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Research Article

The Convergence of a Class of Parallel Newton-Type Iterative Methods

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A general iterative process is proposed, from which a class of parallel Newton-type iterative methods can be derived. A unified convergence theorem for the general iterative process is established. The convergence of these Newton-type iterative methods is obtained from the unified convergence theorem. The results of efficiency analyses and numerical example are satisfactory.

1. Introduction

Attempts to improve Newton method are the subject of many papers [1–10].

Consider the following polynomial of degree n:

$$f(x) = \prod_{i=1}^{n} (x - r_i), \tag{1}$$

with simple zeros $r_1, r_2, \ldots, r_n$.

In paper [1], a parallel iterative method for simultaneously finding all zeros of $f(x)$ was suggested; that is,

$$x_i^{(k+1)} = x_i^{(k)} + \frac{\alpha_i^{(k)}}{1 + \alpha_i^{(k)} \sum_{j=1, j\neq i}^{n} \left(1/\left(x_i^{(k)} - x_j^{(k)}\right)\right)}, \tag{2}$$

$$\alpha_i^{(k)} = -\frac{f\left(x_i^{(k)}\right)}{f'\left(x_i^{(k)}\right)}, \tag{3}$$

where $i = 1, 2, \ldots, n$; $k = 0, 1, 2, \ldots$.

$x_i^{(0)}$ $(i = 1, 2, \ldots, n)$ are distinct initial approximations for zeros r_i $(i = 1, 2, \ldots, n)$ of polynomial $f(x)$.

For appropriate starting values $x_i^{(0)}$, method (2) is of convergence order three.

Suppose that $\varphi(x)$ is some iteration function and $x_j^{(k+1)} = \varphi(x_j^{(k)})$ converges to zeros r_j $(j = 1, 2, \ldots, n)$ of $f(x)$ with convergence order m.

From (2), we obtain the following parallel iterative process:

$$x_i^{(k+1)} = x_i^{(k)} + \frac{\alpha_i^{(k)}}{1 + \alpha_i^{(k)} \sum_{j=1, j\neq i}^{n} \left(1/\left(x_i^{(k)} - u_j^{(k)}\right)\right)}, \tag{4}$$

$$u_j^{(k)} = \varphi\left(x_j^{(k)}\right), \tag{5}$$

where $i, j = 1, 2, \ldots, n$; $k = 0, 1, 2, \ldots$. $\alpha_i^{(k)}$ is defined by (3). We call $\varphi(x)$ correction iterative function.

In particular, if $u_j^{(k)} = \varphi(x_j^{(k)}) = x_j^{(k)}$, then (4) is process (2) derived in paper [1]. If Newton iterative function is chosen as φ, that is, $u_j^{(k)} = \varphi(x_j^{(k)}) = x_j^{(k)} + \alpha_j^{(k)}$ and $\alpha_j^{(k)}$ are defined by (3), then (4) is the method discussed in paper [3]. Because (2) is a modification of Newton method and (4) is an improvement to (2), so we call (4) modified Newton-type iteration method.

In this paper, a unified convergence theorem for the general modified process (4) is established in Section 2 (Theorem 2).

Moreover, in Section 3, three special iterative methods are derived from process (4) according to the choices of φ. These special methods are all modifications to process (2); their convergence and convergence order are obtained via the unified general convergence Theorem 2.

All these special modified methods are convergent with higher order and are more efficient than both Newton method and process (2).

In Section 4, the method is extended to find the multiple zeros of polynomial.

Finally, in Section 5, we give several numerical examples and the computation results are satisfactory.

2. General Convergence Theorem

In this section, we discuss the convergence of the general modified process (4).

