

Research Article

Entire Functions of Bounded L-Index: Its Zeros and Behavior of Partial Logarithmic Derivatives

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In this paper, we obtain new sufficient conditions of boundedness of L-index in joint variables for entire function in \mathbb{C}^n functions. They give an estimate of maximum modulus of an entire function by its minimum modulus on a skeleton in a polydisc and describe the behavior of all partial logarithmic derivatives and the distribution of zeros. In some sense, the obtained results are new for entire functions of bounded index and *l*-index in \mathbb{C} too. They generalize known results of Fricke, Sheremeta, and Kuzyk.

1. Introduction

In this paper, we find multidimensional sufficient conditions of boundedness of L-index in joint variables, which describe distribution of zeros and behavior of partial logarithmic derivatives. Recently, we published a paper [1] where some similar restrictions are established. Another approach was used by a slice function $F(z^0 + t\mathbf{b})$, where $z^0 \in \mathbb{C}^n$, $t \in \mathbb{C}$, **b** is a given direction in $\mathbb{C}^n \setminus \{0\}$, $F : \mathbb{C}^n \to \mathbb{C}$ is an entire function. It is a background for concept of function of bounded *L*-index in direction (see definition and properties in [2, 3]). We proved that if an entire function in \mathbb{C}^n function *F* is of bounded l_j -index in every direction $\mathbf{1}_j = (0, \ldots, 0, \underline{1}, 0, \ldots, 0)$, then *F* is of bounded L-index in

joint variables for $\mathbf{L} = (l_1, \ldots, l_n), \ l_j : \mathbb{C}^n \to \mathbb{R}_+$ (Theorem 6, [1]). It helped us to find restrictions by directional logarithmic derivatives and distribution zeros in every direction $\mathbf{1}_j, \ j \in \{1, \ldots, n\}$. We assumed that the logarithmic derivative in direction $\mathbf{1}_j$ is bounded by a function l_j outside some exceptional set, which contains all zeros of entire function F (see definition of $G_r^{\mathbf{b}}(F)$ below). Prof. Chyzhykov paid attention in conversation with authors that this exceptional set is too small because it does not contain neighborhoods of some

zeros of the function in \mathbb{C}^n . Thus, it leads to the following question: *is there sufficient conditions of boundedness of* **L***index in joint variables with larger exceptional sets?* We give a positive answer to this question (Theorem 10). Moreover, we obtain sufficient conditions of boundedness of **L**-index in joint variables by estimating the maximum modulus of an entire function on the skeleton in polydisc by minimum modulus (Theorem 7). Theorems 9 and 10 present restrictions by a measure of zero set of an entire function *F*, under which *F* has bounded **L**-index in joint variables. Nevertheless, we do not know whether the obtained conditions in Theorems 7–10 are necessary too in \mathbb{C}^n , $(n \ge 2)$. Note that these propositions are new even for entire functions of bounded index in joint variables, i. e. $\mathbf{L} = (1, ..., 1)$ (see definition and properties in [4–8]).

It is known [9] that for every entire function f with bounded multiplicities of zeros there exists a positive continuity on $[0; +\infty)$ function l(r) (r = |z|) such that f is of bounded *l*-index. This result can be easily generalized for entire functions in \mathbb{C}^n . Thus, the concept of bounded **L**-index in joint variables allows the study of growth properties of any entire functions with bounded multiplicities of zero points.

It should be noted that the concepts of bounded *L*-index in a direction and bounded **L**-index in joint variables have few

advantages in the comparison with traditional approaches to study properties of entire solutions of differential equations. In particular, if an entire solution has bounded index [10], then it immediately yields its growth estimates, a uniform distribution of its zeros, a certain regular behavior of the solution, and so forth. A full bibliography about application in theory of ordinary and partial differential equations is in [3, 11, 12].

The paper is devoted to two old problems in theory of entire and meromorphic functions. The first problem is the establishment of sharp estimates for the logarithmic derivatives of the functions in the unit disc outside some exceptional set. Chyzhykov et al. [13–16] considered various formulations of the problem. The obtained estimates were used to study properties of holomorphic solutions of differential equations. Instead, the authors assume that partial logarithmic derivative in every variable satisfies some inequalities (28) or (45).

Another interesting considered problem concerns zero sets of holomorphic function in \mathbb{C}^n . The different estimates of measure of zero set and its geometrical properties are investigated in [17–22]. We suppose that zero points of entire functions admit uniform distribution in some sense, that is, (29).

Below we use results from Ukrainian papers [23, 24], but they are also included in English monographs [3, 11].

2. Main Definitions and Notations

We need some standard notations. Let $\mathbb{R}_+ = [0, +\infty)$. Denote $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}_+^n, \mathbf{1} = (1, \dots, 1) \in \mathbb{R}_+^n, \mathbf{2} = (2, \dots, 2) \in \mathbb{R}_+^n,$ $\mathbf{1}_j = (0, \dots, 0, \underbrace{\mathbf{1}}_{j \text{th place}}, 0, \dots, 0) \in \mathbb{R}_+^n.$

For $R = (r_1, ..., r_n) \in \mathbb{R}^n_+$ and $K = (k_1, ..., k_n) \in \mathbb{Z}^n_+$ denote $||R|| = r_1 + \cdots + r_n$, $K! = k_1! \cdots k_n!$. For $= (a_1, ..., a_n) \in \mathbb{C}^n$, $b = (b_1, ..., b_n) \in \mathbb{C}^n$, $z = (z_1, ..., z_n) \in \mathbb{C}^n$, we will use formal notations without violating the existence of these expressions:

$$|a| = (|a_{1}|, |a_{2}|, ..., |a_{n}|),$$

$$A \pm B = (a_{1} \pm b_{1}, ..., a_{n} \pm b_{n}),$$

$$AB = (a_{1}b_{1}, ..., a_{n}b_{n}),$$

$$\frac{A}{B} = \left(\frac{a_{1}}{b_{1}}, ..., \frac{a_{n}}{b_{n}}\right),$$

$$A^{B} = a_{1}^{b_{1}}a_{2}^{b_{2}} \cdot ... \cdot a_{n}^{b_{n}},$$

$$dz = dz_{1}dz_{2} \cdots dz_{n}.$$
(1)

If $a, b \in \mathbb{R}^n$ the notation a < b means that $a_j < b_j$ (j = 1, ..., n); similarly, the relation $a \le b$ is defined.

The polydisc $\{z \in \mathbb{C}^n : |z_j - z_j^0| < r_j, j = 1, ..., n\}$ is denoted by $\mathbb{D}^n(z^0, R)$, its skeleton $\{z \in \mathbb{C}^n : |z_j - z_j^0| = r_j, j = 1, ..., n\}$ is denoted by $\mathbb{T}^n(z^0, R)$, and the closed polydisc $\{z \in \mathbb{C}^n : |z_j - z_j^0| \le r_j, j = 1, ..., n\}$ is denoted by $\mathbb{D}^n[z^0, R]$.

