Hindawi
International Journal of Analysis
Volume 2017, Article ID 9721671, 3 pages
https://doi.org/10.1155/2017/9721671

Letter to the Editor

Comment on "Some Congruence Properties of a Restricted Bipartition Function $c_N(n)$"

Qing Zou

Department of Mathematics, The University of Iowa, Iowa City, IA 52242-1419, USA

Correspondence should be addressed to Qing Zou; zou-qing@uiowa.edu

Received 9 January 2017; Accepted 12 June 2017; Published 13 July 2017

Academic Editor: Ahmed Zayed

1. Introduction

This commentary is intended to correct the errors that happened in [1]. Recall that a partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n. A bipartition (λ, μ) of a positive integer n is a pair of partitions (λ, μ) such that the sum of all the parts is n. Let $p_{-2}(n)$ be the number of bipartitions of n. This function has some beautiful arithmetic properties (see, e.g., [2]). Recently, there is more and more research on the bipartitions with certain restrictions on each partition, for example, [3–7]. In this short review, we consider the number of bipartitions (λ, μ) with the restriction that each part of μ is divisible by N. Let $c_N(n)$ denote the number of bipartitions (λ, μ) of n such that part of μ is divisible by N. The generating function for $c_N(n)$ is given by [8]

$$\sum_{n=0}^{\infty} c_N(n) q^n = \frac{1}{(q;q)_\infty (q^N;q^N)_\infty}, \tag{1}$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n). \tag{2}$$

We will mainly focus on the case when $N = 5\ell$, where ℓ is any positive integer. In [1], the discussion on congruence identities for $c_{5\ell}(n)$ was based on an incorrect lemma. We will point out all the incorrect points in that paper in the next section.

2. Comments on the Results of Saikia and Boruah

In [1], Saikia and Boruah considered the congruence identities for $c_{5\ell}(n)$ by employing the following lemma.

Lemma 13 of [1]

$$\begin{aligned}\frac{1}{(q;q)_\infty} = {} & \frac{(q^{25};q^{25})_\infty^6}{(q^5;q^5)_\infty^6} \left(F^4(q^5) + qF^3(q^5)\right. \\ & + 2q^2F^2(q^5) + 3q^3F(q^5) + 5q^4 - 3q^5F^{-1}(q^5) \\ & \left. + 2q^6F^{-2}(q^5) - q^7F^{-3}(q^5) + q^8F^{-4}(q^5)\right),\end{aligned} \tag{3}$$

where

$$R(q) := \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cdots}}}}, \quad |q| < 1 \tag{4}$$

is the Rogers-Ramanujan continued fraction and

$$F(q) := q^{-1/5} R(q). \tag{5}$$

However, the lemma of this version is wrong, The correct version should be as follows.

Lemma 1. *Let $F(q)$ be defined as above:*

$$\frac{1}{(q;q)_\infty} = \frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6}\left(\frac{1}{F^4(q^5)} + \frac{q}{F^3(q^5)} + \frac{2q^2}{F^2(q^5)} + \frac{3q^3}{F(q^5)} + 5q^4 - 3q^5F(q^5) + 2q^6F^2(q^5) - q^7F^3(q^5) + q^8F^4(q^5)\right). \tag{6}$$

There are several proofs of this famous identity (see, e.g., [9, 10]).

In [1], all the discussion in Section 5 was based on Lemma 13 of [1] which has been shown above.

Since Lemma 13 of [1] is not correct, the discussion in Section 5 is not correct.

First, we can say that the conclusion shown in Theorem 24 of [1], which says

$$c_{5\ell}(5n+4) \equiv 0 \pmod 5, \tag{7}$$

is correct. However, in the course of the proof, there is one mistake. The equation

$$\sum_{n=0}^{\infty} c_{5\ell}(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^6}{(q^\ell;q^\ell)_\infty(q;q)_\infty^6}, \tag{8}$$

which is equation (79) of [1], is not correct. It should be

$$\sum_{n=0}^{\infty} c_{5\ell}(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q^\ell;q^\ell)_\infty(q;q)_\infty^6}. \tag{9}$$

