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Research Article

New Subclasses concerning Some Analytic and Univalent Functions

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Considering a function $f(z) = z/(1-z^2)$ which is analytic and starlike in the open unit disc U and a function $f(z) = z/(1-z)$ which is analytic and convex in U, we introduce two new classes $\mathcal{S}_\alpha^*(\beta)$ and $\mathcal{K}_\alpha(\beta)$ concerning $f_\alpha(z) = z/(1-z^\alpha)$ $(\alpha > 0)$. The object of the present paper is to discuss some interesting properties for functions in the classes $\mathcal{S}_\alpha^*(\beta)$ and $\mathcal{K}_\alpha(\beta)$.

1. Introduction and Preliminaries

Let $\mathcal{A}$ be the class of functions $f(z)$ which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ with $f(0) = 0$ and $f'(0) = 1$.

Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z) \in \mathcal{A}$ which are univalent in U. Also, let $\mathcal{S}^*(\beta)$ be the subclass of $\mathcal{S}$ consisting of $f(z)$ which are starlike of order β $(0 \le \beta < 1)$ in U. Further, we say that $f(z) \in \mathcal{K}(\beta)$ if $f(z) \in \mathcal{S}$ satisfies $zf'(z) \in \mathcal{S}^*(\beta)$. A function $f(z) \in \mathcal{K}(\beta)$ is said to be convex of order β in U (cf. [1–3]).

With the above definitions for classes $\mathcal{K}(\beta)$, $\mathcal{S}^*(\beta)$, $\mathcal{S}$, and $\mathcal{A}$, it is known that

$$\mathcal{K}(\beta) \subset \mathcal{S}^*(\beta) \subset \mathcal{S} \subset \mathcal{A} \tag{1}$$

and $f(z) \in \mathcal{S}^*(\beta)$ if and only if $\int_0^z (f(t)/t)\,dt \in \mathcal{K}(\beta)$.

The function $f(z)$ given by

$$f(z) = \frac{z}{1-z^2} = z + z^3 + z^5 + \cdots \quad (z \in U) \tag{2}$$

is in the class $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ and the function $f(z)$ given by

$$f(z) = \frac{z}{1-z} = z + z^2 + z^3 + \cdots \quad (z \in U) \tag{3}$$

is in the class $\mathcal{K}(0) \equiv \mathcal{K}$.

If we consider the function $f(z)$ given by

$$f_\alpha(z) = \frac{z}{1-z^\alpha} = z + \sum_{n=1}^{\infty} z^{1+n\alpha} \quad (z \in U) \tag{4}$$

for some real α $(0 < \alpha \le 2)$, we discuss some properties between functions $f(z)$ in (2) and (3), where we consider the principal value for $z^{n\alpha}$.

With the function $f(z)$ given by (4), we introduce a class $\mathcal{A}_\alpha$ of analytic functions $f(z)$ with series expansion in U such that

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^{1+n\alpha} \quad (z \in U) \tag{5}$$

for some real α $(0 < \alpha \le 2)$, where we take the principal value for $z^{n\alpha}$. If $f(z) \in \mathcal{A}_\alpha$ satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta \quad (z \in U) \tag{6}$$

for some real β $(0 \le \beta < 1)$, then we say that $f(z) \in \mathcal{S}_\alpha^*(\beta)$.

Also, if $f(z) \in \mathscr{A}_\alpha$ satisfies

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta \quad (z \in U) \tag{7}$$

for some real β $(0 \leq \beta < 1)$, then we say that $f(z) \in \mathscr{K}_\alpha(\beta)$.

With the above definitions for the classes $\mathscr{S}^*_\alpha(\beta)$ and $\mathscr{K}_\alpha(\beta)$, we have that $f(z) \in \mathscr{K}_\alpha(\beta)$ if and only if $zf'(z) \in \mathscr{S}^*_\alpha(\beta)$ and that $f(z) \in \mathscr{S}^*_\alpha(\beta)$ if and only if $\int_0^z (f(t)/t)dt \in \mathscr{K}_\alpha(\beta)$.

