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Research Article

Riordan Matrix Representations of Euler's Constant γ and Euler's Number e

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We show that the Euler-Mascheroni constant γ and Euler's number e can both be represented as a product of a Riordan matrix and certain row and column vectors.

Dedicated to David Harold Blackwell (April 24, 1919–July 8, 2010)

1. Introduction

It was shown by Kenter [1] that the Euler-Mascheroni constant

$$\gamma = \lim_{n\to\infty}\left[\left(\sum_{m=1}^{n}\frac{1}{m}\right) - \ln n\right] = 0.5772156649\cdots \tag{1}$$

can be represented as a product of an infinite-dimensional row vector, the inverse of a lower triangular matrix, and an infinite-dimensional column vector:

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} & \cdots \end{pmatrix} \begin{pmatrix} 1 & & & & & \\ \frac{1}{2} & 1 & & & & \\ \frac{1}{3} & \frac{1}{2} & 1 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \cdots & 1 & \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots \end{pmatrix}^{-1} \cdot \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ \vdots \\ \frac{1}{n+1} \\ \vdots \end{pmatrix}. \tag{2}$$

Kenter's proof uses induction, definite integrals, convergence of power series, and Abel's Theorem. In this paper, we recast this statement using the language of Riordan matrices. We exhibit another proof as well as a generalization. Our main result is the following.

Theorem 1. *Consider sequences $\{a_0, a_1, \ldots, a_n, \ldots\}$, $\{b_0, b_1, \ldots, b_n, \ldots\}$, and $\{c_0, c_1, \ldots, c_n, \ldots\}$ of complex numbers such that $a_0, b_0, c_0 \neq 0$, as well as an integer exponent d. Assume that*

(i) *the power series* $a(x) = \sum_n a_n x^n$, $b(x) = \sum_n b_n x^n$, $c(x) = \sum_n c_n x^n$, *and* $b(x)^d$ *are convergent in the interval* $|x| < 1$;

(ii) *the following complex residue exists:*

$$\operatorname*{Res}_{z=0}\left[\frac{a(z)\,b\left(z^{-1}\right)^{d} c\left(z^{-1}\right)}{z}\right] = \frac{1}{2\pi i}\oint_{|z|=1} a(z)\,b\left(z^{-1}\right)^{d} c\left(z^{-1}\right)\frac{1}{z}dz. \tag{3}$$

Then, the matrix product

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n & \cdots \end{pmatrix}\begin{pmatrix} b_0 & & & & & \\ b_1 & b_0 & & & & \\ b_2 & b_1 & b_0 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ b_n & b_{n-1} & b_{n-2} & \cdots & b_0 & \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots \end{pmatrix}^{d} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \\ \vdots \end{pmatrix} \tag{4}$$

is equal to the above residue.

The infinite-dimensional lower triangular matrix is an example of a Riordan matrix. Specifically, it is that Riordan matrix associated with the power series $b(x)^d$. Kenter's result follows by careful analysis of the power series:

$$\begin{aligned} a(x) &= -\frac{\log(1-x)}{x} \\ &= 1 + \frac{1}{2}x + \frac{1}{3}x^2 + \cdots + \frac{1}{n+1}x^n + \cdots, \\ b(x)^{-1} &= -\frac{x}{\log(1-x)} \\ &= 1 - \frac{1}{2}x - \frac{1}{12}x^2 - \frac{1}{24}x^3 - \cdots - L_n x^n - \cdots, \\ c(x) &= \frac{a(x)-1}{x} \\ &= \frac{1}{2} + \frac{1}{3}x + \frac{1}{4}x^2 + \cdots + \frac{1}{n+2}x^n + \cdots. \end{aligned} \tag{5}$$

The coefficients L_n are sometimes called the "logarithmic numbers" or the "Gregory coefficients"; these are basically the Bernoulli numbers of the second kind up to a choice of sign. (Kenter employs the coefficients $c_k = -L_k$.) The idea of this paper is that we have the matrix product

$$\begin{pmatrix} 1 & & & & & \\ \frac{1}{2} & 1 & & & & \\ \frac{1}{3} & \frac{1}{2} & 1 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \cdots & 1 & \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots \end{pmatrix}^{-1}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ \vdots \\ \frac{1}{n+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{12} \\ \frac{1}{24} \\ \vdots \\ L_n \\ \vdots \end{pmatrix}, \tag{6}$$

which is equivalent to the recursive identity $\sum_{m=0}^{n-1} L_m/(n-m) = 0$, which is valid whenever $n = 2, 3, 4, \ldots$. The matrix product, and hence the recursive identity, can be *derived* from properties of Riordan matrices. Kenter's result follows from the identity $\sum_{m=1}^{\infty} L_m/m = \gamma$, which in turn follows from an identity involving a definite integral.