For $K = (k_1, ..., k_n) \in \mathbb{Z}_+^n$ and partial derivatives of entire function $F(z) = F(z_1, ..., z_n)$ we will use the notation

$$F^{(K)}(z) = \frac{\partial^{\|K\|}F}{\partial z^{K}} = \frac{\partial^{k_1 + \dots + k_n} f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}.$$
 (2)

Let $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, where $l_j(z)$ are positive continuous functions of $z \in \mathbb{C}^n$, $j \in \{1, 2, \dots, n\}$. An entire function, F(z), $z \in \mathbb{C}^n$, is called a *function of bounded* L*index in joint variables* [1] if there exists a number $m \in \mathbb{Z}_+$ such that for all $z \in \mathbb{C}^n$ and $J = (j_1, j_2, \dots, j_n) \in \mathbb{Z}_+^n$

$$\frac{\left|F^{(J)}(z)\right|}{J!\mathbf{L}^{J}(z)} \le \max\left\{\frac{\left|F^{(K)}(z)\right|}{K!\mathbf{L}^{K}(z)} : K \in \mathbb{Z}_{+}^{n}, \ \|K\| \le m\right\}.$$
 (3)

If $l_j = l_j(|z_j|)$ then we obtain a concept of entire functions of bounded L-index in a sense of definition given in [24]. If $l_j(z_j) \equiv 1, j \in \{1, 2, ..., n\}$, then the entire function is called a *function of bounded index in joint variables* [4–8, 25].

The least integer *m* for which inequality (3) holds is called **L**-*index in joint variables of the function* F and is denoted by $N(F, \mathbf{L})$.

For $R \in \mathbb{R}^n_+$, $j \in \{1, ..., n\}$ and $L(z) = (l_1(z), ..., l_n(z))$ we define

$$\lambda_{1,j}(R) = \inf_{z^0 \in \mathbb{C}^n} \inf \left\{ \frac{l_j(z)}{l_j(z^0)} : z \in \mathbb{D}^n \left[z^0, \frac{R}{\mathbf{L}(z^0)} \right] \right\},$$

$$\lambda_{2,j}(R) = \sup_{z^0 \in \mathbb{C}^n} \sup \left\{ \frac{l_j(z)}{l_j(z^0)} : z \in \mathbb{D}^n \left[z^0, \frac{R}{\mathbf{L}(z^0)} \right] \right\}, \quad (4)$$

$$\Lambda_1(R) = \left(\lambda_{1,j}(R), \dots, \lambda_{1,n}(R) \right),$$

$$\Lambda_2(R) = \left(\lambda_{2,1}(R), \dots, \lambda_{2,n}(R) \right).$$

By Q^n we denote a class of functions $\mathbf{L}(z)$ which for every $R \in \mathbb{R}^n_+$ and $j \in \{1, ..., n\}$ satisfy the condition

$$0 < \lambda_{1,j}(R) \le \lambda_{2,j}(R) < +\infty.$$
(5)

If n = 1 then $Q \equiv Q^1$.

Let $\widetilde{\mathbf{L}}(z) = (\widetilde{l}_1(z), \dots, \widetilde{l}_n(z))$. A notation $\mathbf{L} \simeq \widetilde{\mathbf{L}}$ means that there exist $\Theta_1 = (\theta_{1,j}, \dots, \theta_{1,n}) \in \mathbb{R}^n_+, \Theta_2 = (\theta_{2,j}, \dots, \theta_{2,n}) \in \mathbb{R}^n_+$ such that $\forall z \in \mathbb{C}^n \ \theta_{1,j} \widetilde{l}_j(z) \le l_j(z) \le \theta_{2,j} \widetilde{l}_j(z)$.

3. Auxiliary Propositions

We need the following theorems.

Theorem 1 ([11, p. 158, Th. 4.2], see also [23]). Let $\mathbf{L} \in Q^n$ and $\mathbf{L} \approx \tilde{\mathbf{L}}$. An entire function $F : \mathbb{C}^n \to \mathbb{C}$ has bounded $\tilde{\mathbf{L}}$ -index in joint variables if and only if F has bounded \mathbf{L} -index in joint variables.

Theorem 2 (see [1]). Let $\mathbf{L} \in Q^n$. An entire function F is of bounded \mathbf{L} -index in joint variables if and only if, for any R^l ,

R'', 0 < R' < R'', there exists a number $p_1 = p_1(R', R'') \ge 1$ such that for every $z^0 \in \mathbb{C}^n$ inequality

$$\max\left\{ |F(z)| : z \in \mathbb{T}^{n}\left(z^{0}, \frac{R''}{\mathbf{L}(z^{0})}\right) \right\}$$

$$\leq p_{1} \max\left\{ |F(z)| : z \in \mathbb{T}^{n}\left(z^{0}, \frac{R'}{\mathbf{L}(z^{0})}\right) \right\}.$$
(6)

holds.

Remark 3. It was also proved that the condition "for any R', R'', $\mathbf{0} < R' < R''$, there exists a number $p_1 = p_1(R', R'') \ge 1$ " in Theorem 2 can be replaced by the condition "there exist R', R'', $\mathbf{0} < R' < \mathbf{1} < R''$, and $p_1 = p_1(R', R'') \ge 1$ ". It is Theorem 5 in [1].

Now we relax the restriction R' < 1 < R'' in sufficient conditions.

Theorem 4. Let $\mathbf{L} \in Q^n$, $F : \mathbb{C} \to \mathbb{C}^n$ be an entire function. If there exist R', R'', $\mathbf{0} < R' < R''$, and $p_1 = p_1(R', R'') \ge 1$ such that for every $z^0 \in \mathbb{C}^n$ inequality (6) holds; then the function F has bounded **L**-index in joint variables.

Proof. From (6) with $\mathbf{0} < R' < R''$ it follows that

$$\max\left\{ |F(z)| : z \in \mathbb{T}^{n}\left(z^{0}, \frac{2R''}{R' + R''} \frac{R' + R''}{2L(z^{0})}\right) \right\}$$

$$\leq P_{1} \max\left\{ |F(z)| : z \in \mathbb{T}^{n}\left(z^{0}, \frac{2R'}{R' + R''} \frac{R' + R''}{2L(z^{0})}\right) \right\}.$$
(7)

Denoting $\tilde{\mathbf{L}}(z) = 2\mathbf{L}(z)/(R' + R'')$, we obtain

$$\max\left\{|F(z)|: z \in \mathbb{T}^{n}\left(z^{0}, \frac{2R''}{(R'+R'')\widetilde{\mathbf{L}}(z^{0})}\right)\right\}$$

$$\leq P_{1} \max\left\{|F(z)|: z \in \mathbb{T}^{n}\left(z^{0}, \frac{2R'}{(R'+R'')\widetilde{\mathbf{L}}(z^{0})}\right)\right\},$$
(8)

where $\mathbf{0} < 2R'/(R' + R'') < \mathbf{1} < 2R''/(R' + R'')$. In view of Remark 3, *F* has bounded $\tilde{\mathbf{L}}$ -index in joint variables. By Theorem 1, the function *F* is bounded \mathbf{L} -index in joint variables.

Note that Theorem 4 is new even if $L(z) \equiv 1$.

