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Research Article

Some Stochastic Functional Differential Equations with Infinite Delay: A Result on Existence and Uniqueness of Solutions in a Concrete Fading Memory Space

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This paper is devoted to existence and uniqueness of solutions for some stochastic functional differential equations with infinite delay in a fading memory phase space.

1. Introduction

Let $|\cdot|$ denote the Euclidian norm in $\mathbb{R}^n$. If A is a vector or a matrix, its transpose is denoted by A' and its trace norm is represented by $|A| = (\mathrm{Trace}(A'A))^{1/2}$. Let $a \wedge b$ $(a \vee b)$ be the minimum (maximum) for $a, b \in \mathbb{R}$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions; that is, it is right continuous and $\mathcal{F}_{t_0}$ contains all P-null sets.

$\mathcal{M}^2((-\infty, T]; \mathbb{R}^n)$ denotes the family of all $\mathcal{F}_t$-measurable $\mathbb{R}^n$ valued processes $x(t)$, $t \in (-\infty, T]$ such that $E(\int_{-\infty}^{T} |x(t)|^2 dt) < \infty$.

Assume that $W(t)$ is an m-dimensional Brownian motion which is defined on $(\Omega, \mathcal{F}, P)$; that is, $W(t) = (W_1(t), W_2(t), \ldots, W_m(t))'$.

Let $C^\mu = \{\varphi \in C(-\infty; 0]; \mathbb{R}^n) : \lim_{\theta\to-\infty} e^{\mu\theta}\varphi(\theta)$ exists in $\mathbb{R}^n\}$ denote the family of continuous functions φ defined on $(-\infty, 0]$ with norm $|\varphi|_\mu = \sup_{\theta\leq 0} e^{\mu\theta}|\varphi(\theta)|$.

Consider the n-dimensional stochastic functional differential equation

$$dx(t) = f(x_t, t)\,dt + g(x_t, t)\,dW(t), \quad t_0 \leq t \leq T, \tag{1}$$

where $x_t : (-\infty, 0] \to \mathbb{R}^n; \theta \mapsto x_t(\theta) = x(t+\theta); -\infty < \theta \leq 0$ can be regarded as a C^μ-value stochastic process, and $f : C^\mu \times [t_0, T] \to \mathbb{R}^n$ and $g : C^\mu \times [t_0, T] \to \mathbb{R}^{n\times m}$ are Borel measurable.

The initial data of the stochastic process is defined on $(-\infty, t_0]$. That is, the initial value $x_{t_0} = \xi = \{\xi(\theta) : -\infty < \theta \leq 0\}$ is a $\mathcal{F}_{t_0}$-measurable and C^μ-value random variable such that $\xi \in \mathcal{M}^2(C^\mu)$.

Our aim, in this paper, is to study existence and uniqueness of solutions to stochastic functional differential equations with infinite delay of type (1) in a fading memory phase space.

2. Preliminary

The theory of partial functional differential equations with delay has attracted widespread attention. However, when the delay is infinite, one of the fundamental tasks is the choice of a suitable phase space $\mathcal{B}$. A large variety of phase spaces could be utilized to build an appropriate theory for any class of functional differential equations with infinite delay. One of the reasons for a best choice is to guarantee that the history function $t \to x_t$ is continuous if $x : (-\infty, a] \to \mathbb{R}^n$ is continuous (where $a > 0$). In general, the selection of the phase space plays an important role in the study of both qualitative and quantitative analysis of solutions. Sometimes, it becomes desirable to approach the problem purely axiomatically. The first axiomatic approach was introduced by Coleman and Mizel in [1]. After this paper, many contributions have been published by various authors until 1978 when Hale and Kato organized the study of functional differential equations with infinite delay in [2]. They assumed that $\mathscr{B}$ is a normed linear space of functions mapping $(-\infty, 0]$ into a Banach space $(X, |\cdot|)$, endowed with a norm $|\cdot|_{\mathscr{B}}$ and satisfying the following axioms.

(A_1) There exist a positive constant H and functions $K(\cdot), M(\cdot) : [0, +\infty) \to [0, +\infty)$, with K being continuous and M being locally bounded, such that for any $\sigma \in \mathbb{R}$ and $a > 0$, if $x : (-\infty, \sigma + a] \to X$, $x_\sigma \in \mathscr{B}$, and $x(\cdot)$ is continuous on $[\sigma, \sigma+a]$, then for all t in $[\sigma, \sigma+a]$, the following conditions hold:

(i) $x_t \in \mathscr{B}$,

(ii) $|x(t)| \le H|x_t|_{\mathscr{B}}$,

(iii) $|x_t|_{\mathscr{B}} \le K(t-\sigma)\sup_{\sigma \le s \le t}|x(s)| + M(t-\sigma)|x_\sigma|_{\mathscr{B}}$.

(A_2) For the function $x(\cdot)$ in (A_1), $t \to x_t$ is a $\mathscr{B}$-valued continuous function for t in $[\sigma, \sigma + a]$.

(A_3) The space $\mathscr{B}$ is complete.

Later on, the concept of fading and uniform fading memory spaces has been adopted as the best choice.

For $\varphi \in \mathscr{B}$, $t \ge 0$ and $\theta \le 0$, we define the linear operator $\mathscr{O}(t)$ by

$$[\mathscr{O}(t)\varphi](\theta) = \begin{cases} \varphi(0) & \text{if } t + \theta \ge 0, \\ \varphi(t+\theta) & \text{if } t + \theta < 0. \end{cases} \tag{2}$$

$(\mathscr{O}(t))_{t \ge 0}$ is exactly the solution semigroup associated with the following trivial equation:

$$\frac{d}{dt}u(t) = 0, \qquad u_0 = \varphi. \tag{3}$$

We define

$$\mathscr{O}_0(t) = \frac{\mathscr{O}(t)}{\mathscr{B}_0}, \quad \text{where } \mathscr{B}_0 := \{\varphi \in \mathscr{B}: \varphi(0) = 0\}. \tag{4}$$

Let $\mathscr{C}_{00}$ be the set of continuous functions $\varphi : (-\infty, 0] \to X$ with compact support. We recall the following axiom.

