



Impulsive Differential Equations and Inclusions

M. BENCHOHRA, J. HENDERSON, and S. NTOUYAS

Impulsive Differential Equations and Inclusions

Contemporary Mathematics and Its Applications, Volume 2

Impulsive Differential Equations and Inclusions

M. Benchohra, J. Henderson, and S. Ntouyas

Contemporary Mathematics and Its Applications
Series Editors: Ravi P. Agarwal and Donal O'Regan

Hindawi Publishing Corporation
410 Park Avenue, 15th Floor, #287 pmb, New York, NY 10022, USA
Nasr City Free Zone, Cairo 11816, Egypt
Fax: +1-866-HINDAWI (USA Toll-Free)

© 2006 Hindawi Publishing Corporation

All rights reserved. No part of the material protected by this copyright notice may be reproduced or utilized in any form or by any means, electronic or mechanical, including photocopying, recording, or any information storage and retrieval system, without written permission from the publisher.

ISBN 977-5945-50-X

Dedication

We dedicate this book to our family members who complete us. In particular, M. Benchohra's dedication is to his wife, Kheira, and his children, Mohamed, Maroua, and Abdelillah; J. Henderson dedicates to his wife, Darlene, and his descendants, Kathy, Dirk, Katie, David, and Jana Beth; and S. Ntouyas makes his dedication to his wife, Ioanna, and his daughter, Myrto.

Contents

Preface	xi
1. Preliminaries	1
1.1. Definitions and results for multivalued analysis	1
1.2. Fixed point theorems	4
1.3. Semigroups	7
1.4. Some additional lemmas and notions	9
2. Impulsive ordinary differential equations & inclusions	11
2.1. Introduction	11
2.2. Impulsive ordinary differential equations	12
2.3. Impulsive ordinary differential inclusions	24
2.4. Ordinary damped differential inclusions	49
2.5. Notes and remarks	62
3. Impulsive functional differential equations & inclusions	63
3.1. Introduction	63
3.2. Impulsive functional differential equations	63
3.3. Impulsive neutral differential equations	74
3.4. Impulsive functional differential inclusions	80
3.5. Impulsive neutral functional DIs	95
3.6. Impulsive semilinear functional DIs	107
3.7. Notes and remarks	118
4. Impulsive differential inclusions with nonlocal conditions	119
4.1. Introduction	119
4.2. Nonlocal impulsive semilinear differential inclusions	119
4.3. Existence results for impulsive functional semilinear differential inclusions with nonlocal conditions	136
4.4. Notes and remarks	145
5. Positive solutions for impulsive differential equations	147
5.1. Introduction	147
5.2. Positive solutions for impulsive functional differential equations	147
5.3. Positive solutions for impulsive boundary value problems	154
5.4. Double positive solutions for impulsive boundary value problems	159
5.5. Notes and remarks	165

6.	Boundary value problems for impulsive differential inclusions	167
6.1.	Introduction	167
6.2.	First-order impulsive differential inclusions with periodic boundary conditions	167
6.3.	Upper- and lower-solutions method for impulsive differential inclusions with nonlinear boundary conditions	184
6.4.	Second-order boundary value problems	191
6.5.	Notes and remarks	198
7.	Nonresonance impulsive differential inclusions	199
7.1.	Introduction	199
7.2.	Nonresonance first-order impulsive functional differential inclusions with periodic boundary conditions	199
7.3.	Nonresonance second-order impulsive functional differential inclusions with periodic boundary conditions	209
7.4.	Nonresonance higher-order boundary value problems for impulsive functional differential inclusions	217
7.5.	Notes and remarks	227
8.	Impulsive differential equations & inclusions with variable times	229
8.1.	Introduction	229
8.2.	First-order impulsive differential equations with variable times	229
8.3.	Higher-order impulsive differential equations with variable times	235
8.4.	Boundary value problems for differential inclusions with variable times	241
8.5.	Notes and remarks	252
9.	Nondensely defined impulsive differential equations & inclusions	253
9.1.	Introduction	253
9.2.	Nondensely defined impulsive semilinear differential equations with nonlocal conditions	253
9.3.	Nondensely defined impulsive semilinear differential inclusions with nonlocal conditions	264
9.4.	Nondensely defined impulsive semilinear functional differential equations	276
9.5.	Nondensely defined impulsive semilinear functional differential inclusions	285
9.6.	Notes and remarks	290
10.	Hyperbolic impulsive differential inclusions	291
10.1.	Introduction	291
10.2.	Preliminaries	292
10.3.	Main results	293
10.4.	Notes and remarks	309

Contents	ix
11. Impulsive dynamic equations on time scales	311
11.1. Introduction	311
11.2. Preliminaries	311
11.3. First-order impulsive dynamic equations on time scales	313
11.4. Impulsive functional dynamic equations on time scales with infinite delay	318
11.5. Second-order impulsive dynamic equations on time scales	325
11.6. Existence results for second-order boundary value problems of impulsive dynamic equations on time scales	333
11.7. Double positive solutions of impulsive dynamic boundary value problems	339
11.8. Notes and remarks	344
12. On periodic boundary value problems of first-order perturbed impulsive differential inclusions	345
12.1. Introduction	345
12.2. Existence results	346
12.3. Notes and remarks	353
Bibliography	355
Index	365

Preface

Since the late 1990s, the authors have produced an extensive portfolio of results on differential equations and differential inclusions undergoing impulse effects. Both initial value problems and boundary value problems have been dealt with in their work. The primary motivation for this book is in gathering under one cover an encyclopedic resource for many of these recent results. Having succinctly stated the motivation of the book, there is certainly an obligation to include mentioning some of the all important roles of modelling natural phenomena with impulse problems.

The dynamics of evolving processes is often subjected to abrupt changes such as shocks, harvesting, and natural disasters. Often these short-term perturbations are treated as having acted instantaneously or in the form of “impulses.” Impulsive differential equations such as

$$x' = f(t, x), \quad t \in [0, b] \setminus \{t_1, \dots, t_m\}, \quad (1)$$

subject to impulse effects

$$\Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, \dots, m, \quad (2)$$

with $f : ([0, b] \setminus \{t_1, \dots, t_m\}) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and I_k an impulse operator, have been developed in modelling impulsive problems in physics, population dynamics, biotechnology, pharmacokinetics, industrial robotics, and so forth; in the case when the right-hand side of (1) has discontinuities, differential inclusions such as

$$x'(t) \in F(t, x(t)), \quad t \in [0, b] \setminus \{t_1, \dots, t_m\}, \quad (3)$$

subject to the impulse conditions (2), where $F : ([0, b] \setminus \{t_1, \dots, t_m\}) \times \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$, have played an important role in modelling phenomena, especially in scenarios involving automatic control systems. In addition, when these processes involve hereditary phenomena such as biological and social macrosystems, some of the modelling is done via impulsive functional differential equations such as

$$x' = f(t, x_t), \quad t \in [0, b] \setminus \{t_1, \dots, t_m\}, \quad (4)$$

subject to (2), and an initial value

$$x(s) = \phi(s), \quad s \in [-r, 0], \quad t \in [0, b], \quad (5)$$

where $x_t(\theta) = x(t + \theta)$, $t \in [0, b]$, and $-r \leq \theta \leq 0$, and $f : ([0, b] \setminus \{t_1, \dots, t_m\}) \times D \rightarrow \mathbb{R}^n$, and D is a space of functions from $[-r, 0]$ into \mathbb{R}^n which are continuous except for a finite number of points. When the dynamics is multivalued, the hereditary phenomena are modelled via impulsive functional differential inclusions such as

$$x'(t) \in F(t, x_t), \quad t \in [0, b] \setminus \{t_1, \dots, t_m\}, \quad (6)$$

subject to the impulses (2) and the initial condition (5).

An outline of the book as it is devoted to articles published by the authors evolves in a somewhat natural way around addressing issues relating to initial value problems and boundary value problems for both impulsive differential equations and differential inclusions, as well as for both impulsive functional differential equations and functional differential inclusions. Chapter 1 contains fundamental results from multivalued analysis and differential inclusions. In addition, this chapter contains a number of fixed point theorems on which most of the book's existence results depend. Included among these fixed point theorems are those recognized their names: Avery-Henderson, Bohnenblust-Karlin, Covitz and Nadler, Krasnosel'skii, Leggett-Williams, Leray-Schauder, Martelli, and Schaefer. Chapter 1 also contains background material on semigroups that is necessary for the book's presentation of impulsive semilinear functional differential equations.

Chapter 2 is devoted to impulsive ordinary differential equations and scalar differential inclusions, given, respectively, by

$$y' - Ay = By + f(t, y), \quad y' \in F(t, y), \quad (7)$$

each subject to (2), and each satisfies an initial condition $y(0) = y_0$, where A is an infinitesimal generator of a family of semigroups, B is a bounded linear operator from a Banach space E back to itself, and $F : [0, b] \times E \rightarrow 2^E$. Chapter 3 deals with functional differential equations and functional differential inclusions, with each undergoing impulse effects. Also, neutral functional differential equations and neutral functional differential inclusions are addressed in which the derivative of the state variable undergoes a delay. Chapter 4 is directed toward impulsive semilinear ordinary differential inclusions and functional differential inclusions satisfying nonlocal boundary conditions such as $g(y) = \sum_{k=1}^n c_k y(t_k)$, with each $c_k > 0$ and $0 < t_1 < \dots < t_n < b$. Such problems are used to describe the diffusion phenomena of a small amount of gas in a transport tube.

Chapter 5 is focused on positive solutions and multiple positive solutions for impulsive ordinary differential equations and functional differential equations,

including initial value problems as well as boundary value problems for second-order problems such as

$$y'' = f(t, y_t), \quad t \in [0, b] \setminus \{t_1, \dots, t_m\}, \quad (8)$$

subject to impulses

$$\Delta y(t_k) = I_k(y(t_k)), \quad \Delta y'(t_k) = J_k(y(t_k)), \quad k = 1, \dots, m, \quad (9)$$

and initial conditions

$$y(t) = \phi(t), \quad t \in [-r, 0], \quad y'(0) = \eta. \quad (10)$$

Chapter 6 is primarily concerned with boundary value problems for periodic impulsive differential inclusions. Upper- and lower-solution methods are developed for first-order systems and then for second-order systems of functional differential inclusions, $y''(t) \in F(t, y_t)$. For Chapter 7, impulsive differential inclusions satisfying periodic boundary conditions are studied. The problems of interest are termed as being *nonresonant*, because the linear operators involved are invertible in the absence of impulses. The chapter deals with first-order and higher-order nonresonance impulsive inclusions.