Lemma 5. If $L : \mathbb{C}^n \to \mathbb{R}_+$ is a continuous function such that $(\forall R \in \mathbb{R}^n_+) \Lambda_2(R) < \infty$ then $(\forall R \in \mathbb{R}^n_+) \Lambda_1(R) \ge 1/\Lambda_2(R\Lambda_2(R)) > 0.$

Proof. Let $(\forall R \in \mathbb{R}^n_+) \Lambda_2(R) < \infty$ i.e. $\forall j \in \{1, ..., n\} \lambda_{2,j}(R) < +\infty$. Hence, we have $l_j(z) \leq \lambda_{2,j}(R) l_j(z^0)$ for

 $z \in D^n(z^0, R/L(z^0))$. This means that $|z_j - z_j^0| \le r_j/l_j(z^0) \le r_j\lambda_{2,j}(R)/l_j(z)$. Using definition of $\lambda_{1,j}(R)$, we deduce

$$\inf \left\{ \frac{l_{j}(z)}{l_{j}(z^{0})} : z \in \mathbb{D}^{n} \left(z^{0}, \frac{R}{L(z^{0})} \right) \right\}$$

$$= \frac{1}{\sup \left\{ l_{j}(z^{0}) / l_{j}(z) : z \in \mathbb{D}^{n}(z^{0}, R/L(z^{0})) \right\}}$$

$$\geq \frac{1}{\sup \left\{ l_{j}(z^{0}) / l_{j}(z) : \left| z_{j}^{0} - z_{j} \right| \leq r_{j}\lambda_{2,j}(R) / l_{j}(z), j \in \{1, ..., n\} \right\}}$$

$$\geq \frac{1}{\lambda_{2,j}(R\Lambda_{2}(R))}.$$
(9)
Ihus, $\lambda_{1,j}(R) \geq 1/\lambda_{2,j}(R\Lambda_{2}(R)).$

Remark 6. By Lemma 5 the left inequality in (5) is excessive because the condition $\lambda_{2,j}(R) < +\infty$ implies $\lambda_{1,j}(R) > 0$. But in our considerations we will use so $\Lambda_1(R)$ as $\Lambda_2(R)$. It is convenient.

4. Estimate Maximum Modulus on a Skeleton in Polydisc

Let Z_F be a zero set of entire function F. We denote

$$G_{R}(F) = \bigcup_{z^{0} \in Z_{F}} \left\{ z \in \mathbb{C}^{n} : \left| z_{j} - z_{j}^{0} \right| < \frac{r_{j}}{l_{j}(z^{0})} \quad \forall j$$

$$\in \{1, 2, \dots, n\} \right\} = \bigcup_{z^{0} \in Z_{F}} \mathbb{D}^{n} \left(z^{0}, \frac{R}{\mathbf{L}(z^{0})} \right).$$
(10)

Theorem 7. Let $\mathbf{L} \in Q^n$, F be an entire in \mathbb{C}^n function. If $\exists R > \mathbf{0} \ \exists p_2 \ge 1 \ \exists \Theta \in \mathbb{R}^n_+, \mathbf{0} < \Theta < R, \ \exists R' > \mathbf{0}, \ (R' = \mathbf{0} \ for \ Z_F = \emptyset)$ such that $\forall z^0 \in \mathbb{C}^n \ \exists R^0 = R^0(z^0) \in \mathbb{R}^n_+, \ \Theta \le R^0 \le R$, for which

$$\max \left\{ \mathbb{T}^{n}\left(z^{0}, \frac{R^{0}}{\mathbf{L}(z^{0})}\right) \cap G_{R'}(F) \right\}$$

$$< \left(\frac{2\pi}{3}\right)^{n} \prod_{j=1}^{n} \frac{\theta_{j}}{\lambda_{2,j}\left(2\left(R+1\right)\right) l_{j}\left(z^{0}\right)},$$

$$\max \left\{ |F\left(z\right)| : z \in \mathbb{T}^{n}\left(z^{0}, \frac{R^{0}}{\mathbf{L}(z^{0})}\right) \right\}$$

$$\leq p_{2} \min \left\{ |F\left(z\right)| : z \in \mathbb{T}^{n}\left(z^{0}, \frac{R^{0}}{\mathbf{L}(z^{0})}\right) \setminus G_{R'}(F) \right\}$$
(12)

then the function F has bounded L-index in joint variables (meas is the Lebesgue measure on the skeleton in the polydisc).

Proof. By Theorem 4, we will show that $\exists p_1 > 0 \ \forall z^0 \in \mathbb{C}^n$

$$\max\left\{|F(z)|: z \in \mathbb{T}^{n}\left(z^{0}, \frac{R+1}{\mathbf{L}(z^{0})}\right)\right\}$$

$$\leq p_{1} \max\left\{|F(z)|: z \in \mathbb{T}^{n}\left(z^{0}, \frac{R}{\mathbf{L}(z^{0})}\right)\right\}.$$
(13)

Denote $l_j^* = \max\{l_j(z) : z \in \mathbb{D}^n[z^0, 2(R+1)/L(z^0)]\}, \rho_{j,0} =$ $r_{j}/l_{j}(z^{0}), \rho_{j,k} = \rho_{j,0} + (k \cdot \theta_{j})/l_{j}^{*}, k \in \mathbb{N}, j \in \{1, \dots, n\}.$ The following estimate holds

$$\frac{\theta_{j}}{l_{j}^{*}} < \frac{r_{j}}{l_{j}^{*}} \le \frac{r_{j}}{l_{j}(z^{0})} < \frac{2r_{j}+2}{l_{j}(z^{0})} - \frac{r_{j}+1}{l_{j}(z^{0})}.$$
(14)

Hence, there exists $S^* = (s_1^*, \dots, s_n^*) \in \mathbb{N}$ independent of z^0 such that

$$\rho_{j,m_{j}-1} < \frac{r_{j}+1}{l_{j}(z^{0})} < \rho_{j,m_{j}} \le \frac{2r_{j}+2}{l_{j}(z^{0})}$$
(15)

for some $m_i = m_i(z^0) \le s_i^*$ because $\mathbf{L} \in Q^n$. Indeed,

$$\frac{\left(\left(2r_{j}+2\right)/l_{j}\left(z^{0}\right)-\rho_{j,0}\right)}{\theta_{j}/l_{j}^{*}} = \left(2r_{j}+2-r_{j}\right)\frac{l_{j}^{*}}{\theta_{j}l_{j}(z^{0})} \\
= \frac{r_{j}+2}{\theta_{j}}\max\left\{\frac{l_{j}(z)}{l_{j}(z^{0})}: z \in \mathbb{D}^{n}\left[z^{0}, \frac{2\left(R+1\right)}{L\left(z^{0}\right)}\right]\right\} \quad (16) \\
\leq \frac{r_{j}+2}{\theta_{j}}\lambda_{2,j}\left(2\left(R+1\right)\right).$$

Thus, $s_j^* = [((r_j+2)/\theta_j)\lambda_{2,j}(2(R+1))]$, where [x] is the integer part of $x \in \mathbb{R}$.

Let $M_0 = (m_1, \ldots, m_n)$ and τ_K^{**} be such a point in \mathbb{C}^n that

$$\left|F\left(\tau_{K}^{**}\right)\right| = \max\left\{\left|F\left(z\right)\right| : z \in \mathbb{T}^{n}\left(z^{0}, \mathcal{R}_{K}\right)\right\}, \qquad (17)$$

where $K = (k_1, \ldots, k_n)$, $\mathcal{R}_K = (\rho_{1,k_1}, \ldots, \rho_{n,k_n})$ and $\tau_{j,K}^*$ be the intersection point in \mathbb{C} of the segment $[z_i^0, \tau_{i,K}^{**}]$ with $|z_i - z_i^0| =$ ρ_{j,k_i-1} . We construct a sequence of polydisc $\mathbb{D}^n(z^0, \mathscr{R}_K)$ with $\begin{aligned} &K \leq M_0, \, \mathcal{R}_0 = R/\mathbf{L}(z^0) = (\rho_{1,0}, \dots, \rho_{n,0}) \text{ and } \Theta/\mathbf{L}(z^0) = \\ &(\theta_1/l_1^*, \dots, \theta_n/l_n^*) \text{ (see Figures 1 and 2).} \\ &\text{Denote } \alpha_K^{(j)} = (\tau_{1,K}^{**}, \dots, \tau_{j-1,K}^{**}, \tau_{j,K}^*, \tau_{j+1,K}^{**}, \dots, \tau_{n,K}^{**}). \\ &\text{Hence, for every } r_j > \theta_j \text{ and } K \leq S^* : |\tau_{j,K}^* - \tau_{j,K}^{**}| = \theta_j/l_j^* \leq Q_j^* + Q_j$