2. Some Properties

In this section, we consider some properties of functions with series expansion given by (4).

Theorem 1. *If $f(z)$ is given by (4), then $f(z) \in \mathscr{S}^*_\alpha((2-\alpha)/2)$ for $0 < \alpha \leq 2$ and $f(z) \in \mathscr{K}_\alpha(\alpha)$ for $0 < \alpha < 1$.*

Proof. For $f(z)$ given by (4), we see that $zf'(z)/f(z) = 1$ for $z = 0$ and

$$\begin{aligned}\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) &= \operatorname{Re}\left(\frac{1+(\alpha-1)z^\alpha}{1-z^\alpha}\right) \\ &= 1-\alpha+\alpha\operatorname{Re}\left(\frac{1}{1-z^\alpha}\right) \\ &= 1-\alpha+\alpha\operatorname{Re}\left(\frac{1}{1-e^{i\alpha\theta}}\right) = \frac{2-\alpha}{2} < 1\end{aligned} \tag{8}$$

for $z = e^{i\theta}$ $(0 < \theta < 2\pi)$. This shows that $f(z) \in \mathscr{S}^*_\alpha((2-\alpha)/2)$ for $0 < \alpha \leq 2$. Further, we have that $1 + zf''(z)/f'(z) = 1$ for $z = 0$ and

$$\begin{aligned}&\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \\ &\quad = \operatorname{Re}\left(\frac{1+(2\alpha-1)z^\alpha}{1-z^\alpha} + \frac{\alpha(\alpha-1)z^\alpha}{1+(\alpha-1)z^\alpha}\right) \\ &\quad = 3\alpha-1+2(1-\alpha)\operatorname{Re}\left(\frac{1}{1-z^\alpha}\right) \\ &\qquad -\alpha\operatorname{Re}\left(\frac{1}{1+(\alpha-1)z^\alpha}\right) \\ &\quad = 3\alpha-1+2(1-\alpha)\operatorname{Re}\left(\frac{1}{1-e^{i\alpha\theta}}\right) \\ &\qquad -\alpha\operatorname{Re}\left(\frac{1}{1+(\alpha-1)e^{i\alpha\theta}}\right) \\ &\quad = 2\alpha-\alpha\frac{1+(\alpha-1)\cos(\alpha\theta)}{1+(\alpha-1)^2+2(\alpha-1)\cos(\alpha\theta)}\end{aligned} \tag{9}$$

for $z = e^{i\theta}$ $(0 < \theta < 2\pi)$. Letting

$$g(t) = \frac{1+(\alpha-1)t}{1+(\alpha-1)^2+2(\alpha-1)t} \quad (t = \cos(\alpha\theta)), \tag{10}$$

we have that

$$g'(t) = \frac{\alpha(\alpha-1)(\alpha-2)}{\left(1+(\alpha-1)^2+2(\alpha-1)t\right)^2} > 0 \qquad (0 < \alpha < 1). \tag{11}$$

Thus, we see that

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in U) \tag{12}$$

for $0 < \alpha < 1$. This completes the proof of the theorem. □

Corollary 2. *A function*

$$f(z) = \frac{z}{1-\sqrt{z}} \quad (z \in U) \tag{13}$$

belongs to the class $\mathscr{S}^*_{1/2}(3/4)$ and $\mathscr{K}_{1/2}(1/2)$.

Next, we discuss some properties of functions $f(z)$ for $\mathscr{A}_\alpha$.

Theorem 3. *If $f(z)$ given by (5) satisfies*

$$\sum_{n=1}^{\infty}(n\alpha+1-\beta)\left|a_n\right| \leq 1-\beta \tag{14}$$

for some β $(0 \leq \beta < 1)$, then $f(z) \in \mathscr{S}^*_\alpha(\beta)$.