Chapter 8 extends the theory of some of the previous chapters to functional differential equations and functional differential inclusions under impulses for which the impulse effects vary with time; that is, $y(t_k^+) = I_k(y(t))$, $t = \tau_k(y(t))$, $k = 1, \dots, m$. Chapter 9, as well, extends several results of previous chapters on semilinear problems now to semilinear functional differential equations and functional differential inclusions for operators that are nondensely defined on a Banach space.

Chapter 10 ventures into results for second-order impulsive hyperbolic differential inclusions,

$$\begin{aligned} \frac{\partial^2 u(t, x)}{\partial t \partial x} &\in F(t, x, u(t, x)) \quad \text{a.e. } (t, x) \in ([0, a] \setminus \{t_1, \dots, t_m\}) \times [0, b], \\ \Delta u(t_k, x) &= I_k(u(t_k, x)), \quad k = 1, \dots, m, \\ u(t, 0) &= \psi(t), \quad t \in [0, a] \setminus \{t_1, \dots, t_m\}, \quad u(0, x) = \phi(x), \quad x \in [0, b]. \end{aligned} \quad (11)$$

Such models arise especially for problems in biological or medical domains.

The next to last chapter, Chapter 11, addresses some questions for impulsive dynamic equations on time scales. The methods constitute adjustments from those for impulsive ordinary differential equations to dynamic equations on time scales, but these results are the first such results in the direction of impulsive problems on time scales. The final chapter, Chapter 12, is a brief chapter dealing with periodic

boundary value problems for first-order perturbed impulsive systems,

$$\begin{aligned} x' &\in F(t, x(t)) + G(t, x(t)), \quad t \in [0, b] \setminus \{t_1, \dots, t_m\}, \\ x(t_j^+) &= x(t_j^-) + I_j(x(t_j^-)), \quad j = 1, \dots, m, \quad x(0) = x(b), \end{aligned} \quad (12)$$

where both $F, G : ([0, b] \setminus \{t_1, \dots, t_m\}) \times \mathbb{R} \rightarrow 2^{\mathbb{R}}$.

We express our appreciation and thanks to R. I. Avery, A. Boucherif, B. C. Dhage, E. Gatsori, L. Górniewicz, J. R. Graef, J. J. Nieto, A. Ouahab, and Y. G. Sficas for their collaboration in research and to E. Gatsori and A. Ouahab for their careful typing of some parts of this manuscript. We are especially grateful to the Editors-in-Chief of the *Contemporary Mathematics and Applications* book series, R. P. Agarwal and D. O'Regan, for their encouragement of us during the preparation of this volume for inclusion in the series.

M. Benchohra
J. Henderson
S. Ntouyas

1

Preliminaries

1.1. Definitions and results for multivalued analysis

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis, which are used throughout this book.

Let (X, d) be a metric space and let Y be a subset of X . We denote

- (i) $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$;
- (ii) $\mathcal{P}_p(X) = \{Y \in \mathcal{P}(X) : Y \text{ has the property "p"}\}$, where p could be $\text{cl} =$ closed, $b =$ bounded, $\text{cp} =$ compact, $\text{cv} =$ convex, and so forth.

Thus

- (i) $\mathcal{P}_{\text{cl}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$,
- (ii) $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$,
- (iii) $\mathcal{P}_{\text{cv}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ convex}\}$,
- (iv) $\mathcal{P}_{\text{cp}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$,
- (v) $\mathcal{P}_{\text{cv, cp}}(X) = \mathcal{P}_{\text{cv}}(X) \cap \mathcal{P}_{\text{cp}}(X)$, and so forth.

In what follows, by E we will denote a Banach space over the field of real numbers \mathbb{R} and by J a closed interval in \mathbb{R} . We let

$$C(J, E) = \{y : J \rightarrow E \mid y \text{ is continuous}\}. \quad (1.1)$$

We consider the Tchebyshev norm

$$\|\cdot\|_\infty : C(J, E) \rightarrow [0, \infty), \quad (1.2)$$

defined by

$$\|y\|_\infty = \max \{|y(t)|, t \in J\}, \quad (1.3)$$

where $|\cdot|$ stands for the norm in E . Then $(C(J, E), \|\cdot\|_\infty)$ is a Banach space.

Let $N : E \rightarrow E$ be a linear map. N is called *bounded* provided there exists $r > 0$ such that

$$|N(x)| \leq r|x|, \quad \text{for every } x \in E. \quad (1.4)$$

The following result is classical.

2 Impulsive ordinary differential equations & inclusions

2.1. Introduction

For well over a century, differential equations have been used in modeling the dynamics of changing processes. A great deal of the modeling development has been accompanied by a rich theory for differential equations.

The dynamics of many evolving processes are subject to abrupt changes, such as shocks, harvesting and natural disasters. These phenomena involve short-term perturbations from continuous and smooth dynamics, whose duration is negligible in comparison with the duration of an entire evolution. In models involving such perturbations, it is natural to assume these perturbations act instantaneously or in the form of “impulses.” As a consequence, impulsive differential equations have been developed in modeling impulsive problems in physics, population dynamics, ecology, biological systems, biotechnology, industrial robotics, pharmacokinetics, optimal control, and so forth. Again, associated with this development, a theory of impulsive differential equations has been given extensive attention. Works recognized as landmark contributions include [29, 30, 180, 217], with [30] devoted especially to impulsive periodic systems of differential equations.

Some processes, especially in areas of population dynamics, ecology, and pharmacokinetics, involve hereditary issues. The theory and applications addressing such problems have heavily involved functional differential equations as well as impulsive functional differential equations. The literature devoted to this study is also extensive, with [6, 12–14, 25, 27, 28, 38, 42, 46, 49, 52, 53, 55, 57, 70, 71, 75, 85, 89–91, 94, 95, 117, 130–132, 134, 136, 147, 152, 159, 167, 176, 181, 183, 189, 191, 194, 195, 212, 214, 216, 228] providing a good view of the panorama of work that has been done.

Much attention has also been devoted to modeling natural phenomena with differential equations, both ordinary and functional, for which the part governing the derivative(s) is not known as a single-valued function; for example, a dynamic process governing the derivative $x'(t)$ of a state $x(t)$ may be known only within a set $S(t, x(t)) \subset \mathbb{R}$, and given by $x'(t) \in S(t, x(t))$. A common example of this is observed in a so-called differential inequality such as $x'(t) \leq f(t, x(t))$,

3 Impulsive functional differential equations & inclusions

3.1. Introduction

While the previous chapter was devoted to ordinary differential equations and inclusions involving impulses, our attention in this chapter is turned to functional differential equations and inclusions each undergoing impulse effects. These equations and inclusions have played an important role in areas involving hereditary phenomena for which a delay argument arises in the modelling equation or inclusion. There are also a number of applications in which the delayed argument occurs in the derivative of the state variable, which are sometimes modelled by neutral differential equations or neutral differential inclusions.

This chapter presents a theory for the existence of solutions of impulsive functional differential equations and inclusions, including scenarios of neutral equations, as well as semilinear models. The methods used throughout the chapter range over applications of the Leray-Schauder nonlinear alternative, Schaefer's fixed point theorem, a Martelli fixed point theorem for multivalued condensing maps, and a Covitz-Nadler fixed point theorem for multivalued maps.

3.2. Impulsive functional differential equations

In this section, we will establish existence theory for first- and second-order impulsive functional differential equations. The section will be divided into parts. In the first part, by a nonlinear alternative of Leray-Schauder type, we will present an existence result for the first-order initial value problem

$$y'(t) = f(t, y_t), \quad \text{a.e. } t \in J := [0, T], \quad t \neq t_k, \quad k = 1, \dots, m, \quad (3.1)$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \quad (3.2)$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \quad (3.3)$$

where $f : J \times \mathcal{D} \rightarrow E$ is a given function, $\mathcal{D} = \{\psi : [-r, 0] \rightarrow E \mid \psi \text{ is continuous everywhere except for a finite number of points } s \text{ at which } \psi(s) \text{ and the right limit } \psi(s^+) \text{ exist and } \psi(s^-) = \psi(s)\}$, $\phi \in \mathcal{D}$, $(0 < r < \infty)$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $I_k \in C(E, E)$ ($k = 1, 2, \dots, m$), and E a real separable Banach space with norm $|\cdot|$. Also, throughout, $J' = J \setminus \{t_1, \dots, t_m\}$.

4

Impulsive differential inclusions with nonlocal conditions

4.1. Introduction

In this chapter, we will prove existence results for impulsive semilinear ordinary and functional differential inclusions, with nonlocal conditions. Often, nonlocal conditions are motivated by physical problems. For the importance of nonlocal conditions in different fields we refer to [112]. As indicated in [112, 113, 126] and the references therein, the nonlocal condition $y(0) + g(y) = y_0$ can be more descriptive in physics with better effect than the classical initial condition $y(0) = y_0$. For example, in [126], the author used

$$g(y) = \sum_{k=1}^p c_k y(t_k), \quad (4.1)$$

where $c_i, i = 1, \dots, p$ are given constants and $0 < t_1 < t_2 < \dots < t_p \leq b$, to describe the diffusion phenomenon of a small amount of gas in a transparent tube. In this case, (4.1) allows the additional measurements at $t_i, i = 1, \dots, p$.

Nonlocal Cauchy problems for ordinary differential equations have been investigated by several authors, (see, e.g., [103, 113, 114, 202–204, 206, 207]). Nonlocal Cauchy problems, in the case where F is a multivalued map, were studied by Benchohra and Ntouyas [77–79], and Boucherif [103]. Akça et al. [14] initiated the study of a class of first-order semilinear functional differential equations for which the nonlocal conditions and the impulse effects are combined. Again, in this chapter, we will invoke some of our fixed point theorems in establishing solutions for these nonlocal impulsive differential inclusions.

4.2. Nonlocal impulsive semilinear differential inclusions

In this section, we begin the study of nonlocal impulsive initial value problems by proving existence results for the problem

$$y'(t) \in Ay(t) + F(t, y(t)), \quad t \in J := [0, b], \quad t \neq t_k, \quad k = 1, 2, \dots, m, \quad (4.2)$$

5 Positive solutions for impulsive differential equations

5.1. Introduction

Positive solutions and multiple positive solutions of differential equations have received a tremendous amount of attention. Studies have involved initial value problems, as well as boundary value problems, for both ordinary and functional differential equations. In some cases, impulse effects have also been present. The methods that have been used include multiple applications of the Guo-Krasnosel'skii fixed point theorem [158], the Leggett-Williams multiple fixed point theorem [187], and extensions such as the Avery-Henderson double fixed point theorem [26]. Many such multiple-solution works can be found in the papers [6, 8–10, 19, 52, 94, 95, 137, 159, 194].

