Impulsive Differential Equations and Inclusions

M. BENCHOHRA, J. HENDERSON, and S. NTOUYAS

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Contemporary Mathematics and Its Applications, Volume 2

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Contemporary Mathematics and Its Applications
Series Editors: Ravi P. Agarwal and Donal O'Regan

Hindawi Publishing Corporation
410 Park Avenue, 15th Floor, #287 pmb, New York, NY 10022, USA
Nasr City Free Zone, Cairo 11816, Egypt
Fax: +1-866-HINDAWI (USA Toll-Free)

ISBN 977-5945-50-X

Dedication

We dedicate this book to our family members who complete us. In particular, M. Benchohra's dedication is to his wife, Kheira, and his children, Mohamed, Maroua, and Abdelillah; J. Henderson dedicates to his wife, Darlene, and his descendants, Kathy, Dirk, Katie, David, and Jana Beth; and S. Ntouyas makes his dedication to his wife, Ioanna, and his daughter, Myrto.

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Preface

Since the late 1990s, the authors have produced an extensive portfolio of results on differential equations and differential inclusions undergoing impulse effects. Both initial value problems and boundary value problems have been dealt with in their work. The primary motivation for this book is in gathering under one cover an encyclopedic resource for many of these recent results. Having succinctly stated the motivation of the book, there is certainly an obligation to include mentioning some of the all important roles of modelling natural phenomena with impulse problems.

The dynamics of evolving processes is often subjected to abrupt changes such as shocks, harvesting, and natural disasters. Often these short-term perturbations are treated as having acted instantaneously or in the form of "impulses." Impulsive differential equations such as

$$x' = f(t, x), \quad t \in [0, b] \setminus \{t_1, \dots, t_m\}, \tag{1}$$

subject to impulse effects

$$\Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, \dots, m, \tag{2}$$

with $f : ([0, b] \setminus \{t_1, \dots, t_m\}) \times \mathbb{R}^n \to \mathbb{R}^n$ and I_k an impulse operator, have been developed in modelling impulsive problems in physics, population dynamics, biotechnology, pharmacokinetics, industrial robotics, and so forth; in the case when the right-hand side of (1) has discontinuities, differential inclusions such as

$$x'(t) \in F(t, x(t)), \quad t \in [0, b] \setminus \{t_1, \dots, t_m\}, \tag{3}$$

subject to the impulse conditions (2), where $F : ([0, b] \setminus \{t_1, \dots, t_m\}) \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$, have played an important role in modelling phenomena, especially in scenarios involving automatic control systems. In addition, when these processes involve hereditary phenomena such as biological and social macrosystems, some of the modelling is done via impulsive functional differential equations such as

$$x' = f(t, x_t), \quad t \in [0, b] \setminus \{t_1, \dots, t_m\}, \tag{4}$$

subject to (2), and an initial value

$$x(s) = \phi(s), \quad s \in [-r, 0], \ t \in [0, b], \tag{5}$$

where $x_t(\theta) = x(t+\theta)$, $t \in [0,b]$, and $-r \leq \theta \leq 0$, and $f : ([0,b] \setminus \{t_1, \dots, t_m\}) \times D \to \mathbb{R}^n$, and D is a space of functions from $[-r, 0]$ into $\mathbb{R}^n$ which are continuous except for a finite number of points. When the dynamics is multivalued, the hereditary phenomena are modelled via impulsive functional differential inclusions such as

$$x'(t) \in F(t, x_t), \quad t \in [0, b] \setminus \{t_1, \dots, t_m\}, \tag{6}$$

subject to the impulses (2) and the initial condition (5).

An outline of the book as it is devoted to articles published by the authors evolves in a somewhat natural way around addressing issues relating to initial value problems and boundary value problems for both impulsive differential equations and differential inclusions, as well as for both impulsive functional differential equations and functional differential inclusions. Chapter 1 contains fundamental results from multivalued analysis and differential inclusions. In addition, this chapter contains a number of fixed point theorems on which most of the book's existence results depend. Included among these fixed point theorems are those recognized their names: Avery-Henderson, Bohnenblust-Karlin, Covitz and Nadler, Krasnosel'skii, Leggett-Williams, Leray-Schauder, Martelli, and Schaefer. Chapter 1 also contains background material on semigroups that is necessary for the book's presentation of impulsive semilinear functional differential equations.

Chapter 2 is devoted to impulsive ordinary differential equations and scalar differential inclusions, given, respectively, by

$$y' - Ay = By + f(t, y), \quad y' \in F(t, y), \tag{7}$$

each subject to (2), and each satisfies an initial condition $y(0) = y_0$, where A is an infinitesimal generator of a family of semigroups, B is a bounded linear operator from a Banach space E back to itself, and $F : [0, b] \times E \to 2^E$. Chapter 3 deals with functional differential equations and functional differential inclusions, with each undergoing impulse effects. Also, neutral functional differential equations and neutral functional differential inclusions are addressed in which the derivative of the state variable undergoes a delay. Chapter 4 is directed toward impulsive semilinear ordinary differential inclusions and functional differential inclusions satisfying nonlocal boundary conditions such as $g(y) = \sum_{k=1}^{n} c_i y(t_i)$, with each $c_i > 0$ and $0 < t_1 < \cdots < t_n < b$. Such problems are used to describe the diffusion phenomena of a small amount of gas in a transport tube.

Chapter 5 is focused on positive solutions and multiple positive solutions for impulsive ordinary differential equations and functional differential equations, including initial value problems as well as boundary value problems for second-order problems such as

$$y'' = f(t, y_t), \quad t \in [0, b] \setminus \{t_1, \dots, t_m\}, \tag{8}$$

subject to impulses

$$\Delta y(t_k) = I_k(y(t_k)), \quad \Delta y'(t_k) = J_k(y(t_k)), \quad k = 1, \dots, m, \tag{9}$$

and initial conditions

$$y(t) = \phi(t), \quad t \in [-r, 0], \qquad y'(0) = \eta. \tag{10}$$

Chapter 6 is primarily concerned with boundary value problems for periodic impulsive differential inclusions. Upper- and lower-solution methods are developed for first-order systems and then for second-order systems of functional differential inclusions, $y''(t) \in F(t, y_t)$. For Chapter 7, impulsive differential inclusions satisfying periodic boundary conditions are studied. The problems of interest are termed as being *nonresonant*, because the linear operators involved are invertible in the absence of impulses. The chapter deals with first-order and higher-order nonresonance impulsive inclusions.

Chapter 8 extends the theory of some of the previous chapters to functional differential equations and functional differential inclusions under impulses for which the impulse effects vary with time; that is, $y(t_k^+) = I_k(y(t))$, $t = \tau_k(y(t))$, $k = 1, \dots, m$. Chapter 9, as well, extends several results of previous chapters on semilinear problems now to semilinear functional differential equations and functional differential inclusions for operators that are nondensely defined on a Banach space.

Chapter 10 ventures into results for second-order impulsive hyperbolic differential inclusions,

$$\begin{gathered}
\frac{\partial^2 u(t,x)}{\partial t \partial x} \in F(t, x, u(t,x)) \quad \text{a.e. } (t,x) \in ([0,a] \setminus \{t_1, \dots, t_m\}) \times [0,b], \\
\Delta u(t_k, x) = I_k(u(t_k, x)), \quad k = 1, \dots, m, \\
u(t,0) = \psi(t), \quad t \in [0,a] \setminus \{t_1, \dots, t_m\}, \qquad u(0,x) = \phi(x), \quad x \in [0,b].
\end{gathered} \tag{11}$$

Such models arise especially for problems in biological or medical domains.

The next to last chapter, Chapter 11, addresses some questions for impulsive dynamic equations on time scales. The methods constitute adjustments from those for impulsive ordinary differential equations to dynamic equations on time scales, but these results are the first such results in the direction of impulsive problems on time scales. The final chapter, Chapter 12, is a brief chapter dealing with periodic boundary value problems for first-order perturbed impulsive systems,

$$\begin{aligned} x' &\in F(t,x(t)) + G(t,x(t)), \quad t \in [0,b] \setminus \{t_1,\ldots,t_m\}, \\ x(t_j^+) &= x(t_j^-) + I_j(x(t_j^-)), \quad j = 1,\ldots,m, \; x(0) = x(b), \end{aligned} \tag{12}$$

where both $F, G : ([0,b] \setminus \{t_1,\ldots,t_m\}) \times \mathbb{R} \to 2^{\mathbb{R}}$.

We express our appreciation and thanks to R. I. Avery, A. Boucherif, B. C. Dhage, E. Gatsori, L. Górniewicz, J. R. Graef, J. J. Nieto, A. Ouahab, and Y. G. Sficas for their collaboration in research and to E. Gatsori and A. Ouahab for their careful typing of some parts of this manuscript. We are especially grateful to the Editors-in-Chief of the *Contemporary Mathematics and Applications* book series, R. P. Agarwal and D. O'Regan, for their encouragement of us during the preparation of this volume for inclusion in the series.

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1 Preliminaries

1.1. Definitions and results for multivalued analysis

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis, which are used throughout this book.

Let (X,d) be a metric space and let Y be a subset of X. We denote

(i) $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$;

(ii) $\mathcal{P}_p(X) = \{Y \in P(X) : Y$ has the property "p"$\}$, where p could be cl = closed, b = bounded, cp = compact, cv = convex, and so forth.

Thus

(i) $\mathcal{P}_{\mathrm{cl}}(X) = \{Y \in P(X) : Y \text{ closed}\}$,

(ii) $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$,

(iii) $\mathcal{P}_{\mathrm{cv}}(X) = \{Y \in P(X) : Y \text{ convex}\}$,

(iv) $\mathcal{P}_{\mathrm{cp}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$,

(v) $\mathcal{P}_{\mathrm{cv,cp}}(X) = \mathcal{P}_{\mathrm{cv}}(X) \cap \mathcal{P}_{\mathrm{cp}}(X)$, and so forth.

In what follows, by E we will denote a Banach space over the field of real numbers $\mathbb{R}$ and by J a closed interval in $\mathbb{R}$. We let

$$C(J,E) = \{y : J \longrightarrow E \mid y \text{ is continuous}\}. \tag{1.1}$$

We consider the Tchebyshev norm

$$\|\cdot\|_\infty : C(J,E) \longrightarrow [0,\infty), \tag{1.2}$$

defined by

$$\|y\|_\infty = \max\{|y(t)|,\ t \in J\}, \tag{1.3}$$

where $|\cdot|$ stands for the norm in E. Then $(C(J,E), \|\cdot\|)$ is a Banach space.

Let $N : E \to E$ be a linear map. N is called *bounded* provided there exists $r > 0$ such that

$$|N(x)| \leq r|x|, \quad \text{for every } x \in E. \tag{1.4}$$

The following result is classical.

Proposition 1.1. *A linear map $N : E \to E$ is continuous if and only if N is bounded.*

We let

$$B(E) = \{N : E \longrightarrow E \mid N \text{ is linear bounded}\}, \tag{1.5}$$

and for $N \in B(E)$, we define

$$\|N\|_{B(E)} = \inf \{r > 0 \mid \forall x \in E \ |N(x)| < r|x|\}. \tag{1.6}$$

Then $(B(E), \|\cdot\|_{B(E)})$ is a Banach space.

We also have

$$\|N\|_{B(E)} = \sup \{|N(x)| \mid |x| = 1\}. \tag{1.7}$$

A function $y : J \to E$ is called *measurable* provided that for every open $U \subset E$, the set

$$y^{-1}(U) = \{t \in J \mid y(t) \in U\} \tag{1.8}$$

is Lebesgue measurable.

We will say that a measurable function $y : J \to E$ is *Bochner integrable* (for details, see [230]) provided that the function $|y| : J \to [0, \infty)$ is a Lebesgue integrable function.

We let

$$L^1(J, E) = \{y : J \longrightarrow E \mid y \text{ is Bochner integrable}\}. \tag{1.9}$$

Let us add that two functions $y_1, y_2 : J \to E$ such that the set $\{y_1(t) \neq y_2(t) \mid t \in J\}$ has Lebesgue measure equal to zero are considered as equal.

Then we are able to define

$$\|y\|_{L^1} = \int_0^b |y(t)| \, dt, \quad \text{for } J = [0, b]. \tag{1.10}$$

It is well known that

$$(L^1(J, E), \|\cdot\|_{L^1}) \tag{1.11}$$

is a Banach space.

Let $(X, \|\cdot\|)$ be a Banach space. A multivalued map $G : X \to \mathcal{P}(X)$ has *convex (closed) values* if $G(x)$ is convex (closed) for all $x \in X$. We say that G is *bounded on bounded sets* if $G(B)$ is bounded in X for each bounded set B of X, that is, $\sup_{x \in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$. The map G is called *upper semicontinuous (u.s.c.)* on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood

M of x_0 such that $G(M) \subseteq N$. Also, G is said to be *completely continuous* if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply that $y_* \in G(x_*)$). Finally, we say that G has a *fixed point* if there exists $x \in X$ such that $x \in G(x)$.

A multivalued map $G : J \to \mathcal{P}_{\text{cl}}(X)$ is said to be *measurable* if for each $x \in E$, the function $Y : J \to X$ defined by

$$Y(t) = \operatorname{dist}(x, G(t)) = \inf\{\|x - z\| : z \in G(t)\} \tag{1.12}$$

is Lebesgue measurable.