As another consequence of our main result, we can also show that Euler's number

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = 2.7182818284\cdots \tag{7}$$

can be represented as a product of an infinite-dimensional row vector, a lower triangular matrix, and an infinite-dimensional column vector.

Corollary 2. *For any integers* p, q, *and* d *with* $pq > 1$, *the number*

$$\frac{pq}{pq-1}\sqrt[p]{e^d} = \lim_{n\to\infty}\left[\frac{pq}{pq-1}\left(1 + \frac{1}{pn}\right)^{dn}\right] \tag{8}$$

is equal to the matrix product

$$\begin{pmatrix} 1 & \frac{1}{p} & \frac{1}{p^2} & \cdots & \frac{1}{p^n} & \cdots \end{pmatrix} \begin{pmatrix} 1 & & & & & \\ \frac{1}{1!} & 1 & & & & \\ \frac{1}{2!} & \frac{1}{1!} & 1 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \cdots & 1 & \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots \end{pmatrix}^{d} \begin{pmatrix} 1 \\ \frac{1}{q} \\ \frac{1}{q^2} \\ \vdots \\ \frac{1}{q^n} \\ \vdots \end{pmatrix}. \tag{9}$$

In the process of proving these generalizations, we present a representation theoretic view of Riordan matrices. That is, we consider the matrices as representations $\pi : G \to GL(V)$ of a certain group G, namely, the Riordan group, acting on an infinite-dimensional vector space V, namely, the collection of those formal power series $h(x)$ in $\mathbb{C}[\![x]\!]$, where $h(0) = 0$.

2. Introduction to Riordan Matrices

We wish to list several key results in the theory of Riordan matrices. To do so, we recast this theory using techniques from representation theory very much in the spirit of Bacher [2]. Our ultimate goal in this section is to explain how Riordan matrices are connected to a permutation representation $\pi : G \to GL(V)$ of a certain group G acting on an infinite-dimensional vector space V. Some of the notation in the sequel will differ from standard notation such as that given by Shapiro et al. [3] and Sprugnoli [4, 5], but we will explain the connection.

2.1. Group Actions. Before developing the representation theoretic view, we give the definition of a Riordan matrix and few related useful properties. Let k be a field; it is customary to set $k = \mathbb{C}$ as the set of complex numbers, but, in practice, $k = \mathbb{Q}$ is the set of rational numbers. Set $k[\![x]\!]$ as the collection of formal power series in an indeterminate x; we will view this as a k-vector space with countable basis $\{1, x, x^2, \dots, x^n, \dots\}$. For most of this article, we will not be concerned with regions of convergence for these series.

There are three binary operations $k[\![x]\!] \times k[\![x]\!] \to k[\![x]\!]$ which will be of importance to us, namely, multiplication $\bullet$, composition $\circ$, and addition $+$. Explicitly, if we write

$$f(x) = \sum_{n=0}^{\infty} f_n x^n, \qquad g(x) = \sum_{n=0}^{\infty} g_n x^n, \tag{10}$$

then we have the formal power series

$$(f \bullet g)(x) = \sum_{n=0}^{\infty} \left[\sum_{m=0}^{n} f_m g_{n-m} \right] x^n,$$
$$(f \circ g)(x) = \sum_{n=0}^{\infty} \left[\sum_{m=0}^{\infty} f_m \left(\sum_{n_1+\cdots+n_m=n} g_{n_1} \cdots g_{n_m} \right) \right] x^n,$$
$$(f + g)(x) = \sum_{n=0}^{\infty} [f_n + g_n] x^n. \tag{11}$$

There are three subsets of the vector space $k[\![x]\!]$ which will be of interest to us in the sequel.