The first result of Theorem 25 of [1] is also correct, but the proof has a problem. In the course of the proof, equation (81) of [1] should be

$$\sum_{n=0}^{\infty} c_{15}(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q^3;q^3)_\infty(q;q)_\infty^6}, \tag{10}$$

rather than

$$\sum_{n=0}^{\infty} c_{15}(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^6}{(q^3;q^3)_\infty(q;q)_\infty^6}. \tag{11}$$

However, the second conclusion and the third conclusion of Theorem 25 of [1] are not correct. The proof of Theorem 5 (ii, iii) of [1] relies on the claim that

$$(q^5;q^5)_\infty^6 \equiv (q^{15};q^{15})_\infty^2 \pmod 3. \tag{12}$$

However, as we have pointed out above, we do not have the factor $(q^5;q^5)_\infty^6$ in $\sum_{n=0}^{\infty} c_{15}(5n+4)q^n$. This is the reason why the second conclusion and the third conclusion of Theorem 25 of [1] cannot hold.

Actually, we do have some 3-dissection formulas to find how $\sum_{n=0}^{\infty} c_{15}(15n+9)q^n$ and $\sum_{n=0}^{\infty} c_{15}(15n+14)q^n$ modulo-3 look like. Let us show them now.

Thanks to [11], we have

$$(q^5;q^5)_\infty^2(q;q)_\infty^2 \equiv (q^3;q^3)_\infty^4 + q(q^3;q^3)_\infty^2(q^{15};q^{15})_\infty^2 - q^2(q^{15};q^{15})_\infty^4 \pmod 3. \tag{13}$$

If we let $a = -q$, $b = q^2$ and $n = 3$ in Entry 31 of [12, p. 48], we can obtain

$$(q;q)_\infty = J_{12,27} - qJ_{6,27} - q^2J_{3,27}, \tag{14}$$

where

$$J_{a,b} = (q^a;q^b)_\infty(q^{b-a};q^b)_\infty(q^b;q^b)_\infty. \tag{15}$$

According to these two formulas, we get that

$$\begin{aligned}
\sum_{n=0}^{\infty} c_{15}(5n+4)q^n &\equiv 5\frac{(q^5;q^5)_\infty^5}{(q^3;q^3)_\infty(q;q)_\infty^6} \equiv 2 \\
&\cdot \frac{(q^5;q^5)_\infty^3(q^5;q^5)_\infty^2}{(q^3;q^3)_\infty(q;q)_\infty^6} \equiv 2 \\
&\cdot \frac{(q^{15};q^{15})_\infty(q^5;q^5)_\infty^2(q;q)_\infty^2}{(q^3;q^3)_\infty^4}\cdot(q;q)_\infty \equiv 2 \\
&\cdot \left[(q^{15};q^{15})_\infty + q\frac{(q^{15};q^{15})_\infty^3}{(q^3;q^3)_\infty^2} - q^2\frac{(q^{15};q^{15})_\infty^5}{(q^3;q^3)_\infty^4}\right] \\
&\times (J_{12,27} - qJ_{6,27} - q^2J_{3,27}) \equiv 2 \\
&\cdot \left[J_{12,27}(q^{15};q^{15})_\infty + qJ_{12,27}\frac{(q^{15};q^{15})_\infty^3}{(q^3;q^3)_\infty^2}\right. \\
&- q^2J_{12,27}\frac{(q^{15};q^{15})_\infty^5}{(q^3;q^3)_\infty^4} - qJ_{6,27}(q^{15};q^{15})_\infty \\
&- q^2J_{6,27}\frac{(q^{15};q^{15})_\infty^3}{(q^3;q^3)_\infty^2} + q^3J_{6,27}\frac{(q^{15};q^{15})_\infty^5}{(q^3;q^3)_\infty^4} \\
&- q^2J_{3,27}(q^{15};q^{15})_\infty - q^3J_{3,27}\frac{(q^{15};q^{15})_\infty^3}{(q^3;q^3)_\infty^2} \\
&\left.+ q^4J_{3,27}\frac{(q^{15};q^{15})_\infty^5}{(q^3;q^3)_\infty^4}\right] \pmod 3.
\end{aligned} \tag{16}$$