This chapter is devoted to positive solutions and multiple positive solutions of impulsive differential equations.

5.2. Positive solutions for impulsive functional differential equations

Throughout this section, let $J = [0, b]$, and the points $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$ are fixed. This section is concerned with the existence of three non-negative solutions for initial value problems for first- and second-order functional differential equations with impulsive effects. In Section 5.2.1, we consider the first-order IVP

$$\begin{aligned} y'(t) &= f(t, y_t), \quad t \in J = [0, b], \quad t \neq t_k, \quad k = 1, \dots, m, \\ \Delta y|_{t=t_k} &= I_k(y(t_k^-)), \quad k = 1, \dots, m, \\ y(t) &= \phi(t), \quad t \in [-r, 0], \end{aligned} \tag{5.1}$$

where $f : J \times \mathcal{D} \rightarrow \mathbb{R}$ is a given function, $\mathcal{D} = \{\psi : [-r, 0] \rightarrow \mathbb{R}_+ \mid \psi \text{ is continuous everywhere except for a finite number of points } s \text{ at which } \psi(s) \text{ and the right limit } \psi(s^+) \text{ exist and } \psi(s^-) = \psi(s)\}$, $\phi \in \mathcal{D}$, $0 < r < \infty$, $I_k : \mathbb{R} \rightarrow \mathbb{R}_+$ ($k = 1, 2, \dots, m$), $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, and $J' = J \setminus \{t_1, \dots, t_m\}$.

6

Boundary value problems for impulsive differential inclusions

6.1. Introduction

The method of upper and lower solutions has been successfully applied to study the existence of solutions for first-order impulsive initial value problems and boundary value problems. This method generates solutions of such problems, with the solutions located in an order interval with the upper and lower solutions serving as bounds. Moreover, this method, coupled with some monotonicity-type hypotheses, leads to monotone iterative techniques which generate in a constructive way (amenable to numerical treatment) the extremal solutions within the order interval determined by the upper and lower solutions.

This method has been used only in the context of single-valued impulsive differential equations. We refer to the monographs of Lakshmikantham et al. [180], Samoilenko and Perestyuk [217], the papers of Cabada and Liz [117], Frigon and O'Regan [151], Heikkilä and Lakshmikantham [163], Liu [188], Liz [192, 193], Liz and Nieto [194], and Pierson-Gorez [212]. However, this method has been used recently by Benchohra and Boucherif [35] for the study of first-order initial value problems for impulsive differential inclusions.

Let us mention that other methods like the nonlinear alternative, such as in the papers of Benchohra and Boucherif [34, 35], Frigon and O'Regan [150], and the topological transversality theorem Erbe and Krawcewicz [140], have been used to analyze first- and second-order impulsive differential inclusions. The first part of this chapter presents existence results using upper- and lower-solutions methods to obtain solutions of first-order impulsive differential inclusions with periodic boundary conditions and nonlinear boundary conditions. The last section of the chapter deals with boundary value problems for second-order impulsive differential inclusions.

6.2. First-order impulsive differential inclusions with periodic boundary conditions

This section is devoted to the existence of solutions for the impulsive periodic multivalued problem

7 Nonresonance impulsive differential inclusions

7.1. Introduction

This chapter is devoted to impulsive differential inclusions satisfying periodic boundary conditions. These problems are termed as being *nonresonant*, because the linear operator involved will be invertible in the absence of impulses. The first problem addressed concerns first-order problems. A result from [51] that generalizes a paper by Nieto [199] is presented. The methods used involve the Martelli fixed point theorem (Theorem 1.7) and the Covitz-Nadler fixed point theorem (Theorem 1.11).

The second part of the chapter is focused on a second-order problem, and a result of [55] is obtained which is an extension of the first-order result. Again the method used involves an application of Theorem 1.7. Then, the final section of the chapter is a successful extension of these results to n th order nonresonance problems, which were first established in [63]. Also, an initial value function is introduced for the higher-order consideration.

7.2. Nonresonance first-order impulsive functional differential inclusions with periodic boundary conditions

This section is concerned with the existence of solutions for the nonresonance problem for functional differential inclusions with impulsive effects as

$$y'(t) - \lambda y(t) \in F(t, y_t), \quad t \in J = [0, T], \quad t \neq t_k, \quad k = 1, \dots, m, \quad (7.1)$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \quad (7.2)$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \quad (7.3)$$

$$\phi(0) = y(0) = y(T), \quad (7.4)$$

where $\lambda \neq 0$ and λ is not an eigenvalue of y' , $F : J \times \mathcal{D} \rightarrow \mathcal{P}(E)$ is a compact convex-valued multivalued map, $\mathcal{D} = \{\psi : [-r, 0] \rightarrow E \mid \psi \text{ is continuous everywhere except for a finite number of points } s \text{ at which } \psi(s) \text{ and the right limit } \psi(s^+) \}$

8

Impulsive differential equations & inclusions with variable times

8.1. Introduction

The theory of impulsive differential equations with variable time is relatively less developed due to the difficulties created by the state-dependent impulses. Recently, some interesting extensions to impulsive differential equations with variable times have been done by Bajo and Liz [31], Frigon and O'Regan [150, 151], Kaul [173], Kaul et al. [174], and Benchohra et al. [43, 45, 70, 71, 91, 92].

8.2. First-order impulsive differential equations with variable times

This section is concerned with the existence of solutions, for initial value problems (IVP for short), for first-order functional differential equations with impulsive effects

$$\begin{aligned}y'(t) &= f(t, y_t), \quad \text{a.e. } t \in J = [0, T], \quad t \neq \tau_k(y(t)), \quad k = 1, \dots, m, \\y(t^+) &= I_k(y(t)), \quad t = \tau_k(y(t)), \quad k = 1, \dots, m, \\y(t) &= \phi(t), \quad t \in [-r, 0],\end{aligned}\tag{8.1}$$

where $f : J \times \mathcal{D} \rightarrow \mathbb{R}^n$ is a given function, $\mathcal{D} = \{\psi : [-r, 0] \rightarrow \mathbb{R}^n : \psi \text{ is continuous everywhere except for a finite number of points } \bar{t} \text{ at which } \psi(\bar{t}) \text{ and } \psi(\bar{t}^+) \text{ exist, and } \psi(\bar{t}^-) = \psi(\bar{t})\}$, $\phi \in D$, $0 < r < \infty$, $\tau_k : \mathbb{R}^n \rightarrow \mathbb{R}$, $I_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $k = 1, 2, \dots, m$, are given functions satisfying some assumptions that will be specified later.

The main theorem of this section extends the problem (8.1) considered by Benchohra et al. [46] when the impulse times are constant. Our approach is based on Schaefer's fixed point theorem.

Let us start by defining what we mean by a solution of problem (8.1).

Definition 8.1. A function $y \in \Omega \cap AC((t_k, t_{k+1}), \mathbb{R})$, $k = 0, \dots, m$, is said to be a solution of (8.1) if y satisfies the equation $y'(t) = f(t, y_t)$ a.e. on J , $t \neq \tau_k(y(t))$, $k = 1, \dots, m$, and the conditions $y(t^+) = I_k(y(t))$, $t = \tau_k(y(t))$, $k = 1, \dots, m$, and $y(t) = \phi(t)$ on $[-r, 0]$.

9

Nondensely defined impulsive differential equations & inclusions

9.1. Introduction

This chapter deals with semilinear functional differential equations and functional differential inclusions involving linear operators that are nondensely defined on a Banach space. This chapter extends several previous results of this book that were devoted to semilinear problems with densely defined operators. Some of the results of this chapter were first presented in the work by Benchohra et al. [76].

9.2. Nondensely defined impulsive semilinear differential equations with nonlocal conditions

In this section, we will prove existence results for an evolution equation with nonlocal conditions of the form

$$y'(t) = Ay(t) + F(t, y(t)), \quad t \in J := [0, T], \quad t \neq t_k, \quad k = 1, \dots, m, \quad (9.1)$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \quad (9.2)$$

$$y(0) + g(y) = y_0, \quad (9.3)$$

where $A : D(A) \subset E \rightarrow E$ is a nondensely defined closed linear operator, $F : J \times E \rightarrow E$ is continuous, $g : C(J', E) \rightarrow E$, $(J' = J \setminus \{t_1, \dots, t_m\})$, $I_k : E \rightarrow \overline{D(A)}$, $k = 1, \dots, m$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h \rightarrow 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h \rightarrow 0^+} y(t_k - h)$, and E is a separable Banach space with norm $|\cdot|$.

As indicated in [112, 115, 126] and the references therein, the nonlocal condition $y(0) + g(y) = y_0$ can be applied to physics with better effect than the classical initial condition $y(0) = y_0$. For example, in [126], the author used

$$g(y) = \sum_{k=1}^p c_i y(t_i), \quad (9.4)$$

where c_i , $i = 1, \dots, p$, are given constants and $0 < t_1 < t_2 < \dots < t_p \leq T$, to describe the diffusion phenomenon of a small amount of gas in a transparent tube. In this case, (9.4) allows the additional measurements at t_i , $i = 1, \dots, p$.

10 Hyperbolic impulsive differential inclusions

10.1. Introduction

In this chapter, we will be concerned with the existence of solutions for second-order impulsive hyperbolic differential inclusions in a separable Banach space. More precisely, we will consider impulsive hyperbolic differential inclusions of the form

$$\begin{aligned} \frac{\partial^2 u(t, x)}{\partial t \partial x} &\in F(t, x, u(t, x)), \quad \text{a.e. } (t, x) \in J_a \times J_b, \quad t \neq t_k, \quad k = 1, \dots, m, \\ \Delta u(t_k, x) &= I_k(u(t_k, x)), \quad k = 1, \dots, m, \\ u(t, 0) &= \psi(t), \quad t \in J_a, \quad u(0, x) = \phi(x), \quad x \in J_b, \end{aligned} \quad (10.1)$$

where $J_a = [0, a]$, $J_b = [0, b]$, $F : J_a \times J_b \times E \rightarrow \mathcal{P}(E)$ is a multivalued map ($\mathcal{P}(E)$ is the family of all nonempty subsets of E), $\phi \in C(J_a, E)$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = a$, $I_k \in C(E, E)$ ($k = 1, \dots, m$), $\Delta u|_{t=t_k} = u(t_k^+, y) - u(t_k^-, y)$, $u(t_k^+, y) = \lim_{(h,x) \rightarrow (0^+, y)} u(t_k + h, x)$ is the right limit and $u(t_k^-, y) = \lim_{(h,x) \rightarrow (0^+, y)} u(t_k - h, x)$ is left limit of $u(t, x)$ at (t_k, x) , and E is a real separable Banach space with norm $|\cdot|$.