Let $k = 0, 1, 2, \ldots$, be the indices of iterations and

$$d = \min_{1\le i<j\le n} \left| r_i - r_j \right|, \tag{6}$$

$$h_i^{(k)} = x_i^{(k)} - r_i, \tag{7}$$

$$h^{(k)} = \max_{1\le i\le n} \left| h_i^{(k)} \right|. \tag{8}$$

By some simple calculation, process (4) can also be expressed as follows:

$$h_i^{(k+1)} = \frac{A_i^{(k)}}{1 + A_i^{(k)}} h_i^{(k)}, \tag{9}$$

where $h_i^{(k)}$ are defined by (7) and

$$A_i^{(k)} = \sum_{\substack{j=1\\ j\ne i}}^{n} \frac{\left(x_i^{(k)} - r_i\right)\left(r_j - u_j^{(k)}\right)}{\left(x_i^{(k)} - r_j\right)\left(x_i^{(k)} - u_j^{(k)}\right)}. \tag{10}$$

Assume that the correction iteration function φ in (5) is locally convergent with convergence order m ($m \ge 1$) for each root r_j of $f(x)$; that is, $x_j^{(k+1)} = \varphi(x_j^{(k)})$ converges to root r_j with convergence order m for sufficiently good starting values $x_j^{(0)}$ ($j = 1, 2, \ldots, n$). Then we have the following Lemma 1.

Lemma 1. *Let $u_j^{(k)}$ be defined by (5); then there exist constants c and δ (independent of j and k) such that*

$$\left| u_j^{(k)} - r_j \right| \le c \left| x_j^{(k)} - r_j \right|^m \quad \textit{if} \quad \left| x_j^{(0)} - r_j \right| \le \delta, \tag{11}$$

where $j = 1, 2, \ldots, n$; $k = 0, 1, 2, \ldots$.

In fact, because $x_j^{(k+1)} = \varphi(x_j^{(k)})$ converges to root r_j with convergence order m for sufficiently good starting values $x_j^{(0)}$ ($j = 1, 2, \ldots, n$), for every j, there exist c_j, δ_j, such that

$$\left| u_j^{(k)} - r_j \right| \le c_j \left| x_j^{(k)} - r_j \right|^m \quad \text{if} \quad \left| x_j^{(0)} - r_j \right| \le \delta_j. \tag{12}$$

Let

$$\begin{aligned} \delta &= \min\left\{\delta_1, \delta_2, \ldots, \delta_n\right\}, \\ c &= \max\left\{c_1, c_2, \ldots, c_n\right\}, \end{aligned} \tag{13}$$

and then Lemma 1 holds.

In the following Theorem 2 and its proof, the constants c and δ are defined in Lemma 1 and n is the degree of $f(x)$.

Theorem 2. *Suppose that initial approximations $x_j^{(0)}$ ($j = 1, 2, \ldots, n$) satisfy $|x_j^{(0)} - r_j| < \min\{\delta, c^{-1/(m-1)}, 2d/(3 + \sqrt{8n-7})\}$. Then the iterative process (4) converges to the zeros r_i ($i = 1, 2, \ldots, n$) of $f(x)$, and the convergence order is $m + 2$.*

Proof. Suppose that $x_j^{(0)}$ ($j = 1, 2, \ldots, n$) satisfy the condition in Theorem 2.

Then there exists a positive constant $s > \max\{d/\delta, c^{1/(m-1)}d, (3 + \sqrt{8n-7})/2\}$ such that

$$\left| x_j^{(0)} - r_j \right| \le \frac{d}{s} \quad (j = 1, 2, \ldots, n). \tag{14}$$

Hence from Lemma 1 we know that, for $k = 0$ and $i \ne j$,

$$\begin{aligned} \left| r_j - u_j^{(0)} \right| &\le c \left| x_j^{(0)} - r_j \right|^m \le \frac{d}{s}, \\ \left| x_i^{(0)} - r_j \right| &\ge \left| r_i - r_j \right| - \left| x_i^{(0)} - r_i \right| \ge (s-1)\frac{d}{s}, \\ \left| u_j^{(0)} - x_i^{(0)} \right| &\ge \left| r_i - r_j \right| - \left| u_j^{(0)} - r_j \right| - \left| x_i^{(0)} - r_i \right| \\ &\ge (s-2)\frac{d}{s}. \end{aligned} \tag{15}$$

By (10), it follows that

$$\left| A_i^{(0)} \right| \le \sum_{\substack{j=1\\ j\ne i}}^{n} \frac{c\left| h_i^{(0)} \right| \cdot \left| h_j^{(0)} \right|^m s^2}{(s-1)(s-2)d^2} \le \frac{n-1}{(s-1)(s-2)} < \frac{1}{2}. \tag{16}$$

Let

$$\begin{aligned} \lambda &= \frac{n-1}{(s-1)(s-2)}, \\ \mu &= \frac{\lambda}{1-\lambda}. \end{aligned} \tag{17}$$

It is evident that $\mu < 1$.

