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Research Article

Bounded Subsets of Smirnov and Privalov Classes on the Upper Half Plane

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Some characterizations of boundedness in $N_*(D)$ and $N^p(D)$ $(1 < p < \infty)$ will be described, where $N_*(D)$ denote the Smirnov class and $N^p(D)$ the Privalov class on the upper half plane $D = \{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}$, respectively.

1. Introduction

Let U and T denote the unit disk and the unit circle in $\mathbf{C}$, respectively. The Privalov class $N^p(U)$, $1 < p < \infty$, is defined as the set of all holomorphic functions f on U, satisfying

$$\sup_{0<r<1} \int_T \left(\log\left(1 + \left|f(r\zeta)\right|\right)\right)^p d\sigma(\zeta) < +\infty, \tag{1}$$

where $d\sigma$ denotes normalized Lebesgue measure on T. The notion of $N^p(U)$ was introduced by Privalov [1] and has been explored by several authors (see [2–4]). Letting $p = 1$, we have the Nevanlinna class $N(U)$. It is well-known that each function f in $N(U)$ has the nontangential limit $f^*(\zeta) = \lim_{r\to 1^-} f(r\zeta)$ (a.e. $\zeta \in T$) and that $\log(1 + |f|)$ (and, hence, $(\log(1+|f|))^p$ for $p > 1$) is subharmonic if f is holomorphic. Define a metric

$$d_{N^p(U)}(f,g) = \left\{\int_T \left(\log\left(1 + \left|f^*(\zeta) - g^*(\zeta)\right|\right)\right)^p d\sigma(\zeta)\right\}^{1/p} \tag{2}$$

for $f, g \in N^p(U)$. With the metric $d_{N^p(U)}(\cdot,\cdot)$ $N^p(U)$ becomes an F-algebra [2]. Recall that an F-algebra is a topological algebra in which the topology arises from a complete metric.

We denote the Smirnov class by $N_*(U)$, which consists of all holomorphic functions f on U such that $\log(1 + |f(z)|) \le Q[\phi](z)$ $(z \in U)$ for some $\phi \in L^1(T)$, $\phi \ge 0$, where the right side denotes the Poisson integral of ϕ on U. It is known that if $f \in N(U)$, f belongs to $N_*(U)$ if and only if

$$\lim_{r\to 1^-} \int_T \log\left(1 + \left|f(r\zeta)\right|\right) d\sigma(\zeta) = \int_T \log\left(1 + \left|f^*(\zeta)\right|\right) d\sigma(\zeta). \tag{3}$$

Under the metric

$$d_{N_*(U)}(f,g) = \int_T \log\left(1 + \left|f^*(\zeta) - g^*(\zeta)\right|\right) d\sigma(\zeta) \tag{4}$$

for $f, g \in N_*(U)$, the class is also an F-algebra (see [5]).

For $0 < p < \infty$, the class $M^p(U)$ is defined as the set of all holomorphic functions f on U such that

$$\int_T \left(\log\left(1 + Mf(\zeta)\right)\right)^p d\sigma(\zeta) < +\infty, \tag{5}$$

where $Mf(\zeta) = \sup_{0\le r<1} |f(r\zeta)|$ is the maximal function. The class $M^1(U)$ was introduced by Kim in [6]. As for $p > 0$, the class was considered in [7, 8]. For $f, g \in M^p(U)$, define a metric

$$d_{M^p(U)}(f,g) = \left\{\int_T \left(\log\left(1 + M(f-g)(\zeta)\right)\right)^p d\sigma(\zeta)\right\}^{\alpha_p/p}, \tag{6}$$

where $\alpha_p = \min(1, p)$. With this metric $M^p(U)$ is also an F-algebra (see [9]).

It is well-known that $H^q(U) \subsetneq N^p(U) \subsetneq M^1(U) \subsetneq N_*(U) \subsetneq N(U)$ $(0 < q \le +\infty,\ p > 1)$, where $H^q(U)$ denotes the Hardy space on U. Moreover, it is known that $N(U) \subsetneq M^p(U)$ $(0 < p < 1)$ [6].