(A_4) If a uniformly bounded sequence $(\varphi_n)_{n \ge 0}$ in $\mathscr{C}_{00}$ converges to a function φ compactly on $(-\infty, 0]$, then $\varphi \in \mathscr{B}$ and $|\varphi_n - \varphi|_{\mathscr{B}} \to 0$.

Definition 1.

(1) $\mathscr{B}$ is called a fading memory space if it satisfies the axioms $(A_1), (A_2), (A_3), (A_4)$ and $|\mathscr{O}_0(t)\varphi| \to 0$ as $t \to +\infty$, for all $\varphi \in \mathscr{B}_0$.

(2) $\mathscr{B}$ is called a uniform fading memory space if it satisfies the axioms $(A_1), (A_2), (A_3), (A_4)$ and $|\mathscr{O}_0(t)| \to 0$ as $t \to +\infty$.

Examples. We recall the definitions of some standard examples of phase spaces $\mathscr{B}$.

We start first with the phase space of X-valued bounded continuous functions φ defined on $(-\infty, 0]$, that is, $\mathscr{BC}((-\infty, 0]; X)$ with norm $|\varphi|_{\mathscr{BC}} = \sup_{-\infty < \theta \le 0}|\varphi(\theta)|$.

(1) Let

$$\mathscr{BU} = \{\varphi \in \mathscr{BC}((-\infty, 0]; X) : \varphi \text{ is bounded uniformly continuous}\}, \tag{5}$$

where $\mathscr{BC}$ is the space of all bounded continuous functions mapping $(-\infty, 0]$ into X provided with the uniform norm topology.

(2) Let $\mu \in \mathbb{R}$ and

$$\mathscr{C}^{\mu} = \left\{\varphi \in \mathscr{C}((-\infty, 0]; X): \lim_{\theta \to -\infty} e^{\mu\theta}\varphi(\theta) \text{ exists in } X\right\}, \tag{6}$$

provided with the norm

$$|\varphi|_{\mu} = \sup_{-\infty < \theta \le 0} e^{\mu\theta}|\varphi(\theta)|. \tag{7}$$

(3) For any continuous function $g : (-\infty, 0] \to [0, +\infty)$, we define

$$\mathscr{C}_g = \left\{\varphi \in \mathscr{C}((-\infty, 0]; X): \frac{|\varphi(\theta)|}{g(\theta)} \text{ is bounded on } (-\infty, 0]\right\}, \tag{8}$$

$$\mathscr{C}_g^0 = \left\{\varphi \in \mathscr{C}_g((-\infty, 0]; X): \lim_{\theta \to -\infty}\frac{|\varphi(\theta)|}{g(\theta)} = 0\right\},$$

endowed with the norm

$$|\varphi|_g = \sup_{-\infty < \theta \le 0}\frac{|\varphi(\theta)|}{g(\theta)}. \tag{9}$$

Consider the following conditions on g:

(g_1) $\sup_{-\infty < \theta \le -t}(g(t+\theta)/g(\theta))$ is locally bounded for $t \ge 0$,

(g_2) $\lim_{\theta \to -\infty} g(\theta) = +\infty$,

(g_3) $\lim_{t \to +\infty}\sup_{-\infty < \theta \le -t}(g(t+\theta)/g(\theta)) = 0$.

Properties of each phase space are summarized in Table 1.

For other examples, properties, and details about phase spaces, we refer to the book by Hino et al. [3].

Fengying and Ke [4] discussed existence and uniqueness of solutions to stochastic functional differential equation with infinite delay in the phase space of bounded continuous functions φ defined on $(-\infty, 0]$ with values in $\mathbb{R}^n$, that is, $\mathscr{BC}((-\infty, 0]; \mathbb{R}^n)$ with norm $|\varphi|_{\mathscr{BC}} = \sup_{-\infty < \theta \le 0}|\varphi(\theta)|$.

Lemma 2 (page 22 in [3]). *If the phase space $\mathscr{B}$ satisfies axiom (A_4), then $\mathscr{BC}((-\infty, 0]; \mathbb{R}^n)$ is included in $\mathscr{B}$.*

Table 1

	(A_1)	(A_2)	(A_3)	(A_4)	Uniform fading memory space
$\mathscr{BC}$	Yes	No	Yes	No	No
$\mathscr{BU}$	Yes	Yes	Yes	No	No
$\mathscr{C}_g$	Under (g_1)	Under (g_1)	Yes	Under (g_2)	Under (g_3)
$\mathscr{C}_g^0$	Under (g_1)	Under (g_1)	Yes	Under (g_2)	Under (g_3)
$\mathscr{C}^\mu$	Yes	Yes	Yes	Under $\mu > 0$	Under $\mu > 0$

3. Existence and Uniqueness

Lemma 3 (see [4]). *If $p \geq 2$, $g \in L^2([t_0,T];\mathbb{R}^{n\times m})$ such that $E\int_{t_0}^T |g(s)|^p ds < \infty$, then*

$$E\left|\int_{t_0}^{T} g(s)\, dW(s)\right|^p \leq \left(\frac{p(p-1)}{2}\right)^{p/2} T^{(p-2)/2} E\int_{t_0}^{T} |g(s)|^p\, ds. \quad (10)$$

Lemma 4 (Borel-Cantelli, page 487 in [5]). *If $\{E_n\}$ is a sequence of events and*

$$\sum_{n=1}^{\infty} P(E_n) < \infty, \quad (11)$$

then

$$P(\{E_n \ i.o.\}) = 0, \quad (12)$$

where i.o. is an abbreviation for "infinitively often."