Theorem 1.2 (Kuratowki, Ryll, and Nardzewski). *Let E be a separable Banach space and let $F : J \to P_{\text{cl}}(E)$ be a measurable map, then there exists a measurable map $f : J \to E$ such that $f(t) \in F(t)$, for every $t \in J$.*

Let $\mathcal{A}$ be a subset of $J \times B$. $\mathcal{A}$ is $\mathcal{L} \otimes \mathcal{B}$ measurable if $\mathcal{A}$ belongs to the σ-algebra generated by all sets of the form $N \times D$, where N is Lebesgue measurable in J and D is Borel measurable in B. A subset $\mathcal{A}$ of $L^1(J, E)$ is decomposable if for all $u, v \in \mathcal{A}$ and $N \subset J$ measurable, the function $u\chi_N + v\chi_{J-N} \in \mathcal{A}$, where χ stands for the characteristic function.

Let X be a nonempty closed subset of E and $G : X \to \mathcal{P}(E)$ a multivalued operator with nonempty closed values. G is *lower semicontinuous (l.s.c.)* if the set $\{x \in X : G(x) \cap B \neq \emptyset\}$ is open for any open set B in E.

Definition 1.3. Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1(J, E))$ be a multivalued operator. Say that N has property (BC) if

(1) N is lower semicontinuous (l.s.c.);

(2) N has nonempty closed and decomposable values.

Let $F : J \times E \to \mathcal{P}(E)$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator

$$\mathcal{F} : C(J, E) \longrightarrow \mathcal{P}(L^1(J, E)) \tag{1.13}$$

by letting

$$\mathcal{F}(y) = \{v \in L^1(J, E) : v(t) \in F(t, y(t)) \text{ for a.e. } t \in J\}. \tag{1.14}$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated to F.

Definition 1.4. Let $F : J \times E \to \mathcal{P}(E)$ be a multivalued function with nonempty compact values. Say that F is of lower semicontinuous type (l.s.c. type) if its associated Niemytzki operator $\mathcal{F}$ is lower semicontinuous and has nonempty closed and decomposable values.

Next, we state a selection theorem due to Bressan and Colombo.

Theorem 1.5 (see [105]). *Let Y be separable metric space and let $N : Y \to \mathcal{P}(L^1(J, E))$ be a multivalued operator which has property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $f : Y \to L^1(J, E)$ such that $f(x) \in N(x)$ for every $x \in Y$.*

For more details on multivalued maps, we refer to the books of Deimling [125], Górniewicz [156], Hu and Papageorgiou [170], and Tolstonogov [225].

1.2. Fixed point theorems

Fixed point theorems play a major role in our existence results. Therefore we state a number of fixed point theorems. We start with Schaefer's fixed point theorem.

Theorem 1.6 (Schaefer's fixed point theorem) (see also [220, page 29]). *Let X be a Banach space and let $N : X \to X$ be a completely continuous map. If the set*

$$\Phi = \{x \in X : \lambda x = Nx \text{ for some } \lambda > 1\} \tag{1.15}$$

is bounded, then N has a fixed point.

The second fixed point theorem concerns multivalued condensing mappings. The upper semicontinuous map G is said to be condensing if for any $\mathcal{B} \in \mathcal{P}_b(X)$ with $\mu(\mathcal{B}) \neq 0$, we have $\mu(G(\mathcal{B})) < \mu(\mathcal{B})$, where μ denotes the Kuratowski measure of noncompactness [32]. We remark that a compact map is the easiest example of a condensing map.

Theorem 1.7 (Martelli's fixed point theorem [196]). *Let X be a Banach space and let $G : X \to \mathcal{P}_{\mathrm{cp,cv}}(X)$ be an upper semicontinuous and condensing map. If the set*

$$\mathcal{M} := \{y \in X : \lambda y \in G(y) \text{ for some } \lambda > 1\} \tag{1.16}$$

is bounded, then G has a fixed point.

Next, we state a well-known result often referred to as the *nonlinear alternative*. By $\overline{U}$ and ∂U, we denote the closure of U and the boundary of U, respectively.

Theorem 1.8 (nonlinear alternative [157]). *Let X be a Banach space with $C \subset X$ closed and convex. Assume U is a relatively open subset of C with $0 \in U$ and $G : \overline{U} \to C$ is a compact map. Then either,*

(i) *G has a fixed point in $\overline{U}$; or*
(ii) *there is a point $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda G(u)$.*

Theorem 1.9 (Bohnenblust and Karlin [98]) (see also [231, page 452]). *Let X be a Banach space and $K \in \mathcal{P}_{\mathrm{cl},c}(X)$ and suppose that the operator $G : K \to \mathcal{P}_{\mathrm{cl,cv}}(X)$*

is upper semicontinuous and the set $G(K)$ is relatively compact in X. Then G has a fixed point in K.

Before stating our next fixed point theorem, we need some preliminaries.

Let (X, d) be a metric space induced from the normed space $(X, |\cdot|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$, given by

$$H_d(\mathcal{A}, \mathcal{B}) = \max\left\{\sup_{a\in\mathcal{A}} d(a, \mathcal{B}), \sup_{b\in\mathcal{B}} d(\mathcal{A}, b)\right\}, \tag{1.17}$$

where $d(\mathcal{A}, b) = \inf_{a\in\mathcal{A}} d(a, b)$, $d(a, \mathcal{B}) = \inf_{b\in\mathcal{B}} d(a, b)$. Then $(\mathcal{P}_{b,\mathrm{cl}}(X), H_d)$ is a metric space and $(\mathcal{P}_{\mathrm{cl}}(X), H_d)$ is a generalized (complete) metric space (see [177]).

Definition 1.10. A multivalued operator $G : X \to \mathcal{P}_{\mathrm{cl}}(X)$ is called

(a) γ-Lipschitz if there exists $\gamma > 0$ such that

$$H_d(G(x), G(y)) \leq \gamma d(x, y), \quad \text{for each } x, y \in X; \tag{1.18}$$

(b) a contraction if it is γ-Lipschitz with $\gamma < 1$.

The next fixed point theorem is the well-known Covitz and Nadler's fixed point theorem for multivalued contractions [123] (see also Deimling [125, Theorem 11.1]).

Theorem 1.11 (Covitz and Nadler [123]). *Let (X, d) be a complete metric space. If $G : X \to \mathcal{P}_{\mathrm{cl}}(X)$ is a contraction, then $\operatorname{fix} G \neq \emptyset$.*

The next theorems concern the existence of multiple positive solutions.

Definition 1.12. Let $(\mathcal{B}, \|\cdot\|)$ be a real Banach space. A nonempty, closed, convex set $\mathcal{P} \subset \mathcal{B}$ is said to be a *cone* provided the following are satisfied:

(a) if $y \in \mathcal{P}$ and $\lambda \geq 0$, then $\lambda y \in \mathcal{P}$;

(b) if $y \in \mathcal{P}$ and $-y \in \mathcal{P}$, then $y = 0$.

Every cone $\mathcal{P} \subset \mathcal{B}$ induces a partial ordering, $\leq$, on $\mathcal{B}$ defined by

$$x \leq y \quad \text{iff } y - x \in \mathcal{P}. \tag{1.19}$$

Definition 1.13. Given a cone $\mathcal{P}$ in a real Banach space $\mathcal{B}$, a functional $\psi : \mathcal{P} \to R$ is said to be *increasing* on $\mathcal{P}$, provided that $\psi(x) \leq \psi(y)$, for all $x, y \in \mathcal{P}$ with $x \leq y$.

Given a nonnegative continuous functional γ on a cone $\mathcal{P}$ of a real Banach space $\mathcal{B}$, (i.e., $\gamma : \mathcal{P} \to [0, \infty)$ continuous), we define, for each $d > 0$, the convex

set

$$\mathcal{P}(\gamma, d) = \{x \in \mathcal{P} \mid \gamma(x) < d\}. \tag{1.20}$$

Theorem 1.14 (Leggett-Williams fixed point theorem [187]). *Let E be a Banach space, $C \subset E$ a cone of E, and $R > 0$ a constant. Let $C_R = \{y \in C : \|y\| < R\}$. Suppose that a concave nonnegative continuous functional ψ exists on the cone C with $\psi(y) \leq \|y\|$ for $y \in \overline{C}_R$, and let $N : \overline{C}_R \to \overline{C}_R$ be a completely continuous operator. Assume there are numbers ρ, L and K with $0 < \rho < L < K \leq R$ such that*

(A1) $\{y \in C(\psi, L, K) : \psi(y) > L\} \neq \emptyset$ *and* $\psi(N(y)) > L$ *for all* $y \in C(\psi, L, K)$;

(A2) $\|N(y)\| < \rho$ *for all* $y \in \overline{C}_\rho$;

(A3) $\psi(N(y)) > L$ *for all* $y \in C(\psi, L, R)$ *with* $\|N(y)\| > K$, *where* $C(\psi, L, K) = \{y \in C : \psi(y) \geq L$ *and* $\|y\| \leq K\}$.

Then N has at least three fixed points y_1, y_2, y_3 in $\overline{C}_R$. Furthermore,

$$\begin{gathered} y_1 \in C_\rho, \quad y_2 \in \{y \in C(\psi, L, R) : \psi(y) > L\}, \\ y_3 \in \overline{C}_R - \{C(\psi, L, R) \cup \overline{C}_\rho\}. \end{gathered} \tag{1.21}$$

Theorem 1.15 (Krasnosel'skii twin fixed point theorem [163]). *Let E be a Banach space, $C \subset E$ a cone of E, and $R > 0$ a constant. Let $C_R = \{y \in C : \|y\| < R\}$ and let $N : C_R \to C$ be a completely continuous operator, where $0 < r < R$. If*

(A1) $\|N(y)\| < \|y\|$ *for all* $y \in \partial C_r$;

(A2) $\|N(y)\| > \|y\|$ *for all* $y \in \partial C_R$.

Then N has at least two fixed points y_1, y_2, in $\overline{C}_R$. Furthermore,

$$||y_1|| < r, \quad r < ||y_2|| \leq R. \tag{1.22}$$

Theorem 1.16 (Avery-Henderson fixed point theorem [26]). *Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{B}$. Let α and γ be increasing, nonnegative, continuous functionals on $\mathcal{P}$, and let θ be a nonnegative continuous functional on $\mathcal{P}$ with $\theta(0) = 0$ such that for some $c > 0$ and $M > 0$,*

$$\gamma(x) \leq \theta(x) \leq \alpha(x), \quad \|x\| \leq M\gamma(x), \tag{1.23}$$

for all $x \in \overline{\mathcal{P}(\gamma, c)}$. Suppose there exist a completely continuous operator $A : \overline{\mathcal{P}(\gamma, c)} \to \mathcal{P}$ and $0 < a < b < c$ such that

$$\theta(\lambda x) \leq \lambda\theta(x), \quad \text{for } 0 \leq \lambda \leq 1,\ x \in \partial\mathcal{P}(\theta, b), \tag{1.24}$$

and

(i) $\gamma(Ax) > c$, *for all* $x \in \partial\mathcal{P}(\gamma, c)$;

(ii) $\theta(Ax) < b$, *for all* $x \in \partial\mathcal{P}(\theta, b)$;

(iii) $\mathcal{P}(\alpha, a) \neq \emptyset$, *and* $\alpha(Ax) > a$, *for all* $x \in \partial\mathcal{P}(\alpha, a)$.

Then A has at least two fixed points x_1 and x_2 belonging to $\overline{\mathcal{P}(\gamma, c)}$ such that

$$\begin{aligned} a < \alpha(x_1), &\quad \text{with } \theta(x_1) < b, \\ b < \theta(x_2), &\quad \text{with } \gamma(x_2) < c. \end{aligned} \tag{1.25}$$

1.3. Semigroups

In this section, we present some concepts and results concerning semigroups. This section will be fundamental to our development of semilinear problems.

1.3.1. C_0-semigroups

Let E be a Banach space and let $B(E)$ be the Banach space of bounded linear operators.

Definition 1.17. A semigroup of class (C_0) is a one-parameter family $\{T(t) \mid t \geq 0\} \subset B(E)$ satisfying the following conditions:

(i) $T(t) \circ T(s) = T(t+s)$, for $t, s \geq 0$,

(ii) $T(0) = I$, (the identity operator in E),

(iii) the map $t \to T(t)(x)$ is strongly continuous, for each $x \in E$, that is,

$$\lim_{t \to 0} T(t)x = x, \quad \forall x \in E. \tag{1.26}$$

A semigroup of bounded linear operators $T(t)$ is uniformly continuous if

$$\lim_{t \to 0} \|T(t) - I\| = 0. \tag{1.27}$$

We note that if a semigroup $T(t)$ is class (C_0), then we have the growth condition

$$\|T(t)\|_{B(E)} \leq M \cdot \exp(\beta t), \quad \text{for } 0 \leq t < \infty, \text{ with some constants } M > 0 \text{ and } \beta. \tag{1.28}$$

If, in particular, $M = 1$ and $\beta = 0$, that is, $\|T(t)\|_{B(E)} \leq 1$, for $t \geq 0$, then the semigroup $T(t)$ is called a *contraction semigroup* (C_0).