 $r_j/l_j(\alpha_K^{(j)})$. Thus, for some $R^0 = R^0(\alpha_K^{(j)}) \in \mathbb{R}^n_+, \Theta \le R^0 \le R$, we deduce

$$\begin{aligned} \left| F\left(\tau_{K}^{**}\right) \right| &\leq \max\left\{ \left| F\left(z\right) \right| : z \in \mathbb{T}^{n} \left(\alpha_{K}^{(j)}, \frac{R^{0}}{\mathbf{L}\left(\alpha_{K}^{(j)}\right)} \right) \right. \\ &\leq p_{2} \min\left\{ \left| F\left(z\right) \right| : z \\ &\in \mathbb{T}^{n} \left(\alpha_{K}^{(j)}, \frac{R^{0}}{\mathbf{L}\left(\alpha_{K}^{(j)}\right)} \right) \setminus G_{R'}\left(F\right) \right\} &\leq p_{2} \end{aligned}$$





FIGURE 2

$$\cdot \min\left\{ |F(z)| : z \\ \in \mathbb{T}^n \left(\alpha_K^{(j)}, \frac{R^0}{\mathbf{L}\left(\alpha_K^{(j)}\right)} \right) \setminus G_{R'}(F), z \\ \in \mathbb{D}^n \left[z^0, \mathscr{R}_{K-1_j} \right] \right\} \le p_2 \max\left\{ |F(z)| : z \\ \in \mathbb{T}^n \left(z^0, \mathscr{R}_{K-1_j} \right) \right\}.$$

$$(18)$$

To deduce (18), we implicitly used that

$$\left(\mathbb{T}^{n}\left(\alpha_{K}^{(j)},\frac{R^{0}}{\mathbf{L}\left(\alpha_{K}^{(j)}\right)}\right)\setminus G_{R'}\left(F\right)\right)\cap\mathbb{D}^{n}\left[z^{0},\mathscr{R}_{K-\mathbf{1}_{j}}\right]$$

$$\neq\emptyset.$$
(19)

Condition (11) provides (19). Indeed, we will find a lower estimate of measure of the set $\mathbb{T}^n(\alpha_K^{(j)}, R^0/\mathbf{L}(\alpha_K^{(j)})) \cap$ $\mathbb{D}^{n}[z^{0},\mathscr{R}_{K-1_{j}}]$ and will show that the measure is not lesser than a left part of inequality (11).



FIGURE 3: With $r' = r_m^0 / l_m(\alpha_K^{(j)})$.



FIGURE 4: With $r' = r_i^0 / l_i(\alpha_K^{(j)})$.

The set $\mathbb{T}^n(\alpha_K^{(j)}, \mathbb{R}^0/\mathbf{L}(\alpha_K^{(j)})) \cap \mathbb{D}^n[z^0, \mathcal{R}_{K-1_j}]$ is a Cartesian product of the following arcs on circles: for every $m \in \{1, ..., n\}, m \neq j$ (see Figure 3)

$$\left\{ z_m \in \mathbb{C} : \left| z_m - \tau_{m,K}^{**} \right| = \frac{r_m^0}{l_m\left(\alpha_K^{(j)}\right)} \right\}$$

$$\cap \left\{ z_m \in \mathbb{C} : \left| z_m - z_m^0 \right| \le \rho_{m,k_m} \right\}$$

$$(20)$$

and for m = j (see Figure 4)

$$\left\{ z_{j} \in \mathbb{C} : \left| z_{j} - \tau_{j,K}^{*} \right| = \frac{r_{j}^{0}}{l_{j} \left(\alpha_{K}^{(j)} \right)} \right\}$$

$$\cap \left\{ z_{j} \in \mathbb{C} : \left| z_{j} - z_{j}^{0} \right| \le \rho_{j,k_{j}-1} \right\}.$$
(21)

It is easy to prove that the length of arc equals

$$\frac{2r_m^0}{l_m\left(\alpha_K^{(j)}\right)} \cdot \arccos\frac{r_m^0}{2l_m\left(\alpha_K^{(j)}\right)\rho_{m,k_m}} \quad \text{for } m \neq j, \qquad (22)$$

$$\frac{2r_j^0}{l_j\left(\alpha_K^{(j)}\right)} \cdot \arccos \frac{r_j^0}{2l_j\left(\alpha_K^{(j)}\right)\rho_{j,k_j-1}} \quad \text{for } m = j.$$
(23)

But for $m \neq j r_m^0/l_m(\alpha_K^{(j)}) \leq \rho_{m,k_m}$ and $r_j^0/l_j(\alpha_K^{(j)}) \leq \rho_{j,k_i-1}$ the argument in arccosine from (23) and (22) does not

exceed 1/2. This means that the length of arc is not lesser than

$$\frac{2r_m^0}{l_m\left(\alpha_K^{(j)}\right)} \arccos \frac{1}{2} \ge \frac{2\theta_m \pi}{3l_m\left(z^0\right)\lambda_{2,m}\left(2\left(R+1\right)\right)}$$
for every $m \in \{1, 2, \dots, n\}$,
$$(24)$$

because $L \in Q^n$. Accordingly, the measure of the set

$$\mathbb{T}^{n}\left(\alpha_{K}^{(j)}, \frac{R^{0}}{\mathbf{L}\left(\alpha_{K}^{(j)}\right)}\right) \cap \mathbb{D}^{n}\left[z^{0}, \mathscr{R}_{K-1_{j}}\right]$$
(25)

on the skeleton of polydisc is always not lesser than $\prod_{m=1}^{n} (2\theta_m \pi/3l_m(z^0)\lambda_{2,m}(2(R + 1))))$. Assuming a strict inequality in (11), we deduce that (19) is valid.

Applying (18) m_i th times in every variable z_i , we obtain

$$\max\left\{|F(z)|: z \in \mathbb{T}^{n}\left(z^{0}, \frac{R+1}{\mathbf{L}(z^{0})}\right)\right\}$$

$$\leq \max\left\{|F(z)|: z \in \mathbb{T}^{n}\left(z^{0}, \mathcal{R}_{M_{0}}\right)\right\} \leq p_{2}$$

$$\cdot \max\left\{|F(z)|: z \in \mathbb{T}^{n}\left(z^{0}, \mathcal{R}_{M_{0}-1_{n}}\right)\right\} \leq p_{2}^{m_{n}}$$

$$\cdot \max\left\{|F(z)|: z \in \mathbb{T}^{n}\left(z^{0}, \mathcal{R}_{M_{0}-m_{n}1_{n}}\right)\right\} \leq \cdots$$

$$\leq p_{2}^{m_{n}+1}$$

$$\cdot \max\left\{|F(z)|: z \in \mathbb{T}^{n}\left(z^{0}, \mathcal{R}_{M_{0}-m_{n}1_{n}-1_{n-1}}\right)\right\}$$

$$\leq p_{2}^{m_{n}+m_{n-1}}$$

$$\cdot \max\left\{|F(z)|: z \in \mathbb{T}^{n}\left(z^{0}, \mathcal{R}_{M_{0}-m_{n}1_{n}-m_{n-1}1_{n-1}}\right)\right\}$$

$$\leq \cdots \leq p_{2}^{\|M_{0}\|} \max\left\{|F(z)|: z \in \mathbb{T}^{n}\left(z^{0}, \mathcal{R}_{0}\right)\right\}$$

$$\leq p_{2}^{\|S^{*}\|} \max\left\{|F(z)|: z \in \mathbb{T}^{n}\left(z^{0}, \frac{R}{\mathbf{L}(z^{0})}\right)\right\}.$$

By Theorem 2 the function F has bounded L-index in joint variables.