Definitions 1. $\mathbb{R}^n$-value stochastic process $x(t)$ defined on $-\infty < t \leq T$ is called the solution of (1) with initial data x_{t_0}, if $x(t)$ has the following properties:

(i) $x(t)$ is continuous and $\{x(t)\}_{t_0\leq t\leq T}$ is $\mathscr{F}_t$-adapted,

(ii) $\{f(x_t,t)\} \in \mathscr{L}^1([t_0,T];\mathbb{R}^n)$ and $\{g(x_t,t)\} \in \mathscr{L}^2([t_0,T];\mathbb{R}^{n\times m})$,

(iii) $x_{t_0} = \xi$, for each $t_0 \leq t \leq T$,

$$x(t) = \xi(0) + \int_{t_0}^{t} f(x_s,s)\, ds + \int_{t_0}^{t} g(x_s,s)\, dW(s) \ \text{ almost surely (a.s.)}. \quad (13)$$

$x(t)$ is called unique solution, if any other solution $\overline{x}(t)$ is distinguishable with $x(t)$; that is,

$$P\{x(t) = \overline{x}(t), \ \text{for any } t_0 \leq t \leq T\} = 1. \quad (14)$$

Now, we establish existence and uniqueness of solutions for (1) with initial data x_{t_0}. We suppose a uniform Lipschitz condition and a weak linear growth condition.

Theorem 5. *Assume that there exist two positive number K and $\overline{K}$ such that,*

(i) for any $\varphi,\psi \in C^\mu$ and $t \in [t_0,T]$, it follows that

$$|f(\varphi,t) - f(\psi,t)|^2 \vee |g(\varphi,t) - g(\psi,t)|^2 \leq \overline{K}|\varphi-\psi|_\mu^2, \quad (15)$$

(ii) for any $t \in [t_0,T]$, it follows that $f(0,t), g(0,t) \in L^2(C^\mu)$ such that

$$|f(0,t)|^2 \vee |g(0,t)|^2 \leq K. \quad (16)$$

Then, problem (1), with initial data $x_{t_0} = \xi \in \mathscr{M}^2((-\infty,0];\mathbb{R}^n)$, has a unique solution $x(t)$. Moreover, $x(t) \in \mathscr{M}^2((-\infty,T];\mathbb{R}^n)$.

Lemma 6. *Let (15) and (16) hold. If $x(t)$ is the solution of (1) with initial data $x_{t_0} = \xi$, then*

$$E\left(\sup_{t_0\leq t\leq T} |x(t)|^2\right) \leq Ce^{6\overline{K}(T-t_0+1)(T-t_0)}, \quad (17)$$

where $C = 3E|\xi|_\mu^2 + 6K(T-t_0+1)(T-t_0) + 6\overline{K}(T-t_0+1)(T-t_0)E|\xi|_\mu^2$.

Moreover, if $\xi \in \mathscr{M}^2((-\infty,0];\mathbb{R}^n)$, then $x(t) \in \mathscr{M}^2((-\infty,T];\mathbb{R}^n)$.

Proof. For each number $q \geq 1$, define the stopping time

$$\tau_q = T \wedge \inf\left\{t \in [t_0,T] : |x_t|_\mu \geq q\right\}. \quad (18)$$

Obviously, as $q \to \infty, \tau_q \nearrow T$ a.s. Let $x^q(t) = x(t\wedge\tau_q)$, $t \in [t_0,T]$, and then $x^q(t)$ satisfy the following equation:

$$x^q(t) = \xi(0) + \int_{t_0}^{t} f(x_s^q,s)\, I_{[t_0,\tau_q]}(s)\, ds + \int_{t_0}^{t} g(x_s^q,s)\, I_{[t_0,\tau_q]}(s)\, dW(s). \quad (19)$$

Using the elementary inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, we get

$$|x^q(t)|^2 \leq 3|\xi|_\mu^2 + 3\left|\int_{t_0}^{t} f(x_s^q,s)\, I_{[t_0,\tau_q]}(s)\, ds\right|^2 + 3\left|\int_{t_0}^{t} g(x_s^q,s)\, I_{[t_0,\tau_q]}(s)\, dW(s)\right|^2. \quad (20)$$

Taking the expectation on both sides and using the Hölder inequality, Lemma 3, and (15) and (16), we get for all t in $[t_0, T]$

$$\begin{aligned}
E\left|x^q(t)\right|^2 &\le 3E\left|\xi\right|_\mu^2 + 3E\left|\int_{t_0}^t f\left(x_s^q, s\right) I_{[t_0,\tau_q]}(s)\, ds\right|^2 \\
&+ 3E\left|\int_{t_0}^t g\left(x_s^q, s\right) I_{[t_0,\tau_q]}(s)\, dW(s)\right|^2 \le 3E\left|\xi\right|_\mu^2 \\
&+ 3\left(t-t_0\right) E\int_{t_0}^t \left|f\left(x_s^q, s\right)\right|^2 ds \\
&+ 3E\int_{t_0}^t \left|g\left(x_s^q, s\right)\right|^2 ds \le 3E\left|\xi\right|_\mu^2 + 3\left(t-t_0\right) \\
&\cdot E\int_{t_0}^t \left|f\left(x_s^q, s\right) - f(0,s) + f(0,s)\right|^2 ds \\
&+ 3E\int_{t_0}^t \left|g\left(x_s^q, s\right) - g(0,s) + g(0,s)\right|^2 ds \\
&\le 3E\left|\xi\right|_\mu^2 + 6\left(t-t_0\right) E\int_{t_0}^t \left|f\left(x_s^q, s\right) - f(0,s)\right|^2 ds \\
&+ 6E\int_{t_0}^t \left|g\left(x_s^q, s\right) - g(0,s)\right|^2 ds + 6\left(t-t_0\right) \\
&\cdot E\int_{t_0}^t \left|f(0,s)\right|^2 ds + 6E\int_{t_0}^t \left|g(0,s)\right|^2 ds \\
&\le 3E\left|\xi\right|_\mu^2 + 6\left(t-t_0\right) E\int_{t_0}^t \left|f\left(x_s^q, s\right) - f(0,s)\right|^2 ds \\
&+ 6E\int_{t_0}^t \left|g\left(x_s^q, s\right) - g(0,s)\right|^2 ds + 6\left(t-t_0\right) \\
&\cdot E\int_{t_0}^t \left|f(0,s)\right|^2 ds + 6E\int_{t_0}^t \left|g(0,s)\right|^2 ds \\
&+ 6E\int_{t_0}^t \left|f\left(x_s^q, s\right) - f(0,s)\right|^2 ds + 6\left(t-t_0\right) \\
&\cdot E\int_{t_0}^t \left|g\left(x_s^q, s\right) - g(0,s)\right|^2 ds \\
&+ 6E\int_{t_0}^t \left|f(0,s)\right|^2 ds + 6\left(t-t_0\right) \\
&\cdot E\int_{t_0}^t \left|g(0,s)\right|^2 ds \le 3E\left|\xi\right|_\mu^2 + 6\left(t-t_0+1\right) \\
&\cdot E\int_{t_0}^t \left|f\left(x_s^q, s\right) - f(0,s)\right|^2 ds + 6\left(t-t_0+1\right) \\
&\cdot E\int_{t_0}^t \left|g\left(x_s^q, s\right) - g(0,s)\right|^2 ds + 6\left(t-t_0+1\right) \\
&\cdot E\int_{t_0}^t \left|f(0,s)\right|^2 ds + 6\left(t-t_0+1\right) \\
&\cdot E\int_{t_0}^t \left|g(0,s)\right|^2 ds \le 3E\left|\xi\right|_\mu^2 + 6\left(t-t_0+1\right) \\
&\cdot E\int_{t_0}^t \left(\left|f\left(x_s^q, s\right) - f(0,s)\right|^2\right. \\
&+ \left|g\left(x_s^q, s\right) - g(0,s)\right|^2 + \left|f(0,s)\right|^2 \\
&+ \left.\left|g(0,s)\right|^2\right) ds \le 3E\left|\xi\right|_\mu^2 + 6\left(t-t_0+1\right) \\
&\cdot E\int_{t_0}^t \left(\overline{K}\left|x_s^q\right|_\mu^2 + K\right) ds \le 3E\left|\xi\right|_\mu^2 + 6K\left(t-t_0\right. \\
&+ \left.1\right)\left(t-t_0\right) + 6\overline{K}\left(t-t_0+1\right) E\int_{t_0}^t \left|x_s^q\right|_\mu^2 ds.
\end{aligned} \tag{21}$$