The equality holds true for $f(z)$ given by

$$f(z) = z + \sum_{n=1}^{\infty}\frac{(1-\beta)e^{i\pi}}{n(n+1)(n\alpha+1-\beta)}z^{1+n\alpha}. \tag{15}$$

Proof. Let the function $f(z)$ be given by (5); then, we have that

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)}-1\right| &= \left|\frac{\sum_{n=1}^{\infty}n\alpha a_n z^{n\alpha}}{1+\sum_{n=1}^{\infty}a_n z^{n\alpha}}\right| \leq \frac{\sum_{n=1}^{\infty}n\alpha\left|a_n\right||z|^{n\alpha}}{1-\sum_{n=1}^{\infty}\left|a_n\right||z|^{n\alpha}} \\ &< \frac{\sum_{n=1}^{\infty}n\alpha\left|a_n\right|}{1-\sum_{n=1}^{\infty}\left|a_n\right|} \leq 1-\beta\end{aligned} \tag{16}$$

if $f(z)$ satisfies (14). This shows that $f(z) \in \mathscr{S}^*_\alpha(\beta)$. Further, if we consider a function $f(z)$ given by (15), then we see that

$$\begin{aligned}\sum_{n=1}^{\infty}(n\alpha+1-\beta)\left|a_n\right| &= \sum_{n=1}^{\infty}\frac{1-\beta}{n(n+1)} \\ &= (1-\beta)\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) \\ &= 1-\beta.\end{aligned} \tag{17}$$

□

Theorem 4. *If $f(z)$ given by (5) satisfies*

$$\sum_{n=1}^{\infty}(n\alpha+1)(n\alpha+1-\beta)\left|a_n\right| \leq 1-\beta \tag{18}$$

for some β $(0 \leq \beta < 1)$, then $f(z) \in \mathscr{K}_\alpha(\beta)$.

The equality in (18) holds true for $f(z)$ given by

$$f(z) = z + \sum_{n=1}^{\infty} \frac{(1-\beta)(n\alpha+1)e^{i\pi}}{n(n+1)(n\alpha+1-\beta)} z^{1+n\alpha}. \quad (19)$$

Further, we obtain the following.

Theorem 5. *Let $f(z)$ be given by (5) with $\arg a_n = \pi - n\alpha\theta$ $(0 < \theta < 2\pi)$. Then, $f(z) \in \mathcal{S}_\alpha^*(\beta)$ if and only if*

$$\sum_{n=1}^{\infty} (n\alpha + 1 - \beta)|a_n| \leq 1 - \beta \quad (20)$$

for some β $(0 \leq \beta < 1)$. The equality holds true for

$$f(z) = z + \sum_{n=1}^{\infty} \frac{(1-\beta)e^{i(\pi - n\alpha\theta)}}{n(n+1)(n\alpha+1-\beta)} z^{1+n\alpha}. \quad (21)$$

Proof. Theorem 3 implies that if $f(z)$ satisfies (20), then $f(z) \in \mathcal{S}_\alpha^*(\beta)$. Next, we suppose that $f(z) \in \mathcal{S}_\alpha^*(\beta)$. Then,

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) = \mathrm{Re}\left(\frac{1 + \sum_{n=1}^{\infty}(n\alpha+1)a_n z^{n\alpha}}{1 + \sum_{n=1}^{\infty} a_n z^{n\alpha}}\right). \quad (22)$$

If we consider $z = re^{i\theta}$, then we have that

$$a_n z^{n\alpha} = |a_n| r^{n\alpha} e^{i\pi} = -|a_n| r^{n\alpha}. \quad (23)$$

Then, we obtain that

$$\begin{aligned} \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) &= \frac{1 - \sum_{n=1}^{\infty}(n\alpha+1)|a_n| r^{n\alpha}}{1 - \sum_{n=1}^{\infty}|a_n| r^{n\alpha}} \\ &= 1 - \frac{\sum_{n=1}^{\infty} n\alpha |a_n| r^{n\alpha}}{1 - \sum_{n=1}^{\infty}|a_n| r^{n\alpha}} > \beta. \end{aligned} \quad (24)$$

This gives us

$$\frac{\sum_{n=1}^{\infty} n\alpha |a_n|}{1 - \sum_{n=1}^{\infty}|a_n|} \leq 1 - \beta, \quad (25)$$

that is,

$$\sum_{n=1}^{\infty} (n\alpha + 1 - \beta)|a_n| \leq 1 - \beta. \quad (26)$$

Thus, $f(z) \in \mathcal{S}_\alpha^*(\beta)$ if and only if the coefficient inequality (20) holds true. □

Further, for the class $\mathcal{K}_\alpha(\beta)$, we have the following.