Proposition 3. *Define the subsets*

$$H = \{ f(x) \in k[\![x]\!] \mid f(0) \neq 0 \},$$
$$K = \{ g(x) \in k[\![x]\!] \mid g(0) = 0 \text{ yet } g'(0) \neq 0 \}, \tag{12}$$
$$V = \{ h(x) \in k[\![x]\!] \mid h(0) = 0 \}.$$

(i) *H is a group under multiplication $\bullet$, K is a group under composition $\circ$, and V is a group under addition $+$. In particular, V is a k-vector space with countable basis $\{x, x^2, \dots, x^n, \dots\}$.*

(ii) *The map $\varphi : K \to \mathrm{Aut}(H)$ which sends $g(x) \in K$ to the automorphism $\varphi_g : f(x) \mapsto (f \circ \overline{g})(x)$ is a group homomorphism, where $\overline{g}(x)$ is the compositional inverse of $g(x)$. In particular, $G = H \rtimes_{\varphi} K$ is a group under the binary operation $* : G \times G \to G$ defined by*

$$(f_1, g_1) * (f_2, g_2) = \left(f_1 \bullet \varphi_{g_1}(f_2), g_1 \circ g_2 \right). \tag{13}$$

(iii) *The map $* : G \times V \to V$ defined by $(f, g) * h = f \bullet (h \circ \overline{g})$ is a group action of G on V.*

We use $\overline{g}(x)$ to denote the compositional inverse $g^{-1}(x)$ so that we will not confuse this with the multiplicative inverse $g(x)^{-1}$. Later, we will show that G is isomorphic to the Riordan group $\mathbf{R}$. Moreover, we will show that H, a normal subgroup of G, is isomorphic to the Appell subgroup of $\mathbf{R}$. The motivation of this result is to use the action of G on V to write down a permutation representation $\pi : G \to GL(V)$ and then use the canonical basis $\{x, x^2, \dots, x^n, \dots\}$ of V to list infinite-dimensional matrices.

Proof. We show (i) to fix some notation to be used in the sequel. Since $(f \bullet g)(0) = f(0)g(0) \neq 0$ for any $f(x), g(x) \in H$, we see that $\bullet : H \times H \to H$ is an associative binary operation. The identity is the constant power series $e(x) = 1$, and the inverse of $f(x)$ is its reciprocal, seen to be a power series by expressing said reciprocal in terms of a formal geometric series:

$$\begin{aligned}\frac{1}{f(x)} &= \frac{1}{f_0} \cdot \frac{1}{1 - \sum_{n=0}^{\infty} (-f_n/f_0) x^n} \\ &= \sum_{n=0}^{\infty} \left[\sum_{m=0}^{\infty} \sum_{n_1+\cdots+n_m=n-m} (-1)^m \frac{f_{n_1+1} \cdots f_{n_m+1}}{f_0^{m+1}} \right] x^n.\end{aligned} \tag{14}$$

Since $(f \circ g)(0) = f(g(0)) = f(0) = 0$ and $(f \circ g)'(0) = f'(g(0))g'(0) = f'(0)g'(0) \neq 0$ for any $f(x), g(x) \in K$, we see that $\circ : K \times K \to K$ is an associative binary operation. The identity is the power series $\mathrm{id}(x) = x$, and the inverse of $g(x)$ is its compositional inverse $\overline{g}(x) = \sum_n \overline{g}_n x^n$ having the implicitly defined coefficients

$$\begin{gathered}\overline{g}_0 = 0, \\ \overline{g}_1 = \frac{1}{g_1}, \\ \sum_{m=0}^{n} \overline{g}_m \left[\sum_{n_1+\cdots+n_m=n} g_{n_1} \cdots g_{n_m} \right] = 0 \quad \text{for } n = 2, 3, \ldots.\end{gathered} \tag{15}$$

Since $(f + g)(0) = f(0) + g(0) = 0$ for any $f(x), g(x) \in V$, we see that $+ : V \times V \to V$ is an associative binary operation. The identity is the constant power series $o(x) = 0$, and the inverse of $h(x)$ is the negation $-h(x)$, seen to be a power series with $(-h)(0) = -h(0) = 0$.

Now, we show (ii). Since $(f \circ \overline{g})(0) = f(\overline{g}(0)) = f(0) \neq 0$ for any $f(x) \in H$ and $g(x) \in K$, we see that $\varphi : K \to \mathrm{Aut}(H)$ is well defined. Given $g(x), h(x) \in K$, we have $\varphi_g \circ \varphi_h = \varphi_{g \circ h}$ because for all $f(x) \in H$ we have

$$\begin{aligned}\left(\varphi_g \circ \varphi_h\right)[f(x)] &= \varphi_g \left[\left(f \circ \overline{h}\right)(x) \right] = \left(f \circ \overline{h} \circ \overline{g}\right)(x) \\ &= \left(f \circ \overline{g \circ h}\right)(x) = \varphi_{g \circ h}[f(x)].\end{aligned} \tag{16}$$