Definition 1.18. Let $T(t)$ be a semigroup of class (C_0) defined on E. The *infinitesimal generator* A of $T(t)$ is the linear operator defined by

$$A(x) = \lim_{h \to 0} \frac{T(h)(x) - x}{h}, \quad \text{for } x \in D(A), \tag{1.29}$$

where $D(A) = \{x \in E \mid \lim_{h \to 0}(T(h)(x) - x)/h \text{ exists in } E\}$.

Proposition 1.19. *The infinitesimal generator A is a closed linear and densely defined operator in E. If $x \in D(A)$, then $T(t)(x)$ is a C^1-map and*

$$\frac{d}{dt}T(t)(x) = A\big(T(t)(x)\big) = T(t)\big(A(x)\big) \quad on\ [0,\infty). \tag{1.30}$$

Theorem 1.20 (Pazy [210]). *Let A be a densely defined linear operator with domain and range in a Banach space E. Then A is the infinitesimal generator of uniquely determined semigroup $T(t)$ of class (C_0) satisfying*

$$||T(t)||_{B(E)} \leq M\exp(\omega t), \quad t \geq 0, \tag{1.31}$$

where $M > 0$ and $\omega \in \mathbb{R}$ if and only if $(\lambda I - A)^{-1} \in B(E)$ and $\|(\lambda I - A)^{-n}\| \leq M/(\lambda-\omega)^n$, $n = 1, 2, \ldots$, for all $\lambda \in \mathbb{R}$.

We say that a family $\{C(t) \mid t \in \mathbb{R}\}$ of operators in $B(E)$ is a *strongly continuous cosine family* if

(i) $C(0) = I$,
(ii) $C(t+s) + C(t-s) = 2C(t)C(s)$, for all $s, t \in \mathbb{R}$,
(iii) the map $t \mapsto C(t)(x)$ is strongly continuous, for each $x \in E$.

The strongly continuous sine family $\{S(t) \mid t \in \mathbb{R}\}$, associated to the given strongly continuous cosine family $\{C(t) \mid t \in \mathbb{R}\}$, is defined by

$$S(t)(x) = \int_0^t C(s)(x)ds, \quad x \in E,\ t \in \mathbb{R}. \tag{1.32}$$

The infinitesimal generator $A : E \to E$ of a cosine family $\{C(t) \mid t \in \mathbb{R}\}$ is defined by

$$A(x) = \frac{d^2}{dt^2}C(t)(x)|_{t=0}. \tag{1.33}$$

For more details on strongly continuous cosine and sine families, we refer the reader to the books of Goldstein [155], Heikkilä and Lakshmikantham [163], Fattorini [145], and to the papers of Travis and Webb [226, 227].

1.3.2. Integrated semigroups

Definition 1.21 (see [21]). Let E be a Banach space. An integrated semigroup is a family of operators $(S(t))_{t\geq 0}$ of bounded linear operators $S(t)$ on E with the following properties:

(i) $S(0) = 0$;
(ii) $t \to S(t)$ is strongly continuous;
(iii) $S(s)S(t) = \int_0^s (S(t+r) - S(r))dr$, for all $t, s \geq 0$.

Definition 1.22 (see [175]). An operator A is called a generator of an integrated semigroup if there exists $\omega \in \mathbb{R}$ such that $(\omega,\infty) \subset \rho(A)(\rho(A)$ is the resolvent set of A), and there exists a strongly continuous exponentially bounded family $(S(t))_{t\geq 0}$ of bounded operators such that $S(0) = 0$ and $R(\lambda, A) := (\lambda I - A)^{-1} = \lambda \int_0^\infty e^{-\lambda t} S(t)dt$ exists for all λ with $\lambda > \omega$.

Proposition 1.23 (see [21]). *Let A be the generator of an integrated semigroup $(S(t))_{t\geq 0}$. Then, for all $x \in E$ and $t \geq 0$,*

$$\int_0^t S(s)x\,ds \in D(A), \qquad S(t)x = A\int_0^t S(s)x\,ds + tx. \tag{1.34}$$

Definition 1.24 (see [175]). (i) An integrated semigroup $(S(t))_{t\geq 0}$ is called locally Lipschitz continuous if for all $\tau > 0$, there exists a constant L such that

$$\left|S(t) - S(s)\right| \leq L|t-s|, \quad t, s \in [0, \tau]. \tag{1.35}$$

(ii) An integrated semigroup $(S(t))_{t\geq 0}$ is called non degenerate if $S(t)x = 0$ for all $t \geq 0$ implies that $x = 0$.

Definition 1.25. Say that the linear operator A satisfies the Hille-Yosida condition if there exist $M \geq 0$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and

$$\sup\left\{(\lambda - \omega)^n \left|(\lambda I - A)^{-n}\right| : n \in \mathbb{N},\ \lambda > \omega\right\} \leq M. \tag{1.36}$$

Theorem 1.26 (see [175]). *The following assertions are equivalent:*

(i) *A is the generator of a nondegenerate, locally Lipschitz continuous integrated semigroup;*
(ii) *A satisfies the Hille-Yosida condition.*

If A is the generator of an integrated semigroup $(S(t))_{t\geq 0}$ which is locally Lipschitz, then from [21], $S(\cdot)x$ is continuously differentiable if and only if $x \in \overline{D(A)}$ and $(S'(t))_{t\geq 0}$ is a C_0 semigroup on $\overline{D(A)}$.

1.4. Some additional lemmas and notions

We include here, for easy references, some auxiliary results, which are crucial in what follows.

Definition 1.27. The multivalued map $F : J \times E \to \mathcal{P}(E)$ is said to be L^1-Carathéodory if

(i) $t \mapsto F(t, u)$ is measurable for each $u \in E$;
(ii) $u \mapsto F(t, u)$ is upper semicontinuous on E for almost all $t \in J$;
(iii) for each $\rho > 0$, there exists $\varphi_\rho \in L^1(J, \mathbb{R}_+)$ such that

$$||F(t,u)||_{\mathcal{P}(E)} = \sup\left\{|v| : v \in F(t,u)\right\} \leq \varphi_\rho(t), \quad \forall \|u\| \leq \rho \text{ and for a.e. } t \in J. \tag{1.37}$$

Lemma 1.28 (see [186]). *Let X be a Banach space. Let $F : J \times X \to \mathcal{P}_{\mathrm{cp},c}(X)$ be an L^1-Carathéodory multivalued map with*

$$S_{F(y)} = \{g \in L^1(J,X) : g(t) \in F(t, y(t)), \text{ for a.e. } t \in J\} \neq \emptyset, \tag{1.38}$$

and let Γ be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$, then the operator

$$\Gamma \circ S_F : C(J,X) \longrightarrow \mathcal{P}_{\mathrm{cp},c}(C(J,X)), \qquad y \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F(y)}) \tag{1.39}$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

Lemma 1.29 (see [148]). *Assume that*

(1.29.1) *$F : J \times E \to \mathcal{P}(E)$ is a nonempty, compact-valued, multivalued map such that*
 (a) *$(t,u) \mapsto F(t,u)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable,*
 (b) *$u \mapsto F(t,u)$ is lower semicontinuous for a.e. $t \in J$;*

(1.29.2) *for each $r > 0$, there exists a function $h_r \in L^1(J,\mathbb{R}^+)$ such that*

$$\begin{aligned}&\|F(t,u)\|_{\mathcal{P}}\\ &\quad := \sup\{|v| : v \in F(t,u)\} \leq h_r(t) \quad \text{for a.e. } t \in J; \text{ and for } u \in E \text{ with } \|u\| \leq r.\end{aligned} \tag{1.40}$$

Then F is of l.s.c. type.

Lemma 1.30 (see [163, Lemma 1.5.3]). *If $p \in L^1(J,\mathbb{R})$ and $\psi : \mathbb{R}_+ \to (0,+\infty)$ is increasing with*

$$\int_0^\infty \frac{du}{\psi(u)} = \infty, \tag{1.41}$$

then the integral equation

$$z(t) = z_0 + \int_0^t p(s)\psi(z(s))\,ds, \quad t \in J, \tag{1.42}$$

has for each $z_0 \in \mathbb{R}$ a unique solution z. If $u \in C(J,E)$ satisfies the integral inequality

$$|u(t)| \leq z_0 + \int_0^t p(s)\psi(|u(s)|)\,ds, \quad t \in J, \tag{1.43}$$

then $|u| \leq z$.

2 Impulsive ordinary differential equations & inclusions

2.1. Introduction

For well over a century, differential equations have been used in modeling the dynamics of changing processes. A great deal of the modeling development has been accompanied by a rich theory for differential equations.

The dynamics of many evolving processes are subject to abrupt changes, such as shocks, harvesting and natural disasters. These phenomena involve short-term perturbations from continuous and smooth dynamics, whose duration is negligible in comparison with the duration of an entire evolution. In models involving such perturbations, it is natural to assume these perturbations act instantaneously or in the form of "impulses." As a consequence, impulsive differential equations have been developed in modeling impulsive problems in physics, population dyamics, ecology, biological systems, biotechnology, industrial robotics, pharmcokinetics, optimal control, and so forth. Again, associated with this development, a theory of impulsive differential equations has been given extensive attention. Works recognized as landmark contributions include [29, 30, 180, 217], with [30] devoted especially to impulsive periodic systems of differential equations.

Some processes, especially in areas of population dynamics, ecology, and pharmacokinetics, involve hereditary issues. The theory and applications addressing such problems have heavily involved functional differential equations as well as impulsive functional differential equations. The literature devoted to this study is also extensive, with [6, 12–14, 25, 27, 28, 38, 42, 46, 49, 52, 53, 55, 57, 70, 71, 75, 85, 89–91, 94, 95, 117, 130–132, 134, 136, 147, 152, 159, 167, 176, 181, 183, 189, 191, 194, 195, 212, 214, 216, 228] providing a good view of the panorama of work that has been done.

Much attention has also been devoted to modeling natural phenomena with differential equations, both ordinary and functional, for which the part governing the derivative(s) is not known as a single-valued function; for example, a dynamic process governing the derivative $x'(t)$ of a state $x(t)$ may be known only within a set $S(t,x(t)) \subset \mathbb{R}$, and given by $x'(t) \in S(t,x(t))$. A common example of this is observed in a so-called differential inequality such as $x'(t) \leq f(t,x(t))$, where say $f : \mathbb{R} \to \mathbb{R}$, which can also be expressed as the differential inclusion, $x'(t) \in (-\infty, f(t, x(t)))$, or $x'(t) \in S(t, x(t)) \equiv \{v \in \mathbb{R} \mid v \leq f(t, x(t))\}$. Differential inclusions arise in models for control systems, mechanical systems, economics systems, game theory, and biological systems to name a few. For a thumbnail sketch of the literature on differential inclusions, we suggest [22, 96, 97, 104, 106–111, 118, 121, 146, 179, 198, 211, 213, 215, 221].

It is natural from both a physical standpoint as well as a theoretical view to give considerable attention to a synthesis involving problems for impulsive differential inclusions. It is these theoretical considerations that have become a rapidly developing field with several prominent works written by Benchohra et al. [36, 39–41, 43–45, 47, 48, 50, 51, 54, 56, 59–64, 58, 65–68, 73, 80, 82, 87, 89, 92, 93], Erbe and Krawcewicz [140], and Frigon and O'Regan [153].

This chapter is devoted to solutions of impulsive ordinary differential equations and to solutions of impulsive differential inclusions. Both first- and second-order problems are treated. This chapter also includes a substantial section on damped differential inclusions.

2.2. Impulsive ordinary differential equations

Throughout, let $J = [0, b]$, let $0 < t_1 < \cdots < t_m < t_{m+1} = b$, and let E be a real separable Banach space with norm $|\cdot|$ (at times $E = \mathbb{R}^n$, but this will be indicated when so restricted). In this section, we will be concerned with the existence of mild solutions for first- and second-order impulsive semilinear damped differential equations in a Banach space. Existence of solutions will arise from applications of some of the fixed point theorems featured in Chapter 1. First, we consider first-order impulsive semilinear differential equations of the form

$$\begin{gathered} y'(t) - Ay(t) = By(t) + f(t, y), \quad \text{a.e. } t \in J := [0, b],\ t \neq t_k,\ k = 1, \dots, m, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \\ y(0) = y_0, \end{gathered} \tag{2.1}$$

where $f : J \times E \to E$ is a given function, A is the infinitesimal generator of a family of semigroups $\{T(t) : t \geq 0\}$, B is a bounded linear operator from E into E, $y_0 \in E$, $I_k \in C(E, E)$ $(k = 1, \dots, m)$, and $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h \to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h \to 0^+} y(t_k - h)$ represent the right and left limits of $y(t)$ at $t = t_k$, respectively, $k = 1, \dots, m$.

Later, we study second-order impulsive semilinear evolution differential equations of the form

$$y''(t) - Ay(t) = By'(t) + f(t, y), \quad \text{a.e. } t \in J,\ t \neq t_k,\ k = 1, \dots, m, \tag{2.2}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{2.3}$$

$$\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{2.4}$$

$$y(0) = y_0, \qquad y'(0) = y_1, \tag{2.5}$$

where f, I_k, B, and y_0 are as in problem (2.1), A is the infinitesimal generator of a famiily of cosine operators $\{C(t) : t \geq 0\}$, $\overline{I}_k \in C(E,E)$, and $y_1 \in E$.

