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Research Article

The Configuration Space of n-Tuples of Equiangular Unit Vectors for $n = 3, 4$, and 5

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Let $M_n(\theta)$ be the configuration space of n-tuples of unit vectors in $\mathbb{R}^3$ such that all interior angles are θ. The space $M_n(\theta)$ is an $(n-3)$-dimensional space. This paper determines the topological type of $M_n(\theta)$ for $n = 3$, 4, and 5.

1. Introduction

Recently, starting in [1], the topology of the configuration space of spatial polygons of arbitrary edge lengths has been considered by many authors. In the equilateral case, the definition is given as follows. For $\ell > 0$, we set

$$P_n(\ell) = \frac{\left\{(a_1,\dots,a_n) \in \left(S^2\right)^n \mid \sum_{i=1}^n \ell a_i = 0\right\}}{SO(3)}. \tag{1}$$

Here $a_i \in S^2$ denote the unit vectors in the directions of the edges of a polygon; the group $SO(3)$ acts diagonally on $(a_1, \dots, a_n)$.

Many topological properties of $P_n(\ell)$ are already known: First, it is clear that there is a homeomorphism

$$P_n(\ell) \cong P_n(1) \quad \forall \ell. \tag{2}$$

Second, it is proved in [2] that $P_5(1)$ is homeomorphic to del Pezzo surface of degree 5.

Third, when n is odd, the integral cohomology ring $H^*(P_n(1); \mathbb{Z})$ was determined in [3]. We refer to [4] for other properties of $P_n(1)$, which is an excellent survey of linkages.

In another direction, we consider the space of n-tuples of equiangular unit vectors in $\mathbb{R}^3$. More precisely, we define the following: We fix $\theta \in [0, \pi]$ and set

$$A_n(\theta) = \left\{(a_1,\dots,a_n) \in \left(S^2\right)^n \mid \langle a_i, a_{i+1}\rangle = \cos\theta \text{ for } 1 \le i \le n-1,\ \langle a_n, a_1\rangle = \cos\theta\right\}, \tag{3}$$

where $\langle , \rangle$ denotes the standard inner product on $\mathbb{R}^3$. Using (3), we define

$$M_n(\theta) = \frac{A_n(\theta)}{SO(3)}. \tag{4}$$

It is expected that the space $M_n(\theta)$ is much more difficult than $P_n(\ell)$. For example, the following trivial observation shows that $M_n(\theta)$ does not admit a similar property to (2): when n is odd, we have $M_n(0) = \{\text{one point}\}$ but $M_n(\pi) = \varnothing$.

We claim that $M_n(\theta)$ is a hypersurface of the torus T^{n-2}. In fact, if we forget the condition $\langle a_n, a_1\rangle = \cos\theta$ in (3), the space corresponding to (4) is T^{n-2} as observed in [5, 6]. Hence the claim follows.

We recall previous results on $M_n(\theta)$. First, [7] considered the case for $\theta = \pi/2$. The main result is that, realizing $M_n(\pi/2)$ as a homotopy colimit of a diagram involving $M_{n-2}(\pi/2)$ and $M_{n-1}(\pi/2)$, we inductively computed $\chi(M_n(\pi/2))$. In particular, we obtained a homeomorphism $M_5(\pi/2) \cong \Sigma_5$, where Σ_5 denotes a connected closed orientable surface of genus 5.

Second, we set

$$X_n(\theta) := P_n(1) \cap M_n(\theta). \tag{5}$$

Note that $X_n(\theta)$ is the configuration space of equilateral and equiangular n-gons. Crippen [8] studied the topological type of $X_n(\theta)$ for $n = 3, 4$, and 5. The result is that $X_n(\theta)$ is either $\varnothing$, one point, or two points depending on θ. Later, O'Hara [9] studied the topological type of $X_6(\theta)$. The result is that $X_6(\theta)$ is disjoint union of a certain number of S^1's and points.

Table 1: The topological type of $M_3(\theta)$.

θ	Topological type
$2\pi/3 < \theta \le \pi$	$\varnothing$
$2\pi/3$	{one point}
$0 < \theta < 2\pi/3$	{two points}
0	{one point}

Table 2: The topological type of $M_4(\theta)$.