We have also for each t in $[t_0, T]$

$$\begin{aligned}
\sup_{t_0\le s\le t}\left|x_s^q\right|_\mu^2 &= \sup_{t_0\le s\le t}\left\{\sup_{-\infty<\theta\le 0} e^{2\mu\theta}\left|x^q(s+\theta)\right|^2\right\} \\
&= \sup_{t_0\le s\le t}\left\{\sup_{-\infty<r\le s} e^{2\mu(r-s)}\left|x^q(r)\right|^2\right\} \\
&\le \sup_{t_0\le s\le t} e^{-2\mu s}\left\{\sup_{-\infty<r\le s} e^{2\mu r}\left|x^q(r)\right|^2\right\} \\
&\le \sup_{t_0\le s\le t} e^{-2\mu s}\left\{\sup_{-\infty<r\le t_0} e^{2\mu r}\left|x^q(r)\right|^2\right. \\
&\vee \left.\sup_{t_0\le r\le s} e^{2\mu r}\left|x^q(r)\right|^2\right\} \\
&\le \sup_{t_0\le s\le t} e^{-2\mu s}\left\{\sup_{-\infty<r-t_0\le 0} e^{2\mu r}\left|x^q\left(r-t_0+t_0\right)\right|^2\right. \\
&\vee \left.\sup_{t_0\le r\le s} e^{2\mu r}\left|x^q(r)\right|^2\right\} \\
&\le \sup_{t_0\le s\le t} e^{-2\mu s}\left\{\sup_{-\infty<r-t_0\le 0} e^{2\mu r}\left|x_{t_0}^q\left(r-t_0\right)\right|^2\right. \\
&\vee \left.\sup_{t_0\le r\le s} e^{2\mu r}\left|x^q(r)\right|^2\right\} \le \sup_{t_0\le s\le t} e^{-2\mu s}\left\{e^{2\mu t_0}\left|\xi\right|_\mu^2\right. \\
&\vee \left.\sup_{t_0\le r\le s} e^{2\mu r}\left|x^q(r)\right|^2\right\} \le \sup_{t_0\le s\le t} e^{-2\mu s}\left\{e^{2\mu t_0}\left|\xi\right|_\mu^2\right. \\
&\vee \left.\sup_{t_0\le r\le s} e^{2\mu s}\left|x^q(r)\right|^2\right\} \le e^{2\mu(t_0-s)}\left|\xi\right|_\mu^2 \\
&+ \sup_{t_0\le r\le t}\left|x^q(r)\right|^2 \le \left|\xi\right|_\mu^2 + \sup_{t_0\le r\le t}\left|x^q(r)\right|^2.
\end{aligned} \tag{22}$$

Letting $t = T$, we get

$$E\left(\sup_{t_0\le s\le T}\left|x^q(s)\right|^2\right) \le 3E|\xi|_\mu^2 + 6K(T-t_0+1)(T-t_0) + 6\overline{K}(T-t_0+1)E\int_{t_0}^{T}\left(|\xi|_\mu^2 + \sup_{t_0\le r\le T}\left|x^q(r)\right|^2\right)dr \le C + 6\overline{K}(T-t_0+1)\int_{t_0}^{T}E\left(\sup_{t_0\le r\le T}\left|x^q(r)\right|^2\right)dr, \tag{23}$$

where $C = 3E|\xi|_\mu^2 + 6K(T-t_0+1)(T-t_0) + 6\overline{K}(T-t_0+1)(T-t_0)E|\xi|_\mu^2$.