The study of the dynamical buckling of the hinged extensible beam, which is either stretched or compressed by axial force in a Hilbert space, can be modeled by the hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} - \left(\alpha + \beta \int_0^L \left|\frac{\partial u}{\partial t}(\xi,t)\right|^2 d\xi\right)\frac{\partial^2 u}{\partial x^2} + g\left(\frac{\partial u}{\partial t}\right) = 0, \qquad (E_1)$$

where $\alpha, \beta, L > 0$, $u(t,x)$ is the deflection of the point x of the beam at the time t, g is a nondecreasing numerical function, and L is the length of the beam.

Equation (E_1) has its analogue in $\mathbb{R}^n$ and can be included in a general mathematical model:

$$u'' + A^2 u + M\left(||A^{1/2}u||_H^2\right)Au + g(u') = 0, \qquad (E_2)$$

where A is a linear operator in a Hilbert space H, and M and g are real functions. Equation (E_1) was studied by Patcheu [209], and (E_2) by Matos and Pereira [197]. These equations are special cases of (2.2), (2.5).

In the following, we introduce first some notations. Let $J_0 = [0,t_1]$, $J_1 = (t_1,t_2],\dots,J_m = (t_m,b]$, $(J_k = (t_k,t_{k+1}], k = 1,2,\dots,m)$, $J' = J \setminus \{t_1,t_2,\dots,t_m\}$, $(t_0 = 0,\ t_{m+1} = b)$, $\mathrm{PC}(J,E) = \{y : J \to E : y(t)$ is continuous everywhere except for some t_k at which $y(t_k^-)$ and $y(t_k^+)$, $k = 1,\dots,m$ exist and $y(t_k^-) = y(t_k)\}$, and $\mathrm{PC}^1(J,E) = \{y : J \longrightarrow E : y(t)$ is continuously differentiable everywhere except for some t_k at which $y'(t_k^-)$ and $y'(t_k^+)$, $k = 1,\dots,m$, exist and $y'(t_k^-) = y'(t_k)\}$. Evidently, $\mathrm{PC}(J,E)$ is a Banach space with norm

$$\|y\|_{\mathrm{PC}} = \sup\{|y(t)| : t \in J\}. \qquad (2.6)$$

It is also clear that $\mathrm{PC}^1(J,E)$ is a Banach space with norm

$$\|y\|_{\mathrm{PC}^1} = \max\{\|y\|_{\mathrm{PC}}, \|y'\|_{\mathrm{PC}}\}. \qquad (2.7)$$

Let us start by defining what we mean by a mild solution of problem (2.1).

Definition 2.1. A function $y \in \mathrm{PC}(J,E)$ is said to be a mild solution of (2.1) if y is the solution of the impulsive integral equation

$$\begin{aligned} y(t) = T(t)y_0 &+ \int_0^t T(t-s)B(y(s))\,ds + \int_0^t T(t-s)f(s,y(s))\,ds \\ &+ \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k)). \end{aligned} \qquad (2.8)$$

Our first existence result makes use of Schaefer's theorem [220].

Theorem 2.2. *Let $f : J \times E \to E$ be an L^1-Carathéodory function. Assume that*

(2.2.1) *there exist constants c_k such that $|I_k(y)| \leq c_k$, $k = 1, \dots, m$ for each $y \in E$;*

(2.2.2) *there exists a constant M such that $\|T(t)\|_{B(E)} \leq M$ for each $t \geq 0$;*

(2.2.3) *there exist a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ and $p \in L^1(J, \mathbb{R}_+)$ such that*

$$|f(t, y)| \leq p(t)\psi(|y|), \quad \text{for a.e. } t \in J \text{ and each } y \in E, \tag{2.9}$$

with

$$\int_0^b m(s)ds < \int_c^\infty \frac{du}{u + \psi(u)}, \tag{2.10}$$

where

$$m(s) = \max\{M\|B\|_{B(E)}, Mp(s)\}, \qquad c = M\left[|y_0| + \sum_{k=1}^{m} c_k\right]; \tag{2.11}$$

(2.2.4) *for each bounded $\mathcal{B} \subseteq \mathrm{PC}(J, E)$ and $t \in J$, the set*

$$\begin{aligned} \Big\{ T(t)y_0 &+ \int_0^t T(t-s)B(y(s))ds + \int_0^t T(t-s)f(s, y(s))ds \\ &+ \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)) : y \in \mathcal{B} \Big\} \end{aligned} \tag{2.12}$$

is relatively compact in E.

Then the impulsive initial (IVP for short) (2.1) *has at least one mild solution.*

Proof. Transform the problem (2.1) into a fixed point problem. Consider the operator $N : \mathrm{PC}(J, E) \to \mathrm{PC}(J, E)$ defined by

$$\begin{aligned} N(y)(t) = T(t)y_0 &+ \int_0^t T(t-s)B(y(s))ds + \int_0^t T(t-s)f(s, y(s))ds \\ &+ \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)). \end{aligned} \tag{2.13}$$

Clearly the fixed points of N are mild solutions to (2.1).

We will show that N is completely continuous. The proof will be given in several steps.

Step 1. N is continuous.

Let y_n be a sequence in $\mathrm{PC}(J,E)$ such that $y_n \to y$. We will prove that $N(y_n) \to N(y)$. For each $t \in J$, we have

$$\begin{aligned} N(y_n)(t) = T(t)y_0 &+ \int_0^t T(t-s)B(y_n(s))ds + \int_0^t T(t-s)f(s,y_n(s))ds \\ &+ \sum_{0<t_k<t} T(t-t_k)I_k(y_n(t_k^-)). \end{aligned} \tag{2.14}$$

Then

$$\begin{aligned} |N(y_n)(t) - N(y)(t)| &\le \int_0^t |T(t-s)|\,|B(y_n(s)) - B(y(s))|\,ds \\ &\quad + \int_0^t |T(t-s)|\,|f(s,y_n(s)) - f(s,y(s))|\,ds \\ &\quad + \sum_{0<t_k<t} |T(t-t_k)|\,|I_k(y_n(t_k^-)) - I_k(y(t_k^-))| \\ &\le bM\|B\|_{B(E)}||y_n - y||_{\mathrm{PC}} \\ &\quad + M\int_0^b |f(s,y_n(s)) - f(s,y(s))|\,ds \\ &\quad + M\sum_{0<t_k<t} |I_k(y_n(t_k^-)) - I_k(y(t_k^-))|. \end{aligned} \tag{2.15}$$

Since I_k, $k = 1,\dots,m$ are continuous, B is bounded and f is an L^1-Carathéodory function, we have by the Lebesgue dominated convergence theorem

$$\begin{aligned} ||N(y_n) - N(y)||_{\mathrm{PC}} &\le bM\|B\|_{B(E)}||y_n - y||_{\mathrm{PC}} \\ &\quad + M\int_0^b |f(s,y_n(s)) - f(s,y(s))|\,ds \\ &\quad + M\sum_{0<t_k<t} |I_k(y_n(t_k^-)) - I_k(y(t_k))| \longrightarrow 0, \end{aligned} \tag{2.16}$$

as $n \to \infty$. Thus N is continuous.

Step 2. N maps bounded sets into bounded sets in $\mathrm{PC}(J,E)$.

Indeed, it is enough to show that for any $q > 0$, there exists a positive constant ℓ such that for each $y \in \mathcal{B}_q = \{y \in \mathrm{PC}(J,E) : \|y\|_{\mathrm{PC}} \le q\}$, one has $\|N(y)\|_{\mathrm{PC}} \le \ell$. Let $y \in \mathcal{B}_q$. By (2.2.1)-(2.2.2) and the fact that f is an L^1-Carathéodory function, we have, for each $t \in J$,

$$\begin{aligned} |N(y)(t)| &\le M|y_0| + M\int_0^b |B(y(s))|\,ds + M\int_0^b \varphi_q(s)ds + M\sum_{k=1}^m c_k \\ &\le M|y_0| + Mbq\|B\|_{B(E)} + M||\varphi_q||_{L^1} + M\sum_{k=1}^m c_k := \ell. \end{aligned} \tag{2.17}$$

Step 3. N maps bounded sets into equicontinuous sets of $PC(J,E)$.

Let $\tau_1,\tau_2 \in J'$, $\tau_1 < \tau_2$, and let $\mathcal{B}_q$ be a bounded set of $PC(J,E)$ as in Step 2. Let $y \in \mathcal{B}_q$, then for each $t \in J$ we have

$$\begin{aligned}
|N(y)(\tau_2) - N(y)(\tau_1)| \le\ & |[T(\tau_2) - T(\tau_1)]y_0| \\
&+ \int_0^{\tau_1} |T(\tau_2 - s) - T(\tau_1 - s)|\,|By(s)|\,ds \\
&+ \int_{\tau_1}^{\tau_2} |T(\tau_2 - s)|\,|By(s)|\,ds \\
&+ \int_0^{\tau_1} |T(\tau_2 - s) - T(\tau_1 - s)|\,\varphi_q(s)ds \\
&+ \int_{\tau_1}^{\tau_2} |T(\tau_2 - s)|\,\varphi_q(s)ds \\
&+ \sum_{\tau_1 < t < \tau_2} c_k |T(\tau_2 - t_k) - T(\tau_1 - t_k)|.
\end{aligned} \tag{2.18}$$

The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$.

This proves the equicontinuity for the case where $t \ne t_i$, $i = 1,\dots,m+1$. It remains to examine the equicontinuity at $t = t_i$. First we prove equicontinuity at $t = t_i^-$. Fix $\delta_1 > 0$ such that $\{t_k : k \ne i\} \cap [t_i - \delta_1, t_i + \delta_1] = \emptyset$.

For $0 < h < \delta_1$, we have that

$$\begin{aligned}
|N(y)(t_i) - N(y)(t_i - h)| \le\ & |(T(t_i) - T(t_i - h))y_0| \\
&+ \int_0^{t_i - h} |(T(t_i - s) - T(t_i - h - s))By(s)ds| \\
&+ \int_{t_i - h}^{t_i} |T(t_i - h)By(s)ds| \\
&+ \int_0^{t_i - h} |[T(t_i - h - s) - T(t_i - s)]\varphi_q(s)|\,ds \\
&+ \int_0^{t_i - h} |T(t_i - h - s)\varphi_q(s)|\,ds \\
&+ \sum_{k=1}^{i-1} |[T(t_i - h - t_k) - T(t_i - t_k)]I(y(t_k^-))|.
\end{aligned} \tag{2.19}$$

The right-hand side tends to zero as $h \to 0$.

Next we prove equicontinuity at $t = t_i^+$. Fix $\delta_2 > 0$ such that $\{t_k : k \neq i\} \cap [t_i - \delta_2, t_i + \delta_2] = \emptyset$. For $0 < h < \delta_2$, we have that

$$\begin{aligned} |N(y)(t_i + h) - N(y)(t_i)| \leq {} & |(T(t_i + h) - T(t_i))y_0| \\ & + \int_0^{t_i} |(T(t_i + h - s) - T(t_i - s))By(s)ds| \\ & + \int_{t_i}^{t_i+h} |T(t_i - h)By(s)ds| \\ & + \int_0^{t_i} |[T(t_i + h - s) - T(t_i - s)]\varphi_q(s)|ds \\ & + \int_{t_i}^{t_i+h} |T(t_i - h)\varphi_q(s)|ds \\ & + \sum_{0<t_k\leq t_i} |[T(t_i - h - t_k) - T(t_i - t_k)]I_k(y(t_k^-))| \\ & + \sum_{t_i<t_k\leq t_i+h} |T(t_i - h - t_k)I_k(y(t_k^-))|. \end{aligned} \tag{2.20}$$

The right-hand side tends to zero as $h \to 0$.

As a consequence of Steps 1 to 3 and (2.2.4) together with the Arzelá-Ascoli theorem we can conclude that $N : \mathrm{PC}(J,E) \to \mathrm{PC}(J,E)$ is a completely continuous operator.