In the last few years impulsive differential and partial differential equations have become the object of increasing investigation in some mathematical models of real world phenomena, especially in biological or medical domain; see the monographs by Bařnov and Simeonov [29], Lakshmikantham et al. [180], Samoilenko and Perestyuk [217].

In the last three decades several papers have been devoted to the study of hyperbolic partial differential equations with local and nonlocal initial conditions; see for instance [113, 115, 182] and the references cited therein. For similar results with set-valued right-hand side, we refer to the papers by Byszewski and Papageorgiou [116], Papageorgiou [208], and Benchohra and Ntouyas [33, 81, 83, 84].

Here we will present three existence results for problem (10.1) in the cases when F has convex and nonconvex values. In the convex case, an existence result will be given by means of the nonlinear alternative of Leray-Schauder type for multivalued maps. In the nonconvex, case two results will be presented. The first

11

Impulsive dynamic equations on time scales

11.1. Introduction

In recent years dynamic equations on time scales have received much attention. We refer to the books by Agarwal and O'Regan [7], Bohner and Peterson [101, 102], and Lakshmikantham et al. [184], and the papers by Anderson [15, 18], Agarwal et al. [2, 3, 5], Bohner and Guseinov [100], Bohner and Eloe [99], and Erbe and Peterson [141, 142].

The time scales calculus has a tremendous potential for applications in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics, neural networks, social sciences, as is pointed out in the monographs of Aulbach and Hilger [24], Bohner and Peterson [101, 102], and Lakshmikantham et al. [184].

The existence of solutions of boundary value problem on a time scale was recently studied by Agarwal and O'Regan [7], Anderson [16, 17], Henderson [166], and Sun and Li [223]. In this chapter, dynamic equations on time scales are considered for both impulsive initial value problems and impulsive boundary value problems. The results here are based on work from [72, 165].

11.2. Preliminaries

We will introduce some basic definitions and facts from the time scale calculus that we will use in the sequel.

A time scale \mathbb{T} is a nonempty closed subset of \mathbb{R} . It follows that the jump operators $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{T}$ defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\} \quad (11.1)$$

(supplemented by $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$) are well defined. The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. If \mathbb{T} has a right-scattered minimum m , define $\mathbb{T}_k := \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_k = \mathbb{T}$. If \mathbb{T} has a left-scattered maximum M , define $\mathbb{T}^k := \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^k = \mathbb{T}$. The notations $[c, d]$, $[c, d)$, and so

12

On periodic boundary value problems of first-order perturbed impulsive differential inclusions

12.1. Introduction

In this chapter, we study the existence of solutions to periodic nonlinear boundary value problems for first-order Carathéodory impulsive ordinary differential inclusions with convex multifunctions. Given a closed and bounded interval $J := [0, T]$ in \mathbb{R} , and given the impulsive moments t_1, t_2, \dots, t_p with $0 = t_0 < t_1 < t_2 < \dots < t_p < t_{p+1} = T$, $J' = J \setminus \{t_1, t_2, \dots, t_p\}$, $J_j = (t_j, t_{j+1})$, consider the following periodic boundary value problem for impulsive differential inclusions (IDI):

$$x'(t) \in F(t, x(t)) + G(t, x(t)) \quad \text{a.e. } t \in J', \quad (12.1)$$

$$x(t_j^+) = x(t_j^-) + I_j(x(t_j^-)), \quad (12.2)$$

$$x(0) = x(T), \quad (12.3)$$

where $F, G : J \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ are impulsive multifunctions, $I_j : \mathbb{R} \rightarrow \mathbb{R}$, $j = 1, 2, \dots, p$, are the impulse functions, and $x(t_j^+)$ and $x(t_j^-)$ are, respectively, the right and the left limits of x at $t = t_j$.

Let $C(J, \mathbb{R})$ and $L^1(J, \mathbb{R})$ denote the space of continuous and Lebesgue integrable real-valued functions on J . Consider the Banach space

$$X := \{x : J \rightarrow \mathbb{R} : x \in C(J', \mathbb{R}), x(t_j^+), x(t_j^-) \text{ exist}, x(t_j^-) = x(t_j), j = 1, 2, \dots, p\}, \quad (12.4)$$

equipped with the norm $\|x\| = \max\{|x(t)| : t \in J\}$, and the space

$$Y := \{x \in X : x \text{ is differentiable a.e. on } (0, T), x' \in L^1(J, \mathbb{R})\}. \quad (12.5)$$

By a solution of (12.1)–(12.3), we mean a function x in $Y_T := \{v \in Y : v(0) = v(T)\}$ that satisfies the differential inclusion (12.1) and the impulsive conditions (12.2).

Our aim is to provide sufficient conditions to the multifunctions F , G and the impulsive functions I_j that insure the existence of solutions of problem IDI (12.1)–(12.3).

Bibliography

- [1] R. P. Agarwal, M. Benchohra, D. O'Regan, and A. Ouahab, *Second order impulsive dynamic equations on time scales*, Functional Differential Equations **11** (2004), no. 3-4, 223–234.
- [2] R. P. Agarwal and M. Bohner, *Quadratic functionals for second-order matrix equations on time scales*, Nonlinear Analysis **33** (1998), no. 7, 675–692.
- [3] ———, *Basic calculus on time scales and some of its applications*, Results in Mathematics **35** (1999), no. 1-2, 3–22.
- [4] R. P. Agarwal, M. Bohner, and A. C. Peterson, *Inequalities on time scales: a survey*, Mathematical Inequalities & Applications **4** (2001), 535–557.
- [5] R. P. Agarwal, M. Bohner, and P. J. Y. Wong, *Sturm-Liouville eigenvalue problems on time scales*, Applied Mathematics and Computation **99** (1999), no. 2-3, 153–166.
- [6] R. P. Agarwal and D. O'Regan, *Multiple nonnegative solutions for second order impulsive differential equations*, Applied Mathematics and Computation **114** (2000), no. 1, 51–59.
- [7] ———, *Triple solutions to boundary value problems on time scales*, Applied Mathematics Letters **13** (2000), no. 4, 7–11.
- [8] ———, *Existence of three solutions to integral and discrete equations via the Leggett Williams fixed point theorem*, The Rocky Mountain Journal of Mathematics **31** (2001), no. 1, 23–35.
- [9] R. P. Agarwal, D. O'Regan, and P. J. Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic, Dordrecht, 1999.
- [10] Z. Agur, L. Cojocaru, G. Mazaur, R. M. Anderson, and Y. L. Danon, *Pulse mass measles vaccination across age cohorts*, Proceedings of the National Academy of Sciences of the United States of America **90** (1993), 11698–11702.
- [11] N. U. Ahmed, *Semigroup Theory with Applications to Systems and Control*, Pitman Research Notes in Mathematics Series, vol. 246, Longman Scientific & Technical, Harlow; John Wiley & Sons, New York, 1991.
- [12] ———, *Optimal impulse control for impulsive systems in Banach spaces*, International Journal of Differential Equations and Applications **1** (2000), no. 1, 37–52.
- [13] ———, *Systems governed by impulsive differential inclusions on Hilbert spaces*, Nonlinear Analysis **45** (2001), no. 6, 693–706.
- [14] H. Akça, A. Boucherif, and V. Covachev, *Impulsive functional-differential equations with nonlocal conditions*, International Journal of Mathematics and Mathematical Sciences **29** (2002), no. 5, 251–256.
- [15] D. R. Anderson, *Eigenvalue intervals for a second-order mixed conditions problem on time scale*, International Journal on Nonlinear Differential Equations **7** (2002), 97–104.
- [16] ———, *Eigenvalue intervals for a two-point boundary value problem on a measure chain*, Journal of Computational and Applied Mathematics **141** (2002), no. 1-2, 57–64.
- [17] ———, *Solutions to second-order three-point problems on time scales*, Journal of Difference Equations and Applications **8** (2002), no. 8, 673–688.
- [18] ———, *Eigenvalue intervals for even-order Sturm-Liouville dynamic equations*, Communications on Applied Nonlinear Analysis **12** (2005), no. 4, 1–13.
- [19] D. R. Anderson, R. I. Avery, and A. C. Peterson, *Three positive solutions to a discrete focal boundary value problem*, Journal of Computational and Applied Mathematics **88** (1998), no. 1, 103–118.
- [20] W. Arendt, *Resolvent positive operators and integrated semigroup*, Proceedings of the London Mathematical Society, Third Series **54** (1987), no. 2, 321–349.
- [21] ———, *Vector-valued Laplace transforms and Cauchy problems*, Israel Journal of Mathematics **59** (1987), no. 3, 327–352.
- [22] J.-P. Aubin and A. Cellina, *Differential Inclusions*, Springer, New York; Birkhäuser, Basel, 1984.