By the Gronwall inequality, we infer

$$E\left(\sup_{t_0\le s\le T}\left|x^q(s)\right|^2\right) \le Ce^{6\overline{K}(T-t_0+1)(T-t_0)}. \tag{24}$$

That is,

$$E\left(\sup_{t_0\le s\le T}\left|x(s\wedge\tau_q)\right|^2\right) \le Ce^{6\overline{K}(T-t_0+1)(T-t_0)}. \tag{25}$$

Consequently

$$E\left(\sup_{t_0\le s\le \tau_q}|x(s)|^2\right) \le Ce^{6\overline{K}(T-t_0+1)(T-t_0)}. \tag{26}$$

Letting $q\to\infty$, that implies the following inequality

$$E\left(\sup_{t_0\le s\le T}|x(s)|^2\right) \le Ce^{6\overline{K}(T-t_0+1)(T-t_0)}. \tag{27}$$

Now, to prove the second part or the lemma, suppose that $\xi \in \mathscr{M}^2((-\infty,0];\mathbb{R}^n)$. Then

$$\begin{aligned}
E\left(\sup_{-\infty\le t\le T}|x(t)|^2\right) &= E\left(\sup_{-\infty\le t\le t_0}|x(t)|^2\right) + E\left(\sup_{t_0\le t\le T}|x(t)|^2\right) \\
&\le E\left(\sup_{-\infty\le t\le t_0}|x(t)|^2\right) + Ce^{6\overline{K}(T-t_0+1)(T-t_0)} \\
&\le E\left(\sup_{-\infty\le t-t_0\le 0}\left|x(t-t_0+t_0)\right|^2\right) + Ce^{6\overline{K}(T-t_0+1)(T-t_0)} \\
&\le E\left(\sup_{-\infty\le s\le 0}\left|x_{t_0}(s)\right|^2\right) + Ce^{6\overline{K}(T-t_0+1)(T-t_0)} \\
&\le E|\xi|^2 + Ce^{6\overline{K}(T-t_0+1)(T-t_0)} \\
&< \infty.
\end{aligned} \tag{28}$$

The demonstration of the lemma is complete. □

Proof of Theorem 5. We begin by checking uniqueness of solution. Let $x(t)$ and $\overline{x}(t)$ be two solutions of (1), by Lemma 6 $x(t)$ and $\overline{x}(t) \in \mathscr{M}^2((-\infty,T];\mathbb{R}^n)$. Note that

$$x(t) - \overline{x}(t) = \int_{t_0}^{t}\left[f(x_s,s) - f(\overline{x}_s,s)\right]ds + \int_{t_0}^{t}\left[g(x_s,s) - g(\overline{x}_s,s)\right]dW(s). \tag{29}$$

By the elementary inequality, $(a+b)^2 \le 2(a^2+b^2)$, one then gets

$$|x(t) - \overline{x}(t)|^2 \le 2\left|\int_{t_0}^{t}\left[f(x_s,s) - f(\overline{x}_s,s)\right]ds\right|^2 + 2\left|\int_{t_0}^{t}\left[g(x_s,s) - g(\overline{x}_s,s)\right]dW(s)\right|^2. \tag{30}$$

By Hölder inequality, Lemma 3, and (15) and (16), we have

$$\begin{aligned}
E|x(t) - \overline{x}(t)|^2 &\le 2(t-t_0)E\int_{t_0}^{t}\left|f(x_s,s) - f(\overline{x}_s,s)\right|^2 ds + 2E\int_{t_0}^{t}\left|g(x_s,s) - g(\overline{x}_s,s)\right|^2 ds \\
&\le 2\overline{K}(t-t_0)E\int_{t_0}^{t}\left|x_s - \overline{x}_s\right|_\mu^2 ds + 2\overline{K}E\int_{t_0}^{t}\left|x_s - \overline{x}_s\right|_\mu^2 ds \\
&\le 2\overline{K}(t-t_0+1)E\int_{t_0}^{t}\left|x_s - \overline{x}_s\right|_\mu^2 ds.
\end{aligned} \tag{31}$$

From the fact $x_{t_0}(s) = \overline{x}_{t_0}(s) = \xi(s)$, $s \in (-\infty,0]$, and

$$\sup_{t_0\le s\le t}\left|x_s - \overline{x}_s\right|_\mu^2 = \sup_{t_0\le s\le t}\left\{\sup_{-\infty<\theta\le 0}e^{2\mu\theta}\left|x(s+\theta) - \overline{x}(s+\theta)\right|^2\right\} = \sup_{t_0\le s\le t}\left\{\sup_{-\infty<r\le s}e^{2\mu(r-s)}\left|x(r) - \overline{x}(r)\right|^2\right\}$$

$$
\begin{aligned}
&\leq \sup_{t_0\leq s\leq t} e^{-2\mu s}\left\{\sup_{-\infty<r\leq s} e^{2\mu r}\left|x(r)-\overline{x}(r)\right|^2\right\}\\
&\leq \sup_{t_0\leq s\leq t} e^{-2\mu s}\left\{\sup_{-\infty<\theta\leq 0} e^{2\mu(\theta+t_0)}\left|x_{t_0}(\theta)-\overline{x}_{t_0}(\theta)\right|^2\right.\\
&\left.\vee \sup_{t_0\leq r\leq s} e^{2\mu r}\left|x(r)-\overline{x}(r)\right|^2\right\}\\
&\leq \sup_{t_0\leq s\leq t} e^{-2\mu s}\left\{e^{2t_0\mu}\left|\xi-\xi\right|_\mu^2\right.\\
&\left.\vee \sup_{t_0\leq r\leq s} e^{2\mu r}\left|x(r)-\overline{x}(r)\right|^2\right\}\\
&\leq \sup_{t_0\leq s\leq t} e^{-2\mu s}\left\{\sup_{t_0\leq r\leq s} e^{2\mu s}\left|x(r)-\overline{x}(r)\right|^2\right\}\\
&\leq \sup_{t_0\leq r\leq t}\left|x(r)-\overline{x}(r)\right|^2.
\end{aligned}
\tag{32}
$$

We have

$$
\begin{aligned}
&E\left(\sup_{t_0\leq s\leq t}\left|x(s)-\overline{x}(s)\right|^2\right)\\
&\quad\leq 2\overline{K}\left(t-t_0+1\right)\int_{t_0}^{t} E\left(\sup_{t_0\leq r\leq s}\left|x(r)-\overline{x}(r)\right|^2\right)ds.
\end{aligned}
\tag{33}
$$

Applying the Gronwall inequality yields

$$
E\left(\left|x(t)-\overline{x}(t)\right|^2\right)=0,\quad t_0\leq t\leq T. \tag{34}
$$