θ	Topological type
π	{one point}
$\pi/2 < \theta < \pi$	Figure 1(a)
$\pi/2$	Figure 1(b)
$0 < \theta < \pi/2$	Figure 1(a)
0	{one point}

The purpose of this paper is to determine the topological type of $M_n(\theta)$ for $n = 3, 4$, and 5. In contrast to the fact that at most one-dimensional spaces appear in the results of [8, 9], surfaces appear in our results.

This paper is organized as follows. In Section 2, we state our main results and in Section 3 we prove them.

2. Main Results

Theorem A. *The topological type of $M_3(\theta)$ is given in Table 1.*

Theorem B. *(i) The topological type of $M_4(\theta)$ is given in Table 2.*

(ii) As θ approaches $\pi/2$, point A in Figure 1(a) approaches point B.

Theorem C. *(i) The topological type of $M_5(\theta)$ is given in Table 3. Let Σ_g be a connected closed orientable surface of genus g.*

(ii) (a) Let θ satisfy that $2\pi/5 < \theta < 2\pi/3$. We study the situation where θ approaches $2\pi/3$. We identify the torus Σ_1 with the Dupin cyclide, which we denote by D. (See Figure 2.)

Using this, we identify Σ_5 with $\#_5 D$, where the connected sum is formed by cutting a small circular hole away from the narrow part of D. As θ approaches $2\pi/3$, the center of each narrow part pinches to a point. Thus the five singular points appear.

(b) We consider the situation where θ increases from $2\pi/3$. Then each pinched point of $M_5(2\pi/3)$ separates. Thus we obtain S^2.

(c) Let θ satisfy that $2\pi/5 < \theta < 2\pi/3$. We consider the situation where θ approaches $2\pi/5$. In contrast to (a), the center of exactly one narrow part pinches to a point. Thus one singular point appears.

Corollary D. *As a subspace of $((S^2)^5 \times [0, \pi])/SO(3)$, we define the space*

$$Y := \bigcup_{0 \le \theta \le \pi} M_5(\theta). \tag{6}$$

Then $M_5(0)$ is a singular point of Y and has a neighborhood $C\Sigma_4$, where C denotes the cone.

Remark 1. Cone-type singularities appear in Theorems B and C and Corollary D. We note that singularities of configuration spaces of mechanical linkages have been studied extensively by Blanc and Shvalb [10].

3. Proofs of the Main Results

We fix $\theta \in [0, \pi]$ and set

$$\begin{aligned} e_1 &= (1, 0, 0), \\ p &= (\cos\theta, \sin\theta, 0). \end{aligned} \tag{7}$$

Normalizing a_1 and a_2 to be e_1 and p, respectively, we have the following description:

$$\begin{aligned} M_n(\theta) = \{ &(a_1, \dots, a_n) \in (S^2)^n \mid a_1 = e_1, \\ &a_2 = p,\ \langle a_i, a_{i+1} \rangle = \cos\theta \text{ for } 2 \le i \le \\ &n-1,\ \langle a_n, a_1 \rangle = \cos\theta \}. \end{aligned} \tag{8}$$

Hereafter we use (8).

In order to prove our main results, we use the following fact, whose proof is left to the reader.

Fact 2. Let $(\alpha, \beta, \gamma) \in (S^2)^3$ satisfy that

$$\begin{aligned} \langle \alpha, \beta \rangle &= 0, \\ \langle \alpha, \gamma \rangle &= \cos\theta. \end{aligned} \tag{9}$$

Then, there exists $\phi \in \mathbb{R}$ such that

$$\gamma = (\cos\theta)\,\alpha + (\sin\theta\cos\phi)\,\beta + (\sin\theta\sin\phi)(\alpha \times \beta). \tag{10}$$

Now we first consider the case $n = 5$. Consider Fact 2 for $\alpha = p$, $\beta = (-\sin\theta, \cos\theta, 0)$, and $\gamma = a_3$. Then there exists $x \in \mathbb{R}$ such that

$$\begin{aligned} a_3 = {} &(\cos\theta)\,p + (\sin\theta\cos x)(-\sin\theta, \cos\theta, 0) \\ &+ (\sin\theta\sin x)(0, 0, 1). \end{aligned} \tag{11}$$