Mochizuki [10] introduced the Nevanlinna class $N_0(D)$ and the Smirnov class $N_*(D)$ on the upper half plane $D := \{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}$: the class $N_0(D)$ is the set of all holomorphic functions f on D satisfying

$$\sup_{y>0} \int_{\mathbf{R}} \log\left(1 + \left|f\left(x+iy\right)\right|\right) dx < +\infty \tag{7}$$

and $N_*(D)$ the set of all holomorphic functions f on D satisfying $\log(1 + |f(z)|) \le P[\phi](z)$ $(z \in D)$ for some $\phi \in L^1(\mathbf{R})$, $\phi \ge 0$, where the right side denotes the Poisson integral of ϕ on D. It is well-known that each function f in $N_0(D)$ has the nontangential limit $f^*(x) = \lim_{y\to 0^+} f(x+iy)$ (a.e. $x \in \mathbf{R}$). Let $f \in N_0(D)$. Then $f \in N_*(D)$ if and only if

$$\begin{aligned} &\lim_{y\to 0^+} \int_{\mathbf{R}} \log\left(1 + \left|f\left(x+iy\right)\right|\right) dx \\ &\quad = \int_{\mathbf{R}} \log\left(1 + \left|f^*\left(x\right)\right|\right) dx \end{aligned} \tag{8}$$

(see [10]). Moreover, under the metric

$$d_{N_*(D)}\left(f, g\right) = \int_{\mathbf{R}} \log\left(1 + \left|f^*\left(x\right) - g^*\left(x\right)\right|\right) dx, \tag{9}$$

the class $N_*(D)$ becomes an F-algebra [10].

The class $M^p(D)$ $(0 < p < \infty)$ is defined as the set of all holomorphic functions f on D such that

$$\int_{\mathbf{R}} \left(\log\left(1 + Mf\left(x\right)\right)\right)^p dx < +\infty, \tag{10}$$

where $Mf(x) = \sup_{y>0}|f(x+iy)|$. The class $M^p(X)$ with $p = 1$ was introduced by Ganzhula in [11]. As for $p > 0$, Efimov and Subbotin investigated this class [12]. For $f, g \in M^p(D)$, define a metric

$$\begin{aligned} &d_{M^p(D)}\left(f, g\right) \\ &\quad = \left\{\int_{\mathbf{R}} \left(\log\left(1 + M\left(f-g\right)\left(x\right)\right)\right)^p dx\right\}^{\alpha_p/p}, \end{aligned} \tag{11}$$

where $\alpha_p = \min(1, p)$. With this metric $M^p(D)$ is also an F-algebra (see [11, 12]).

In [13], the class $N^p(D)$ was introduced, analogous to $N^p(U)$; that is, we denote by $N^p(D)$ $(p > 1)$ the set of all holomorphic functions f on D such that

$$\sup_{y>0} \int_{\mathbf{R}} \left(\log\left(1 + \left|f\left(x+iy\right)\right|\right)\right)^p dx < +\infty. \tag{12}$$

Each $f \in N^p(D)$ has the nontangential limit $f^*(x)$ for a.e. $x \in \mathbf{R}$, and under the metric,

$$\begin{aligned} &d_{N^p(D)}\left(f, g\right) \\ &\quad = \left\{\int_{\mathbf{R}} \left(\log\left(1 + \left|f^*\left(x\right) - g^*\left(x\right)\right|\right)\right)^p dx\right\}^{1/p}, \end{aligned} \tag{13}$$

the class $N^p(D)$ becomes an F-algebra [13].

A subset L of a linear topological space A is said to be *bounded* if for any neighborhood U of zero in A there exists a real number α, $0 < \alpha < 1$, such that $\alpha L = \{\alpha f;\ f \in L\} \subset U$. Yanagihara characterized bounded subsets of $N_*(U)$ [14]. As for $M^p(U)$ with $p = 1$, Kim described some characterizations of boundedness (see [6]). For $p > 1$, these characterizations were considered by Meštrović [15]. As for $M^p(D)$ with $p = 1$, Ganzhula investigated the properties of boundedness [11] and Efimov characterized bounded subsets of $M^p(D)$ in the case $0 < p < \infty$ [16]. In recent paper [17], the author described bounded subsets of $M^p(U)$ in the case $0 < p < 1$.

In this paper, we consider some characterizations of boundedness in $N_*(D)$ and $N^p(D)$ $(p > 1)$.