Thus, from (9), we obtain that, for all i,

$$\left| h_i^{(1)} \right| \le \frac{\left| A_i^{(0)} \right|}{1 - \left| A_i^{(0)} \right|} \left| h_i^{(0)} \right| \le \mu \left| h_i^{(0)} \right| \le \frac{d}{s}. \tag{18}$$

Generally, if $|x_j^{(k)} - r_j| \le d/s$ ($j = 1, 2, \ldots, n$), then we can obtain analogously that

$$\left| A_i^{(k)} \right| = \sum_{\substack{j=1\\ j\ne i}}^{n} \frac{cs^2 \left| h_i^{(k)} \right| \cdot \left| h_j^{(k)} \right|^m}{(s-1)(s-2)d^2} \le \lambda < \frac{1}{2}, \tag{19}$$

$$\left| h_i^{(k+1)} \right| \le \mu \left| h_i^{(k)} \right| \le \frac{d}{s}. \tag{20}$$

By mathematical induction, we know that (19) and (20) are valid for $i = 1, 2, \ldots, n$; $k = 0, 1, 2, \ldots$.

From (20), we have

$$\left|h_i^{(k)}\right| \le \mu^k \left|h_i^{(0)}\right| \le \left(\frac{d}{s}\right)\mu^k, \quad \text{for } i = 1, 2, \ldots, n;\ k = 0, 1, 2, \ldots. \tag{21}$$

It is evident that $h_i^{(k)} \to 0$ $(k \to \infty)$. That is, $x_i^{(k)} \to r_i$ $(k \to \infty)$ for $i = 1, 2, \ldots, n$.

Making use of (8) and (19), we have

$$\left|A_i^{(k)}\right| \le \frac{c\,(n-1)\,s^2}{(s-1)\,(s-2)\,d^2}\left|h^{(k)}\right|^{m+1} \le \frac{\lambda c s^2}{d^2}\left|h^{(k)}\right|^{m+1}. \tag{22}$$

Further, by (9) and $|A_i^{(k)}| < 1/2$, we have

$$\left|h_i^{(k+1)}\right| \le \frac{2\lambda c s^2}{d^2}\left|h^{(k)}\right|^{m+2}. \tag{23}$$

Hence, the convergence order of method (4) with (5) is $m + 2$. □

3. Some Special Modified Newton Methods Derived from Formula (4)

For the correction function φ in (5), we will make several kinds of choice and derive some special modified Newton methods from (4). Furthermore, by the convergence Theorem 2, we give the convergence and efficiency of these special modified methods.

Definition 3. For an iteration method, we define the efficiency

$$e = \frac{\log k}{w}, \tag{24}$$

where k is the convergence order; w is the amount of computation required in every step of iteration.

Since $f(x), f'(x), f''(x)$ are all polynomials, computational efficiency requires that the evaluation of these functions be done by Horner's method [8]. Then only n multiplications and n additions will be required to evaluate an arbitrary polynomial of degree n. Since $f(x)$ defined by (1) is a polynomial of degree n, we take n multiplications or divisions as a unit of the amount of computation and take no count of additions in the following. As a consequence, the evaluation of $f(x_i^{(k)})$, $f'(x_i^{(k)})$, $f''(x_i^{(k)})$ and $\sum_{\substack{j=1\\ j\ne i}}^{n} (1/(x_i^{(k)} - x_j^{(k)}))$ require approximately one unit, respectively. Now the convergence and efficiency analyses of these special modified methods can be given as follows.

(i) Newton iterative function is chosen as φ; that is,

$$u_j^{(k)} = \varphi\left(x_j^{(k)}\right) = x_j^{(k)} + \alpha_j^{(k)}. \tag{25}$$

We obtain the iterative method (4) with (25) which has been considered in [3].

Because Newton iterative function is second-order convergent ($m = 2$), the convergence and convergence order of method (4) with (25) can be concluded from Theorem 2 directly.