The above expression means that $x(t)=\overline{x}(t)$ a.s. for $t_0\leq t\leq T$. Therefore, for all $-\infty<t\leq T$, $x(t)=\overline{x}(t)$ a.s., the proof of uniqueness is complete. □

Next, to check the existence, define $x_{t_0}^0=\xi$ and $x^0(t)=\xi(0)$ for $t_0\leq t\leq T$. Let $x_{t_0}^k=\xi$, $k=1,2,\ldots$, and define Picard sequence

$$
\begin{aligned}
x^k(t)&=\xi(0)+\int_{t_0}^{t} f\left(x_s^{k-1},s\right)ds\\
&\quad+\int_{t_0}^{t} g\left(x_s^{k-1},s\right)dW(s),\quad t_0\leq t\leq T.
\end{aligned}
\tag{35}
$$

Obviously $x^0(t)\in\mathscr{M}^2((-\infty,T];\mathbb{R}^n)$. By induction, we can see that $x^k(t)\in\mathscr{M}^2((-\infty,T];\mathbb{R}^n)$.

In fact, by elementary inequality $(a+b+c)^2\leq 3(a^2+b^2+c^2)$

$$
\begin{aligned}
\left|x^k(t)\right|^2&\leq 3\left|\xi(0)\right|^2+3\left|\int_{t_0}^{t} f\left(x_s^{k-1},s\right)ds\right|^2\\
&\quad+3\left|\int_{t_0}^{t} g\left(x_s^{k-1},s\right)dW(s)\right|^2.
\end{aligned}
\tag{36}
$$

From the Hölder inequality and Lemma 3, we have

$$
\begin{aligned}
E\left|x^k(t)\right|^2&\leq 3E\left|\xi\right|_\mu^2+3\left(t-t_0\right)\\
&\quad\cdot E\int_{t_0}^{t}\left|f\left(x_s^{k-1},s\right)-f(0,s)+f(0,s)\right|^2ds\\
&\quad+3E\int_{t_0}^{t}\left|g\left(x_s^{k-1},s\right)-g(0,s)+g(0,s)\right|^2ds.
\end{aligned}
\tag{37}
$$

Again the elementary inequality $(a+b)^2\leq 2a^2+2b^2$, (22), (15), and (16) imply that

$$
\begin{aligned}
&E\left|x^k(t)\right|^2\leq 3E\left|\xi\right|_\mu^2\\
&\quad+3\left(t-t_0+1\right)E\int_{t_0}^{t}\left(2\overline{K}\left|x_s^{k-1}\right|_\mu^2+2K\right)ds\\
&\quad\leq 3E\left|\xi\right|_\mu^2+6K\left(t-t_0+1\right)E\int_{t_0}^{t}ds\\
&\quad+6\overline{K}\left(t-t_0+1\right)E\int_{t_0}^{t}\left|x_s^{k-1}\right|_\mu^2ds\leq C_1\\
&\quad+6\overline{K}\left(t-t_0+1\right)E\int_{t_0}^{t}\left|x_s^{k-1}\right|_\mu^2ds\leq C_1\\
&\quad+6\overline{K}\left(t-t_0+1\right)E\int_{t_0}^{t}\left(\left|\xi\right|_\mu^2+\sup_{t_0\leq r\leq s}\left|x^{k-1}(r)\right|^2\right)ds\\
&\quad\leq C_1+C_2\\
&\quad+6\overline{K}\left(t-t_0+1\right)\int_{t_0}^{t}E\left(\sup_{t_0\leq r\leq s}\left|x^{k-1}(r)\right|^2\right)ds,
\end{aligned}
\tag{38}
$$

where $C_1=3E|\xi|_\mu^2+6K(t-t_0+1)(t-t_0)$ and $C_2=6\overline{K}(t-t_0+1)(t-t_0)E|\xi|_\mu^2$.

Hence, for any $\ell\geq 1$, one can derive that

$$
\begin{aligned}
&\max_{1\leq k\leq\ell} E\left(\sup_{t_0\leq s\leq t}\left|x^k(s)\right|^2\right)\\
&\leq C_1+C_2\\
&\quad+6\overline{K}\left(t-t_0+1\right)\int_{t_0}^{t}\max_{1\leq k\leq\ell}E\left(\sup_{t_0\leq r\leq s}\left|x^{k-1}(r)\right|^2\right)ds.
\end{aligned}
\tag{39}
$$

Note that

$$
\begin{aligned}
&\max_{1\leq k\leq\ell} E\left|x^{k-1}(s)\right|^2=\max\left\{E\left|\xi(0)\right|^2,E\left|x^1(s)\right|^2,\ldots,\right.\\
&\quad\left.E\left|x^{\ell-1}(s)\right|^2\right\}\leq\max\left\{E\left|\xi\right|_\mu^2,E\left|x^1(s)\right|^2,\ldots,\right.\\
&\quad\left.E\left|x^{\ell-1}(s)\right|^2,E\left|x^{\ell}(s)\right|^2\right\}=\max\left\{E\left|\xi\right|_\mu^2,\right.\\
&\quad\left.\max_{1\leq k\leq\ell}E\left|x^k(s)\right|^2\right\}\leq E\left|\xi\right|_\mu^2+\max_{1\leq k\leq\ell}E\left|x^k(s)\right|^2,
\end{aligned}
\tag{40}
$$

and then

$$\max_{1\le k\le \ell} E\left(\sup_{t_0\le s\le t}\left|x^k(s)\right|^2\right) \le C_1 + C_2 + 6\overline{K}(t-t_0+1) \cdot \int_{t_0}^{t}\left(E|\xi|_\mu^2 + \max_{1\le k\le \ell} E\left(\sup_{t_0\le r\le s}\left|x^k(r)\right|^2\right)\right) ds \le C_3 + 6\overline{K}(t-t_0+1)\int_{t_0}^{t}\max_{1\le k\le \ell} E\left(\sup_{t_0\le r\le s}\left|x^k(r)\right|^2\right) ds, \quad (41)$$

where $C_3 = C_1 + 2C_2$.

