and it is denoted simply by cat(X); this is the "normalized" version of the Lusternik-Schnirelmann category.

A lot about the integers cat and secat is collected in [8]. The integer relcat is introduced in [4] and further studied in [9, 10].

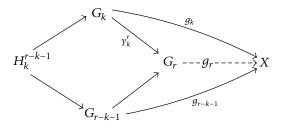
**Proposition 4.** For any map  $\iota_X : A \to X$ , we have

secat 
$$g_k(\iota_X) = \left\lfloor \frac{\operatorname{secat} \iota_X}{k+1} \right\rfloor.$$
 (7)

*Proof.* By definition, secat  $g_k(\iota_X)$  is the least integer n such that  $g_n(g_k(\iota_X))$ , that is,  $g_{kn+k+n}(\iota_X)$ , has a homotopy section. Thus, if secat  $\iota_X = m$ , n will be such that  $kn + k + n \ge m$  and k(n-1) + k + (n-1) < m: that is,  $n \ge m/(k+1) - k/(k+1)$  and n < m/(k+1) + 1/(k+1), so  $n = \lfloor m/(k+1) \rfloor$ .

# 3. Higher Relative Category

For any map  $\iota_X : A \to X$  and two integers  $0 \le k < r$ , consider the following homotopy commutative diagram:



where the outside square is a homotopy pullback and the inside square is a homotopy pushout.

Because of the associativity of the join, we also have  $\gamma_k^r \simeq \gamma_{r-1} \circ \gamma_{r-2} \circ \cdots \circ \gamma_{k+1} \circ \gamma_k$ . For coherence, let  $\gamma_k^k = id_{G_k}$ .

Definition 5. Let  $\iota_X : A \to X$  be any map. The *relative category* of order k of  $\iota_X$  is the least integer  $n \ge k$  such that the map  $g_n : G_n(\iota_X) \to X$  has a homotopy section  $\sigma$  and  $\sigma \circ g_k \simeq \gamma_k^n$ .

We denote this integer by  $\operatorname{relcat}_k \iota_X$ . In order to avoid the prefix "rel" when  $A \simeq *$ , we write  $\operatorname{cat}_k X = \operatorname{relcat}_k \iota_X$  in this case.

*Remark 6.* Notice that  $\operatorname{relcat}_{0}\iota_X = \operatorname{relcat}\iota_X$  and that, clearly,  $k \leq \operatorname{relcat}_{k}\iota_X \leq \operatorname{relcat}_{k+1}\iota_X$  for any k. Also notice that  $\operatorname{relcat}_{k}\iota_X = k$  if and only if  $g_k(\iota_X)$  is a homotopy equivalence. In particular,  $\operatorname{cat}_k * = k$  for any k.

Following the same reasoning as in Proposition 4, we have the following.

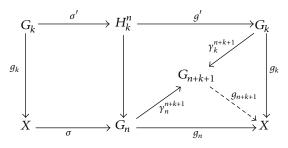
**Proposition 7.** For any map  $\iota_X : A \to X$ , we have

relcat 
$$g_k(\iota_X) = \left\lfloor \frac{\operatorname{relcat}_k \iota_X}{k+1} \right\rfloor.$$
 (8)

**Proposition 8.** For any map  $\iota_X : A \to X$ , any k, we have

secat 
$$\iota_X \leq \operatorname{relcat}_k \iota_X \leq \operatorname{secat} \iota_X + k + 1.$$
 (9)

*Proof.* Only the second inequality needs a proof. Let  $n = \text{secat } \iota_X$  and let  $\sigma$  be a homotopy section of  $g_n$ . Consider the following homotopy commutative diagram:

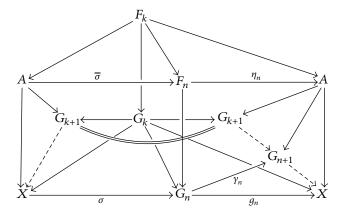


where  $\sigma' = (\sigma \circ g_k, \operatorname{id}_{G_k})$  is the whisker map induced by the right homotopy pullback. We have  $g' \circ \sigma' \simeq \operatorname{id}_{G_k}$  and the left square is a homotopy pullback by the Prism lemma (see [7, Lemma 1.3], for instance). The map  $\sigma^+ = \gamma_n^{n+k+1} \circ \sigma$ 

**Theorem 9.** For any map  $\iota_X : A \to X$ , any k, we have

$$k \leq \operatorname{relcat}_k \iota_X \leq \operatorname{relcat}_{k+1} \iota_X \leq \operatorname{relcat}_k \iota_X + 1.$$
 (10)

*Proof.* The first two inequalities are our Remark 6; only the third needs a proof. Let  $n = \operatorname{relcat}_k \iota_X$  and let  $\sigma$  be a homotopy section of  $g_n$  such that  $\sigma \circ g_k \simeq \gamma_k^n$ . Consider the following homotopy commutative diagram:



The map  $\sigma^+ = \gamma_n \circ \sigma$  is a homotopy section of  $g_{n+1}$  and  $\sigma^+ \circ g_{k+1} \simeq \gamma_{k+1}^{n+1}$ , so relcat<sub>k+1</sub> $\iota_X \leq n+1$ .