- [23] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Systems & Control: Foundations & Applications, vol. 2, Birkhäuser Boston, Massachusetts, 1990.
- [24] B. Aubach and S. Hilger, *Linear dynamic processes with inhomogeneous time scale*, Nonlinear Dynamics and Quantum Dynamical Systems (Gaussig, 1990), Math. Res., vol. 59, Akademie, Berlin, 1990, pp. 9–20.
- [25] R. I. Avery, M. Benchohra, J. Henderson, and S. Ntouyas, *Double solutions of boundary value problems for ordinary differential equations with impulse*, Dynamics of Continuous, Discrete & Impulsive Systems **10** (2003), no. 1–3, 1–10.
- [26] R. I. Avery and J. Henderson, *Two positive fixed points of nonlinear operators on ordered Banach spaces*, Communications on Applied Nonlinear Analysis **8** (2001), no. 1, 27–36.
- [27] N. V. Azbelev and A. I. Domoshnitskii, *On the question of linear differential inequalities. I*, Differentsial'nye Uravneniya **27** (1991), no. 3, 376–384, 547, translation in Differential Equations **27** (1991), no. 3, 257–263.
- [28] ———, *On the question of linear differential inequalities. II*, Differentsial'nye Uravneniya **27** (1991), no. 6, 923–931, 1098, translation in Differential Equations **27** (1991), no. 6, 641–647.
- [29] D. D. Bañov and P. S. Simeonov, *Systems with Impulse Effect*, Ellis Horwood Series: Mathematics and Its Applications, Ellis Horwood, Chichester, 1989.
- [30] ———, *Impulsive Differential Equations: Periodic Solutions and Applications*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 66, Longman Scientific & Technical, Harlow; John Wiley & Sons, New York, 1993.
- [31] I. Bajo and E. Liz, *Periodic boundary value problem for first order differential equations with impulses at variable times*, Journal of Mathematical Analysis and Applications **204** (1996), no. 1, 65–73.
- [32] J. Banaś and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lecture Notes in Pure and Applied Mathematics, vol. 60, Marcel Dekker, New York, 1980.
- [33] M. Benchohra, *A note on an hyperbolic differential inclusion in Banach spaces*, Bulletin of the Belgian Mathematical Society. Simon Stevin **9** (2002), no. 1, 101–107.
- [34] M. Benchohra and A. Boucherif, *On first order initial value problems for impulsive differential inclusions in Banach spaces*, Dynamic Systems and Applications **8** (1999), no. 1, 119–126.
- [35] ———, *Initial value problems for impulsive differential inclusions of first order*, Differential Equations and Dynamical Systems **8** (2000), no. 1, 51–66.
- [36] M. Benchohra, A. Boucherif, and A. Ouahab, *On nonresonance impulsive functional differential inclusions with nonconvex valued right-hand side*, Journal of Mathematical Analysis and Applications **282** (2003), no. 1, 85–94.
- [37] M. Benchohra, E. P. Gatsori, L. Górniewicz, and S. Ntouyas, *Existence results for impulsive semilinear neutral functional differential inclusions with nonlocal conditions*, Nonlinear Analysis and Applications, Kluwer Academic, Dordrecht, 2003, pp. 259–288.
- [38] ———, *Nondensely defined evolution impulsive differential equations with nonlocal conditions*, Fixed Point Theory **4** (2003), no. 2, 185–204.
- [39] M. Benchohra, E. P. Gatsori, J. Henderson, and S. Ntouyas, *Nondensely defined evolution impulsive differential inclusions with nonlocal conditions*, Journal of Mathematical Analysis and Applications **286** (2003), no. 1, 307–325.
- [40] M. Benchohra, E. P. Gatsori, S. Ntouyas, and Y. G. Sficas, *Nonlocal Cauchy problems for semilinear impulsive differential inclusions*, International Journal of Differential Equations and Applications **6** (2002), no. 4, 423–448.
- [41] M. Benchohra, L. Górniewicz, S. Ntouyas, and A. Ouahab, *Existence results for impulsive hyperbolic differential inclusions*, Applicable Analysis **82** (2003), no. 11, 1085–1097.
- [42] ———, *Existence results for nondensely defined impulsive semilinear functional differential equations*, Nonlinear Analysis and Applications, Kluwer Academic, Dordrecht, 2003, pp. 289–300.
- [43] ———, *Impulsive hyperbolic differential inclusions with variable times*, Topological Methods in Nonlinear Analysis **22** (2003), no. 2, 319–329.
- [44] M. Benchohra, J. R. Graef, J. Henderson, and S. Ntouyas, *Nonresonance impulsive higher order functional nonconvex-valued differential inclusions*, Electronic Journal of Qualitative Theory of Differential Equations **2002** (2002), no. 13, 1–13.

- [45] M. Benchohra, J. R. Graef, S. Ntouyas, and A. Ouahab, *Upper and lower solutions method for impulsive differential inclusions with nonlinear boundary conditions and variable times*, Dynamics of Continuous, Discrete & Impulsive Systems **12** (2005), no. 3-4, 383–396.
- [46] M. Benchohra, J. Henderson, and S. Ntouyas, *An existence result for first-order impulsive functional differential equations in Banach spaces*, Computers & Mathematics with Applications **42** (2001), no. 10-11, 1303–1310.
- [47] ———, *Existence results for first order impulsive semilinear evolution inclusions*, Electronic Journal of Qualitative Theory of Differential Equations **2001** (2001), no. 1, 1–12.
- [48] ———, *Existence results for impulsive multivalued semilinear neutral functional differential inclusions in Banach spaces*, Journal of Mathematical Analysis and Applications **263** (2001), 763–780.
- [49] ———, *Impulsive neutral functional differential equations in Banach spaces*, Applicable Analysis **80** (2001), 353–365.
- [50] ———, *On a periodic boundary value problem for first order impulsive differential inclusions*, Dynamic Systems and Applications **10** (2001), no. 4, 477–488.
- [51] ———, *On nonresonance impulsive functional differential inclusions with periodic boundary conditions*, International Journal of Applied Mathematics **5** (2001), no. 4, 377–391.
- [52] ———, *On positive solutions for a boundary value problem for second order impulsive functional differential equations*, Panamerican Mathematical Journal **11** (2001), no. 4, 61–69.
- [53] ———, *On second-order multivalued impulsive functional differential inclusions in Banach spaces*, Abstract and Applied Analysis **6** (2001), no. 6, 369–380.
- [54] ———, *Existence results for impulsive functional differential inclusions in Banach spaces*, Mathematical Sciences Research Journal **6** (2002), no. 1, 47–59.
- [55] ———, *Existence results for impulsive semilinear neutral functional differential equations in Banach spaces*, Memoirs on Differential Equations and Mathematical Physics **25** (2002), 105–120.
- [56] ———, *Impulsive neutral functional differential inclusions in Banach spaces*, Applied Mathematics Letters **15** (2002), no. 8, 917–924.
- [57] ———, *Multivalued impulsive neutral functional differential inclusions in Banach spaces*, Tamkang Journal of Mathematics **33** (2002), no. 1, 77–88.
- [58] ———, *On a boundary value problem for second order impulsive functional differential inclusions in Banach spaces*, International Journal of Nonlinear Differential Equations, Theory, Methods & Applications **7** (2002), no. 1 & 2, 65–75.
- [59] ———, *On first order impulsive differential inclusions with periodic boundary conditions*, Dynamics of Continuous, Discrete & Impulsive Systems **9** (2002), no. 3, 417–427.
- [60] ———, *On nonresonance impulsive functional nonconvex valued differential inclusions*, Commentationes Mathematicae Universitatis Carolinae **43** (2002), no. 4, 595–604.
- [61] ———, *Semilinear impulsive neutral functional differential inclusions in Banach spaces*, Applicable Analysis **81** (2002), no. 4, 951–963.
- [62] ———, *Impulsive functional differential inclusions in Banach spaces*, Communications in Applied Analysis **7** (2003), no. 2-3, 253–264.
- [63] ———, *Nonresonance higher order boundary value problems for impulsive functional differential inclusions*, Radovi Matematički **11** (2003), no. 2, 205–214.
- [64] ———, *On first order impulsive semilinear functional differential inclusions*, Archivum Mathematicum (Brno) **39** (2003), no. 2, 129–139.
- [65] ———, *On nonresonance second order impulsive functional differential inclusions with nonlinear boundary conditions*, to appear in The Canadian Applied Mathematics Quarterly.
- [66] M. Benchohra, J. Henderson, S. Ntouyas, and A. Ouahab, *Existence results for impulsive functional and neutral functional differential inclusions with lower semicontinuous right hand side*, Electronic Journal of Mathematical and Physical Sciences **1** (2002), no. 1, 72–91.
- [67] ———, *Existence results for impulsive lower semicontinuous differential inclusions*, International Journal of Pure and Applied Mathematics **1** (2002), no. 4, 431–443.
- [68] ———, *Impulsive functional semilinear differential inclusions with lower semicontinuous right hand side*, International Journal of Applied Mathematics **11** (2002), no. 2, 171–196.
- [69] ———, *Existence results for impulsive semilinear damped differential inclusions*, Electronic Journal of Qualitative Theory of Differential Equations **2003** (2003), no. 11, 1–19.

- [70] ———, *Higher order impulsive functional differential equations with variable times*, Dynamic Systems and Applications **12** (2003), no. 3-4, 383–392.
- [71] ———, *Impulsive functional differential equations with variable times*, Computers & Mathematics with Applications **47** (2004), no. 10-11, 1659–1665.
- [72] ———, *On first order impulsive dynamic equations on time scales*, Journal of Difference Equations and Applications **10** (2004), no. 6, 541–548.
- [73] ———, *Upper and lower solutions method for first-order impulsive differential inclusions with nonlinear boundary conditions*, Computers & Mathematics with Applications **47** (2004), no. 6-7, 1069–1078.
- [74] ———, *Impulsive functional dynamic equations on time scales*, Dynamic Systems and Applications **14** (2005), 1–10.
- [75] ———, *Multiple solutions for impulsive semilinear functional and neutral functional differential equations in Hilbert space*, Journal of Inequalities and Applications **2005** (2005), no. 2, 189–205.
- [76] ———, *Existence results for nondensely defined impulsive semilinear functional differential inclusions*, preprint.
- [77] M. Benchohra and S. Ntouyas, *An existence result for semilinear delay integrodifferential inclusions of Sobolev type with nonlocal conditions*, Communications on Applied Nonlinear Analysis **7** (2000), no. 3, 21–30.
- [78] ———, *Existence of mild solutions of semilinear evolution inclusions with nonlocal conditions*, Georgian Mathematical Journal **7** (2000), no. 2, 221–230.
- [79] ———, *Existence of mild solutions on noncompact intervals to second-order initial value problems for a class of differential inclusions with nonlocal conditions*, Computers & Mathematics with Applications **39** (2000), no. 12, 11–18.
- [80] ———, *Existence theorems for a class of first order impulsive differential inclusions*, Acta Mathematica Universitatis Comenianae. New Series **70** (2001), no. 2, 197–205.
- [81] ———, *Hyperbolic functional differential inclusions in Banach spaces with nonlocal conditions*, Functiones et Approximatio Commentarii Mathematici **29** (2001), 29–39.
- [82] ———, *On first order impulsive differential inclusions in Banach spaces*, Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie. Nouvelle Série **44(92)** (2001), no. 2, 165–174.
- [83] ———, *An existence theorem for an hyperbolic differential inclusion in Banach spaces*, Discussiones Mathematicae. Differential Inclusions, Control and Optimization **22** (2002), no. 1, 5–16.
- [84] ———, *On an hyperbolic functional differential inclusion in Banach spaces*, Fasciculi Mathematici (2002), no. 33, 27–35.
- [85] ———, *On second order impulsive functional differential equations in Banach spaces*, Journal of Applied Mathematics and Stochastic Analysis **15** (2002), no. 1, 47–55.
- [86] ———, *On first order impulsive semilinear differential inclusions*, Communications in Applied Analysis **7** (2003), no. 2-3, 349–358.
- [87] M. Benchohra, S. Ntouyas, and A. Ouahab, *Existence results for impulsive semilinear damped differential equations*, International Journal of Applied Mathematics **11** (2002), no. 1, 77–93.
- [88] ———, *Existence results for second order boundary value problem of impulsive dynamic equations on time scales*, Journal of Mathematical Analysis and Applications **296** (2004), no. 1, 65–73.
- [89] ———, *Existence results for impulsive functional semilinear differential inclusions with nonlocal conditions*, to appear in Dynamics of Continuous, Discrete & Impulsive Systems.
- [90] M. Benchohra and A. Ouahab, *Some uniqueness results for impulsive semilinear neutral functional differential equations*, Georgian Mathematical Journal **9** (2002), no. 3, 423–430.
- [91] ———, *Impulsive neutral functional differential equations with variable times*, Nonlinear Analysis **55** (2003), no. 6, 679–693.
- [92] ———, *Impulsive neutral functional differential inclusions with variable times*, Electronic Journal of Differential Equations **2003** (2003), no. 67, 1–12.
- [93] ———, *Initial and boundary value problems for second order impulsive functional differential inclusions*, Electronic Journal of Qualitative Theory of Differential Equations **2003** (2003), no. 3, 1–10.