Step 4. Now it remains to show that the set

$$\mathcal{E}(N) := \{y \in \mathrm{PC}(J,E) : y = \lambda N(y), \text{ for some } 0 < \lambda < 1\} \tag{2.21}$$

is bounded. Let $y \in \mathcal{E}(N)$. Then $y = \lambda N(y)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$,

$$\begin{aligned} y(t) = \lambda\bigg[& T(t)y_0 + \int_0^t T(t-s)By(s)ds + \int_0^t T(t-s)f(s,y(s))ds \\ & + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-))\bigg]. \end{aligned} \tag{2.22}$$

This implies by (2.2.1)–(2.2.3) that for each $t \in J$ we have

$$|y(t)| \leq M|y_0| + \int_0^t m(s)(|y(s)| + \psi(|y(s)|))ds + M\sum_{k=1}^m c_k. \tag{2.23}$$

Let us denote the right-hand side of the above inequality as $v(t)$. Then we have

$$\begin{gathered} |y(t)| \leq v(t), \quad \forall t \in J, \qquad v(0) = M\bigg[|y_0| + \sum_{k=1}^m c_k\bigg], \\ v'(t) = m(t)(|y(t)| + \psi(|y(t)|)), \quad \text{for a.e. } t \in J. \end{gathered} \tag{2.24}$$

Using the increasing character of ψ, we get

$$v'(t) \leq m(t)(v(t) + \psi(v(t))), \quad \text{for a.e. } t \in J. \tag{2.25}$$

Then for each $t \in J$ we have

$$\int_{v(0)}^{v(t)} \frac{du}{u + \psi(u)} \leq \int_0^b m(s)ds < \int_{v(0)}^{\infty} \frac{du}{u + \psi(u)}. \tag{2.26}$$

Consequently, there exists a constant $\overline{d}$ such that $v(t) \leq \overline{d}$, $t \in J$, and hence $\|y\|_{\text{PC}} \leq \overline{d}$ where $\overline{d}$ depends only on the functions p and ψ. This shows that $\mathcal{E}(N)$ is bounded.

Set $X := \text{PC}(J, E)$. As a consequence of Schaefer's fixed point theorem (Theorem 1.6) we deduce that N has a fixed point which is a mild solution of (2.1). □

Remark 2.3. We mention that the condition (2.2.1), (i.e., $|I_k(y)| \leq c_k$), is not fulfilled in some important cases, such as for the linear impulse, $I_k(y) = \alpha_k(y(t_i^-))$. However, the boundedness condition can be weakened by assuming, for example, that I_k is sublinear, or in some cases by invoking Cauchy function arguments as in [7, 8]. In many results that appear later in this book, it is sometimes assumed that the impulses, I_k, are bounded. In each such case, this remark could be made.

Now we present a uniqueness result for the problem (2.1). Our considerations are based on the Banach fixed point theorem.

Theorem 2.4. *Assume that f is an L^1-Carathéodory function and suppose (2.2.2) holds. In addition assume the following conditions are satisfied.*

(2.4.1) *There exists a constant d such that*

$$|f(t,y) - f(t,\overline{y})| \leq d|y - \overline{y}|, \quad \textit{for each } t \in J, \ \forall y, \overline{y} \in E. \tag{2.27}$$

(2.4.2) *There exist constants c_k such that*

$$|I_k(y) - I_k(\overline{y})| \leq c_k|y - \overline{y}|, \quad \textit{for each } k = 1, \ldots, m, \ \forall y, \overline{y} \in E. \tag{2.28}$$

If

$$Mb\|B\|_{B(E)} + Mbd + M\sum_{k=1}^{m} c_k < 1, \tag{2.29}$$

then the IVP (2.1) *has a unique mild solution.*

Proof. Transform the problem (2.1) into a fixed point problem. Let the operator $N : \mathrm{PC}(J,E) \to \mathrm{PC}(J,E)$ be defined as in Theorem 2.2. We will show that N is a contraction. Indeed, consider $y, \overline{y} \in \mathrm{PC}(J,E)$. Then we have, for each $t \in J$,

$$\begin{aligned}
|N(y)(t) - N(\overline{y})(t)| &\le \int_0^t M|B(y(s)) - B(\overline{y}(s))|\,ds \\
&\quad + \int_0^t M|f(s,y(s)) - f(s,\overline{y}(s))|\,ds \\
&\quad + M\sum_{k=1}^m |I_k(y(t_k^-)) - I_k(\overline{y}(t_k^-))| \\
&\le M\|B\|_{B(E)}\int_0^t |y(s) - \overline{y}(s)|\,ds \\
&\quad + Md\int_0^t |y(s) - \overline{y}(s)|\,ds + M\sum_{k=1}^m c_k|y(t_k^-) - \overline{y}(t_k^-)| \\
&\le M\|B\|_{B(E)}\int_0^b |y(s) - \overline{y}(s)|\,ds \\
&\quad + Md\int_0^b |y(s) - \overline{y}(s)|\,ds + M\sum_{k=1}^m c_k\|y - \overline{y}\|_{\mathrm{PC}} \\
&\le Mb\|B\|_{B(E)}\|y - \overline{y}\|_{\mathrm{PC}} + Mbd\|y - \overline{y}\|_{\mathrm{PC}} \\
&\quad + M\sum_{k=1}^m c_k\|y - \overline{y}\|_{\mathrm{PC}} \\
&= \left(Mb\|B\|_{B(E)} + Mbd + M\sum_{k=1}^m c_k\right)\|y - \overline{y}\|_{\mathrm{PC}}.
\end{aligned} \tag{2.30}$$

Then

$$\|N(y) - N(\overline{y})\|_{\mathrm{PC}} \le \left(Mb\|B\|_{B(E)} + Mbd + M\sum_{k=1}^m c_k\right)\|y - \overline{y}\|_{\mathrm{PC}}, \tag{2.31}$$

showing that N is a contraction, and hence it has a unique fixed point which is a mild solution to (2.1). □

Now we study the problem (2.2)–(2.5). We give first the definition of mild solution of the problem (2.2)–(2.5).

Definition 2.5. A function $y \in \mathrm{PC}^1(J,E)$ is said to be a mild solution of (2.2)–(2.5) if $y(0) = y_0$, $y'(0) = y_1$, and y is a solution of the impulsive integral equation

$$\begin{aligned} y(t) = {} & (C(t) - S(t)B)y_0 + S(t)y_1 + \int_0^t C(t-s)B(y(s))\,ds \\ & + \int_0^t S(t-s)f(s,y(s))\,ds \\ & + \sum_{0<t_k<t} \big[C(t-t_k)I_k(y(t_k)) - S(t-t_k)\overline{I}_k(y(t_k))\big]. \end{aligned} \tag{2.32}$$

Theorem 2.6. *Let $f : J \times E \to E$ be an L^1-Carathéodory function. Assume (2.2.1) and the following conditions are satisfied:*

(2.6.1) *there exist constants $\overline{d}_k$ such that $|\overline{I}_k(y)| \le \overline{d}_k$ for each $y \in E$, $k = 1,\dots,m$;*

(2.6.2) *there exists a constant $M_1 > 0$ such that $\|C(t)\|_{B(E)} < M_1$ for all $t \in \mathbb{R}$;*

(2.6.3) *there exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and $p \in L^1(J,\mathbb{R}_+)$ such that*

$$|f(t,y)| \le p(t)\psi(|y|), \quad \text{for a.e. } t \in J \text{ and each } y \in E \tag{2.33}$$

with

$$\int_1^\infty \frac{d\tau}{\tau + \psi(\tau)} = +\infty. \tag{2.34}$$

(2.6.4) *for each bounded $\mathcal{B} \subseteq \mathrm{PC}^1(J,E)$ and $t \in J$, the set*

$$\begin{aligned} \Big\{ & (C(t) - S(t)B)y_0 + S(t)y_1 + \int_0^t C(t-s)B(y(s))\,ds + \int_0^t S(t-s)f(s,y(s))\,ds \\ & + \sum_{0<t_k<t} \Big[C(t-t_k)I_k(y(t_k^-)) + S(t-t_k)\overline{I}_k(y(t_k^-))\Big] : y \in \mathcal{B} \Big\} \end{aligned} \tag{2.35}$$

is relatively compact in E.

Then the IVP (2.2)–(2.5) *has at least one mild solution.*

Proof. Transform the problem (2.2)–(2.5) into a fixed point problem. Consider the operator $\overline{N} : \mathrm{PC}^1(J,E) \to \mathrm{PC}^1(J,E)$ defined by

$$\begin{aligned} \overline{N}(y)(t) = {} & (C(t) - S(t)B)y_0 + S(t)y_1 + \int_0^t C(t-s)B(y(s))\,ds \\ & + \int_0^t S(t-s)f(s,y(s))\,ds \\ & + \sum_{0<t_k<t} \big[C(t-t_k)I_k(y(t_k^-)) + S(t-t_k)\overline{I}_k(y(t_k^-))\big]. \end{aligned} \tag{2.36}$$

As in the proof of Theorem 2.2 we can show that $\overline{N}$ is completely continuous. Now we prove only that the set

$$\mathcal{E}(\overline{N}) := \{y \in \mathrm{PC}^1(J,E) : y = \lambda\overline{N}(y), \text{ for some } 0 < \lambda < 1\} \tag{2.37}$$

is bounded. Let $y \in \mathcal{E}(\overline{N})$. Then for each $t \in J$ we have

$$\begin{aligned} y(t) = \lambda\bigg[&(C(t) - S(t)B)y_0 + S(t)y_1 + \int_0^t C(t-s)B(y(s))ds \\ &+ \int_0^t S(t-s)f(s,y(s))ds\bigg] \\ &+ \lambda \sum_{0<t_k<t} [C(t-t_k)I_k(y(t_k^-)) + S(t-t_k)\overline{I}_k(y(t_k^-))]. \end{aligned} \tag{2.38}$$

Also

$$\begin{aligned} y'(t) = \lambda\bigg[&(AS(t) - C(t)B)y_0 + C(t)y_1 + By(t) \\ &+ \int_0^t AS(t-s)By(s)ds + \int_0^t C(t-s)f(s,y(s))ds\bigg] \\ &+ \lambda \sum_{0<t_k<t} [AS(t-t_k)I_k(y(t_k^-)) + C(t-t_k)\overline{I}_k(y(t_k^-))]. \end{aligned} \tag{2.39}$$

This implies by (2.2.1) and (2.6.1)–(2.6.3) that for each $t \in J$ we have

$$\begin{aligned} |y(t)| \leq{}& M_1(1 + b\|B\|_{B(E)})|y_0| + bM_1|y_1| \\ &+ M_1\|B\|_{B(E)}\int_0^t |y(s)|ds \\ &+ M_1 b\int_0^t p(s)\psi(|y(s)|)ds + M_1\sum_{k=1}^m [c_k + \overline{d}_k] \\ \leq{}& M_1(1 + b\|B\|_{B(E)})|y_0| + bM_1|y_1| \\ &+ \int_0^t \widehat{m}(s)(|y(s)| + \psi(|y(s)|))ds \\ &+ M_1\sum_{k=1}^m [c_k + \overline{d}_k], \end{aligned} \tag{2.40}$$

where

$$\widehat{m}(t) = \max\{M_1\|B\|_{B(E)}, bM_1 p(t)\}. \tag{2.41}$$

Let us take the right-hand side of (2.40) as $w(t)$, then we have

$$\begin{gathered} w(0)=M_1\big(1+b\|B\|_{B(E)}\big)\,|\,y_0\,|+bM_1\,|\,y_1\,|+M_1\sum_{k=1}^{m}\big(c_k+\overline{d}_k\big), \\ |\,y(t)\,|\leq w(t),\quad t\in J, \\ w'(t)=\widehat{m}(t)\big(w(t)+\psi(w(t))\big),\quad \text{for a.e. } t\in J. \end{gathered} \tag{2.42}$$

From (2.39) we have

$$\begin{aligned} |\,y'(t)\,| \leq{}& M_1\big(\|A\|_{B(E)}b+\|B\|_{B(E)}\big)\,|\,y_0\,|+M_1\,|\,y_1\,|+\|B\|_{B(E)}w(t) \\ &+bM_1\|A\|_{B(E)}\|B\|_{B(E)}\int_0^t |\,y(s)\,|\,ds \\ &+M_1\int_0^t p(s)\psi(|\,y(s)\,|)ds+M_1\sum_{k=1}^{m}\big[\|A\|_{B(E)}bc_k+\overline{d}_k\big]. \end{aligned} \tag{2.43}$$

If we take the right-hand side of (2.43) as $z(t)$, we have

$$\begin{aligned} w(t)&\leq z(t),\quad t\in J, \\ |\,y'(t)\,|&\leq z(t),\quad t\in J, \\ z(0)&=M_1\big(\|A\|_{B(E)}b+\|B\|_{B(E)}\big)\,|\,y_0\,|+M_1\,|\,y_1\,|+\|B\|_{B(E)}w(0), \\ z'(t)&=\|B\|_{B(E)}w'(t)+bM_1\|A\|_{B(E)}\|B\|_{B(E)}\,|\,y(t)\,|+M_1p(t)\psi(|\,y(t)\,|) \\ &\leq \|B\|_{B(E)}w'(t)+bM_1\|A\|_{B(E)}\|B\|_{B(E)}w(t)+M_1p(t)\psi(w(t)) \\ &\leq \|B\|_{B(E)}\widehat{m}(t)\big(w(t)+\psi(w(t))\big) \\ &\quad +bM_1\|A\|_{B(E)}\|B\|_{B(E)}w(t)+M_1p(t)\psi(w(t)) \\ &\leq m_1(t)\big[w(t)+\psi(w(t))\big] \\ &\leq m_1(t)\big[z(t)+\psi(z(t))\big], \end{aligned} \tag{2.44}$$

where

$$m_1(t)=\max\big\{\|B\|_{B(E)}\big(\widehat{m}(t)+bM_1\|A\|_{B(E)}\|B\|_{B(E)}\big),\|B\|_{B(E)}\widehat{m}(t)+M_1p(t)\big\}. \tag{2.45}$$

This implies for each $t\in J$ that

$$\int_{z(0)}^{z(t)}\frac{d\tau}{\tau+\psi(\tau)}\leq\int_0^b m_1(s)ds<\infty. \tag{2.46}$$

This inequality implies that there exists a constant b^* such that $z(t) \leq b^*$ for each $t \in J$, and hence

$$\begin{aligned} |y'(t)| &\leq z(t) \leq b^*, \\ |y(t)| &\leq w(t) \leq z(t) \leq b^*. \end{aligned} \tag{2.47}$$

Consequently $\|y\|_* \leq b^*$.