So relcat $_k t_X$  increases at most by one when k increases by one.

**Corollary 10.** For any map  $\iota_X : A \to X$ , any k, we have

$$\operatorname{relcat} \iota_X \leq \operatorname{relcat}_k \iota_X \leq \operatorname{relcat} \iota_X + k. \tag{11}$$

*Remark 11.* As a consequence of Theorem 9 and Corollary 10, if  $n = \operatorname{relcat} \iota_X$ , there are at most n integers k for which  $\operatorname{relcat}_{k+1}\iota_X = \operatorname{relcat}_k\iota_X$ .

*Example 12.* If  $\iota_X$  is a homotopy equivalence, then  $g_k$  is a homotopy equivalence for all k. So relcat<sub>k</sub> $\iota_X = k$  for all k.

*Example 13.* Let  $A \neq *$  and consider the map  $\iota_* : A \to *$ . We have secat  $\iota_* = 0$  because  $\iota_*$  has a (unique) section. By Proposition 8, relcat<sub>k</sub> $\iota_* = k$  or 1+k. Indeed, for any k, the map  $\gamma_k^{k+1} : A \bowtie \cdots \bowtie A$  (k + 1 times)  $\rightarrow A \bowtie \cdots \bowtie A$  (k + 2 times) is homotopic to the null map, so  $\sigma \circ g_k \simeq \gamma_k^{k+1}$ , where  $\sigma$  :  $* \rightarrow G_{k+1}(\iota_*)$ . But we cannot have relcat<sub>k</sub> $\iota_* = k$  unless  $g_k(\iota_*)$ :  $A \bowtie \cdots \bowtie A$  (k + 1 times)  $\rightarrow *$  is a homotopy equivalence.

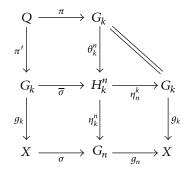
If we choose a space *A* such that  $A \neq *$  but  $\Sigma A \simeq *$  (the 2-skeleton of the Poincaré homology 3 spheres, for instance), then  $A \bowtie A \simeq \Sigma A \land A \simeq *$  and  $g_k$  is a homotopy equivalence for all k > 0, so relcat<sub>0</sub> $\iota_* = 1$  and relcat<sub>k</sub> $\iota_* = k$  for all k > 0. However, if we chose a simply connected CW-complex *A* (in order that  $A \bowtie \cdots \bowtie A \neq *$ ), then relcat<sub>k</sub> $\iota_* = k + 1$  for all *k*.

*Example 14.* Consider any CW-complex X with cat X = 1 and the map  $\iota_X : * \to X$ . We have secat  $\iota_X = relcat \iota_X = cat X = 1$ .

Let us compute  $\operatorname{cat}_1 X = \operatorname{relcat}_1 \iota_X$ . Notice that  $G_1(X) \simeq \Sigma \Omega X$ . By Theorem 9, we know that  $1 \leq \operatorname{cat}_1 X \leq 2$ . But we cannot have  $\operatorname{cat}_1 X = 1$  because  $g_1$  is not a homotopy equivalence, so  $\operatorname{cat}_1 X = 2$ . By the way, we can say that  $\gamma_1^2 : \Sigma \Omega X \to G_2(X)$ factorizes up to homotopy through  $g_1 : \Sigma \Omega X \to X$ .

*Example 15.* More generally, if relcat  $\iota_X = 1$ , we have  $k \leq \text{relcat}_k \iota_X \leq 1 + k$  for any k by Corollary 10. Thus,  $\text{relcat}_k \iota_X = k + 1$  while  $g_k(\iota_X)$  is not a homotopy equivalence (and if any n exists such that  $g_n(\iota_X)$  is a homotopy equivalence, then  $\text{relcat}_k \iota_X = k$  for all  $k \geq n$ ).

Suppose we are given any map  $\iota_X : A \to X$  with secat $(\iota_X) \leq n$  and any homotopy section  $\sigma : X \to G_n$  of  $g_n : G_n \to X$ . For any  $k \leq n$ , consider the following homotopy pullbacks:



where  $\theta_k^n = (\gamma_k^n, \mathrm{id}_{G_k})$  is the whisker map induced by the homotopy pullback  $H_k^n$ . Notice that  $\eta_n^k \circ \theta_k^n \simeq \mathrm{id}_{G_k}$ . By the Prism lemma, we know that the homotopy pullback of  $\sigma$  and  $\eta_k^n$  is indeed  $G_k$  and that  $\eta_n^k \circ \overline{\sigma} \simeq \mathrm{id}_{G_k}$ . Also notice that  $\pi \simeq \pi'$ since  $\pi \simeq \eta_n^k \circ \theta_k^n \circ \pi \simeq \eta_n^k \circ \overline{\sigma} \circ \pi' \simeq \pi'$ .