- [94] ———, *Multiple solutions for nonresonance impulsive functional differential equations*, Electronic Journal of Differential Equations **2003** (2003), no. 52, 1–10.
- [95] M. Benchohra, A. Ouahab, J. Henderson, and S. Ntouyas, *A note on multiple solutions for impulsive functional differential equations*, Communications on Applied Nonlinear Analysis **12** (2005), no. 3, 61–70.
- [96] V. I. Blagodatskih, *Some results on the theory of differential inclusions*, Summer School on Ordinary Differential Equations, part II, Czechoslovakia, Brno, 1974, pp. 29–67.
- [97] V. I. Blagodatskih and A. F. Filippov, *Differential inclusions and optimal control*, Akademiya Nauk SSSR. Trudy Matematicheskogo Instituta imeni V. A. Steklova **169** (1985), 194–252, 255 (Russian).
- [98] H. F. Bohnenblust and S. Karlin, *On a theorem of Ville*, Contributions to the Theory of Games, Annals of Mathematics Studies, no. 24, Princeton University Press, New Jersey, 1950, pp. 155–160.
- [99] M. Bohner and P. W. Eloe, *Higher order dynamic equations on measure chains: wronskians, discontinuity, and interpolating families of functions*, Journal of Mathematical Analysis and Applications **246** (2000), no. 2, 639–656.
- [100] M. Bohner and G. Sh. Guseinov, *Improper integrals on time scales*, Dynamic Systems and Applications **12** (2003), no. 1-2, 45–65.
- [101] M. Bohner and A. C. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser Boston, Massachusetts, 2001.
- [102] M. Bohner and A. C. Peterson (eds.), *Advances in Dynamic Equations on Time Scales*, Birkhäuser Boston, Massachusetts, 2003.
- [103] A. Boucherif, *Nonlocal Cauchy problems for first-order multivalued differential equations*, Electronic Journal of Differential Equations **2002** (2002), no. 47, 1–9.
- [104] A. Bressan, *On a bang-bang principle for nonlinear systems*, Unione Matematica Italiana. Bollettino. Supplemento (1980), no. 1, 53–59.
- [105] A. Bressan and G. Colombo, *Extensions and selections of maps with decomposable values*, Studia Mathematica **90** (1988), no. 1, 69–86.
- [106] A. I. Bulgakov, *Continuous selections of multivalued mappings, and functional-differential inclusions with a nonconvex right-hand side*, Matematicheskii Sbornik **181** (1990), no. 11, 1427–1442.
- [107] ———, *Some problems of differential and integral inclusions with a nonconvex right-hand side*, Functional-Differential Equations (Russian), Perm Politekh. Inst., Perm, 1991, pp. 28–57.
- [108] ———, *Integral inclusions with nonconvex images and their applications to boundary value problems for differential inclusions*, Matematicheskii Sbornik **183** (1992), no. 10, 63–86.
- [109] A. I. Bulgakov, A. A. Efremov, and E. A. Panasenko, *Ordinary differential inclusions with internal and external perturbations*, Differentsial'nye Uravneniya **36** (2000), no. 12, 1587–1598, 1726.
- [110] A. I. Bulgakov and V. V. Skomorokhov, *Approximation of differential inclusions*, Matematicheskii Sbornik **193** (2002), no. 2, 35–52.
- [111] A. I. Bulgakov and L. I. Tkach, *Perturbation of a convex-valued operator by a Hammerstein-type multivalued mapping with nonconvex images, and boundary value problems for functional-differential inclusions*, Matematicheskii Sbornik **189** (1998), no. 6, 3–32.
- [112] L. Byszewski, *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem*, Journal of Mathematical Analysis and Applications **162** (1991), no. 2, 494–505.
- [113] ———, *Existence and uniqueness of mild and classical solutions of semilinear functional-differential evolution nonlocal Cauchy problem*, Selected Problems of Mathematics, 50th Anniv. Cracow Univ. Technol. Anniv. Issue, vol. 6, Cracow University of Technology, Kraków, 1995, pp. 25–33.
- [114] L. Byszewski and H. Akca, *On a mild solution of a semilinear functional-differential evolution nonlocal problem*, Journal of Applied Mathematics and Stochastic Analysis **10** (1997), no. 3, 265–271.
- [115] L. Byszewski and V. Lakshmikantham, *Monotone iterative technique for nonlocal hyperbolic differential problem*, Journal of Mathematical and Physical Sciences **26** (1992), no. 4, 345–359.

- [116] L. Byszewski and N. S. Papageorgiou, *An application of a noncompactness technique to an investigation of the existence of solutions to a nonlocal multivalued Darboux problem*, Journal of Applied Mathematics and Stochastic Analysis **12** (1999), no. 2, 179–190.
- [117] A. Cabada and E. Liz, *Discontinuous impulsive differential equations with nonlinear boundary conditions*, Nonlinear Analysis **28** (1997), no. 9, 1491–1497.
- [118] D. A. Carlson, *Carathéodory's method for a class of dynamic games*, Journal of Mathematical Analysis and Applications **276** (2002), no. 2, 561–588.
- [119] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics, vol. 580, Springer, Berlin, 1977.
- [120] K. C. Chang, *The obstacle problem and partial differential equations with discontinuous nonlinearities*, Communications on Pure and Applied Mathematics **33** (1980), no. 2, 117–146.
- [121] F. H. Clarke, Yu. S. Ledyaeve, and R. J. Stern, *Asymptotic stability and smooth Lyapunov functions*, Journal of Differential Equations **149** (1998), no. 1, 69–114.
- [122] C. Corduneanu and V. Lakshmikantham, *Equations with unbounded delay: a survey*, Nonlinear Analysis **4** (1980), no. 5, 831–877.
- [123] H. Covitz and S. B. Nadler Jr., *Multi-valued contraction mappings in generalized metric spaces*, Israel Journal of Mathematics **8** (1970), 5–11.
- [124] G. Da Prato and E. Sinestrari, *Differential operators with nondense domain*, Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie IV **14** (1987), no. 2, 285–344 (1988).
- [125] K. Deimling, *Multivalued Differential Equations*, de Gruyter Series in Nonlinear Analysis and Applications, vol. 1, Walter de Gruyter, Berlin, 1992.
- [126] K. Deng, *Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions*, Journal of Mathematical Analysis and Applications **179** (1993), no. 2, 630–637.
- [127] B. C. Dhage, *Multi-valued mappings and fixed points II*, Tamkang Journal of Mathematics **37** (2006), no. 1, 27–46.
- [128] B. C. Dhage, A. Boucherif, and S. Ntouyas, *On periodic boundary value problems of first-order perturbed impulsive differential inclusions*, Electronic Journal of Differential Equations **2004** (2004), no. 84, 1–9.
- [129] B. C. Dhage, J. Henderson, and J. J. Nieto, *Periodic boundary value problems for first order functional impulsive differential inclusions*, Communications on Applied Nonlinear Analysis **11** (2004), no. 3, 13–25.
- [130] A. Domoshnitsky, *Factorization of a linear boundary value problem and the monotonicity of the Green operator*, Differentsial'nye Uravneniya **28** (1992), no. 3, 390–394, 546.
- [131] ———, *On periodic boundary value problem for first order impulsive functional-differential non-linear equation*, Functional Differential Equations **4** (1997), no. 1-2, 39–46 (1998).
- [132] A. Domoshnitsky and M. Drakhlin, *Nonoscillation of first order impulse differential equations with delay*, Journal of Mathematical Analysis and Applications **206** (1997), no. 1, 254–269.
- [133] Y. Dong, *Periodic boundary value problems for functional-differential equations with impulses*, Journal of Mathematical Analysis and Applications **210** (1997), no. 1, 170–181.
- [134] Y. Dong and E. Zhou, *An application of coincidence degree continuation theorem in existence of solutions of impulsive differential equations*, Journal of Mathematical Analysis and Applications **197** (1996), no. 3, 875–889.
- [135] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics, vol. 194, Springer, New York, 2000.
- [136] L. H. Erbe, H. I. Freedman, X. Z. Liu, and J. H. Wu, *Comparison principles for impulsive parabolic equations with applications to models of single species growth*, Journal of the Australian Mathematical Society. Series B **32** (1991), no. 4, 382–400.
- [137] L. H. Erbe, S. C. Hu, and H. Wang, *Multiple positive solutions of some boundary value problems*, Journal of Mathematical Analysis and Applications **184** (1994), no. 3, 640–648.
- [138] L. H. Erbe, Q. Kong, and B. G. Zhang, *Oscillation Theory for Functional-Differential Equations*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 190, Marcel Dekker, New York, 1995.
- [139] L. H. Erbe and W. Krawcewicz, *Nonlinear boundary value problems for differential inclusions $y'' \in F(t, y, y')$* , Annales Polonici Mathematici **54** (1991), no. 3, 195–226.