Theorem 6. *Let $f(z)$ be given by (5) with $\arg a_n = \pi - n\alpha\theta$ $(0 < \theta < 2\pi)$. Then, $f(z) \in \mathcal{K}_\alpha(\beta)$ if and only if*

$$\sum_{n=1}^{\infty} (n\alpha+1)(n\alpha + 1 - \beta)|a_n| \leq 1 - \beta \quad (27)$$

for some β $(0 \leq \beta < 1)$. The equality holds true for

$$f(z) = z + \sum_{n=1}^{\infty} \frac{(n\alpha+1)(1-\beta)e^{i(\pi - n\alpha\theta)}}{n(n+1)(n\alpha+1-\beta)} z^{1+n\alpha}. \quad (28)$$

3. Radius Problems

In this section, we consider

$$g(z) = \frac{z}{1 - z^\alpha} \quad (z \in U) \quad (29)$$

for some real $\alpha > 2$. Then, we say that $g(z) \notin \mathcal{S}_\alpha^*(\beta)$ and $g(z) \notin \mathcal{K}_\alpha(\beta)$ for any real β $(0 \leq \beta < 1)$.

Now, we derive the following.

Theorem 7. *If $g(z)$ is given by (29) with $\alpha > 2$, then*

$$\mathrm{Re}\left(\frac{zg'(z)}{g(z)}\right) > \frac{1 - (\alpha-1)r^\alpha}{1 + r^\alpha} \quad (0 < |z| = r < 1). \quad (30)$$

Proof. For $g(z)$ given by (29), we have that

$$\frac{zg'(z)}{g(z)} = \frac{1 + (\alpha-1)r^\alpha e^{i\alpha\theta}}{1 - r^\alpha e^{i\alpha\theta}} = \frac{e^{-i\alpha\theta} + (\alpha-1)r^\alpha}{e^{-i\alpha\theta} - r^\alpha} \quad (31)$$

for $z = re^{i\theta} \in U$. This gives us

$$\mathrm{Re}\left(\frac{zg'(z)}{g(z)}\right) = \frac{1 + (\alpha-2)r^\alpha \cos\alpha\theta - (\alpha-1)r^{2\alpha}}{1 + r^{2\alpha} - 2r^\alpha \cos\alpha\theta}. \quad (32)$$

Letting

$$h(t) = \frac{1 + (\alpha-2)r^\alpha t - (\alpha-1)r^{2\alpha}}{1 + r^{2\alpha} - 2r^\alpha t} \quad (t = \cos\alpha\theta), \quad (33)$$

we see that $h'(t) > 0$. This gives us

$$\mathrm{Re}\left(\frac{zg'(z)}{g(z)}\right) > \frac{1 - (\alpha-1)r^\alpha}{1 + r^\alpha}. \quad (34)$$

□

Corollary 8. *If $g(z)$ is given by (29) with $\alpha > 2$, then*

$$\mathrm{Re}\left(\frac{zg'(z)}{g(z)}\right) > \beta \quad (0 \leq \beta < 1) \quad (35)$$

for $0 < |z| \leq \sqrt[\alpha]{(1-\beta)/(\beta+\alpha-1)} < 1$.