Hence, $\varphi : K \to \mathrm{Aut}(H)$ is indeed a group homomorphism. The semidirect product $G = H \rtimes_{\varphi} K$ consists of pairs $(f(x), g(x))$ with $f(x) \in H$ and $g(x) \in K$, where the binary operation $* : G \times G \to G$ is defined by

$$\begin{aligned}&\left(f_1(x), g_1(x)\right) * \left(f_2(x), g_2(x)\right) \\ &\quad = \left(f_1(x) f_2\left(\overline{g_1}(x)\right), g_1\left(g_2(x)\right)\right).\end{aligned} \tag{17}$$

Finally, we show (iii). The map $* : G \times V \to V$ is defined as the formal identity

$$\left(f(x), g(x)\right) * h(x) = f(x) h\left(\overline{g}(x)\right). \tag{18}$$

Since $[(f, g) * h](0) = f(0)h(\overline{g}(0)) = f(0)h(0) = 0$, we see that the map $* : G \times V \to V$ is well defined. As the identity element of G is $(e(x), \mathrm{id}(x)) = (1, x)$, we see that $(e(x), \mathrm{id}(x)) * h(x) = h(x)$ so that it acts trivially on V. Given two elements $(f_1, g_1), (f_2, g_2) \in G$ and $h(x) \in V$, we have the identity

$$\begin{aligned}&\left(f_1(x), g_1(x)\right) * \left[\left(f_2(x), g_2(x)\right) * h(x)\right] \\ &\quad = \left(f_1(x), g_1(x)\right) * \left[f_2(x) h\left(\overline{g_2}(x)\right)\right] \\ &\quad = f_1(x) f_2\left(\overline{g_1}(x)\right) h\left(\overline{g_2} \circ \overline{g_1}(x)\right) \\ &\quad = f_1(x) f_2\left(\overline{g_1}(x)\right) h\left(\overline{g_1 \circ g_2}(x)\right) \\ &\quad = \left(f_1(x) f_2\left(\overline{g_1}(x)\right), g_1\left(g_2(x)\right)\right) * h(x) \\ &\quad = \left[\left(f_1(x), g_1(x)\right) * \left(f_2(x), g_2(x)\right)\right] * h(x).\end{aligned} \tag{19}$$

Similarly, given two elements $h_1(x), h_2(x) \in V$ and $(f, g) \in G$, we have the identity

$$\begin{aligned}&\left(f(x), g(x)\right) * \left[h_1(x) + h_2(x)\right] \\ &\quad = f(x)\left[h_1\left(\overline{g}(x)\right) + h_2\left(\overline{g}(x)\right)\right] \\ &\quad = \left(f(x), g(x)\right) * h_1(x) + \left(f(x), g(x)\right) \\ &\qquad * h_2(x).\end{aligned} \tag{20}$$

Hence, $* : G \times V \to V$ is indeed a group action. □

2.2. Riordan Matrices. Recall that the set

$$V = \{h(x) \in k[\![x]\!] \mid h(0) = 0\} \tag{21}$$

is a k-vpng $\{x, x^2, \ldots, x^n, \ldots\}$. Since the semidirect product $G = H \rtimes_{\varphi} K$ acts on V, we have a "permutation" representation $\pi : G \to GL(V)$. Explicitly, this representation is defined on the basis elements of V via the formal identity

$$\begin{gathered}\left(f(x), g(x)\right) * x^m = f(x)\left[\overline{g}(x)\right]^m = \sum_{n=1}^{\infty} l_{n,m} x^n \\ \text{for } m = 1, 2, 3, \ldots.\end{gathered} \tag{22}$$

(Recall that $\overline{g}(x)$ is the compositional inverse of $g(x)$.) The matrix with respect to the basis $\{x, x^2, \ldots, x^n, \ldots\}$ is given by the lower triangular matrix

$$\pi\left(f(x), g(x)\right) = \begin{pmatrix} l_{1,1} & & & & & \\ l_{2,1} & l_{2,2} & & & & \\ l_{3,1} & l_{3,2} & l_{3,3} & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ l_{n,1} & l_{n,2} & l_{n,3} & \cdots & l_{n,n} & \\ \vdots & \vdots & \vdots & \ldots & \vdots & \ddots \end{pmatrix}. \tag{23}$$

Recall that $g(0) = 0$ yet $f(0), g'(0) \neq 0$. The following result explains the main multiplicative property of these matrices.