2. The Results

Theorem 1. *Let $p > 1$. $L \subset N^p(D)$ is bounded if and only if*

(i) there exists a $K < \infty$ such that

$$\int_{\mathbf{R}} \left(\log\left(1 + \left|f^*\left(x\right)\right|\right)\right)^p dx < K \tag{14}$$

for all $f \in L$;

(ii) for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_E \left(\log\left(1 + \left|f^*\left(x\right)\right|\right)\right)^p dx < \varepsilon, \quad \forall f \in L, \tag{15}$$

for any measurable set $E \subset \mathbf{R}$ with the Lebesgue measure $|E| < \delta$.

Proof. We follow [16, Theorem 1].

Necessity. Let L be a bounded subset of $N^p(D)$.

(i) For any number $\eta > 0$ there exists $\alpha = \alpha(\eta)$, $0 < \alpha < 1$, such that

$$\left(d_{N^p(D)}\left(\alpha f, 0\right)\right)^p = \int_{\mathbf{R}} \left(\log\left(1 + \alpha\left|f^*\left(x\right)\right|\right)\right)^p dx < \eta^p \tag{16}$$

for all $f \in L$. Utilizing the inequality $(1+x)^\alpha \le 1 + \alpha x$ $(0 < \alpha < 1,\ x \ge 0)$, it follows that, from (16),

$$\begin{aligned} &\int_{\mathbf{R}} \left(\log\left(1 + \left|f^*\left(x\right)\right|\right)\right)^p dx \\ &\quad \le \int_{\mathbf{R}} \left(\log\left(1 + \alpha\left|f^*\left(x\right)\right|\right)^{1/\alpha}\right)^p dx \\ &\quad = \frac{1}{\alpha^p} \int_{\mathbf{R}} \left(\log\left(1 + \alpha\left|f^*\left(x\right)\right|\right)\right)^p dx < \left(\frac{\eta}{\alpha}\right)^p = K \\ &\quad = \text{constant} \end{aligned} \tag{17}$$

for all $f \in L$. Therefore, condition (i) holds.

(ii) For any number $\varepsilon > 0$, we take η as $\eta < \varepsilon^{1/p}/2$. Choose a number $\alpha = \alpha(\varepsilon)$, $0 < \alpha < 1$, such that equality (16) holds for all $f \in L$. Then for any measurable set $E \subset \mathbf{R}$, using Minkowski's inequality, we have the estimate

$$\begin{aligned} &\int_E \left(\log\left(1 + \left|f^*\left(x\right)\right|\right)\right)^p dx \\ &\quad < \int_E \left(\log\left(\frac{1}{\alpha} + \left|f^*\left(x\right)\right|\right)\right)^p dx \end{aligned}$$

$$
\begin{aligned}
&= \int_E \left(\log\frac{1}{\alpha} + \log\left(1+\alpha\left|f^*(x)\right|\right)\right)^p dx \\
&\le \left(\left(|E|\left(\log\frac{1}{\alpha}\right)^p\right)^{1/p}\right. \\
&\quad \left. + \left(\int_{\mathbf{R}} \left(\log\left(1+\alpha\left|f^*(x)\right|\right)\right)^p dx\right)^{1/p}\right)^p \\
&< \left(|E|^{1/p}\log\frac{1}{\alpha} + \eta\right)^p .
\end{aligned} \tag{18}
$$

If we take $\delta > 0$ as $\delta < \varepsilon/(2^p(\log(1/\alpha))^p)$, then

$$
\begin{aligned}
\int_E \left(\log\left(1+\left|f^*(x)\right|\right)\right)^p dx &< \left(\delta^{1/p}\log\frac{1}{\alpha} + \frac{\varepsilon^{1/p}}{2}\right)^p \\
&< \left(\frac{\varepsilon^{1/p}}{2} + \frac{\varepsilon^{1/p}}{2}\right)^p = \varepsilon
\end{aligned} \tag{19}
$$

for all $f \in L$ and any measurable set $E \subset \mathbf{R}$, $|E| < \delta$. Thus condition (ii) holds.