Let us denote $c(z', r) = \{z \in \mathbb{C} : |z - z'| = r/l(z')\}$. For n = 1 Theorem 7 implies the following corollary.

Corollary 8. Let $l \in Q$, f be an entire function. If $\exists r > 0$, $\exists r' \geq 0$, $\exists p_2 \geq 1 \exists \theta \in (0, r)$, such that $\forall z^0 \in \mathbb{C} \exists r^0 = r^0(z^0) \in [\theta; r]$, and $\operatorname{meas}\{c(z^0, r^0) \cap G_{R'}(F)\} < 2\pi\theta/3l(z^0)\lambda_2(2r+2)$ and

$$\max\left\{\left|f\left(z\right)\right|:z\in c\left(z^{0},r^{0}\right)\right\}$$

$$\leq p_{2}\min\left\{\left|f\left(z\right)\right|:z\in c\left(z^{0},r^{0}\right)\setminus G_{r'}\left(f\right)\right\}$$
(27)

then the function f has bounded l-index (here meas means the Lebesgue measure on the circle).

In a some sense, this corollary is new even for an entire function of one variable because the circle $c(z^0, r^0)$ can contain zeros of the function f. Meanwhile, in corresponding theorem from [26, 27] the circle $c(z^0, r^0)$ is chosen such that $f(z) \neq 0$ for all $z \in c(z^0, r^0)$.

5. Behavior of Partial Logarithmic Derivatives

Denote $\mathscr{J} = \{(j_1, ..., j_n) : j_i \in \{0, 1\}, i \in \{1, ..., n\}\} \setminus \mathbf{0}.$

Theorem 9. Let $\mathbf{L} \in Q^n$. If an entire function F satisfies the following conditions

(1) for every R > 0 there exists $p_1 = p_1(R) > 0$ such that for all $z \in \mathbb{C}^n \setminus G_R(F)$ and for all $J \in \mathcal{J}$

$$\left| \left(\ln F(z) \right)^{(J)} \right| \le p_1 \mathbf{L}^J(z),$$
 (28)

where $\ln F(z)$ is the principal value of logarithm.

- (2) for every R > 0 and $R' \ge 0$ exists $p_2 = p_2(R, R') \ge 1$ that for all $z^0 \in \mathbb{C}^n$ such that $T^n(z^0, R/\mathbb{L}(z^0)) \setminus G_{R'}(F) = \bigcup_i C_i \ne \emptyset$, where the sets C_i are connected disjoint sets, and either (a) $\max_{i} \min_{z \in C_i} |F(z)| \le p_2 \min_i \min_{z \in C_i} |F(z)|$, or (b) $\max_{i} \max_{z \in C_i} |F(z)| \le p_2 \min_i \max_{z \in C_i} |F(z)|$, or (c) $|F(z^*)| = \max_i \max_{z \in C_i} |F(z)|$, $|F(z^{**})| = \min_i \min_{z \in C_i} |F(z)|$, and z^*, z^{**} belong to the same set C_{i_0}
- (3) for every R > 0 there exists $n^*(R) > 0$ such that for all $z \in \mathbb{C}^n$

meas
$$\left\{Z_F \cap \mathbb{D}^n\left[z, \frac{R}{\mathbf{L}(z)}\right]\right\} \le n^*(R)$$
. (29)

then F has bounded L-index in joint variables (here meas is (2n - 2)-dimensional of the Lebesgue measure).

Proof. Let $z^0 \in \mathbb{C}^n$ be arbitrarily chosen point. In view of Theorem 7 we need to prove that

$$\max\left\{\mathbb{T}^{n}\left(z^{0}, \frac{R^{0}}{\mathbf{L}(z^{0})}\right) \cap G_{R'}(F)\right\}$$

$$< \left(\frac{2\pi}{3}\right)^{n} \prod_{j=1}^{n} \frac{\theta_{j}}{\lambda_{2,j}\left(2\left(R+1\right)\right) l_{j}\left(z^{0}\right)}$$
(30)
for some $R^{0} = R^{0}\left(z^{0}\right)$.

Let R > 0 be arbitrary radius. We choose Θ , $R' \in \mathbb{R}^n_+$ such that $\theta_j < 2r_j/(2 + 3\lambda_{2,j}(2(R + 1)))$,

$$r'_{j} < \min \left\{ \theta_{j}, \\ \sqrt{\frac{2\lambda_{1,j}^{2} \left(R + R\Lambda_{2} \left(R \right) / \Lambda_{1} \left(R \right) \right) \theta_{j} \left(r_{j} - \theta_{j} \right)}{3 \left(n^{*} \left(R + R\Lambda_{2} \left(R \right) / \Lambda_{1} \left(R \right) \right) \right)^{1/n} \lambda_{2,j} \left(2 \left(R + 1 \right) \right)}} \right\},$$
(31)
$$j \in \{1, \dots, n\}.$$

Let $dS = ds_1 \cdot \ldots \cdot ds_n$, $S = (s_1, \ldots, s_n)$, ω_z be a volume measure in \mathbb{R}^{2n} . Clearly, (see [28, p. 75-76])

$$\int_{\mathbb{D}^{n}(z^{0},R)} u(z) d\omega_{z} = \int_{0}^{r_{1}} \cdots \int_{0}^{r_{n}} s_{1} \cdots s_{n}$$

$$\cdot \left(\int_{0}^{2\pi} \cdots \int_{0}^{2\pi} u(z^{0} + Se^{i\Theta}) d\theta_{1} \cdots d\theta_{n}\right) ds_{1} \cdots ds_{n} \qquad (32)$$

$$= \int_{0}^{R} \left(\int_{0}^{2\pi} \cdots \int_{0}^{2\pi} u(z^{0} + Se^{i\Theta}) d(s_{1}\theta_{1}) \cdots d(s_{n}\theta_{n})\right) dS,$$

where u is plurisubharmonic function. Hence,

$$\int_{0}^{R/\mathbf{L}(z^{0})} \operatorname{meas}\left\{\mathbb{T}^{n}\left(z^{0},S\right) \cap G_{R'}\left(F\right)\right\} dS$$

$$= \operatorname{meas}\left\{\mathbb{D}^{n}\left[z^{0},\frac{R}{\mathbf{L}\left(z^{0}\right)}\right] \cap G_{R'}\left(F\right)\right\}.$$
(33)

Obviously, there can exist points $z' \in Z_F \setminus \mathbb{D}^n[z^0, R/\mathbf{L}(z^0)]$ such that

$$\mathbb{D}^{n}\left[z^{0}, \frac{R}{\mathbf{L}(z^{0})}\right] \cap \mathbb{D}^{n}\left[z', \frac{R'}{\mathbf{L}(z')}\right] \neq \emptyset.$$
(34)

Let z''_j be the intersection point of the segment $[z_j^0, z'_j]$ and the circle $|z_j - z_j^0| = r_j/l_j(z^0)$, $j \in \{1, ..., n\}$. Then $|z''_j - z'_j| \le r'_j/l_j(z')$ and $z'' \in \mathbb{T}^n(z^0, R/L(z^0))$. Using $L \in Q^n$, we estimate maximum distance between z_j^0 and z'_j :

$$\begin{split} l_{j}(z') &\geq \frac{l_{j}(z')}{l_{j}(z'')} \cdot \frac{l_{j}(z'')}{l_{j}(z^{0})} \cdot l_{j}(z^{0}) \\ &\geq \frac{\lambda_{1,j}(R)}{\lambda_{2,j}(R')} \cdot l_{j}(z^{0}) \geq \frac{\lambda_{1,j}(R)}{\lambda_{2,j}(R')} l_{j}(z^{0}), \\ &|z_{j}^{0} - z_{j}'| \leq |z^{0} - z_{j}''| + |z_{j}'' - z_{j}'| \leq \frac{r_{j}}{l_{j}(z^{0})} + \frac{r_{j}'}{l_{j}(z'_{j})} \\ &\leq \frac{r_{j}}{l_{j}(z^{0})} \left(1 + \frac{\lambda_{2,j}(R)}{\lambda_{1,j}(R')}\right). \end{split}$$
(35)