Set $X := \mathrm{PC}^1(J,E)$. As a consequence of Schaefer's theorem we deduce that $\overline{N}$ has a fixed point which is a mild solution of (2.2)–(2.5). □

In this last part of this section we present a uniqueness result for the solutions of the problem (2.2)–(2.5) by means of the Banach fixed point principle.

Theorem 2.7. *Suppose that hypotheses (2.2.1), (2.4.1), (2.4.2), (2.6.2), and the following are satisfied:*

(2.7.1) *there exist constants $\overline{c}_k$ such that*

$$|\overline{I}_k(y) - \overline{I}_k(\overline{y})| \leq \overline{c}_k |y - \overline{y}|, \quad \text{for each } k = 1,\dots,m, \ \forall y, \overline{y} \in E. \tag{2.48}$$

If

$$\theta = \max\{\theta_1, \theta_2\} < 1, \tag{2.49}$$

where

$$\begin{aligned} \theta_1 &= M_1 b \|B\|_{B(E)} + b^2 M_1 d + M_1 [c_k + b\overline{c}_k], \\ \theta_2 &= \|B\|_{B(E)} + M_1 b^2 \|A\|_{B(E)} \|B\|_{B(E)} + b M_1 d + M_1 [b \|A\|_{B(E)} c_k + \overline{c}_k], \end{aligned} \tag{2.50}$$

then the IVP (2.2)–(2.5) *has a unique mild solution.*

Proof. Transform the problem (2.2)–(2.5) into a fixed point problem. Consider the operator $\overline{N}$ defined in Theorem 2.6. We will show that $\overline{N}$ is a contraction. Indeed, consider $y, \overline{y} \in \mathrm{PC}^1(J,E)$. Thus, for $t \in J$,

$$\begin{aligned} |\overline{N}(y)(t) - \overline{N}(\overline{y})(t)| \leq{}& M_1 \int_0^t |B(y(s)) - B(\overline{y}(s))| \, ds \\ &+ M_1 b \int_0^t |f(s, y(s)) - f(s, \overline{y}(s))| \, ds \\ &+ \sum_{k=1}^m |C(t - t_k)| \, |I_k(y(t_k^-)) - I_k(\overline{y}(t_k^-))| \\ &+ \sum_{k=1}^m |S(t - t_k)| \, |\overline{I}_k(y(t_k^-)) - \overline{I}_k(\overline{y}(t_k^-))| \end{aligned}$$

$$
\begin{aligned}
&\le M_1\|B\|_{B(E)}\int_0^b |y(s)-\overline{y}(s)|\,ds\\
&\quad + bM_1 d\int_0^b |y(s)-\overline{y}(s)|\,ds\\
&\quad + M_1\sum_{k=1}^{m}[c_k+\overline{c}_k]\|y-\overline{y}\|_{\mathrm{PC}}\\
&\le M_1 b\|B\|_{B(E)}\|y-\overline{y}\|_{\mathrm{PC}} + M_1 b^2 d\|y-\overline{y}\|_{\mathrm{PC}}\\
&\quad + M_1\sum_{k=1}^{m}[c_k+\overline{c}_k]\|y-\overline{y}\|_{\mathrm{PC}}.
\end{aligned}
\tag{2.51}
$$

Similarly we have

$$
\begin{aligned}
|\overline{N}(y)'(t)-\overline{N}(\overline{y})'(t)| \le{}& \|B\|_{B(E)}\|y-\overline{y}\|_{\mathrm{PC}}\\
&+\|A\|_{B(E)}b^2M_1\|B\|_{B(E)}\|y-\overline{y}\|_{\mathrm{PC}}\\
&+M_1 db\|y-\overline{y}\|_{\mathrm{PC}}\\
&+\sum_{k=1}^{m}\big[\|A\|_{B(E)}bM_1c_k+M_1\overline{c}_k\big].
\end{aligned}
\tag{2.52}
$$

Then

$$
\|\overline{N}(y)-\overline{N}(\overline{y})\|_{\mathrm{PC}^1} \le \theta\|y-\overline{y}\|_{\mathrm{PC}^1}. \tag{2.53}
$$

Then $\overline{N}$ is a contraction and hence it has a unique fixed point which is a mild solution to (2.2)–(2.5). □

2.3. Impulsive ordinary differential inclusions

Again, let $J=[0,b]$ and let $0=t_0<t_1<\cdots<t_m<t_{m+1}=b$. In this section, we will be concerned with the existence of solutions of the first-order initial value problem for the impulsive differential inclusion:

$$
\begin{gathered}
y'(t)\in F(t,y(t)),\quad t\in J,\ t\neq t_k,\ k=1,\dots,m,\\
y(t_k^+)=I_k(y(t_k^-)),\quad k=1,\dots,m,\\
y(0)=y_0,
\end{gathered}
\tag{2.54}
$$

where $F: J\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a compact convex-valued multivalued map defined from a single-valued function, $y_0\in\mathbb{R}$, and $I_k\in C(\mathbb{R},\mathbb{R})$ $(k=1,2,\dots,m)$, and $y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t=t_k$, respectively. In addition, let $\mathrm{PC}(J,E)$ be as defined in Section 2.2, with $E=\mathbb{R}$ and let $\mathrm{AC}(J,\mathbb{R})$ be the space of all absolutely continuous functions $y: J\to\mathbb{R}$.

Definition 2.8. By a solution to (2.54), we mean a function $y \in \mathrm{PC}(J, E) \cap \mathrm{AC}((t_k, t_{k+1}), \mathbb{R})$, $0 \le k \le m$, that satisfies the differential inclusion

$$y'(t) \in F(t, y(t)), \quad \text{a.e. on } J \setminus \{t_k\},\ k = 1, \dots, m, \tag{2.55}$$

and for each $k = 1, \dots, m$, the function y satisfies the equations $y(t_k^+) = I_k(y(t_k^-))$ and $y(0) = y_0$.

For local purposes, we repeat here the definition of a Carathéodory function.

Definition 2.9. A function $f : J \times \mathbb{R} \to \mathbb{R}$ is said to be Carathéodory if
(i) $t \mapsto f(t, y)$ is measurable for each $y \in \mathbb{R}$;
(ii) $y \mapsto f(t, y)$ is continuous for almost all $t \in J$.

Definition 2.10. A function $f : J \times \mathbb{R} \to \mathbb{R}$ is said to be of type $\mathcal{M}$ if for each measurable function $y : J \to \mathbb{R}$, the function $t \mapsto f(t, y(t))$ is measurable.

Notice that a Carathéodory map is of type $\mathcal{M}$.
Let $f : J \times \mathbb{R} \to \mathbb{R}$ be a given function. Define

$$\underline{f}(t, y) = \liminf_{u \to y} f(t, u), \qquad \overline{f}(t, y) = \limsup_{u \to y} f(t, u). \tag{2.56}$$

Also, notice that for all $t \in J$, $\underline{f}$ is lower semicontinuous (l.s.c.) (i.e., for all $t \in J$, $\{y \in \mathbb{R} : \underline{f}(t, y) > \alpha\}$ is open for each $\alpha \in \mathbb{R}$), and $\overline{f}$ is upper semicontinuous (u.s.c.) (i.e., for all $t \in J$, $\{y \in \mathbb{R} : \overline{f}(t, y) < \alpha\}$ is open for each $\alpha \in \mathbb{R}$).
Let $f : J \times \mathbb{R} \to \mathbb{R}$. We define the multivalued map $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ by

$$F(t, y) = [\underline{f}(t, y), \overline{f}(t, y)]. \tag{2.57}$$

We say that F is of "type $\mathcal{M}$" if $\underline{f}$ and $\overline{f}$ are of type $\mathcal{M}$.
The following result is crucial in the proofs of our main results.

Theorem 2.11 (see [148, Proposition (VI.1), page 40]). *Assume that F is of type $\mathcal{M}$ and for each $k \ge 0$, there exists $\phi_k \in L^2(J, \mathbb{R})$ such that*

$$\|F(t, y)\| = \sup\{|v| : v \in F(t, y)\} \le \phi_k(t), \quad \text{for } |y| \le k. \tag{2.58}$$

Then the operator $\mathcal{F} : C(J, \mathbb{R}) \to \mathcal{P}(L^2(J, \mathbb{R}))$ defined by

$$\mathcal{F} y := \{h : J \longrightarrow \mathbb{R} \text{ measurable} : h(t) \in F(t, y(t)) \text{ a.e. } t \in J\} \tag{2.59}$$

is well defined, u.s.c., bounded on bounded sets in $C(J, \mathbb{R})$ and has convex values.

We are now in a position to state and prove our first existence result for the impulsive IVP (2.54). The proof involves a Martelli fixed point theorem.

Theorem 2.12. *Assume that* $F : J \times \mathbb{R} \to \mathcal{P}_{\text{cp,cv}}(\mathbb{R})$ *is of type* $\mathcal{M}$. *Suppose that the following hypotheses hold:*

(2.12.1) *there exist* $\{r_i\}_{i=0}^m$ *and* $\{s_i\}_{i=0}^m$ *with* $s_0 \leq y_0 \leq r_0$ *and*

$$s_{i+1} \leq \min_{[s_i,r_i]} I_{i+1}(y) \leq \max_{[s_i,r_i]} I_{i+1}(y) \leq r_{i+1}; \tag{2.60}$$

(2.12.2)

$$\overline{f}(t,r_i) \leq 0, \quad \underline{f}(t,s_i) \geq 0, \quad \textit{for } t \in [t_i,t_{i+1}],\ i = 1,\dots,m; \tag{2.61}$$

(2.12.3) *there exists* $\psi : [0,\infty) \to (0,\infty)$ *continuous such that* $\psi \in L^2_{\text{loc}}([0,\infty))$ *and*

$$\|F(t,y)\| = \sup\{|v| : v \in F(t,y)\} \leq \psi(|y|), \quad \forall t \in J. \tag{2.62}$$

Then the impulsive IVP (2.54) *has at least one solution.*

Proof. This proof will be given in several steps.
Step 1. We restrict our attention to the problem on $[0,t_1]$, that is, the initial value problem

$$\begin{gathered} y'(t) \in F(t,y(t)), \quad t \in (0,t_1), \\ y(0) = y_0. \end{gathered} \tag{2.63}$$

Define the modified function $f_1 : [0,t_1] \times \mathbb{R} \to \mathbb{R}$ relative to r_0 and s_0 by

$$f_1(t,y) = \begin{cases} f(t,r_0) & \text{if } y > r_0, \\ f(t,y) & \text{if } s_0 \leq y \leq r_0, \\ f(t,s_0) & \text{if } y < s_0, \end{cases} \tag{2.64}$$

and the correponding multivalued map

$$F_1(t,y) = \begin{cases} [\underline{f}(t,r_0), \overline{f}(t,r_0)] & \text{if } y > r_0, \\ [\underline{f}(t,y), \overline{f}(t,y)] & \text{if } s_0 \leq y \leq r_0, \\ [\underline{f}(t,s_0), \overline{f}(t,s_0)] & \text{if } y < s_0. \end{cases} \tag{2.65}$$

Consider the modified problem

$$\begin{gathered} y' \in F_1(t,y), \quad t \in [0,t_1), \\ y(0) = y_0. \end{gathered} \tag{2.66}$$

We transform the problem into a fixed point problem. For this, consider the operators $L : H^1([0,t_1],\mathbb{R}) \to L^2([0,t_1],\mathbb{R})$ (where $H^1([0,t_1],\mathbb{R})$ is the standard

Sobolev space) defined by $Ly = y'$, $j : H^1([0,t_1],\mathbb{R}) \to C([0,t_1],\mathbb{R})$, the completely continuous imbedding, and

$$\mathcal{F} : C([0,t_1],\mathbb{R}) \longrightarrow \mathcal{P}(L^2([0,t_1],\mathbb{R})) \tag{2.67}$$

defined by

$$\mathcal{F}y = \{v : [0,t_1] \longrightarrow \mathbb{R} \text{ measurable} : v(t) \in F_1(t,y(t)) \text{ for a.e. } t \in [0,t_1]\}. \tag{2.68}$$

Clearly, L is linear, continuous, and invertible. It follows from the open mapping theorem that L^{-1} is a bounded linear operator. $\mathcal{F}$ is by Theorem 2.11 well-defined, bounded on bounded subsets of $C([0,t_1],\mathbb{R})$, u.s.c. and has convex values. Thus, the problem (2.66) is equivalent to $y \in L^{-1}\mathcal{F}jy := G_1 y$. Consequently, G_1 is compact, u.s.c., and has convex closed values. Therefore G_1 is a condensing map.

Now we show that the set

$$M_1 := \{y \in C([0,t_1],\mathbb{R}) : \lambda y \in G_1 y \text{ for some } \lambda > 1\} \tag{2.69}$$

is bounded.