Theorem 4. *Continue notation as above.*

(i) $\pi : G \to GL(V)$ *is a group homomorphism. That is,*

$$\begin{aligned}\pi\left(f_1(x), g_1(x)\right)\pi\left(f_2(x), g_2(x)\right) \\ = \pi\left(f_1(x) f_2\left(\overline{g_1}(x)\right), g_1\left(g_2(x)\right)\right).\end{aligned} \tag{24}$$

(ii) *For a generating function* $t(x) = t_0 + t_1 x + t_2 x^2 + \cdots$ *with* $t_0 \neq 0$,

$$\begin{aligned}\pi\left(f(x), g(x)\right)\pi\left(t(x), \mathrm{id}(x)\right) \\ = \left(\sum_{p=1}^{m} l_{n,p} t_{p-m}\right)_{n,m\geq 1}.\end{aligned} \tag{25}$$

Such matrices $\pi(f, g)$ are called the *Riordan matrices associated with the pair* (f, g). The collection **R** of Riordan matrices is a group which is isomorphic to $G = H \rtimes_{\varphi} K$; this is the *Riordan group*. The collection of matrices $\pi(f, \mathrm{id})$ is a group which is isomorphic to H; this normal subgroup is the *Appell subgroup of* **R**.

Proof. We show (i). In the proof of Proposition 3, we found that for each $h(x) \in V$ we have the following formal identity involving power series as elements of $k[\![x]\!]$:

$$\begin{aligned}&\left(f_1(x), g_1(x)\right) * \left[\left(f_2(x), g_2(x)\right) * h(x)\right] \\ &\quad = \left[\left(f_1(x), g_1(x)\right) * \left(f_2(x), g_2(x)\right)\right] * h(x) \\ &\quad = \left(f_1(x) f_2\left(\overline{g_1}(x)\right), g_1\left(g_2(x)\right)\right) * h(x).\end{aligned} \tag{26}$$

In particular, this holds for the basis elements $h(x) = x^n$, so the result follows.

Now, we show (ii). For a generating function $t(x) = t_0 + t_1 x + t_2 x^2 + \cdots$, we have the product

$$\left(t(x), \mathrm{id}(x)\right) * x^m = t(x) x^m = \sum_{n=1}^{\infty} t_{n-m} x^n; \tag{27}$$

so matrices in the Appell subgroup are in the form

$$\pi\left(t(x), \mathrm{id}(x)\right) = \begin{pmatrix} t_0 & & & & & \\ t_1 & t_0 & & & & \\ t_2 & t_1 & t_0 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_0 & \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots \end{pmatrix}. \tag{28}$$

This gives the matrix product

$$\begin{aligned}\pi\left(f(x), g(x)\right)\pi\left(t(x), x\right) &= \left(l_{n,p}\right)_{n,p\geq 1}\left(t_{p-m}\right)_{p,m\geq 1} \\ &= \left(\sum_{p=1}^{m} l_{n,p} t_{p-m}\right)_{n,m\geq 1}\end{aligned} \tag{29}$$

so the result follows. □

2.3. Examples. Let $k = \mathbb{Q}$. Using elementary calculus, we find the power series expansions

$$\begin{aligned}-\frac{\ln(1-x)}{x} &= 1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \frac{1}{5}x^4 + \frac{1}{6}x^5 \\ &\quad + \cdots, \\ -\frac{x}{\ln(1-x)} &= 1 - \frac{1}{2}x - \frac{1}{12}x^2 - \frac{1}{24}x^3 - \frac{19}{720}x^4 \\ &\quad - \frac{3}{160}x^5 + \cdots,\end{aligned} \tag{30}$$

which are valid whenever $|x| < 1$. Hence, the formal power series

$$f(x) = 1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \cdots + \frac{1}{n+1}x^n + \cdots \tag{31}$$

is an element of H and has multiplicative inverse

$$\begin{aligned}\frac{1}{f(x)} &= 1 - \frac{1}{2}x - \frac{1}{12}x^2 - \frac{1}{24}x^3 - \frac{19}{720}x^4 - \frac{3}{160}x^5 \\ &\quad + \cdots.\end{aligned} \tag{32}$$

We have the product

$$\left(f(x), \mathrm{id}(x)\right) * x^m = f(x) x^m = \sum_{n=1}^{\infty} \frac{1}{n-m+1} x^n \tag{33}$$

which yields the matrix

$$\pi\left(f, \mathrm{id}\right) = \begin{pmatrix} 1 & & & & & \\ \frac{1}{2} & 1 & & & & \\ \frac{1}{3} & \frac{1}{2} & 1 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \cdots & 1 & \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots \end{pmatrix}. \tag{34}$$