Sufficiency. Let conditions (i) and (ii) hold for a subset L of $N^p(D)$, $p > 1$. Consider a neighborhood

$$
V = \left\{g \in N^p(D) : d_{N^p(D)}(g, 0) < \eta\right\}. \tag{20}
$$

Take $\varepsilon > 0$ as $\varepsilon < \eta^p/3$. According to (ii), there exists a number $\delta > 0$ such that

$$
\int_E \left(\log\left(1+\left|f^*(x)\right|\right)\right)^p dx < \varepsilon < \frac{\eta^p}{3} \tag{21}
$$

for all $f \in L$ and any measurable set $E \subset \mathbf{R}$, $|E| < \delta$. Next there exists a finite constant $K > 0$ such that condition (i) holds for all $f \in L$. Applying Chebyshev's inequality to the Lebesgue measure of the set $E_f = \{x \in \mathbf{R} \mid (\log(1 + |f^*(x)|))^p > K/\delta\}$ for $f \in L$, the following estimate is valid:

$$
\left|E_f\right| \le \frac{\delta}{K}\int_{\mathbf{R}} \left(\log\left(1+\left|f^*(x)\right|\right)\right)^p dx < \delta. \tag{22}
$$

Then we may assume $E = E_f$ and $|f^*(x)| > \exp(K/\delta)^{1/p} - 1 = C$ in inequality (21); that is, $|f^*(x)|/C < 1$ for all $x \in \mathbf{R}\backslash E_f$. Therefore, for any number α $(0 < \alpha < 1)$ and all $f \in L$, we have the following:

$$
\begin{aligned}
&\int_{\mathbf{R}} \left(\log\left(1+\alpha\left|f^*(x)\right|\right)\right)^p dx \\
&\quad = \int_{E_f} \left(\log\left(1+\alpha\left|f^*(x)\right|\right)\right)^p dx \\
&\qquad + \int_{\mathbf{R}\backslash E_f} \left(\log\left(1+\alpha\left|f^*(x)\right|\right)\right)^p dx \\
&\quad < \int_{E_f} \left(\log\left(1+\left|f^*(x)\right|\right)\right)^p dx \\
&\qquad + \int_{E_1} \left(\log\left(1+\alpha\left|f^*(x)\right|\right)\right)^p dx \\
&\qquad + \int_{E_2} \left(\log\left(1+\alpha\left|f^*(x)\right|\right)\right)^p dx,
\end{aligned} \tag{23}
$$

where $\mathbf{R}\backslash E_f = E_1 \cup E_2$, $E_1 = \{x \in \mathbf{R} \mid |f^*(x)| < 1\}$, and $E_2 = \{x \in \mathbf{R} \mid 1 \le |f^*(x)| < C\}$. By using the elementary inequality $1 + \beta t \le (1+t)^{2\beta}$ $(0 \le t < 1,\ 0 < \beta < 1/2)$ to the second integral in (23), using (21) and taking

$$
\alpha = \min\left(\frac{1}{2}, \frac{1}{2}\left(\frac{\eta^p}{3K}\right)^{1/p}, \frac{1}{C}\left(2^{(\eta^p/3K)^{1/p}} - 1\right)\right), \tag{24}
$$

we have the following estimate

$$
\begin{aligned}
&\int_{\mathbf{R}} \left(\log\left(1+\alpha\left|f^*(x)\right|\right)\right)^p dx \\
&\quad < \frac{\eta^p}{3} + (2\alpha)^p K + \frac{\eta^p}{3K}\int_{E_2} \left(\log(1+1)\right)^p dx \\
&\quad \le \frac{\eta^p}{3} + \frac{\eta^p}{3} \\
&\qquad + \frac{\eta^p}{3K}\int_{\mathbf{R}} \left(\log\left(1+\left|f^*(x)\right|\right)\right)^p dx < \eta^p.
\end{aligned} \tag{25}
$$

Therefore, $\alpha L \subset V$ and the set L is bounded in $N^p(D)$ by definition.

The proof of the theorem is complete. □

Next we consider some characterizations of boundedness in $N_*(D)$. Proof of the following theorem can be obtained by taking $p = 1$ in the whole proof of Theorem 1; therefore, this proof may be omitted.

Theorem 2. *$L \subset N_*(D)$ is bounded if and only if*

(i) there exists $K < \infty$ such that

$$
\int_{\mathbf{R}} \log\left(1+\left|f^*(x)\right|\right) dx < K \tag{26}
$$

for all $f \in L$;

(ii) for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$
\int_E \log\left(1+\left|f^*(x)\right|\right) dx < \varepsilon, \quad \forall f \in L, \tag{27}
$$

for any measurable set $E \subset \mathbf{R}$ with the Lebesgue measure $|E| < \delta$.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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