Denote $R'' = R + R\Lambda_2(R)/\Lambda_1(R)$. Let V_{2n-2} be a (2n - 2)-dimensional volume, $\chi_F(z)$ a characteristic function of zero set of the function *F*. Now we replace the measure in (33) by integrating on zero set in polydisc $\mathbb{D}^n[z^0, R''/\mathbf{L}(z^0)]$:

$$\begin{split} &\int_{0}^{R/L(z^{0})} \max\left\{\mathbb{T}^{n}\left(z^{0},S\right) \cap G_{R'}\left(F\right)\right\} dS \leq \int_{Z_{F} \cap \mathbb{D}^{n}[z^{0},R''/L(z^{0})]} \chi_{F}\left(z\right) \pi^{n} \prod_{j=1}^{n} \frac{\left(r_{j}'\right)^{2}}{l_{j}^{2}\left(z\right)} dV_{2n-2} \\ &\leq \pi^{n} \prod_{j=1}^{n} \frac{\left(r_{j}'\right)^{2}}{\lambda_{1,j}^{2}\left(R''\right) l_{j}^{2}\left(z^{0}\right)} \int_{Z_{F} \cap \mathbb{D}^{n}[z^{0},R''/L(z^{0})]} \chi_{F}\left(z\right) dV_{2n-2} \leq n^{*}\left(R''\right) \pi^{n} \prod_{j=1}^{n} \frac{\left(r_{j}'\right)^{2}}{\lambda_{1,j}^{2}\left(R''\right) l_{j}^{2}\left(z^{0}\right)} \\ &< n^{*}\left(R''\right) \pi^{n} \prod_{j=1}^{n} \frac{2\lambda_{1,j}^{2}\left(R + R\Lambda_{2}\left(R\right) / \Lambda_{1}\left(R\right)\right) \theta_{j}\left(r_{j} - \theta_{j}\right)}{3\left(n^{*}\left(R + R\Lambda_{2}\left(R\right) / \Lambda_{1}\left(R\right)\right)\right)^{1/n} \lambda_{2,j}\left(2\left(R + 1\right)\right) \lambda_{1,j}^{2}\left(R''\right) l_{j}^{2}\left(z^{0}\right)} \end{aligned}$$
(36)
$$&= \left(\frac{2\pi}{3}\right)^{n} \prod_{j=1}^{n} \frac{\theta_{j}\left(r_{j} - \theta_{j}\right)}{\lambda_{2,j}\left(2\left(R + 1\right)\right) l_{j}^{2}\left(z^{0}\right)}.$$

Besides, we have that

$$\int_{0}^{\Theta/\mathbf{L}(z^{0})} \operatorname{meas}\left\{\mathbb{T}^{n}\left(z^{0},S\right) \cap G_{R'}\left(F\right)\right\} dS$$
$$= \operatorname{meas}\left\{\mathbb{D}^{n}\left[z^{0},\frac{\Theta}{\mathbf{L}\left(z^{0}\right)}\right] \cap G_{R'}\left(F\right)\right\} \qquad (37)$$
$$\leq \pi^{n} \prod_{j=1}^{n} \frac{\theta_{j}^{2}}{l_{j}^{2}\left(z^{0}\right)}.$$

Hence, the following difference is positive

$$\left(\frac{2\pi}{3}\right)^{n}\prod_{j=1}^{n}\frac{\theta_{j}\left(r_{j}-\theta_{j}\right)}{\lambda_{2,j}\left(2\left(R+1\right)\right)l_{j}^{2}\left(z^{0}\right)}$$
$$-\int_{0}^{\Theta/L(z^{0})}\max\left\{\mathbb{T}^{n}\left(z^{0},S\right)\cap G_{R'}\left(F\right)\right\}dS$$
$$\geq \left(\frac{2\pi}{3}\right)^{n}\prod_{j=1}^{n}\frac{\theta_{j}\left(r_{j}-\theta_{j}\right)}{\lambda_{2,j}\left(2\left(R+1\right)\right)l_{j}^{2}\left(z^{0}\right)}$$
$$(38)$$
$$-\pi^{n}\prod_{j=1}^{n}\frac{\theta_{j}^{2}}{l_{j}^{2}\left(z^{0}\right)}$$

$$= \pi^{n} \prod_{j=1}^{n} \frac{\theta_{j}}{l_{j}^{2}(z^{0})} \frac{2r_{j} - \theta_{j} \left(2 + 3\lambda_{2,j} \left(2 \left(R + 1\right)\right)\right)}{3\lambda_{2,j} \left(2 \left(R + 1\right)\right)} > 0$$

because $\theta_j < 2r_j/(2 + 3\lambda_{2,j}(2(R + 1)))$. From (36) it follows that

$$\begin{cases} {}^{R/\mathbf{L}(z^{0})}_{\Theta/\mathbf{L}(z^{0})} \operatorname{meas}\left\{ \mathbb{T}^{n}\left(z^{0},S\right) \cap G_{R'}\left(F\right)\right\} dS \\ < \left(\frac{2\pi}{3}\right)^{n} \prod_{j=1}^{n} \frac{\theta_{j}\left(r_{j}-\theta_{j}\right)}{\lambda_{2,j}\left(2\left(R+1\right)\right) l_{j}^{2}\left(z^{0}\right)} \end{cases}$$

$$-\int_{0}^{\Theta/\mathbf{L}(z^{0})} \operatorname{meas}\left\{\mathbb{T}^{n}\left(z^{0},S\right)\cap G_{R'}\left(F\right)\right\}dS$$
$$\leq \left(\frac{2\pi}{3}\right)^{n}\prod_{j=1}^{n}\frac{\theta_{j}\left(r_{j}-\theta_{j}\right)}{\lambda_{2,j}\left(2\left(R+1\right)\right)l_{j}^{2}\left(z^{0}\right)}.$$
(39)

By mean value theorem there exists $R^0 = R^0(z^0)$ with $r_j \in [\theta_j, r_j]$ such that

$$\int_{\Theta/\mathbf{L}(z^0)}^{R'\mathbf{L}(z^0)} \operatorname{meas}\left\{\mathbb{T}^n\left(z^0,S\right) \cap G_{R'}\left(F\right)\right\} dS$$
$$= \operatorname{meas}\left\{\mathbb{T}^n\left(z^0,\frac{R^0}{\mathbf{L}\left(z^0\right)}\right) \cap G_{R'}\left(F\right)\right\} \prod_{j=1}^n \frac{r_j - \theta_j}{l_j\left(z^0\right)}.$$
(40)

Hence, in view of (39) we obtain a desired inequality

$$\max\left\{\mathbb{T}^{n}\left(z^{0}, \frac{R^{0}}{\mathbf{L}(z^{0})}\right) \cap G_{R'}(F)\right\}$$

$$< \left(\frac{2\pi}{3}\right)^{n} \prod_{j=1}^{n} \frac{\theta_{j}}{\lambda_{2,j}\left(2\left(R+1\right)\right) l_{j}\left(z^{0}\right)}.$$
(41)