Similarly, we have the product

$$\begin{aligned}\left(\frac{1}{f(x)}, \mathrm{id}(x)\right) * x^m &= \frac{1}{f(x)} x^m \\ &= x^m - \frac{1}{2}x^{m+1} - \frac{1}{12}x^{m+2} \\ &\quad - \frac{1}{24}x^{m+3} - \frac{19}{720}x^{m+4} + \cdots.\end{aligned} \tag{35}$$

Since we may use Theorem 4 to conclude that $\pi(f,\mathrm{id})^{-1} = \pi(1/f,\mathrm{id})$, we find the identity

$$\begin{pmatrix} 1 & & & & & \\ \frac{1}{2} & 1 & & & & \\ \frac{1}{3} & \frac{1}{2} & 1 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \cdots & 1 & \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots \end{pmatrix}^{-1} = \begin{pmatrix} 1 & & & & & \\ -\frac{1}{2} & 1 & & & & \\ -\frac{1}{12} & -\frac{1}{2} & 1 & & & \\ -\frac{1}{24} & -\frac{1}{12} & -\frac{1}{2} & 1 & & \\ -\frac{19}{720} & -\frac{1}{24} & -\frac{1}{12} & -\frac{1}{2} & 1 & \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots \end{pmatrix}. \tag{36}$$

These matrices are elements of the Appell subgroup of $\mathbf{R}$.

2.4. Relation with Standard Notation. Standard references for Riordan matrices are Shapiro et al. [3] and Sprugnoli [4, 5]. The notation $\pi(f,g)$ employed above is not the typical one, so we explain the connection. Consider sequences $\{G_0, G_1, G_2, \ldots, G_n, \ldots\}$ and $\{F_1, F_2, F_3, \ldots, F_n, \ldots\}$ of complex numbers $k = \mathbb{C}$, where $G_0, F_1 \neq 0$. Upon associating generating functions $G(x) = G_0 + G_1 x + G_2 x^2 + \cdots$ and $F(x) = F_1 x + F_2 x^2 + F_3 x^3 + \cdots$ with these sequences, respectively, the standard notation for a Riordan matrix is that infinite-dimensional matrix given by

$$L = [G(x), F(x)] = \pi\left(G(x), \overline{F}(x)\right) = (l_{n,m})_{n,m\geq 1} \tag{37}$$

in terms of the compositional inverse $\overline{F}(x)$ of $F(x)$. Indeed, the entry $l_{n,m}$ in the nth row and mth column satisfies the relation

$$G(x)[F(x)]^m = \sum_{n=1}^{\infty} l_{n,m} x^n \quad \text{for } m = 1, 2, 3, \ldots \tag{38}$$

as formal power series in $\mathbb{C}[\![x]\!]$. Equivalently, a Riordan matrix L can be defined by a pair $(G(x), F(x))$ of generating functions.

Corollary 5 (fundamental theorem of the Riordan group [3, 5, 6]). *Continue notation as above.*

(i) *The product of Riordan matrices is again a Riordan matrix. Explicitly, their product satisfies the relation*

$$\begin{aligned} &[G_1(x), F_1(x)][G_2(x), F_2(x)] \\ &\quad = [G_1(x) G_2(F_1(x)), F_2(F_1(x))]. \end{aligned} \tag{39}$$

(ii) *For a generating function $T(x) = T_0 + T_1 x + T_2 x^2 + \cdots$ with $T_0 \neq 0$, one has the product*

$$[G(x), F(x)][T(x), x] = \left(\sum_{p=1}^{m} l_{n,p} T_{p-m}\right)_{n,m\geq 1}. \tag{40}$$

Proof. Statement (i) is shown in [3, Eq. 5] and [6, Proof of Thm. 2.1], but we give an alternate proof. Upon denoting $f_i(x) = G_i(x)$ and $g_i(x) = \overline{F}_i(x)$ for $i = 1$ and 2, we find the matrix product

$$\begin{aligned} &[G_1(x), F_1(x)][G_2(x), F_2(x)] \\ &\quad = \pi(f_1(x), g_1(x))\, \pi(f_2(x), g_2(x)) \\ &\quad = \pi(f_1(x) f_2(\overline{g_1}(x)), g_1(g_2(x))) \\ &\quad = [G_1(x) G_2(F_1(x)), F_2(F_1(x))] \end{aligned} \tag{41}$$

which follows directly from Theorem 4. Statement (ii) is also shown in [6], but it follows directly from Theorem 4 as well. □