Corollary 4. *Suppose that initial approximations $x_j^{(0)}$ $(j = 1, 2, \ldots, n)$ satisfy $|x_j^{(0)} - r_j| < \min\{\delta, c^{-1}, 2d/(3 + \sqrt{8n-7})\}$. Then the iterative process (4) with (25) converges to the zeros r_i $(i = 1, 2, \ldots, n)$ of $f(x)$, and the convergence order is 4; the efficiency $e_1 = \log 4/3$.*

(ii) Let φ be the Halley iterative function; that is,

$$u_j^{(k)} = \varphi\left(x_j^{(k)}\right) = x_j^{(k)} + \frac{\alpha_j^{(k)}}{1 + (1/2)\left(f''\left(x_j^{(k)}\right)/f'\left(x_j^{(k)}\right)\right)\alpha_j^{(k)}}. \tag{26}$$

Halley iterative function is of convergence order 3; therefore we have the following conclusion from Theorem 2.

Corollary 5. *Suppose that initial approximations $x_j^{(0)}$ $(j = 1, 2, \ldots, n)$ satisfy $|x_j^{(0)} - r_j| < \min\{\delta, c^{-1/2}, 2d/(3 + \sqrt{8n-7})\}$. Then the iterative process (4) with (26) converges to the zeros r_i $(i = 1, 2, \ldots, n)$ of $f(x)$, and the convergence order is 5; the efficiency $e_2 = \log 5/4$.*

(iii) Let

$$u_j^{(k)} = x_j^{(k)} + \frac{\alpha_j^{(k)}}{1 + \alpha_j^{(k)} \sum_{\substack{l=1\\ l\ne j}}^{n} \left(1/\left(x_j^{(k)} - v_l^{(k)}\right)\right)}, \tag{27}$$

where $v_l^{(k)} = x_l^{(k)} + \alpha_l^{(k)}$.

From Corollary 4, we know (27) is 4th-order convergent, so we obtain the following conclusion from Theorem 2.

Corollary 6. *Suppose that initial approximations $x_j^{(0)}$ $(j = 1, 2, \ldots, n)$ satisfy $|x_j^{(0)} - r_j| < \min\{\delta, c^{-1/3}, 2d/(3 + \sqrt{8n-7})\}$. Then the iterative process (4) with (27) converges to the zeros r_i $(i = 1, 2, \ldots, n)$ of $f(x)$, and the convergence order is 6; the efficiency $e_3 = \log 6/4$.*

In particular, if we let $\varphi(x_j^{(k)}) = x_j^{(k)}$, then (4) is the modified Newton method (2) (see [1]). The convergence of (2) was not proven in [1], but now its convergence follows directly from Theorem 2, and the convergence order is 3; therefore the efficiency $e_4 = \log 3/3$.

By the way, according to our definition, the computational efficiency of Newton iterative method is $\log 2/2$.

For simultaneously finding polynomial zeros, it is evident that these modified Newton-type methods discussed in Corollaries 4–6 are convergent with higher order and are more efficient than both Newton method and process (2).

4. Extending the Iterative Method (4) to Find Multiple Zeros

In complex number field polynomial $f(x)$ of degree n can be factored as

$$f(x) = (x - r_1)^{\mu_1}(x - r_2)^{\mu_2} \cdots (x - r_m)^{\mu_m}; \tag{28}$$

$r_1, r_2, \ldots, r_m$ are multiple zeros of polynomial $f(x)$.

Here $r_i \neq r_j$ $(i \neq j)$ and $\sum_{i=1}^{m} \mu_i = n$.

By logarithmic derivation, we know that

$$\frac{f'(x)}{f(x)} = \sum_{j=1}^{m} \frac{\mu_j}{x - r_j},$$

$$r_i = x - \frac{\left[\mu_i\left(f(x)/f'(x)\right)\right]}{\left[1 - \left(f(x)/f'(x)\right)\sum_{j=1, j\neq i}^{m}\left(\mu_j/\left(x - r_j\right)\right)\right]}, \quad i = 1, 2, \ldots, m. \tag{29}$$

So we get the iterative method for simultaneously finding all zeros of $f(x)$.

$$x_i^{(k+1)} = x_i^{(k)} - \frac{\left[\mu_i\left(f\left(x_i^{(k)}\right)/f'\left(x_i^{(k)}\right)\right)\right]}{\left[1 - \left(f\left(x_i^{(k)}\right)/f'\left(x_i^{(k)}\right)\right)\sum_{j=1, j\neq i}^{m}\left(\mu_j/\left(x_i^{(k)} - x_j^{(k)}\right)\right)\right]}, \tag{30}$$

where $i = 1, 2, \ldots, m$; $k = 0, 1, 2, \ldots$. $x_i^{(0)}$ $(i = 1, 2, \ldots, m)$ were distinct initial approximations for zeros r_i $(i = 1, 2, \ldots, m)$ of $f(x)$.