Clearly, for every point $z^0 \in \mathbb{C}^n$ we have $\mathbb{T}^n(z^0, R^0/\mathbf{L}(z^0)) \setminus Z_F = \bigcup_i C'_i$, where C'_i are connected disjoint sets, $C'_i \supset C_i$ and C_i is defined in condition (2). Let $z^* \in \mathbb{T}^n(z^0, R/\mathbf{L}(z^0))$ be such that $|F(z^*) = \max\{|F(z)| : z \in \mathbb{T}^n(z^0, R^0/\mathbf{L}(z^0))\}$. Then there exists i_0 such that $z^* \in C'_{i_0}$. Let $z^{**} \in C_{i_0} \subset C'_{i_0}$ be such that $|F(z^{**})| = \min_{z \in C_{i_0}} |F(z)|$. We choose $J = (j_1, \ldots, j_n) \in \mathcal{J}$, where

$$j_i = \begin{cases} 1, & z_i^* \neq z_i^{**}, \\ 0, & z_i^* = z_i^{**}, \end{cases} \quad i \in \{1, \dots, n\}, \qquad (42)$$

and deduce

$$\ln \left| \frac{F(z^*)}{F(z^{**})} \right| \leq \int_{z^{**}}^{z^*} \left| (\ln F(z))^{(J)} \right| \left| dz^J \right|$$

$$\leq \int_{z^{**}}^{z^*} p_1 \mathbf{L}^J(z) \left| dz^J \right|$$

$$\leq p_1 \mathbf{L}^J(z^0) \Lambda_2^J(R) \cdot \frac{(\pi R)^J}{\mathbf{L}^J(z^0)}$$

$$\leq \pi^n p_1 R^J \Lambda_2^J(R).$$
(43)

Hence,

$$\max\left\{|F(z)|: z \in \mathbb{T}^{n}\left(z^{0}, \frac{R^{0}}{\mathbf{L}(z^{0})}\right)\right\} = |F(z^{*})|$$

$$\leq \exp\left\{\pi^{n}p_{1}R^{J}\Lambda_{2}^{J}(R)\right\}|F(z^{**})|$$

$$= \exp\left\{\pi^{n}p_{1}R^{J}\Lambda_{2}^{J}(R)\right\}\min_{z \in C_{i_{0}}}|F(z)|$$

$$\leq \exp\left\{\pi^{n}p_{1}R^{J}\Lambda_{2}^{J}(R)\right\}p_{2}\min_{i}\min_{z \in C_{i}}|F(z)|$$

$$= \exp\left\{\pi^{n}p_{1}R^{J}\Lambda_{2}^{J}(R)\right\}p_{2}$$

$$\cdot\min\left\{|F(z)|: z \in \mathbb{T}^{n}\left(z^{0}, \frac{R^{0}}{\mathbf{L}(z^{0})}\right)\setminus G_{R'}(F)\right\}.$$
(44)

By Theorem 7 the function F has bounded L-index in joint variables.

Let us to denote Δ as Laplace operator. We will consider $\Delta \ln |F|$ as generalized function. Using some known results from potential theory, we can rewrite Theorem 9 as follows.

Theorem 10. Let $\mathbf{L} \in Q^n$. If an entire function F satisfies the following conditions

(1) for every R > 0 there exists $p_1 = p_1(R) > 0$ such that for all $z \in \mathbb{C}^n \setminus G_R(F)$ and for every $j \in \{1, ..., n\}$

$$\left|\frac{\partial \ln F(z)}{\partial z_{j}}\right| \le p_{1}l_{j}(z), \qquad (45)$$

where $\ln F(z)$ is the principal value of logarithm.

- (2) for every R > 0 and $R' \ge 0$ exists $p_2 = p_2(R, R') \ge 1$ that for all $z^0 \in \mathbb{C}^n$ such that $T^n(z^0, R/\mathbb{L}(z^0)) \setminus G_{R'}(F) = \bigcup_i C_i \ne \emptyset$, where the sets C_i are connected disjoint sets, and either (a) $\max_{i} \min_{z \in C_i} |F(z)| \le p_2 \min_i \max_{z \in C_i} |F(z)|$, or (b) $\max_{i} \max_{z \in C_i} |F(z)| \le p_2 \min_i \max_{z \in C_i} |F(z)|$, or (c) $|F(z^*)| = \max_i \max_{z \in C_i} |F(z)|$, $|F(z^{**})| = \min_i \min_{z \in C_i} |F(z)|$, and z^* , z^{**} belong to the same set C_{i_0}
- (3) for every R > 0 there exists $n^*(R) > 0$ such that for all $z \in \mathbb{C}^n$

$$\int_{\mathbb{D}^{n}[z^{0}, R/\mathbf{L}(z^{0})]} \Delta \ln |F| \, dV_{2n} \le n^{*} (R) \tag{46}$$

then F has bounded L-index in joint variables.

Proof. Ronkin [28, p. 230] deduced the following formula for entire function:

$$\int_{\mathbb{D}^{n}[\mathbf{0},R^{*}]} \Delta \ln |F| \, dV_{2n} = 2\pi \int_{Z_{F} \cap \mathbb{D}^{n}[\mathbf{0},R^{*}]} \gamma_{F}(z) \, dV_{2n-2}, \quad (47)$$

where $\gamma_F(z)$ is a multiplicity of zero point of the function F at point z, $R^* \in \mathbb{R}^n_+$ is arbitrary radius. Let $\chi_F(z)$ be a characteristic function of zero set of F. Then $\chi_F(z) \leq \gamma_F(z)$. Hence,

$$\max \left\{ Z_F \cap \mathbb{D}^n \left[z^0, \frac{R}{\mathbf{L}(z^0)} \right] \right\}$$

$$= \int_{Z_F \cap \mathbb{D}^n[z^0, R/\mathbf{L}(z^0)]} \chi_F(z) \, dV_{2n-2}$$

$$\le \int_{Z_F \cap \mathbb{D}^n[z^0, R/\mathbf{L}(z^0)]} \gamma_F(z) \, dV_{2n-2}$$

$$= \frac{1}{2\pi} \int_{\mathbb{D}^n[z^0, R/\mathbf{L}(z^0)]} \Delta \ln |F| \, dV_{2n} \le \frac{n^*(R)}{2\pi};$$

$$(48)$$

that is, inequality (29) holds.

Now we want to prove that (45) implies (28). For every $J \in \mathscr{J} \setminus \bigcup_{k=1}^{n} \mathbf{1}_k$ and $z^0 \in \mathbb{C}^n \setminus G_R(F)$, Cauchy's integral formula can be written in the following form

$$\left(\ln F\left(z^{0}\right)\right)^{(J)} = \frac{J!}{\left(2\pi i\right)^{n}} \int_{\mathbb{T}^{n}(z^{0}, R/\mathbf{L}(z^{0}))} \frac{\left(\ln F\left(z\right)\right)^{(\mathbf{1}_{m})}}{\left(z-z^{0}\right)^{J-\mathbf{1}_{m}+1}} dz,$$
(49)

where *m* is such that $j_m = 1$.