3. Proof of Kenter's Result and Generalizations

3.1. Main Result. We now prove Theorem 1.

Proof of Theorem 1. With the three power series $a(x) = \sum_n a_n x^n$, $b(x) = \sum_n b_n x^n$, and $c(x) = \sum_n c_n x^n$ convergent in the interval $|x| < 1$, consider the power series

$$f(x) = b(x)^d c(x) = \sum_{n=0}^{\infty} f_n x^n \quad \text{where } |x| < 1. \tag{42}$$

As elements of the Appell subgroup of $\mathbf{R}$, we invoke Theorem 4 to see that we have the matrix product $\pi(f(x), x) = \pi(b(x), x)^d \pi(c(x), x)$. In particular, the first column is given by

$$\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_n \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} b_0 & & & & & \\ b_1 & b_0 & & & & \\ b_2 & b_1 & b_0 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ b_n & b_{n-1} & b_{n-2} & \cdots & b_0 & \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots \end{pmatrix}^d \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \\ \vdots \end{pmatrix}. \tag{43}$$

Hence, the matrix product

$$(a_0\ a_1\ a_2\ \cdots\ a_n\ \cdots) \begin{pmatrix} b_0 & & & & & \\ b_1 & b_0 & & & & \\ b_2 & b_1 & b_0 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ b_n & b_{n-1} & b_{n-2} & \cdots & b_0 & \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots \end{pmatrix}^d \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \\ \vdots \end{pmatrix} \tag{44}$$

is equal to the sum $\sum_n a_n f_n$. We wish to evaluate this sum using complex analysis.

By assumption, the power series $a(x)$, $b(x)$, and $c(x)$ are convergent in the interval $|x| < 1$. Hence, for each fixed real number r satisfying $0 < r < 1$, the functions $a(z)$ and $f(z)$ are *uniformly* convergent inside a closed disk $|z| \leq r$. Hence, we can interchange summation and integration to find the integral around the boundary to be equal to

$$\begin{aligned} &\frac{1}{2\pi i} \oint_{|z|=r} a(z)\, b(z^*)^d c(z^*)\, \frac{1}{z} dz \\ &= \frac{1}{2\pi i} \oint_{|z|=r} a(z)\, f(z^*)\, \frac{1}{z} dz \\ &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} a_{n_1} f_{n_2} r^{n_1+n_2} \cdot \frac{1}{2\pi} \int_0^{2\pi} e^{i(n_1-n_2)\theta} d\theta \\ &= \sum_{n=0}^{\infty} a_n f_n r^{2n}. \end{aligned} \tag{45}$$

Here, z^* is the complex conjugate of z. As $r \to 1$, the integral exists, so by Cauchy's Residue Theorem it must be equal to

$$\begin{aligned} &\operatorname*{Res}_{z=0} \left[\frac{a(z)\, b(z^{-1})^d c(z^{-1})}{z} \right] \\ &= \lim_{r\to 1} \left[\frac{1}{2\pi i} \oint_{|z|=1} a(z)\, b(z^{-1})^d c(z^{-1})\, \frac{1}{z} dz \right] \\ &= \lim_{r\to 1} \left[\sum_{n=0}^{\infty} a_n f_n r^{2n} \right] = \sum_{n=0}^{\infty} a_n f_n. \end{aligned} \tag{46}$$

The theorem follows upon equating this with (44). □

3.2. Applications. We explain how to use Theorem 1 in order to express Euler's number $e = 2.7182818284\cdots$ in terms of Riordan matrices.

Proof of Corollary 2. The coefficients of the matrices in (9) correspond to the three power series

$$\begin{aligned} a(x) &= \frac{1}{1 - \dfrac{x}{p}} = \sum_{n=0}^{\infty} \frac{x^n}{p^n}, \\ b(x) &= e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \\ c(x) &= \frac{1}{1 - x/q} = \sum_{n=0}^{\infty} \frac{x^n}{q^n} \\ &\qquad \text{where } |x| < 1. \end{aligned} \tag{47}$$

For a complex number z with $|z| < 1$, we have the identity

$$\begin{aligned} &\frac{a(z)\, b(z^{-1})^d c(z^{-1})}{z} \\ &= \left[\frac{1}{1 - z/p} \right] \left[e^{dz^{-1}} \right] \left[\frac{1}{1 - z^{-1}/q} \right] \\ &= \sum_{n=-\infty}^{\infty} \left[\sum_{n_1-n_2-n_3=n+1} \frac{d^{n_2}}{p^{n_1} n_2! q^{n_3}} \right] z^n. \end{aligned} \tag{48}$$