When $\mu_i = 1$ for all $i = 1, 2, \ldots, m$, the iterative method (30) shall be the iterative method (2) in Section 1.

Using the same technique as in formula (4), we obtain

$$x_i^{(k+1)} = x_i^{(k)} - \frac{\left[\mu_i\left(f\left(x_i^{(k)}\right)/f'\left(x_i^{(k)}\right)\right)\right]}{\left[1 - \left(f\left(x_i^{(k)}\right)/f'\left(x_i^{(k)}\right)\right)\sum_{j=1, j\neq i}^{m}\left(\mu_j/\left(x_i^{(k)} - u_j^{(k)}\right)\right)\right]}. \tag{31}$$

Here,

$$u_j^{(k)} = \varphi_j\left(x_1^{(k)}, x_2^{(k)}, \ldots, x_m^{(k)}\right). \tag{32}$$

For appropriate starting values $x_i^{(0)}$, we suppose that $x_j^{(k+1)} = \varphi_j(x_1^{(k)}, x_2^{(k)}, \ldots, x_m^{(k)})$ converges to zeros r_j $(j = 1, 2, \ldots, m)$ of $f(x)$ with convergence order p.

By some simple calculation, formula (31) can be expressed as follows:

$$h_i^{(k+1)} = \frac{B_i^{(k)}}{1 + B_i^{(k)}} h_i^{(k)}, \tag{33}$$

where

$$\begin{aligned} h_i^{(k)} &= x_i^{(k)} - r_i, \quad i = 1, 2, \ldots, m, \ k = 0, 1, 2, \ldots, \\ B_i^{(k)} &= \frac{1}{\mu_i}\sum_{j=1, j\neq i}^{m} \frac{\mu_j\left(x_i^{(k)} - r_i\right)\left(r_j - u_j^{(k)}\right)}{\left(x_i^{(k)} - r_j\right)\left(x_i^{(k)} - u_j^{(k)}\right)}, \\ d &= \min_{1\le i<j\le m}\left\{\left|r_i - r_j\right|\right\}, \\ \mu &= \min_{1\le i\le m}\left\{\mu_i\right\}. \end{aligned} \tag{34}$$

Lemma 7. *Let $u_j^{(k)}$ be defined by (32); then there exist constants c and δ (independent of j and k), such that*

$$\left|u_j^{(k)} - r_j\right| \le c\left|x_j^{(k)} - r_j\right|^p \quad \text{if} \quad \left|x_j^{(0)} - r_j\right| \le \delta. \tag{35}$$

The proof is similar to Lemma 1.

In the following Theorem 8 and its proof, the constants c and δ are defined in Lemma 7 and n is the degree of $f(x)$.

Take the constant $\theta > \max\{4, n/\mu, d/\delta\}$ and $\theta^{p-1} \ge cd^{p-1}$; we have the following Theorem 8.

Theorem 8. *Suppose that initial approximations $x_j^{(0)}$ $(j = 1, 2, \ldots, m)$ satisfy $|x_j^{(0)} - r_j| \le d/\theta$ $(j = 1, 2, \ldots, m)$, and $x_j^{(k+1)} = \varphi_j(x_1^{(k)}, x_2^{(k)}, \ldots, x_m^{(k)})$ converges to zeros r_j with convergence order p. Then the iterative process (31) with (32) converges to zeros r_i $(i = 1, 2, \ldots, m)$ with convergence order $p + 2$.*