When extracting terms involving q^{3n+1}, dividing by q and replacing q^3 by q, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} c_{15}(15n+9)q^n &\equiv 2\cdot\left[J_{4,9}\frac{(q^5;q^5)_\infty^3}{(q;q)_\infty^2}\right.\\ &\left.-J_{2,9}(q^5;q^5)_\infty+qJ_{1,9}\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^4}\right]\\ &\equiv J_{2,9}(q^5;q^5)_\infty+2J_{4,9}\frac{(q^5;q^5)_\infty^3}{(q;q)_\infty^2}+2qJ_{1,9}\\ &\cdot\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^4}\pmod 3.\end{aligned}\tag{17}$$

When extracting terms involving q^{3n+2}, dividing by q^2 and replacing q^3 by q, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} c_{15}(15n+14)q^n &\equiv -2\\ &\cdot\left[J_{4,9}\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^4}+J_{2,9}\frac{(q^5;q^5)_\infty^3}{(q;q)_\infty^3}+J_{1,9}(q^5;q^5)_\infty\right]\\ &\equiv J_{4,9}\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^4}+J_{2,9}\frac{(q^5;q^5)_\infty^3}{(q;q)_\infty^3}+J_{1,9}(q^5;q^5)_\infty\\ &\pmod 3.\end{aligned}\tag{18}$$

Up to now, we have obtained the forms of $\sum_{n=0}^{\infty} c_{15}(15n+9)q^n$ and $\sum_{n=0}^{\infty} c_{15}(15n+14)q^n$ modulo-3. However, we cannot get any congruence properties of $c_{15}(15n+9)$ or $c_{15}(15n+14)$ by the forms of $\sum_{n=0}^{\infty} c_{15}(15n+9)q^n$ and $\sum_{n=0}^{\infty} c_{15}(15n+14)q^n$ modulo-3 above.

Conflicts of Interest

The author declares no conflicts of interest.

References

[1] N. Saikia and C. Boruah, "Some congruence properties of a restricted bipartition function $c_N(n)$," *International Journal of Analysis*, vol. 2016, Article ID 9037692, 7 pages, 2016.

[2] S. H. Chan and R. Mao, "Pairs of partitions without repeated odd parts," *Journal of Mathematical Analysis and Applications*, vol. 394, no. 1, pp. 408–415, 2012.

[3] W. Y. Chen and B. L. Lin, "Congruences for bipartitions with odd parts distinct," *Ramanujan Journal. An International Journal Devoted to the Areas of Mathematics Influenced by Ramanujan*, vol. 25, no. 2, pp. 277–293, 2011.

[4] W. Y. Chen and B. L. Lin, "Arithmetic properties of overpartition pairs," *Acta Arithmetica*, vol. 151, no. 3, pp. 263–277, 2012.

[5] B. L. Lin, "Arithmetic properties of bipartitions with even parts distinct," *Ramanujan Journal*, vol. 33, no. 2, pp. 269–279, 2014.

[6] B. L. Lin, "Some results on bipartitions with 3-core," *Journal of Number Theory*, vol. 139, pp. 44–52, 2014.

[7] B. L. S. Lin, "Arithmetic of the 7-regular bipartition function modulo 3," *Ramanujan Journal*, vol. 37, no. 3, pp. 469–478, 2015.

[8] J. Sinick, "Ramanujan congruences for a class of eta quotients," *International Journal of Number Theory*, vol. 6, no. 4, pp. 835–847, 2010.

[9] B. C. Berndt, *Number theory in the spirit of Ramanujan*, vol. 34 of *Student Mathematical Library*, American Mathematical Society, 2006.

[10] M. D. Hirschhorn, "An identitiy of Ramanujan, and applications," in *Contemporary Mathematics*, pp. 229–234, 2000.

[11] J. Liu and A. Y. Z. Wang, "Arithmetic properties of a restricted bipartition function," *The Electronic Journal of Combinatorics*, vol. 22, 2015.

[12] B. C. Berndt, *Ramanujan's Notebooks. Part III*, Springer, New York, NY, USA, 1991.