The residue corresponds to the coefficient of the z^{-1} term, so we consider the terms where $n = -1$:

$$\begin{aligned} &\operatorname*{Res}_{z=0} \left[\frac{a(z)\, b(z^{-1})^d c(z^{-1})}{z} \right] = \sum_{n_1=n_2+n_3} \frac{d^{n_2}}{p^{n_1} n_2! q^{n_3}} \\ &= \left[\sum_{n_2=0}^{\infty} \frac{1}{n_2!} \left(\frac{d}{p} \right)^{n_2} \right] \left[\sum_{n_3=0}^{\infty} \frac{1}{(pq)^{n_3}} \right] = e^{d/p} \frac{pq}{pq-1}. \end{aligned} \tag{49}$$

The corollary follows now from Theorem 1. □

Kenter's result is also an application of Theorem 1.

Corollary 6 (see [1]). *The Euler-Mascheroni constant*

$$\gamma = \lim_{n\to\infty}\left[\left(\sum_{m=1}^{n}\frac{1}{m}\right) - \ln n\right] = 0.5772156649\cdots \tag{50}$$

is equal to the matrix product

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} & \cdots \end{pmatrix} \begin{pmatrix} 1 & & & & & \\ \frac{1}{2} & 1 & & & & \\ \frac{1}{3} & \frac{1}{2} & 1 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \cdots & 1 & \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots \end{pmatrix}^{-1} \cdot \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ \vdots \\ \frac{1}{n+1} \\ \vdots \end{pmatrix}. \tag{51}$$

Proof. The coefficients of the matrices above correspond to the three power series

$$\begin{aligned} a(x) &= -\frac{\log(1-x)}{x} = \sum_{n=0}^{\infty}\frac{x^n}{n+1}, \\ b(x) &= -\frac{\log(1-x)}{x} = \sum_{n=0}^{\infty}\frac{x^n}{n+1}, \\ c(x) &= \frac{a(x)-1}{x} = \sum_{n=0}^{\infty}\frac{x^n}{n+2} \\ &\qquad \text{where } |x| < 1. \end{aligned} \tag{52}$$

We will choose the exponent $d = -1$. We will express the reciprocal as the power series

$$\begin{aligned} \frac{x}{\log(1-x)} &= -1 + \frac{1}{2}x + \frac{1}{12}x^2 + \frac{1}{24}x^3 + \frac{19}{720}x^4 \\ &\quad + \frac{3}{160}x^5 + \cdots = \sum_{n=0}^{\infty} L_n x^n \end{aligned} \tag{53}$$

which is also convergent in the interval $|x| < 1$. (Recall that the coefficients L_n are sometimes called the "logarithmic numbers" or the "Gregory coefficients.") For a complex number z with $|z| < 1$, we have the identity

$$\begin{aligned} &\frac{a(z)\,b\left(z^{-1}\right)^d c\left(z^{-1}\right)}{z} \\ &\quad = -\frac{\log(1-z)}{z} + \left[-\frac{\log(1-z)}{z}\right] \\ &\quad \cdot \left[\frac{z^{-1}}{\log\left(1-z^{-1}\right)}\right] \\ &\quad = \sum_{n=0}^{\infty}\frac{z^n}{n+1} + \sum_{n=-\infty}^{\infty}\left[\sum_{m=-n}^{\infty}\frac{L_m}{n+m+1}\right]z^n. \end{aligned} \tag{54}$$

The residue corresponds to the coefficient of the z^{-1} term, so we consider the terms where $n = -1$:

$$\begin{aligned} \operatorname*{Res}_{z=0}\left[\frac{a(z)\,b\left(z^{-1}\right)^d c\left(z^{-1}\right)}{z}\right] &= \sum_{m=1}^{\infty}\frac{L_m}{m} \\ &= \int_0^1\left[\sum_{m=1}^{\infty}L_m x^{m-1}\right]dx \\ &= \int_0^1\left[\frac{1}{x} + \frac{1}{\log(1-x)}\right]dx = \gamma. \end{aligned} \tag{55}$$

The corollary follows now from Theorem 1. □

We conclude by stating that Theorem 1 can also be used to show Riordan matrix representations for $\ln 2$ and $\pi^2/6$. Finding matrix representations of other constants, like $\sqrt{2}$, π, and the Golden Ratio ϕ, is of interest.

Disclosure

Both authors gave the recent annual Blackwell Lectures, organized by the National Association of Mathematicians (NAM) as part of the MAA MathFest. The first author gave his presentation during the summer of 2009, whereas the second gave his during the summer of 2010.

Competing Interests

The authors declare that they have no competing interests.

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