Next, we consider Fact 2 for $\alpha = e_1$, $\beta = (0, 1, 0)$, and $\gamma = a_5$. Then there exists $z \in \mathbb{R}$ such that

$$a_5 = (\cos\theta, \sin\theta\cos z, \sin\theta\sin z). \tag{12}$$

Finally, we consider Fact 2 for $\alpha = a_5$ in (12),

$$\beta = (-\sin\theta, \cos\theta\cos z, \cos\theta\sin z), \tag{13}$$

and $\gamma = a_4$. Then there exists $y \in \mathbb{R}$ such that

$$\begin{aligned} a_4 = {} &(\cos\theta)\,\alpha + (\sin\theta\cos y)\,\beta \\ &+ (\sin\theta\sin y)(\alpha \times \beta). \end{aligned} \tag{14}$$

Now we define the function $f : (\mathbb{R}/2\pi\mathbb{Z})^3 \times [0, \pi] \to \mathbb{R}$ by

$$f(x, y, z, \theta) := \langle (11), (14) \rangle - \cos\theta. \tag{15}$$

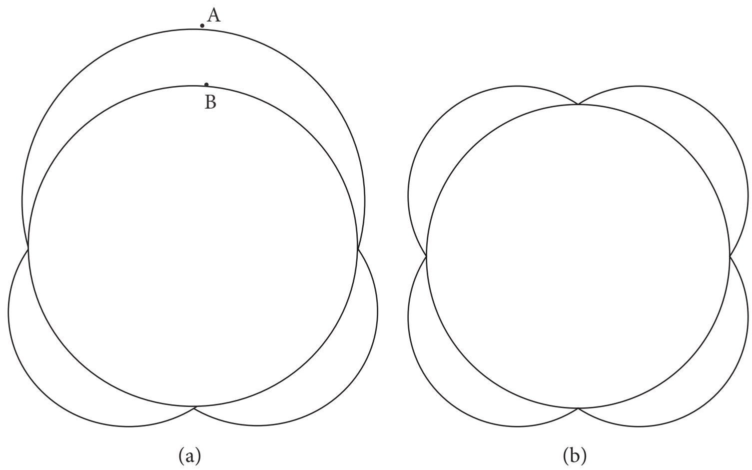

Figure 1: (a) $M_4(\theta)$ for $0 < \theta < \pi/2$ or $\pi/2 < \theta < \pi$. (b) $M_4(\pi/2)$.

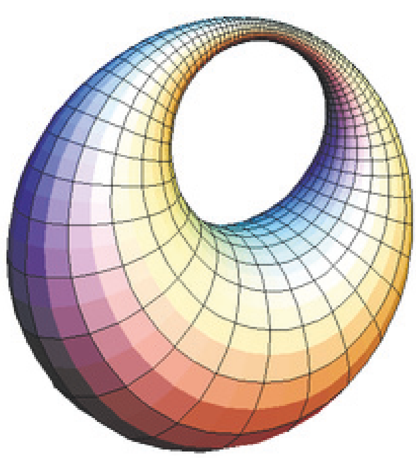

Figure 2: The Dupin cyclide.

We can understand $M_5(\theta)$ as a level set. More precisely, we define the function

$$h : f^{-1}(0) \longrightarrow \mathbb{R} \tag{16}$$

by $h(x, y, z, \theta) = \theta$. Then we have

$$M_5(\theta) = h^{-1}(\theta) \tag{17}$$

if $0 < \theta \leq \pi$.

Remark 3. Since $f(x, y, z, 0) = 0$ for all x, y, and z, we have $h^{-1}(0) = (\mathbb{R}/2\pi\mathbb{Z})^3$. On the other hand, it is clear that $M_5(0) = \{\text{one point}\}$. Hence (17) does not hold for $\theta = 0$. Apart from this point, there is an identification

$$Y \setminus M_5(0) = f^{-1}(0) \setminus h^{-1}(0), \tag{18}$$

where Y is defined in (6).