Proof. Suppose that $x_j^{(0)}$ $(j = 1, 2, \ldots, m)$ satisfy the condition in Theorem 8. Then

$$\begin{aligned} \left|x_i^{(0)} - r_j\right| &\ge \frac{\theta - 1}{\theta} d, \\ \left|u_j^{(0)} - r_j\right| &\le c\left|x_j^{(0)} - r_j\right|^p \le c\left(\frac{d}{\theta}\right)^p \le \frac{d}{\theta}. \end{aligned} \tag{36}$$

Therefore

$$\begin{aligned} \left|x_i^{(0)} - u_j^{(0)}\right| &\ge \left|r_i - r_j\right| - \left|u_j^{(0)} - r_j\right| - \left|x_i^{(0)} - r_i\right| \\ &\ge \frac{\theta - 2}{\theta} d. \end{aligned} \tag{37}$$

Further,

$$\left|B_i^{(0)}\right| \le \frac{1}{\mu_i}\sum_{j=1, j\neq i}^{m} \frac{c\mu_j\theta^2\left|h_i^{(0)}\right|\cdot\left|h_j^{(0)}\right|^p}{(\theta - 1)(\theta - 2)d^2}. \tag{38}$$

Note that $\theta > \max\{4, n/\mu, d/\delta\}$ and $\theta^{p-1} \ge cd^{p-1}$; we have

$$\left|B_i^{(0)}\right| \le \frac{1}{\theta - 2} < \frac{1}{2}. \tag{39}$$

Let

$$q = \frac{1}{\theta - 3} \quad (< 1). \tag{40}$$

Then

$$\left|h_i^{(1)}\right| \le \frac{\left|B_i^{(0)}\right|\left|h_i^{(0)}\right|}{1 - \left|B_i^{(0)}\right|} \le q\left|h_i^{(0)}\right| \le \frac{d}{\theta} \quad (i = 1, 2, \ldots, m). \tag{41}$$

Generally, we can obtain analogously that

$$\left|B_i^{(k)}\right| \le \frac{1}{\mu_i}\sum_{j=1, j\neq i}^{m} \frac{c\theta^2\mu_j\left|h_i^{(k)}\right|\cdot\left|h_j^{(k)}\right|^p}{(\theta - 1)(\theta - 2)d^2} \le \frac{1}{\theta - 2} < \frac{1}{2}, \tag{42}$$

$$\left|h_i^{(k+1)}\right| \le q\left|h_i^{(k)}\right| \le \frac{d}{\theta}. \tag{43}$$

By mathematical induction, we know that (43) is valid for $i = 1, 2, \ldots, m$; $k = 0, 1, 2, \ldots$.

From (43), we get $|h_i^{(k)}| \leq q^k (d/\theta) \to 0$ (when $k \to \infty$).

Let $h^{(k)} = \max_{1\leq j\leq m}\{|h_j^{(k)}|\}$, and from (42) it is inferred that

$$\left|B_i^{(k)}\right| \leq \frac{c\theta^2}{\mu_i (\theta - 1)(\theta - 2) d^2} \left(\sum_{\substack{j=1 \\ j\neq i}}^{m} \mu_j \right) \cdot \left|h^{(k)}\right|^{p+1} \leq \frac{c\theta^2}{2d^2} \left|h^{(k)}\right|^{p+1}. \tag{44}$$

Because $|B_i^{(k)}| < 1/2$, $|h_i^{(k+1)}| \leq 2|B_i^{(k)}| \cdot |h_i^{(k)}| \leq c\theta^2/d^2 \cdot |h^{(k)}|^{p+2}$.

Hence, the convergence order of method (31) is $p+2$. The proof is completed. □

Let

$$u_j^{(k)} = x_j^{(k)} - \frac{\mu_j f\left(x_j^{(k)}\right)}{f'\left(x_j^{(k)}\right)}. \tag{45}$$

Combine (31) and (45); we have the following Corollary 9.

Corollary 9. *Suppose that initial approximations $x_j^{(0)}$ ($j = 1, 2, \ldots, m$) satisfy $|x_j^{(0)} - r_j| < \min\{1/4, \mu/n\}d$ ($j = 1, 2, \ldots, m$). Then the iterative process (31) with (45) converges to the zeros r_i ($i = 1, 2, \ldots, m$) of $f(x)$, and the convergence order is 4.*

5. Numerical Example

In this section, we will report on three numerical examples. The computations were performed on Lenovo computer using MATLAB.