From the Gronwall inequality, we have

$$\max_{1\le k\le \ell} E\left|x^k(t)\right|^2 \le C_3 e^{6\overline{K}(T-t_0+1)(T-t_0)}. \quad (42)$$

Since k is arbitrary

$$E\left|x^k(t)\right|^2 \le C_3 e^{6\overline{K}(T-t_0+1)(T-t_0)} \quad t_0 \le t \le T,\ k \ge 1. \quad (43)$$

From the Hölder inequality, Lemma 3, and (15) and (16), as in a similar earlier inequality, one then has

$$\begin{aligned} E\left|x^1(t)-x^0(t)\right|^2 &\le 2E\left|\int_{t_0}^{t} f\left(x_s^0,s\right) ds\right|^2 + 2E\left|\int_{t_0}^{t} g\left(x_s^0,s\right) dW(s)\right|^2 \\ &\le 2(t-t_0)E\int_{t_0}^{t}\left|f\left(x_s^0,s\right)\right|^2 ds + 2E\int_{t_0}^{t}\left|g\left(x_s^0,s\right)\right|^2 ds \\ &\le 2(t-t_0+1)E\int_{t_0}^{t}\left(2\overline{K}\left|x_s^0\right|_\mu^2 + 2K\right) ds \\ &\le 4K(t-t_0+1)(t-t_0) + 4\overline{K}(t-t_0+1)(t-t_0)E|\xi|_\mu^2. \end{aligned} \quad (44)$$

That is,

$$E\left(\sup_{t_0\le s\le t}\left|x^1(t)-x^0(t)\right|^2\right) \le 4(t-t_0+1)(t-t_0)\left(K+\overline{K}E|\xi|_\mu^2\right) := R. \quad (45)$$

By similar arguments as above, we also have

$$\begin{aligned} E\left|x^2(t)-x^1(t)\right|^2 &\le 2E\left|\int_{t_0}^{t}\left[f\left(x_s^1,s\right)-f\left(x_s^0,s\right)\right] ds\right|^2 + 2E\left|\int_{t_0}^{t}\left[g\left(x_s^1,s\right)-g\left(x_s^0,s\right)\right] dW(s)\right|^2 \\ &\le 2(t-t_0)E\int_{t_0}^{t}\left|f\left(x_s^1,s\right)-f\left(x_s^0,s\right)\right|^2 ds + 2E\int_{t_0}^{t}\left|g\left(x_s^1,s\right)-g\left(x_s^0,s\right)\right|^2 ds \\ &\le 2(t-t_0+1)E\int_{t_0}^{t}\left[\left|f\left(x_s^1,s\right)-f\left(x_s^0,s\right)\right|^2 + \left|g\left(x_s^1,s\right)-g\left(x_s^0,s\right)\right|^2\right] ds \\ &\le 2\overline{K}(t-t_0+1)E\int_{t_0}^{t}\left|x_s^1-x_s^0\right|_\mu^2 ds. \end{aligned} \quad (46)$$

Then

$$\begin{aligned} E\left(\sup_{t_0\le s\le t}\left|x^2(s)-x^1(s)\right|^2\right) &\le 2\overline{K}(t-t_0+1)\int_{t_0}^{t} E\left(\sup_{t_0\le r\le s}\left|x^1(r)-x^0(r)\right|^2\right) ds \\ &\le 2R\overline{K}(t-t_0+1)(t-t_0) = RM(t-t_0), \end{aligned} \quad (47)$$

where $M = 2\overline{K}(t-t_0+1)$. Similarly

$$\begin{aligned} E\left(\sup_{t_0\le s\le t}\left|x^3(s)-x^2(s)\right|^2\right) &\le M\int_{t_0}^{t} E\left(\sup_{t_0\le r\le s}\left|x^2(r)-x^1(r)\right|^2\right) ds \\ &\le M\int_{t_0}^{t} RM(s-t_0) ds \le \frac{R[M(t-t_0)]^2}{2}. \end{aligned} \quad (48)$$

Continue this process to find that

$$\begin{aligned} E\left(\sup_{t_0\le s\le t}\left|x^4(s)-x^3(s)\right|^2\right) &\le M\int_{t_0}^{t} E\left(\sup_{t_0\le r\le s}\left|x^3(r)-x^2(r)\right|^2\right) ds \\ &\le M\int_{t_0}^{t}\frac{R[M(s-t_0)]^2}{2} ds \le \frac{R[M(t-t_0)]^3}{6}. \end{aligned} \quad (49)$$

Now we claim that for any $k \geq 0$

$$E\left(\sup_{t_0\leq s\leq t}\left|x^{k+1}(s)-x^{k}(s)\right|^2\right)\leq \frac{R\left[M\left(t-t_0\right)\right]^k}{k!}, \quad t_0\leq t\leq T. \tag{50}$$

So, for $k = 0, 1, 2, 3$, inequality (50) holds. We suppose that (50) holds for some k, and check (50) for $k+1$. In fact

$$\begin{aligned} &E\left(\sup_{t_0\leq s\leq t}\left|x^{k+2}(s)-x^{k+1}(s)\right|^2\right)\\ &\leq 2\overline{K}\left(t-t_0+1\right)\int_{t_0}^{t}E\left(\sup_{t_0\leq s\leq t}\left|x_s^{k+1}-x_s^{k}\right|_{\mu}^2\right)ds\\ &\leq M\int_{t_0}^{t}\left(\sup_{t_0\leq r\leq s}\left|x^{k+1}(r)-x^{k}(r)\right|^2\right)ds. \end{aligned} \tag{51}$$

From (50)

$$\begin{aligned} &E\left(\sup_{t_0\leq s\leq t}\left|x^{k+2}(s)-x^{k+1}(s)\right|^2\right)\\ &\leq M\int_{t_0}^{t}\frac{R\left[M\left(s-t_0\right)\right]^k}{k!}ds\leq\frac{R\left[M\left(t-t_0\right)\right]^{k+1}}{(k+1)!}, \end{aligned} \tag{52}$$

which means that (50) holds for $k+1$. Therefore, by induction (50) holds for any $k \geq 0$.