- [140] ———, *Existence of solutions to boundary value problems for impulsive second order differential inclusions*, The Rocky Mountain Journal of Mathematics **22** (1992), no. 2, 519–539.
- [141] L. H. Erbe and A. C. Peterson, *Green's functions and comparison theorems for differential equations on measure chains*, Dynamics of Continuous, Discrete and Impulsive Systems **6** (1999), no. 1, 121–137.
- [142] ———, *Positive solutions for a nonlinear differential equation on a measure chain*, Mathematical and Computer Modelling **32** (2000), no. 5–6, 571–585.
- [143] K. Ezzinbi and J. H. Liu, *Nondensely defined evolution equations with nonlocal conditions*, Mathematical and Computer Modelling **36** (2002), no. 9–10, 1027–1038.
- [144] ———, *Periodic solutions of non-densely defined delay evolution equations*, Journal of Applied Mathematics and Stochastic Analysis **15** (2002), no. 2, 113–123.
- [145] H. O. Fattorini, *Second Order Linear Differential Equations in Banach Spaces*, North-Holland Mathematics Studies, vol. 108, North-Holland, Amsterdam, 1985.
- [146] A. F. Filippov, *Classical solutions of differential equations with the right-hand side multi-valued*, Vestnik Moskovskogo Universiteta. Serija I. Matematika, Mehanika **22** (1967), no. 3, 16–26 (Russian).
- [147] D. Franco, E. Liz, J. J. Nieto, and Y. V. Rogovchenko, *A contribution to the study of functional differential equations with impulses*, Mathematische Nachrichten **218** (2000), 49–60.
- [148] M. Frigon, *Application de la théorie de la transversalité topologique à des problèmes non linéaires pour des équations différentielles ordinaires*, Dissertationes Mathematicae **296** (1990), 75.
- [149] M. Frigon and A. Granas, *Théorèmes d'existence pour des inclusions différentielles sans convexité*, Comptes Rendus de l'Académie des Sciences. Série I. Mathématique **310** (1990), no. 12, 819–822.
- [150] M. Frigon and D. O'Regan, *Boundary value problems for second order impulsive differential equations using set-valued maps*, Applicable Analysis **58** (1995), no. 3–4, 325–333.
- [151] ———, *Existence results for first-order impulsive differential equations*, Journal of Mathematical Analysis and Applications **193** (1995), no. 1, 96–113.
- [152] ———, *Impulsive differential equations with variable times*, Nonlinear Analysis **26** (1996), no. 12, 1913–1922.
- [153] ———, *First order impulsive initial and periodic problems with variable moments*, Journal of Mathematical Analysis and Applications **233** (1999), no. 2, 730–739.
- [154] A. Goldbeter, Y. X. Li, and G. Dupont, *Pulsatile signalling in intercellular communication: experimental and theoretical aspects*, Mathematics Applied to Biology and Medicine, Wuerz, Winnipeg, 1993, pp. 429–439.
- [155] J. A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford Mathematical Monographs, Oxford University Press, New York, 1985.
- [156] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings*, Mathematics and Its Applications, vol. 495, Kluwer Academic, Dordrecht, 1999.
- [157] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer Monographs in Mathematics, Springer, New York, 2003.
- [158] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Notes and Reports in Mathematics in Science and Engineering, vol. 5, Academic Press, Massachusetts, 1988.
- [159] D. J. Guo and X. Liu, *Multiple positive solutions of boundary-value problems for impulsive differential equations*, Nonlinear Analysis **25** (1995), no. 4, 327–337.
- [160] J. K. Hale, *Ordinary Differential Equations*, Pure and Applied Mathematics, John Wiley & Sons, New York, 1969.
- [161] J. K. Hale and J. Kato, *Phase space for retarded equations with infinite delay*, Funkcialaj Ekvacioj. Serio Internacia **21** (1978), no. 1, 11–41.
- [162] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional-Differential Equations*, Applied Mathematical Sciences, vol. 99, Springer, New York, 1993.
- [163] S. Heikkilä and V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 181, Marcel Dekker, New York, 1994.

- [164] J. Henderson (ed.), *Boundary Value Problems for Functional-Differential Equations*, World Scientific, New Jersey, 1995.
- [165] J. Henderson, *Double solutions of impulsive dynamic boundary value problems on a time scale*, Journal of Difference Equations and Applications **8** (2002), no. 4, 345–356.
- [166] ———, *Nontrivial solutions to a nonlinear boundary value problem on a time scale*, Communications on Applied Nonlinear Analysis **11** (2004), no. 1, 65–71.
- [167] E. Hernández Morales, *A second-order impulsive Cauchy problem*, International Journal of Mathematics and Mathematical Sciences **31** (2002), no. 8, 451–461.
- [168] S. Hilger, *Ein Maßkettenakül mit Anwendung auf Zentrumsannigfaltigkeiten*, Ph.D. thesis, Universität Würzburg, Würzburg, 1988.
- [169] Y. Hino, S. Murakami, and T. Naito, *Functional-Differential Equations with Infinite Delay*, Lecture Notes in Mathematics, vol. 1473, Springer, Berlin, 1991.
- [170] Sh. Hu and N. S. Papageorgiou, *Handbook of Multivalued Analysis. Vol. I. Theory*, Mathematics and Its Applications, vol. 419, Kluwer Academic, Dordrecht, 1997.
- [171] M. Kamenskii, V. Obukhovskii, and P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*, de Gruyter Series in Nonlinear Analysis and Applications, vol. 7, Walter de Gruyter, Berlin, 2001.
- [172] F. Kappel and W. Schappacher, *Some considerations to the fundamental theory of infinite delay equations*, Journal of Differential Equations **37** (1980), no. 2, 141–183.
- [173] S. K. Kaul, *Monotone iterative technique for impulsive differential equations with variable times*, Nonlinear World **2** (1995), no. 3, 341–354.
- [174] S. Kaul, V. Lakshmikantham, and S. Leela, *Extremal solutions, comparison principle and stability criteria for impulsive differential equations with variable times*, Nonlinear Analysis **22** (1994), no. 10, 1263–1270.
- [175] H. Kellerman and M. Hieber, *Integrated semigroups*, Journal of Functional Analysis **84** (1989), no. 1, 160–180.
- [176] M. Kirane and Y. V. Rogovchenko, *Comparison results for systems of impulse parabolic equations with applications to population dynamics*, Nonlinear Analysis **28** (1997), no. 2, 263–276.
- [177] M. Kisielewicz, *Differential Inclusions and Optimal Control*, Mathematics and Its Applications, vol. 44, Kluwer Academic, Dordrecht, 1991.
- [178] E. Klein and A. C. Thompson, *Theory of Correspondences*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons, New York, 1984.
- [179] N. N. Krasovskii and A. I. Subbotin, *Game-Theoretical Control Problems*, Springer Series in Soviet Mathematics, Springer, New York, 1988.
- [180] V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, Series in Modern Applied Mathematics, vol. 6, World Scientific, New Jersey, 1989.
- [181] V. Lakshmikantham, S. Leela, and S. K. Kaul, *Comparison principle for impulsive differential equations with variable times and stability theory*, Nonlinear Analysis **22** (1994), no. 4, 499–503.
- [182] V. Lakshmikantham and S. G. Pandit, *The method of upper, lower solutions and hyperbolic partial differential equations*, Journal of Mathematical Analysis and Applications **105** (1985), no. 2, 466–477.
- [183] V. Lakshmikantham, N. S. Papageorgiou, and J. Vasundhara, *The method of upper and lower solutions and monotone technique for impulsive differential equations with variable moments*, Applicable Analysis **51** (1993), no. 1–4, 41–58.
- [184] V. Lakshmikantham, S. Sivasundaram, and B. Kaymakçalan, *Dynamic Systems on Measure Chains*, Mathematics and Its Applications, vol. 370, Kluwer Academic, Dordrecht, 1996.
- [185] V. Lakshmikantham, L. Z. Wen, and B. G. Zhang, *Theory of Differential Equations with Unbounded Delay*, Mathematics and Its Applications, vol. 298, Kluwer Academic, Dordrecht, 1994.
- [186] A. Lasota and Z. Opial, *An Application of the Kakutani—Ky Fan Theorem in the Theory of Ordinary Differential Equations*, Bulletin de l'Académie Polonaise des Sciences. Série des Sciences Mathématiques, Astronomiques et Physiques **13** (1965), 781–786.
- [187] R. W. Leggett and L. R. Williams, *Multiple positive fixed points of nonlinear operators on ordered Banach spaces*, Indiana University Mathematics Journal **28** (1979), no. 4, 673–688.