Let $\lambda y \in G_1 y$ for some $\lambda > 1$. Then $y \in \lambda^{-1}G_1 y$, where

$$G_1 y := \left\{g \in C([0,t_1],\mathbb{R}) : g(t) = y_0 + \int_0^t h(s)ds : h \in \mathcal{F}y\right\}. \tag{2.70}$$

Let $y \in \lambda^{-1}G_1 y$. Then there exists $h \in \mathcal{F}y$ such that, for each $t \in J$,

$$y(t) = \lambda^{-1}y_0 + \lambda^{-1}\int_0^t h(s)ds. \tag{2.71}$$

Thus

$$|y(t)| \le |y_0| + \|h\|_{L^2} \quad \text{for each } t \in [0,t_1]. \tag{2.72}$$

Now since $h(t) \in F_1(t,y(t))$, it follows from the definition of $F_1(t,y)$ and assumption (2.12.3) that there exists a positive constant h_0 such that $\|h\|_{L^2} \le h_0$. In fact

$$h_0 = \max\left\{|r_0|,\ |s_0|,\ \sup_{s_0 \le y \le r_0} |\psi(y)|\right\}. \tag{2.73}$$

We then have

$$\|y\|_\infty \le |y_0| + h_0 < +\infty. \tag{2.74}$$

Hence the theorem of Martelli, Theorem 1.7 applies and so G_1 has at least one fixed point which is a solution on $[0,t_1]$ to problem (2.66).

We will show that the solution y of (2.63) satisfies

$$s_0 \leq y(t) \leq r_0, \quad \forall t \in [0, t_1]. \tag{2.75}$$

Let y be a solution to (2.66). We prove that

$$s_0 \leq y(t), \quad \forall t \in [0, t_1]. \tag{2.76}$$

Suppose not. Then there exist $\sigma_1, \sigma_2 \in [0, t_1]$, $\sigma_1 < \sigma_2$ such that $y(\sigma_1) = s_0$ and

$$s_0 > y(t), \quad \forall t \in (\sigma_1, \sigma_2). \tag{2.77}$$

This implies that

$$\begin{gathered} f_1(t, y(t)) = f(t, s_0), \quad \forall t \in (\sigma_1, \sigma_2), \\ y'(t) \in [\underline{f}(t, s_0), \overline{f}(t, s_0)]. \end{gathered} \tag{2.78}$$

Then

$$y'(t) \geq \underline{f}(t, s_0), \quad \forall t \in (\sigma_1, \sigma_2). \tag{2.79}$$

This implies that

$$y(t) \geq y(t_1) + \int_{t_1}^{t} \underline{f}(s, s_0)\, ds, \quad \forall t \in (\sigma_1, \sigma_2). \tag{2.80}$$

Since $\underline{f}(t, s_0) \geq 0$ for $t \in [0, t_1]$, we get

$$0 > y(t) - y(\sigma_1) \geq \int_{\sigma_1}^{t} \underline{f}(s, s_0)\, ds \geq 0, \quad \forall t \in (\sigma_1, \sigma_2), \tag{2.81}$$

which is a contradiction. Thus $s_0 \leq y(t)$ for $t \in [0, t_1]$.

Similarly, we can show that $y(t) \leq r_0$ for $t \in [0, t_1]$. This shows that the problem (2.66) has a solution y on the interval $[0, t_1]$, which we denote by y_1. Then y_1 is a solution of (2.63).

Step 2. Consider now the problem

$$\begin{gathered} y' \in F_2(t, y), \quad t \in (t_1, t_2), \\ y(t_1^+) = I_1(y_1(t_1^-)), \end{gathered} \tag{2.82}$$

where

$$F_2(t, y) = \begin{cases} [\underline{f}(t, r_1), \overline{f}(t, r_1)] & \text{if } y > r_1, \\ [\underline{f}(t, y), \overline{f}(t, y)] & \text{if } s_1 \leq y \leq r_1, \\ [\underline{f}(t, s_1), \overline{f}(t, s_1)] & \text{if } y < s_1. \end{cases} \tag{2.83}$$

Analogously, we can show that the set

$$M_2 := \{y \in C([t_1,t_2],\mathbb{R}) : \lambda y \in G_2 y \text{ for some } \lambda > 1\} \tag{2.84}$$

is bounded. Here the operator G_2 is defined by $G_2 := L^{-1}\mathcal{F} j$ where $L^{-1} : L^2([t_1,t_2],\mathbb{R}) \to H^1([t_1,t_2],\mathbb{R})$, $j : H^1([t_1,t_2],\mathbb{R}) \to C([t_1,t_2],\mathbb{R})$ the completely continuous imbedding, and $\mathcal{F} : C([t_1,t_2],\mathbb{R}) \to \mathcal{P}(L^2([t_1,t_2],\mathbb{R}))$ is defined by

$$\mathcal{F} y = \{v : [t_1,t_2] \longrightarrow \mathbb{R} \text{ measurable} : v(t) \in F_2(t,y(t)) \text{ for a.e. } t \in [t_1,t_2]\}. \tag{2.85}$$

We again apply the theorem of Martelli to show that G_2 has a fixed point, which we denote by y_2, and so is a solution of problem (2.82) on the interval $(t_1,t_2]$.

We now show that

$$s_1 \leq y_2(t) \leq r_1, \quad \forall t \in [t_1,t_2]. \tag{2.86}$$

Since $y_1(t_1^-) \in [s_0,r_0]$, then (2.12.1) implies that

$$s_1 \leq I_1(y(t_1^-)) \leq r_1, \quad \text{i.e.} \quad s_1 \leq y(t_1^+) \leq r_1. \tag{2.87}$$

Since $\overline{f}(t,r_1) \leq 0$ and $\underline{f}(t,s_1) \geq 0$, we can show that

$$s_1 \leq y_2(t) \leq r_1, \quad \text{for } t \in [t_1,t_2], \tag{2.88}$$

and hence y_2 is a solution to

$$\begin{gathered} y' \in F(t,y), \quad t \in (t_1,t_2), \\ y(t_1^+) = I_1(y_1(t_1^-)). \end{gathered} \tag{2.89}$$

Step 3. We continue this process and we construct solutions y_k on $[t_{k-1},t_k]$, with $k = 3,\dots,m+1$, to

$$\begin{gathered} y' \in F(t,y), \quad t \in (t_{k-1},t_k), \\ y(t_{k-1}^+) = I_{k-1}(y_{k-1}(t_{k-1}^-)), \end{gathered} \tag{2.90}$$

with $s_{k-1} \leq y_k(t) \leq r_{k-1}$ for $t \in [t_{k-1},t_k]$. Then

$$y(t) = \begin{cases} y_1(t), & t \in [0,t_1], \\ y_2(t), & t \in (t_1,t_2], \\ \vdots & \\ y_{m+1}(t), & t \in (t_m,T], \end{cases} \tag{2.91}$$

is a solution to (2.54). □

Using the same reasoning as that used in the proof of Theorem 2.12, we can obtain the following result.

Theorem 2.13. *Suppose that $F : J \times \mathbb{R} \to \mathcal{P}_{\text{cp,cv}}(\mathbb{R})$ is of type $\mathcal{M}$. Suppose the following hypotheses hold.*

(2.13.1) *There are functions $\{r_i\}_{i=0}^m$ and $\{s_i\}_{i=0}^m$ with $r_i, s_i \in C([t_i, t_{i+1}])$ and $s_i(t) \leq r_i(t)$ for $t \in [t_i, t_{i+1}]$, $i = 0, \ldots, m$. Also, $s_0 \leq y_0 \leq r_0$ and*

$$
\begin{aligned}
s_{i+1}(t_{i+1}^+) &\leq \min_{[s_i(t_{i+1}^-), r_i(t_{i+1}^-)]} I_{i+1}(y) \\
&\leq \max_{[s_i(t_{i+1}^-), r_i(t_{i+1}^-)]} I_{i+1}(y) \\
&\leq r_{i+1}(t_{i+1}^+), \quad i = 0, \ldots, m-1.
\end{aligned}
\tag{2.92}
$$

(2.13.2)

$$
\begin{aligned}
\int_{z_i}^{w_i} \underline{f}(t, s_i(t))\,dt &\geq s_i(w_i) - s_i(z_i), \\
\int_{z_i}^{w_i} \overline{f}(t, r_i(t))\,dt &\leq r_i(w_i) - r_i(z_i), \quad i = 0, \ldots, m
\end{aligned}
\tag{2.93}
$$

with

$$
z_i < w_i, \quad z_i, w_i \in [t_i, t_{i+1}]. \tag{2.94}
$$

Then the impulsive IVP (2.54) *has at least one solution.*

Consider now the following initial value problem for first-order impulsive differential inclusions of the type

$$
\begin{aligned}
y'(t) &\in F(t, y(t)), \quad \text{a.e. } t \in J,\ t \neq t_k,\ k = 1, \ldots, m, \\
\Delta y|_{t=t_k} &= I_k(y(t_k^-)), \quad k = 1, \ldots, m, \\
y(0) &= y_0,
\end{aligned}
\tag{2.95}
$$

where $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a multivalued map with nonempty compact values, $y_0 \in \mathbb{R}^n$, $\mathcal{P}(R^n)$ is the family of all subsets of $\mathbb{R}^n$, $I_k \in C(\mathbb{R}^n, \mathbb{R}^n)$ ($k = 1, 2, \ldots, m$), $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively. Our existence results in this scenario will involve the Leray-Schauder alternative as well as Schaefer's theorem.

Definition 2.14. A function $y \in \text{PC}(J, \mathbb{R}^n) \cap \text{AC}((t_k, t_{k+1}), \mathbb{R}^n)$, $0 \leq k \leq m$, is said to be a solution of (2.95) if y satisfies the differential inclusion $y'(t) \in F(t, y(t))$ a.e. on $J - \{t_1, \ldots, t_m\}$ and the conditions $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, and $y(0) = y_0$.

The first result of this section concerns the a priori estimates on possible solutions of the problem (2.95).

Theorem 2.15. *Suppose that the following is satisfied:*

(2.15.1) *there exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and $p \in L^1(J,\mathbb{R}_+)$ such that*

$$\|F(t,y)\| \leq p(t)\psi(|y|), \quad \text{for a.e. } t \in J \text{ and each } y \in \mathbb{R}^n, \tag{2.96}$$

with

$$\int_{t_{k-1}}^{t_k} p(s)ds < \int_{N_{k-1}}^{\infty} \frac{du}{\psi(u)}, \quad k = 1,\dots,m+1, \tag{2.97}$$

where

$$\begin{gathered} N_0 = |y_0|, \qquad N_{k-1} = \sup_{y\in[-M_{k-2},M_{k-2}]} |I_{k-1}(y)| + M_{k-2}, \\ M_{k-2} = \Gamma_{k-1}^{-1}\left(\int_{t_{k-2}}^{t_{k-1}} p(s)ds\right), \end{gathered} \tag{2.98}$$

for $k = 1,\dots,m+1$, and

$$\Gamma_l(z) = \int_{N_{l-1}}^{z} \frac{du}{\psi(u)}, \quad z \geq N_{l-1},\ l \in \{1,\dots,m+1\}. \tag{2.99}$$

Then for each $k = 1,\dots,m+1$ there exists a constant M_{k-1} such that

$$\sup\{|y(t)| : t \in [t_k, t_{k-1}]\} \leq M_{k-1}, \tag{2.100}$$

for each solution y of the problem (2.95).