Next to verify $\{x^k(t)\}$ converge to $x(t)$ in L^2 with $x(t)$ in $\mathscr{M}^2((-\infty, T]; \mathbb{R}^n)$ and $x(t)$ is the solution of (1) with initial data x_{t_0}. For (50), taking $t = T$, then

$$E\left(\sup_{t_0\leq t\leq T}\left|x^{k+1}(t)-x^{k}(t)\right|^2\right)\leq\frac{R\left[M\left(T-t_0\right)\right]^k}{k!}. \tag{53}$$

By the Chebyshev inequality

$$\begin{aligned} &P\left\{\sup_{t_0\leq t\leq T}\left|x^{k+1}(t)-x^{k}(t)\right|>\frac{1}{2^k}\right\}\\ &\leq\frac{R\left[4M\left(T-t_0\right)\right]^k}{k!}. \end{aligned} \tag{54}$$

By using Alembert's rule, we show that $\sum_{k=0}^{+\infty}(R[4M(T-t_0)]^k/k!)$ converge.

That is, $\sum_{k=0}^{+\infty}(R[4M(T-t_0)]^k/k!) < \infty$, and by Borel-Cantelli's lemma, for almost all $\omega \in \Omega$, there exists a positive integer $k_0 = k_0(\omega)$ such that

$$\sup_{t_0\leq t\leq T}\left|x^{k+1}(t)-x^{k}(t)\right|\leq\frac{1}{2^k}\quad \text{as } k\geq k_0, \tag{55}$$

and then, $\{x^k(t)\}$ is also a Cauchy sequence in L^2. Hence, $\{x^k(t)\}$ converges uniformly and let $x(t)$ be its limit for any $t \in (-\infty, T]$; since $x^k(t)$ is continuous on $t \in (-\infty, T]$ and $\mathscr{F}_t$ adapted, $x(t)$ is also continuous and $\mathscr{F}_t$ adapted.

So, as $k \to +\infty$, $x^k(t) \to x(t)$ in L^2. That is, $E|x^k(t) - x(t)|^2 \to 0$ as $k \to \infty$.

Then from (43)

$$E\left|x(t)\right|^2\leq C_3e^{6\overline{K}(T-t_0+1)(T-t_0)}\quad \forall t_0\leq t\leq T, \tag{56}$$

and therefore

$$\begin{aligned} E\int_{-\infty}^{T}\left|x(s)\right|^2ds&=E\int_{-\infty}^{t_0}\left|x(s)\right|^2ds+E\int_{t_0}^{T}\left|x(s)\right|^2ds\\ &\leq E\int_{-\infty}^{0}\left|\xi(s)\right|^2ds\\ &\quad+\int_{t_0}^{T}C_3e^{6\overline{K}(T-t_0+1)(T-t_0)}ds<\infty. \end{aligned} \tag{57}$$

That is, $x(t) \in \mathscr{M}^2((-\infty, T]; \mathbb{R}^n)$.

Now, to show that $x(t)$ satisfies (1)

$$\begin{aligned} &E\left|\int_{t_0}^{t}\left[f\left(x_s^k,s\right)-f\left(x_s,s\right)\right]ds\right.\\ &\left.+\int_{t_0}^{t}\left[g\left(x_s^k,s\right)-g\left(x_s,s\right)\right]dW(s)\right|^2\\ &\leq 2E\left|\int_{t_0}^{t}\left[f\left(x_s^k,s\right)-f\left(x_s,s\right)\right]ds\right|^2\\ &+2E\left|\int_{t_0}^{t}\left[g\left(x_s^k,s\right)-g\left(x_s,s\right)\right]dW(s)\right|^2\leq 2(t\\ &-t_0)E\int_{t_0}^{t}\left|f\left(x_s^k,s\right)-f\left(x_s,s\right)\right|^2ds\\ &+2E\int_{t_0}^{t}\left|g\left(x_s^k,s\right)-g\left(x_s,s\right)\right|^2ds\\ &\leq ME\int_{t_0}^{t}\left|x_s^k-x_s\right|_{\mu}^2ds\\ &\leq M\int_{t_0}^{t}E\left(\sup_{t_0\leq r\leq s}\left|x^k(r)-x(r)\right|^2\right)ds\\ &\leq M\int_{t_0}^{T}E\left|x^k(s)-x(s)\right|^2ds. \end{aligned} \tag{58}$$

Noting that the sequence $\{x^k\} \to x(t)$ means that for any given $\varepsilon > 0$ there exists k_0 such that $k \geq k_0$, for any $t \in (-\infty, T]$, one then deduces that

$$\begin{aligned} &E\left|x^k(t)-x(t)\right|^2<\varepsilon,\\ &\int_{t_0}^{T}E\left|x^k(t)-x(t)\right|^2ds<\left(T-t_0\right)\varepsilon, \end{aligned} \tag{59}$$

which means that, for any $t \in [t_0, T]$, one has

$$\begin{aligned} \int_{t_0}^{t} f\left(x_s^k, s\right) ds &\longrightarrow \int_{t_0}^{t} f\left(x_s, s\right) ds, \\ \int_{t_0}^{t} g\left(x_s^k, s\right) dW(s) &\longrightarrow \int_{t_0}^{t} g\left(x_s, s\right) dW(s) \quad \text{in } L^2. \end{aligned} \tag{60}$$

For $t_0 \le t \le T$, taking limits on both sides of (35), we deduce that

$$\begin{aligned} \lim_{k\to\infty} x^k(t) = \xi(0) &+ \lim_{k\to\infty} \int_{t_0}^{t} f\left(x_s^{k-1}, s\right) ds \\ &+ \lim_{k\to\infty} \int_{t_0}^{t} g\left(x_s^{k-1}, s\right) dW(s), \end{aligned} \tag{61}$$

and consequently

$$x(t) = \xi(0) + \int_{t_0}^{t} f\left(x_s, s\right) ds + \int_{t_0}^{t} g\left(x_s, s\right) dW(s) \qquad t_0 \le t \le T. \tag{62}$$

Finally, $x(t)$ is the solution of (1), and the demonstration of existence is complete.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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