**Proposition 16.** For any map  $\iota_X : A \to X$  with secat $(\iota_X) \leq n$  and any homotopy section  $\sigma : X \to G_n$  of  $g_n : G_n \to X$ , with the same definitions and notations as above, the following conditions are equivalent:

(i)  $\sigma \circ g_k \simeq \gamma_k^n$ .

- (ii)  $\pi$  has a homotopy section.
- (iii)  $\pi$  *is a homotopy epimorphism.*
- (iv)  $\theta_k^n \simeq \overline{\sigma}$ .

*Proof.* We have the following sequence of implications:

(i)  $\Rightarrow$  (ii): since  $\sigma \circ g_k \simeq \gamma_k^n \simeq \eta_k^n \circ \theta_k^n \circ \mathrm{id}_{G_k}$ , we have a whisker map  $(g_k, \mathrm{id}_{G_k}) : G_k \to Q$  induced by the homotopy pullback Q which is a homotopy section of  $\pi$ .

(ii)  $\Rightarrow$  (iii): it is obvious.

(iii)  $\Rightarrow$  (iv): we have  $\theta_k^n \circ \pi \simeq \overline{\sigma} \circ \pi' \simeq \overline{\sigma} \circ \pi$  since  $\pi \simeq \pi'$ . Thus,  $\theta_k^n \simeq \overline{\sigma}$  since  $\pi$  is a homotopy epimorphism.

(iv)  $\Rightarrow$  (i): we have  $\sigma \circ g_k \simeq \eta_k^n \circ \overline{\sigma} \simeq \eta_k^n \circ \theta_k^n \simeq \gamma_k^n$ .

**Theorem 17.** Let  $\iota_X : A \to X$  be a (q - 1) connected map. If for some  $k \leq \text{secat } \iota_X$ ,  $G_k$  has the homotopy type of a CWcomplex with dimension strictly less than  $(\text{secat } \iota_X + 1)q - 1$ , then  $\text{relcat}_i \iota_X = \text{secat } \iota_X$  for all  $j \leq k$ .

This is an immediate consequence of the following.

**Proposition 18.** Let  $\iota_X : A \to X$  be a (q-1) connected map with secat  $\iota_X \leq n$ . If for some  $k \leq n$ ,  $G_k$  has the homotopy type of a CW-complex with dimension strictly less than (n+1)q-1, then  $\sigma \circ g_k \simeq \gamma_k^n$  for any homotopy section  $\sigma$  of  $g_n$ , so relcat $_k \iota_X \leq n$ .

*Proof.* Recall that, for any  $i \ge 0$ ,  $g_i$  is the (i + 1)-fold join of  $\iota_X$ . Thus, by [11, Theorem 47], we obtain that  $g_i : G_i \to X$  is (i + 1)q - 1-connected. As  $g_i$  and  $\eta_i^k$  have the same homotopy fiber, which is (i+1)q-2-connected, we see that  $\eta_i^k : H_k^i \to G_k$  is (i + 1)q - 1-connected, too. By [12, Theorem IV.7.16], this means that, for every CW-complex K with dim K < (i + 1)q - 1,  $\eta_i^k$  induces a one-to-one correspondence  $[K, H_k^i] \to [K, G_k]$ . Apply this to  $K \simeq G_k$  and i = n: since  $\theta_k^n$  and  $\overline{\sigma}$  are both homotopy sections of  $\eta_n^k$ , we obtain  $\theta_k^n \simeq \overline{\sigma}$ , and Proposition 16 gives the desired result.

*Example 19.* Let X be the Eilenberg-Mac Lane space  $K(\mathbb{Q}, 1)$ . It is known that  $\operatorname{cat}(X) = 2$  and that  $G_1(X) \approx \Sigma \Omega X$  has the homotopy type of a wedge of circles (see [8, Example 1.9 and Remark 1.62], for instance). By Theorem 9, we know that  $2 \leq \operatorname{cat}_1 X \leq 3$ . Because dim  $G_1(X) = 1 < (\operatorname{cat} X + 1) - 1 = 2$ , we have  $\sigma \circ g_1 \approx \gamma_1^2$  for any homotopy section  $\sigma$  of  $g_2(X)$  and, thus,  $\operatorname{cat}_1 X = 2$ . Moreover,  $g_k$  is never a homotopy equivalence, so  $\operatorname{cat}_k X > k$  for any k; thus,  $\operatorname{cat}_k X = k + 1$  for  $k \geq 1$ .

### **Competing Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

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