Proof. If we consider

$$\mathrm{Re}\left(\frac{zg'(z)}{g(z)}\right) > \frac{1 - (\alpha-1)r^\alpha}{1 + r^\alpha} \geq \beta, \quad (36)$$

then

$$0 < r \leq \sqrt[\alpha]{\frac{1-\beta}{\beta+\alpha-1}} < 1. \quad (37)$$

□

Remark 9. If $\beta = 0$ in (35), then

$$0 < |z| \leq \sqrt[\alpha]{\frac{1}{\alpha-1}} < 1, \quad (38)$$

and if $\beta = 1/2$, then

$$0 < |z| \leq \sqrt[\alpha]{\frac{1}{2\alpha-1}} < 1. \quad (39)$$

4. Partial Sums

Finally, we consider the partial sums of $f(z)$ given by (5). In view of (5), we write

$$f_n(z) = z + a_n z^{1+n\alpha} \quad (n = 1, 2, 3, \ldots) \tag{40}$$

for some real α $(0 < \alpha \leq 2)$. Recently, Darus and Ibrahim [4] and Hayami et al. [5] have shown some interesting results for some partial sums of analytic functions.

Now, we derive the following.

Theorem 10. *Let $f_n(z)$ be given by (40) with $|a_n| \leq 1$. Then,*

$$\operatorname{Re}\left(\frac{zf_n'(z)}{f_n(z)}\right) > \frac{1-(n\alpha+1)|a_n|}{1-|a_n|} \quad (z \in U), \tag{41}$$

$$\operatorname{Re}\left(\frac{zf_n'(z)}{f_n(z)}\right) \geq \frac{1-(n\alpha+1)r^{n\alpha}}{1-r^{n\alpha}} \quad (|z| = r < 1). \tag{42}$$

Proof. It follows that

$$\begin{aligned}\operatorname{Re}\left(\frac{zf_n'(z)}{f_n(z)}\right) &= \operatorname{Re}\left(1+\frac{n\alpha a_n z^{n\alpha}}{1+a_n z^{n\alpha}}\right) = 1 \\ &+ \operatorname{Re}\left(\frac{n\alpha|a_n|r^{n\alpha}(\cos(n\alpha\theta+\varphi)+i\sin(n\alpha\theta+\varphi))}{1+|a_n|r^{n\alpha}\cos(n\alpha\theta+\varphi)+i|a_n|r^{n\alpha}\sin(n\alpha\theta+\varphi)}\right),\end{aligned} \tag{43}$$

where $a_n = |a_n|e^{i\varphi}$ and $z = re^{i\theta}$. This gives us

$$\begin{aligned}&\operatorname{Re}\left(\frac{zf_n'(z)}{f_n(z)}\right) \\ &= 1+\frac{n\alpha|a_n|r^{n\alpha}(|a_n|r^{n\alpha}+\cos(n\alpha\theta+\varphi))}{1+2|a_n|r^{n\alpha}\cos(n\alpha\theta+\varphi)+|a_n|^2r^{n\alpha}}.\end{aligned} \tag{44}$$

Defining $h(t)$ by

$$\begin{aligned}h(t) &= \frac{|a_n|r^{n\alpha}+t}{1+2|a_n|r^{n\alpha}t+|a_n|^2r^{n\alpha}} \\ &\qquad (t = \cos(n\alpha\theta+\varphi)),\end{aligned} \tag{45}$$

we have that $h'(t) > 0$ with $|a_n| \leq 1$.

Thus, we obtain

$$\operatorname{Re}\left(\frac{zf_n'(z)}{f_n(z)}\right) > 1-\frac{n\alpha|a_n|r^{n\alpha}}{1-|a_n|r^{n\alpha}} \quad (0 \leq r < 1). \tag{46}$$

Making $r \to 1$ in (46), we see (41). Also letting $|a_n| = 1$ in (46), we see (42). □

Corollary 11. *Let $f_n(z)$ be given by (40) with $|a_n| \leq (1-\beta)/(n\alpha+1-\beta)$ $(0 \leq \beta < 1)$. Then, $f_n(z) \in \mathcal{S}_\alpha^*(\beta)$.*

Proof. Since $|a_n| < 1$, $f_n(z)$ satisfies (41).

Therefore, for $|a_n| \leq (1-\beta)/(n\alpha+1-\beta)$, (41) gives us $f_n(z) \in \mathcal{S}_\alpha^*(\beta)$. □

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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