If $z^0 \in \mathbb{C}^n \setminus G_R(F)$ and $z' \in Z_F \subset G_R(F)$, then for every $j \in \{1, \ldots, n\}$

$$\left|z_{j}^{0}-z_{j}'\right| \geq \frac{r_{j}}{l_{j}\left(z'\right)} \geq \frac{r_{j}\lambda_{1,j}\left(R\right)}{l_{j}\left(z^{0}\right)} > \frac{r_{j}\lambda_{1,j}\left(R\right)}{2l_{j}\left(z^{0}\right)}.$$
 (50)

Let us consider the set $A = \bigcup_{z^0 \in \mathbb{C}^n \setminus G_R(F)} \mathbb{T}^n(z^0, R\Lambda_1(R)/2\mathbf{L}(z^0))$. We want to find the greatest radius $R^* \in \mathbb{R}^n_+$ such that $G_{R^*}(F) \cap A = \emptyset$:

$$\frac{R}{\mathbf{L}(z')} - \frac{R\Lambda_{1}(R)}{2\mathbf{L}(z^{0})} \geq \frac{R}{\mathbf{L}(z')} - \frac{R\Lambda_{1}(R)}{2\Lambda_{1}(R)\mathbf{L}(z')}$$

$$= \frac{R}{2\mathbf{L}(z')}.$$
(51)

Thus, for $R^* = R/3$ $A \in \mathbb{C}^n \setminus G_{R^*}(F)$. Using (45), we obtain that for every $z^0 \in \mathbb{C}^n \setminus G_R(F)$

$$\begin{split} \left| \left(\ln F\left(z^{0}\right) \right)^{(f)} \right| &\leq \frac{1}{(2\pi)^{n}} \\ &\cdot \int_{\mathbb{T}^{n}(z^{0}, R\Lambda_{1}(R)/2L(z^{0}))} \frac{\left| \left(\ln F\left(z\right) \right)^{(1_{m})} \right|}{\left| z - z^{0} \right|^{f-1_{m}+1}} \left| dz \right| &\leq \frac{1}{(2\pi)^{n}} \\ &\cdot \int_{\mathbb{T}^{n}(z^{0}, R\Lambda_{1}(R)/2L(z^{0}))} \left(\frac{2L\left(z^{0}\right)}{R\Lambda_{1}(R)} \right)^{f-1_{m}+1} p_{1}\left(\frac{1}{3}R\right) \\ &\cdot l_{m}\left(z\right) \left| dz \right| &\leq \frac{1}{(2\pi)^{n}} \\ &\cdot \int_{\mathbb{T}^{n}(z^{0}, R\Lambda_{1}(R)/2L(z^{0}))} \left(\frac{2L\left(z^{0}\right)}{R\Lambda_{1}(R)} \right)^{f-1_{m}+1} p_{1}\left(\frac{1}{3}R\right) \\ &\quad (52) \\ &\cdot \lambda_{2,m}\left(0.5R\Lambda_{1}\left(R\right)\right) l_{m}\left(z^{0}\right) \left| dz \right| \\ &= \frac{r_{m}\lambda_{1,m}\left(R\right)}{2\left(2\pi\right)^{n}} \left(\frac{2L\left(z^{0}\right)}{R\Lambda_{1}\left(R\right)} \right)^{f+1} p_{1}\left(\frac{1}{3}R\right) \\ &\quad \cdot \lambda_{2,m}\left(0.5R\Lambda_{1}\left(R\right)\right) \int_{\mathbb{T}^{n}(z^{0}, R\Lambda_{1}(R)/2L(z^{0}))} \left| dz \right| \\ &= 0.5r_{m}\lambda_{1,m}\left(R\right) p_{1}\left(\frac{1}{3}R\right)\lambda_{2,m}\left(0.5R\Lambda_{1}\left(R\right)\right) \\ &\quad \cdot \left(\frac{2L\left(z^{0}\right)}{R\Lambda_{1}\left(R\right)} \right)^{f} \leq C\left(R\right) L^{f}\left(z^{0}\right), \end{split}$$

where $C(R) = 0.5p_1((1/3)R)\max_{J \in \mathcal{J}} \{\lambda_{2,m}(0.5R\Lambda_1(R))r_m \lambda_{1,m}(R)(\mathbf{2}/R\Lambda_1(R))^J\}$. Thus, we proved that inequality (28) is valid.

For n = 1 Theorem 9 implies the following corollary.

Corollary 11. Let $l \in Q$, f be an entire in \mathbb{C} function, $n(r, z^0, f)$ a number of zeros of the f in the disc $|z - z_0| \leq r/l(z^0)$. If the function f satisfies the following conditions:

(1) for every r > 0 there exists $p_1 = p_1(r) > 0$ such that for all $z \in \mathbb{C} \setminus G_r(f)$

$$\left|\frac{f'(z)}{f(z)}\right| \le p_1 l(z), \tag{53}$$

(2) for every r > 0 and $r' \ge 0$ exists $p_2 = p_2(r, r') \ge 1$ that for all $z^0 \in \mathbb{C}$ such that $\{z \in \mathbb{C} : |z - z^0| = r/l(z^0)\} \setminus G_{r'}(f) = \bigcup_i C_i \neq \emptyset$, where the sets C_i are connected disjoint sets, and either (a) $\max_i \min_{z \in C_i} |f(z)| \le p_2 \min_i \min_{z \in C_i} |f(z)|$, or (b) $\max_i \max_{z \in C_i} |f(z)| \le p_2 \min_i \max_{z \in C_i} |f(z)|$, or (c) $|f(z^*)| = \max_i \max_{z \in C_i} |f(z)|$, $|f(z^{**})| = \min_i \min_{z \in C_i} |f(z)|$, and z^* , z^{**} belong to the same set C_{i_0} (3) for every r > 0 there exists $n^*(r) > 0$ such that for all $z^0 \in \mathbb{C}$ $n(r, z^0, f) \le n^*(r)$,

then f has bounded l-index.

It is known (see [12, 27, 29]) that in one-dimensional case conditions (1) and (3) of Corollary 11 are necessary and sufficient for boundedness of *l*-index or index. Thus, condition (2) is excessive in the case. But for \mathbb{C}^n $(n \ge 2)$, it is required because $\mathbb{D}^n[z^0, R/\mathbf{L}(z^0)] \setminus G_{R'}(F)$ is a multiply connected domain, when $\mathbb{D}^n[z^0, R/\mathbf{L}(z^0)]$ contains zeros of the function *F*.

We need some notations from [1]. Let $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be a given direction. For a given $z^0 \in \mathbb{C}^n$ we denote $g_{z^0}(t) :=$ $F(z^0 + t\mathbf{b})$. If one has $g_{z^0}(t) \neq 0$ for all $t \in \mathbb{C}$, then $G_r^{\mathbf{b}}(F, z^0) :=$ \emptyset ; if $g_{z^0}(t) \equiv 0$, then $G_r^{\mathbf{b}}(F, z^0) := \{z^0 + t\mathbf{b} : t \in \mathbb{C}\}$. And if $g_{z^0}(t) \neq 0$ and a_k^0 are zeros of the function $g_{z^0}(t)$, then $G_r^{\mathbf{b}}(F, z^0) := \bigcup_k \{z^0 + t\mathbf{b} : |t - a_k^0| \leq r/L(z^0 + a_k^0\mathbf{b})\}, r > 0$. Let

$$G_r^{\mathbf{b}}(F) = \bigcup_{z^0 \in \mathbb{C}^n} G_r^{\mathbf{b}}(F, z^0).$$
(54)

Remark 12. In [1, Theorem 8], sufficient conditions of boundedness of L-index in joint variables were obtained, which are similar to Theorem 10. Particularly, we assumed the validity of inequality (45) for all $z \in \mathbb{C}^n \setminus G_{r_j}^{1_j}(F), j \in \{1, 2, ..., n\}$. However, $G_{r_j}^{1_j}(F) \subset G_R(F)$, where $R = (r_1, ..., r_n)$. Thus, condition (1) in Theorem 10 is weaker than the corresponding assumption in Theorem 8 from [1].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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