Example 1. As in [5], we consider complex polynomial of degree 10 ($i = \sqrt{-1}$):

$$f(x) = x^{10} - 20(1+i)x^9 + 400ix^8 + 3\times 10^4 x^6 - 6 \times 10^5 (1+i)x^5 + 12\times 10^6 ix^4 - 4\times 10^8 x^2 + 8\times 10^9 (1+i)x - 16\times 10^{10} i. \tag{46}$$

We want to find the zeros of $f(x)$ by method (4) with (25).

The zeros of $f(x)$ are $r_{1,2} = \pm 10, r_{3,4} = \pm 10i, r_{5,6} = 10 \pm 10i, r_{7,8} = -10 \mp 10i, r_9 = 20, r_{10} = 20i$.

In our computation, we take error $\varepsilon = 10^{-12}$ (in paper [5], error $\varepsilon = 10^{-6}$) and choose the starting values just as paper [5]; that is,

$$\begin{aligned} x_{1,2}^{(0)} &= \pm(10.1 + 0.1i), \\ x_{3,4}^{(0)} &= \pm(0.1 + 10.1i), \\ x_{5,6}^{(0)} &= 10.1(1 \pm i), \\ x_{7,8}^{(0)} &= -10.1(1 \pm i), \\ x_9^{(0)} &= 19.9 + 0.1i, \\ x_{10}^{(0)} &= 0.1 + 19.9i. \end{aligned} \tag{47}$$

The numerical results of method (4) with (25) are listed as follows.

Numerical Results of Example 1

$$\begin{aligned}
x_1^{(1)} &= 9.999998471976 + 0.000002471890i, \\
x_2^{(1)} &= -10.000000671094 - 0.000002530585i, \\
x_3^{(1)} &= 0.000002471890 + 9.999998471976i, \\
x_4^{(1)} &= -0.000002530585 - 10.000000671094i, \\
x_5^{(1)} &= 9.999999158562 + 9.999999158562i, \\
x_6^{(1)} &= 10.000001503999 - 10.000002165629i, \\
x_7^{(1)} &= -9.999999683089 - 9.999999683089i, \\
x_8^{(1)} &= -10.000002165629 + 10.000001503999i, \\
x_9^{(1)} &= 19.999999580699 - 0.000001353811i, \\
x_{10}^{(1)} &= -0.000001353811 + 19.999999580699i, \\
x_1^{(2)} &= 10.000000000000 - 0.000000000000i, \\
x_2^{(2)} &= -10.000000000000 - 0.000000000000i, \\
x_3^{(2)} &= -0.000000000000 + 10.000000000000i, \\
x_4^{(2)} &= -0.000000000000 - 10.000000000000i, \\
x_5^{(2)} &= 10.000000000000 + 10.000000000000i, \\
x_6^{(2)} &= 10.000000000000 - 10.000000000000i, \\
x_7^{(2)} &= -10.000000000000 - 10.000000000000i, \\
x_8^{(2)} &= -10.000000000000 + 10.000000000000i, \\
x_9^{(2)} &= 20.000000000000 - 0.000000000000i, \\
x_{10}^{(2)} &= -0.000000000000 + 20.000000000000i.
\end{aligned} \tag{48}$$

We see from (48) that, for method (4) with (25), after two iterations the numerical results attain the precision.

Example 2. Given a polynomial

$$f(x) = 32x^3 - 56x^2 + 24x - 3, \tag{49}$$

$f(x) = 0$ is the so-called *Rayleigh equation* in theory of earthquake.

Table 1: Numerical results of Example 2.

Iterative method	Number of iterations	Results $x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
Method (2)	1	0.200000000000	0.375000000000	1.176470588235
	2	0.243808087597	0.323805689748	1.183011463275
	3	0.249955665119	0.317035707337	1.183012701892
	4	0.249999999979	0.316987298131	1.183012701892
	5	0.250000000000	0.316987298108	1.183012701892
Method (4) with (25)	1	0.217105263158	0.345588235294	1.184859154930
	2	0.249398039932	0.317688644132	1.183012708464
	3	0.249999999474	0.316987298719	1.183012701892
	4	0.250000000000	0.316987298108	1.183012701892
Method (4) with (26)	1	0.231729055258	0.346042471043	1.183941605839
	2	0.249920728625	0.317052319337	1.183012700566
	3	0.250000000000	0.316987298108	1.183012701892
Method (4) with (27)	1	0.234609565063	0.331231334248	1.182746284452
	2	0.249997316046	0.316989331975	1.183012701890
	3	0.250000000000	0.316987298108	1.183012701892
Newton method	8	0.250000000000	0.316987298108	1.183012701892