Lemma 4. *We set*

$$\begin{aligned} S := \Big\{ & (x, y, z, \theta) \in \left(\frac{\mathbb{R}}{2\pi\mathbb{Z}}\right)^3 \times (0, \pi] \mid f(x, y, z, \theta) \\ & = 0, \\ & \left(\frac{\partial f}{\partial x}(x, y, z, \theta), \frac{\partial f}{\partial y}(x, y, z, \theta), \frac{\partial f}{\partial z}(x, y, z, \theta)\right) \\ & = (0, 0, 0) \Big\}. \end{aligned} \tag{19}$$

Then S is given in Table 4.

Proof. The lemma is proved by direct computations. □

Proof of Theorem C. We consider h in (16) as a Morse function on $f^{-1}(0)$. First, Table 4 and (17) show that $M_5(4\pi/5) = \{\text{one point}\}$.

Second, direct computation shows that

$$\frac{\partial f}{\partial \theta}\left(0, 0, \pi, \frac{4\pi}{5}\right) = \frac{5}{2}\sqrt{\frac{5 - \sqrt{5}}{2}}. \tag{20}$$

Since this is nonzero, the space $f^{-1}(0)$ is smooth at $(0, 0, \pi, 4\pi/5)$. Actually, we can prove that the point is a nondegenerate critical point of the function h. Hence Morse lemma shows that there is a homeomorphism $M_5(\theta) \cong S^2$ for $2\pi/3 < \theta < 4\pi/5$. But if we use [11, Corollary B], we need not check that h is nondegenerate at $(0, 0, \pi, 4\pi/5)$. For our reference, we draw the figure of $M_5(4\pi/5 - 0.1)$ in Figure 3.

Third, the other parts of Table 3 follow from Table 4. This completes the proof of Theorem C. □

Proof of Corollary D. The corollary is an immediate consequence of Theorem C. □

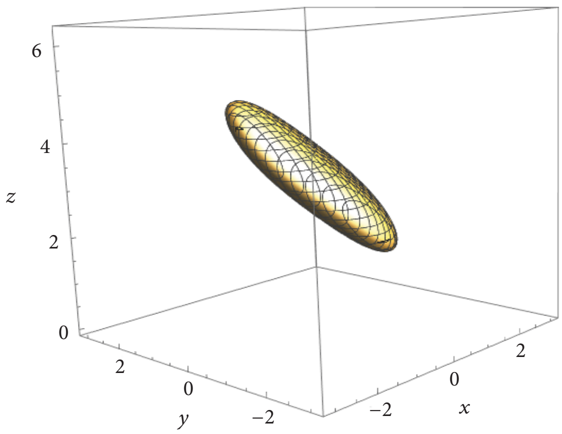

Figure 3: $M_5(4\pi/5 - 0.1)$.

Table 3: The topological type of $M_5(\theta)$.

θ	Topological type
$4\pi/5 < \theta \leq \pi$	$\emptyset$
$4\pi/5$	{one point}
$2\pi/3 < \theta < 4\pi/5$	S^2
$2\pi/3$	Contains five singular points
$2\pi/5 < \theta < 2\pi/3$	Σ_5
$2\pi/5$	Contains one singular point
$0 < \theta < 2\pi/5$	Σ_4
0	{one point}

Table 4: The set S.

θ	(x, y, z)
$4\pi/5$	$(0, 0, \pi)$
$2\pi/3$	$(\pi, 0, 0)$, $(0, \pi, 0)$, $(0, 0, \pi)$, $(\pi, 0, \pi)$, $(0, \pi, \pi)$
$2\pi/5$	$(0, 0, \pi)$

Proof of Theorem B. We define a_3 as in (11). We also define a_4 to be the right-hand side of (12). We define the function $f : (\mathbb{R}/2\pi\mathbb{Z})^2 \times [0, \pi] \to \mathbb{R}$ by

$$f(x, z, \theta) := \langle a_3, a_4 \rangle - \cos\theta. \quad (21)$$

Similarly to (17), we have $M_4(\theta) = h^{-1}(\theta)$. Since $h^{-1}(\theta)$ is one-dimensional, it is easy to draw its figure. Thus Theorem B follows. □

Proof of Theorem A. We define the function $f : (\mathbb{R}/2\pi\mathbb{Z}) \times [0, \pi] \to \mathbb{R}$ by $f(x, \theta) = \langle a_3, e_1 \rangle - \cos\theta$. Since $M_3(\theta) = h^{-1}(\theta)$, Theorem A follows. □

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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