- [188] J. H. Liu, *Nonlinear impulsive evolution equations*, Dynamics of Continuous, Discrete and Impulsive Systems **6** (1999), no. 1, 77–85.
- [189] X. Liu and G. Ballinger, *Existence and continuability of solutions for differential equations with delays and state-dependent impulses*, Nonlinear Analysis **51** (2002), no. 4, 633–647.
- [190] X. Liu, S. Sivaloganathan, and S. Zhang, *Monotone iterative techniques for time-dependent problems with applications*, Journal of Mathematical Analysis and Applications **237** (1999), no. 1, 1–18.
- [191] X. Liu and S. Zhang, *A cell population model described by impulsive PDEs—existence and numerical approximation*, Computers & Mathematics with Applications **36** (1998), no. 8, 1–11.
- [192] E. Liz, *Existence and approximation of solutions for impulsive first order problems with nonlinear boundary conditions*, Nonlinear Analysis **25** (1995), no. 11, 1191–1198.
- [193] ———, *Abstract monotone iterative techniques and applications to impulsive differential equations*, Dynamics of Continuous, Discrete and Impulsive Systems **3** (1997), no. 4, 443–452.
- [194] E. Liz and J. J. Nieto, *Positive solutions of linear impulsive differential equations*, Communications in Applied Analysis **2** (1998), no. 4, 565–571.
- [195] G. Marino, P. Pietramala, and L. Muglia, *Impulsive neutral integrodifferential equations on unbounded intervals*, Mediterranean Journal of Mathematics **1** (2004), no. 1, 93–108.
- [196] M. Martelli, *A Rothe's type theorem for non-compact acyclic-valued maps*, Bollettino della Unione Matematica Italiana (4) **11** (1975), no. 3, suppl., 70–76.
- [197] M. P. Matos and D. C. Pereira, *On a hyperbolic equation with strong damping*, Funkcialaj Ekvacioj **34** (1991), no. 2, 303–311.
- [198] M. D. P. Monteiro Marques, *Differential Inclusions in Nonsmooth Mechanical Problems*, Progress in Nonlinear Differential Equations and Their Applications, Birkhäuser, Basel, 1993.
- [199] J. J. Nieto, *Basic theory for nonresonance impulsive periodic problems of first order*, Journal of Mathematical Analysis and Applications **205** (1997), no. 2, 423–433.
- [200] ———, *Impulsive resonance periodic problems of first order*, Applied Mathematics Letters **15** (2002), no. 4, 489–493.
- [201] ———, *Periodic boundary value problems for first-order impulsive ordinary differential equations*, Nonlinear Analysis **51** (2002), no. 7, 1223–1232.
- [202] S. Ntouyas, *Initial and boundary value problems for functional-differential equations via the topological transversality method: a survey*, Bulletin of the Greek Mathematical Society **40** (1998), 3–41.
- [203] S. Ntouyas and P. Ch. Tsamatos, *Global existence for second order functional semilinear equations*, Periodica Mathematica Hungarica **31** (1995), no. 3, 223–228.
- [204] ———, *Global existence for second order semilinear ordinary and delay integrodifferential equations with nonlocal conditions*, Applicable Analysis **67** (1997), no. 3–4, 245–257.
- [205] ———, *Global existence for semilinear evolution equations with nonlocal conditions*, Journal of Mathematical Analysis and Applications **210** (1997), no. 2, 679–687.
- [206] ———, *Global existence for semilinear evolution integrodifferential equations with delay and non-local conditions*, Applicable Analysis **64** (1997), no. 1–2, 99–105.
- [207] ———, *Global existence for second order functional semilinear integrodifferential equations*, Mathematica Slovaca **50** (2000), no. 1, 95–109.
- [208] N. S. Papageorgiou, *Existence of solutions for hyperbolic differential inclusions in Banach spaces*, Archivum Mathematicum **28** (1992), no. 3–4, 205–213.
- [209] S. K. Patcheu, *On a global solution and asymptotic behaviour for the generalized damped extensible beam equation*, Journal of Differential Equations **135** (1997), no. 2, 299–314.
- [210] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, vol. 44, Springer, New York, 1983.
- [211] G. Pianigiani, *On the fundamental theory of multivalued differential equations*, Journal of Differential Equations **25** (1977), no. 1, 30–38.
- [212] C. Pierson-Gorez, *Problèmes aux Limites Pour des Equations Différentielles avec Impulsions*, Ph.D. thesis, Université Louvain-la-Neuve, Louvain, 1993.
- [213] A. Plis, *On trajectories of orientor fields*, Bulletin de l'Académie Polonaise des Sciences. Série des Sciences Mathématiques **13** (1965), 571–573.

- [214] A. Ponosov, A. Shindiapin, and J. J. Miguel, *The W -transform links delay and ordinary differential equations*, Functional Differential Equations **9** (2002), no. 3-4, 437–469.
- [215] R. T. Rockafellar, *Equivalent subgradient versions of Hamiltonian and Euler-Lagrange equations in variational analysis*, SIAM Journal on Control and Optimization **34** (1996), no. 4, 1300–1314.
- [216] Y. V. Rogovchenko, *Impulsive evolution systems: main results and new trends*, Dynamics of Continuous, Discrete and Impulsive Systems **3** (1997), no. 1, 57–88.
- [217] A. M. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations*, World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises, vol. 14, World Scientific, New Jersey, 1995.
- [218] H. Schaefer, *Über die Methode der a priori-Schranken*, Mathematische Annalen **129** (1955), 415–416.
- [219] G. N. Silva and R. B. Vinter, *Measure driven differential inclusions*, Journal of Mathematical Analysis and Applications **202** (1996), no. 3, 727–746.
- [220] D. R. Smart, *Fixed Point Theorems*, Cambridge University Press, London, 1974.
- [221] G. V. Smirnov, *Introduction to the Theory of Differential Inclusions*, Graduate Studies in Mathematics, vol. 41, American Mathematical Society, Rhode Island, 2002.
- [222] D. E. Stewart, *Existence of solutions to rigid body dynamics and the Painlevé paradoxes*, Comptes Rendus de l'Académie des Sciences. Série I. Mathématique **325** (1997), no. 6, 689–693.
- [223] J. P. Sun, *A new existence theorem for right focal boundary value problems on a measure chain*, Applied Mathematics Letters **18** (2005), no. 1, 41–47.
- [224] H. R. Thieme, *"Integrated semigroups" and integrated solutions to abstract Cauchy problems*, Journal of Mathematical Analysis and Applications **152** (1990), no. 2, 416–447.
- [225] A. A. Tolstonogov, *Differential Inclusions in a Banach Space*, Mathematics and Its Applications, vol. 524, Kluwer Academic, Dordrecht, 2000.
- [226] C. C. Travis and G. F. Webb, *Cosine families and abstract nonlinear second order differential equations*, Acta Mathematica Academiae Scientiarum Hungaricae **32** (1978), no. 1-2, 75–96.
- [227] ———, *Second order differential equations in Banach space*, Nonlinear Equations in Abstract Spaces (Proceedings of Internat. Sympos., University of Texas, Arlington, Tex, 1977), Academic Press, New York, 1978, pp. 331–361.
- [228] A. S. Vatsala and Y. Sun, *Periodic boundary value problems of impulsive differential equations*, Applicable Analysis **44** (1992), no. 3-4, 145–158.
- [229] J. Wu, *Theory and Applications of Partial Functional-Differential Equations*, Applied Mathematical Sciences, vol. 119, Springer, New York, 1996.
- [230] K. Yosida, *Functional Analysis*, 6th ed., Fundamental Principles of Mathematical Sciences, vol. 123, Springer, Berlin, 1980.
- [231] E. Zeidler, *Nonlinear Functional Analysis and Its Applications. I. Fixed-Point Theorems*, Springer, New York, 1986.

Index

Symbols

γ -Lipschitz, 5

A

a priori bounds, 109, 246, 296, 304, 322

absolutely continuous, 24, 95, 102, 168

Avery-Henderson theorem, 6

B

Bochner integrable, 2, 292

Bohnenblust-Karlin theorem, 4

bounded, 1, 2

C

Carathéodory, 9, 25, 292

completely continuous, 3

condensing, 4, 27, 39, 41, 45, 62, 63, 80, 82,

96, 118, 140, 187, 205, 213, 221, 245

cone, 5, 6, 150, 153, 159, 160, 339, 340

contraction, 5, 7, 19, 23, 24, 57, 60, 62, 85, 88,

89, 94, 95, 102, 107, 118, 133, 145, 209,

217, 224, 269, 272, 292, 299, 346,

350–352

cosine family, 8, 13, 58

Covitz and Nadler theorem, 5, 54, 62, 63, 85,

118, 199, 222, 269

D

decomposable, 3, 114, 133, 224, 273, 292,

299, 300

delay, 63, 318

densely defined operator, 8, 253, 254

E

evolution inclusion, 37, 42, 50

G

graininess, 312

Green's function, 159, 218, 339

Gronwall's lemma, 189, 246, 251, 296, 305,

328, 332

H

Hille-Yosida condition, 9, 254, 255, 257, 273

hyperbolic, 13, 291, 301

I

impulsive, 12

infinitesimal, 7, 12, 13, 37, 50, 58, 107, 108,

120, 136

initial value problem, 14, 26, 30, 63, 74, 80, 85,

90, 95, 115, 119, 134, 147, 165, 167, 184,

229, 235, 254, 273, 311, 312, 314, 325

integrable bounded, 347

integral solution, 254, 255, 257, 261, 262,

264–266, 269, 270, 273, 276–278, 282,

283, 285–288, 290

integrated semigroup, 8

K

Krasnosel'skii twin fixed point theorem, 6

Kuratowski measure of noncompactness, 4

L

Leggett-Williams fixed point theorem, 6

Leray-Schauder, 30, 35, 63, 69, 118, 130, 133,

243, 246, 249, 251, 264, 291, 292, 297,

299, 301, 305, 307, 313, 315–317, 320,

323–325, 333, 338, 339

Lipschitz, 9

lower semicontinuous, 3, 25, 33, 84, 92, 99,

103, 114, 115, 133, 224, 225, 273, 286,

292

lower semicontinuous type, 300

lower solution, 43, 62, 167, 168, 184, 198, 242,

247

M

Martelli, 4, 25, 27, 29, 63, 118, 199, 219

measurable, 2, 3

mild solution, 13, 20, 37, 42, 50, 108, 136

modified problem, 26, 44, 48, 169, 185, 243,

249, 251

multiple solutions, 5, 147, 159, 165

N

neutral, 63, 74, 95, 276, 277, 282, 285, 287,

290, 318, 323

Niemitzki operator, 3, 300

nondensely, 253, 254, 264, 276, 277, 285, 290

nonlinear alternative, 4

nonlocal condition, 119, 122, 136, 253, 254,
260, 264, 290
nonresonance, 199, 209, 217, 227

P

partial ordering, 159, 168
Perturbed, 345
precompact, 53, 111, 112, 127, 140, 259, 267,
268, 280, 322

R

regressive, 312–314, 325, 329
right dense, 311–313, 318, 334, 339
right dense continuous, 312

S

Schaefer theorem, 4
selection, 4, 55, 56, 61, 84, 87, 89, 94, 101, 106,
114, 118, 132, 133, 144, 273, 288, 292,
298–300, 347, 351
semigroup, 7, 8, 12, 37, 50, 62, 107, 108, 120,
136, 254, 261
semigroup C_0 , 7
separable Banach space, 3, 12, 50, 63, 108, 120,
136, 192, 200, 253, 264, 277, 285, 291
sine family, 8
solution integral, 255, 278, 282, 286, 287

T

three fixed points, 6, 151, 154
truncation operator, 44, 169
twin fixed point theorem, 6, 165
type \mathcal{M} , 25

U

uniformly continuous, 7
upper semicontinuous, 2, 4, 5, 9, 25, 41, 124,
171, 173, 292, 347, 349
upper solution, 168, 169, 174, 184, 190, 242

V

variable times, 229, 235, 241, 301