The exact roots of the equation or the zeros of $f(x)$ are $x_1 = 1/4, x_2 = (3 - \sqrt{3})/4, x_3 = (3 + \sqrt{3})/4$.

We want to find the zeros of $f(x)$ by Newton method, process (2), and the three modified methods discussed in Corollaries 4–6. We choose starting values $x_1^{(0)} = 0$, $x_2^{(0)} = 0.5$, $x_3^{(0)} = 1$, and we take error $\varepsilon = 10^{-12}$.

The numerical results of process (2), method (4) with (25), method (4) with (26), and method (4) with (27) are listed in Table 1, but for Newton method we only give the final numerical results.

From Table 1, we see that, for Newton method, after eight iterations, the iteration approximations attain the precision; for method (2) with (3), after five iterations, the iteration approximations attain the precision; for method in Corollary 4, after four step iterations, the iteration approximations attain the precision; for methods in Corollary 5 and in Corollary 6, after three iterations, all the iteration approximations attain the precision. Hence, these modified Newton-type methods converge faster than both Newton method and iterative method (2).

Example 3. We consider polynomial $f(x) = x^7 + x^6 + x^5 + 17x^4 - x^3 + 31x^2 - x + 15$. We want to find the zeros of $f(x)$ by iterative process (31) with (45). The exact zeros of $f(x)$ are $r_1 = -3$, $r_{2,3} = \pm i$, $r_{4,5} = 1 \pm 2i$; the corresponding multiplicities are $\mu_1 = 1$, $\mu_2 = 2$, $\mu_3 = 2$, $\mu_4 = 1$, $\mu_5 = 1$. We choose starting values $x_1^{(0)} = -2.5 + 0.5i$, $x_2^{(0)} = 0.5 + 1.5i$, $x_3^{(0)} = 0.5 - 1.5i$, $x_4^{(0)} = 1.5 + 2.5i$, and $x_5^{(0)} = 1.5 - 2.5i$, and we take error $\varepsilon = 10^{-14}$. The numerical results of the first three iterations by iterative process (31) with (45) are listed as follows.

Numerical Results of Example 3

$$
\begin{aligned}
x_1^{(1)} &= -3.00565194346854 - 0.01318777497764i,\\
x_2^{(1)} &= -0.15410479694978 + 0.89034788387744i,\\
x_3^{(1)} &= -0.15107817440832 - 0.88441680259590i,\\
x_4^{(1)} &= 0.96243366036343 + 2.03642298912267i,\\
x_5^{(1)} &= 0.96330847662789 - 2.03255647412651i,\\
x_1^{(2)} &= -2.99999982955636 - 0.00000016455696i,\\
x_2^{(2)} &= -0.00000190344179 + 1.00020769732097i,\\
x_3^{(2)} &= -0.00003765337762 - 1.00020338825104i,\\
x_4^{(2)} &= 1.00004824175549 + 1.99995917074785i,\\
x_5^{(2)} &= 1.00004838408085 - 1.99997115571258i,\\
x_1^{(3)} &= -3.00000000000000 + 0.00000000000000i,\\
x_2^{(3)} &= -0.00000000000002 + 1.00000000000016i,\\
x_3^{(3)} &= 0.00000000000002 - 1.00000000000004i,\\
x_4^{(3)} &= 1.00000000000000 + 2.00000000000000i,\\
x_5^{(3)} &= 1.00000000000000 - 2.00000000000000i.
\end{aligned}
\tag{50}
$$

From (50) we see that, for iterative method (31) with (45), after three iterations, all the iteration approximations attain the precision 10^{-12}.

The numerical results computed by these new parallel Newton-type iterative methods are satisfactory.

Competing Interests

The author declares that there are no competing interests.

Authors' Contributions

The author read and approved the final manuscript.

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