Proof. Let y be a possible solution to (2.95). Then $y|_{[0,t_1]}$ is a solution to

$$y'(t) \in F(t,y(t)), \quad \text{for a.e. } t \in [0,t_1], \qquad y(0) = y_0. \tag{2.101}$$

Since $|y|' \leq |y'|$, we have

$$|y(t)|' \leq p(t)\psi(|y(t)|), \quad \text{for a.e. } t \in [0,t_1]. \tag{2.102}$$

Let $t^* \in [0,t_1]$ such that

$$\sup\{|y(t)| : t \in [0,t_1]\} = |y(t^*)|. \tag{2.103}$$

Then

$$\frac{|y(t)|'}{\psi(|y(t)|)} \leq p(t), \quad \text{for a.e. } t \in [0,t_1]. \tag{2.104}$$

From this inequality, it follows that

$$\int_0^{t^*} \frac{|y(s)|'}{\psi(|y(s)|)} ds \le \int_0^{t^*} p(s)ds. \tag{2.105}$$

Using the change of variable formula, we get

$$\Gamma_1(|y(t^*)|) = \int_{|y(0)|}^{|y(t^*)|} \frac{du}{\psi(u)} \le \int_0^{t^*} p(s)ds \le \int_0^{t_1} p(s)ds. \tag{2.106}$$

In view of (2.15.1), we obtain

$$|y(t^*)| \le \Gamma_1^{-1}\left(\int_0^{t_1} p(s)ds\right). \tag{2.107}$$

Hence

$$|y(t^*)| = \sup\{|y(t)| : t \in [0, t_1]\} \le \Gamma_1^{-1}\left(\int_0^{t_1} p(s)ds\right) := M_0. \tag{2.108}$$

Now $y|_{[t_1,t_2]}$ is a solution to

$$\begin{gathered} y'(t) \in F(t, y(t)), \quad \text{for a.e. } t \in [t_1, t_2], \\ \Delta y|_{t=t_1} = I_1(y(t_1)). \end{gathered} \tag{2.109}$$

Note that

$$|y(t_1^+)| \le \sup_{y\in[-M_0,M_0]} |I_{k-1}(y)| + M_0 := N_1. \tag{2.110}$$

Then

$$|y(t)|' \le p(t)\psi(|y(t)|), \quad \text{for a.e. } t \in [t_1, t_2]. \tag{2.111}$$

Let $t^* \in [t_1, t_2]$ such that

$$\sup\{|y(t)| : t \in [t_1, t_2]\} = |y(t^*)|. \tag{2.112}$$

Then

$$\frac{|y(t)|'}{\psi(|y(t)|)} \le p(t). \tag{2.113}$$

From this inequality, it follows that

$$\int_{t_1}^{t^*} \frac{|y(s)|'}{\psi(|y(s)|)} ds \le \int_{t_1}^{t^*} p(s)ds. \tag{2.114}$$

Proceeding as above, we obtain

$$\Gamma_2(|y(t^*)|) = \int_{N_1}^{|y(t^*)|} \frac{du}{\psi(u)} \leq \int_{t_1}^{t^*} p(s)ds \leq \int_{t_1}^{t_2} p(s)ds. \tag{2.115}$$

This yields

$$|y(t^*)| = \sup\{|y(t)| : t \in [t_1, t_2]\} \leq \Gamma_2^{-1}\left(\int_{t_1}^{t_2} p(s)ds\right) := M_1. \tag{2.116}$$

We continue this process and taking into account that $y|_{[t_m,T]}$ is a solution to the problem

$$\begin{gathered} y'(t) \in F(t, y(t)), \quad \text{for a.e. } t \in [t_m, T], \\ \Delta y|_{t=t_m} = I_m(y(t_m)). \end{gathered} \tag{2.117}$$

We obtain that there exists a constant M_m such that

$$\sup\{|y(t)| : t \in [t_m, T]\} \leq \Gamma_{m+1}^{-1}\left(\int_{t_m}^{T} p(s)ds\right) := M_m. \tag{2.118}$$

Consequently, for each possible solution y to (2.95), we have

$$\|y\|_{\mathrm{PC}} \leq \max\{|y_0|, M_{k-1} : k = 1, \dots, m+1\} := \widehat{b}. \tag{2.119}$$

□

Theorem 2.16. *Suppose (2.15.1) and the following hypotheses are satisfied:*

(2.16.1) $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ *is a nonempty compact-valued multivalued map such that*

(a) $(t, y) \mapsto F(t, y)$ *is* $\mathcal{L} \otimes \mathcal{B}$ *measurable,*

(b) $y \mapsto F(t, y)$ *is lower semicontinuous for a.e.* $t \in J$*;*

(2.16.2) *for each* $r > 0$, *there exists a function* $h_r \in L^1(J, \mathbb{R}^+)$ *such that*

$$\begin{aligned} &\|F(t, y)\| \\ &\quad := \sup\{|v| : v \in F(t, y)\} \leq h_r(t), \quad \text{for a.e. } t \in J \text{ and for } y \in \mathbb{R}^n \text{ with } |y| \leq r. \end{aligned} \tag{2.120}$$

Then the impulsive IVP (2.95) *has at least one solution.*

Proof. Hypotheses (2.16.1) and (2.16.2) imply by Lemma 1.29 that F is of lower semicontinuous type. Then from Theorem 1.5 there exists a continuous function $f : \mathrm{PC}(J, \mathbb{R}^n) \to L^1(J, \mathbb{R}^n)$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in \mathrm{PC}(J, \mathbb{R}^n)$.

Consider the following problem:

$$\begin{gathered} (y'(t)) = f(y(t)), \quad t \in J,\ t \neq t_k,\ k = 1, \dots, m, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \\ y(0) = y_0. \end{gathered} \tag{2.121}$$

Remark 2.17. If $y \in \mathrm{PC}(J,\mathbb{R}^n)$ is a solution of the problem (2.121), then y is a solution to the problem (2.95).

Transform the problem (2.121) into a fixed point problem. Consider the operator $N : \mathrm{PC}(J,\mathbb{R}^n) \to \mathrm{PC}(J,\mathbb{R}^n)$ defined by

$$N(y)(t) := y_0 + \int_0^t f(y(s))ds + \sum_{0<t_k<t} I_k(y(t_k^-)). \tag{2.122}$$

We will show that N is a compact operator.

Step 1. N is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $\mathrm{PC}(J,\mathbb{R}^n)$. Then

$$\begin{aligned}
|N(y_n(t)) - N(y(t))| &\le \int_0^t |f(y_n(s)) - f(y(s))|ds \\
&\quad + \sum_{0<t_k<t} |I_k(y_n(t_k^-)) - I_k(y(t_k^-))| \\
&\le \int_0^b |f(y_n(s)) - f(y(s))|ds \\
&\quad + \sum_{0<t_k<t} |I_k(y_n(t_k^-)) - I_k(y(t_k^-))|.
\end{aligned} \tag{2.123}$$

Since the functions f and I_k, $k = 1,\dots,m$ are continuous, then

$$\|N(y_n) - N(y)\|_{\mathrm{PC}} \le \|f(y_n) - f(y)\|_{L^1} + \sum_{k=1}^m |I_k(y_n(t_k^-)) - I_k(y(t_k^-))| \longrightarrow 0, \tag{2.124}$$

as $n \to \infty$.

Step 2. N maps bounded sets into bounded sets in $\mathrm{PC}(J,\mathbb{R}^n)$.

It is enough to show that there exists a positive constant ℓ such that for each $y \in B_q = \{y \in \mathrm{PC}(J,\mathbb{R}^n) : \|y\|_{\mathrm{PC}} \le q\}$ we have $\|N(y)\|_{\mathrm{PC}} \le \ell$.

Indeed, since I_k $(k = 1,\dots,m)$ are continuous and from (2.16.2), we have

$$\begin{aligned}
|N(y)(t)| &\le |y_0| + \int_0^t |f(y(s))|ds + \sum_{0<t_k<t} |I_k(y(t_k^-))| \\
&\le |y_0| + \|h_q\|_{L^1} + \sum_{k=1}^m |I_k(y(t_k^-))| := \ell.
\end{aligned} \tag{2.125}$$

Step 3. N maps bounded sets into equicontinuous sets of $\mathrm{PC}(J,\mathbb{R}^n)$.

Let $r_1, r_2 \in J'$, and let $B_q = \{y \in \mathrm{PC}(J,\mathbb{R}^n) : \|y\|_{\mathrm{PC}} \le q\}$ be a bounded set of $\mathrm{PC}(J,\mathbb{R}^n)$. Then

$$|N(y)(r_2) - N(y)(r_1)| \le \int_{r_1}^{r_2} h_q(s)ds + \sum_{0<t_k<r_2-r_1} |I_k(y(t_k^-))|. \tag{2.126}$$

As $r_2 \to r_1$, the right-hand side of the above inequality tends to zero. This proves the equicontinuity for the case where $t \neq t_i$, $i = 1, \dots, m$. It remains to examine the equicontinuity at $t = t_i$. The proof is similar to that given in Theorem 2.2. Then $N(B_q)$ is equicontinuous.

Set

$$U = \{y \in \mathrm{PC}(J, \mathbb{R}^n) : \|y\|_{\mathrm{PC}} < \hat{b} + 1\}, \tag{2.127}$$

where $\hat{b}$ is the constant of Theorem 2.15. As a consequence of Steps 1 to 3, together with the Arzelá-Ascoli theorem, we can conclude that $N : \overline{U} \to \mathrm{PC}(J, \mathbb{R}^n)$ is compact.

From the choice of U there is no $y \in \partial U$ such that $y = \lambda N y$ for any $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of the Leray-Schauder type [157] we deduce that N has a fixed point $y \in U$ which is a solution of the problem (2.121) and hence a solution to the problem (2.95). □

We present now a result for the problem (2.95) in the spirit of Schaefer's theorem.

Theorem 2.18. *Suppose that hypotheses (2.2.1), (2.16.1), (2.16.2), and the following are satisfied:*

(2.18.1) *there exist a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ and $p \in L^1(J, \mathbb{R}_+)$ such that*

$$||F(t, y)|| \leq p(t)\psi(|y|) \tag{2.128}$$

for a.e. $t \in J$ and each $y \in \mathbb{R}^n$ with

$$\int_0^b p(s)ds < \int_c^\infty \frac{du}{\psi(u)}, \quad c = |y_0| + \sum_{k=1}^m c_k. \tag{2.129}$$

Then the impulsive IVP (2.95) *has at least one solution.*

Proof. In Theorem 2.16, for the problem (2.121), we proved that the operator N is completely continuous. In order to apply Schaefer's theorem it remains to show that the set

$$\mathcal{E}(N) := \{y \in \mathrm{PC}(J, \mathbb{R}^n) : \lambda y = N(y), \text{ for some } \lambda > 1\} \tag{2.130}$$

is bounded. Let $y \in \mathcal{E}(N)$. Then $\lambda y = N(y)$ for some $\lambda > 1$. Thus

$$y(t) = \lambda^{-1} y_0 + \lambda^{-1} \int_0^t f(y(s))ds + \lambda^{-1} \sum_{0<t_k<t} I_k(y(t_k^-)). \tag{2.131}$$

This implies that for each $t \in J$ we have

$$\begin{aligned} |y(t)| &\le |y_0| + \int_0^t p(s)\psi(|y(s)|)ds + \sum_{k=1}^m |I_k(y(t_k^-))| \\ &\le |y_0| + \int_0^t p(s)\psi(|y(s)|)ds + \sum_{k=1}^m c_k. \end{aligned} \tag{2.132}$$

Let $v(t)$ represent the right-hand side of the above inequality. Then

$$v(0) = |y_0| + \sum_{k=1}^m c_k, \qquad v'(t) = p(t)\psi(|y(t)|), \quad \text{for a.e. } t \in J. \tag{2.133}$$

Since ψ is nondecreasing, we have

$$v'(t) \le p(t)\psi(v(t)), \quad \text{for a.e. } t \in J. \tag{2.134}$$

It follows that

$$\int_0^t \frac{v'(s)}{\psi(v(s))} ds \le \int_0^t p(s)ds. \tag{2.135}$$

We then have

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \le \int_0^t p(s)ds \le \int_0^b p(s)ds < \int_{v(0)}^{\infty} \frac{du}{\psi(u)}. \tag{2.136}$$

This inequality implies that there exists a constant d depending only on the functions p and ψ such that

$$|y(t)| \le d, \quad \text{for each } t \in J. \tag{2.137}$$

Hence

$$\|y\|_{\mathrm{PC}} := \sup\{|y(t)| : 0 \le t \le T\} \le d. \tag{2.138}$$

This shows that $\mathcal{E}(N)$ is bounded. As a consequence of Schaefer's theorem (see [220]) we deduce that N has a fixed point y which is a solution to problem (2.121). Then, from Remark 2.17, y is a solution to the problem (2.95). □

Remark 2.19. We can easily show that the above reasoning with appropriate hypotheses can be applied to obtain existence results for the following second-order impulsive differential inclusion:

$$\begin{gathered} y''(t) \in F(t, y(t)), \quad \text{a.e. } t \in J,\ t \ne t_k,\ k = 1,\dots,m, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m, \\ \Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1,\dots,m, \\ y(0) = y_0, \qquad y'(0) = y_1, \end{gathered} \tag{2.139}$$

where F, I_k $(k = 1,\dots,m)$, y_0 are as in the problem (2.95) and $\overline{I}_k \in C(\mathbb{R}^n,\mathbb{R}^n)$ $(k = 1,\dots,m)$, $y_1 \in \mathbb{R}^n$. The details are left to the reader.

In the next discussion, we extend the above results to the semilinear case. That is, in a fashion similar to the development of the theory for semilinear equations, we deal first with the existence of mild solutions for the impulsive semilinear evolution inclusion:

$$\begin{gathered} y'(t) - Ay(t) \in F(t,y(t)), \quad t \in J,\ t \neq t_k,\ k = 1,\dots,m, \\ y(t_k^+) = I_k(y(t_k^-)), \quad k = 1,\dots,m, \\ y(0) = a, \end{gathered} \tag{2.140}$$

where $F : J \times E \to \mathcal{P}(E)$ is a closed, bounded and convex-valued multivalued map, $a \in E$, A is the infinitesimal generator of a strongly continuous semigroup $T(t)$, $t \geq 0$, $I_k \in C(E,E)$ $(k = 1,2,\dots,m)$, and $y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively.

Again, let us start by defining what we mean by a solution of problem (2.140).

Definition 2.20. A function $y \in \mathrm{PC}(J,E) \cap \mathrm{AC}((t_k,t_{k+1}),E)$, $0 \leq k \leq m$, is said to be a mild solution of (2.140) if there exists a function $v \in L^1(J,E)$ such that $v(t) \in F(t,y(t))$ a.e. on J_k, and

$$y(t) = \begin{cases} T(t)a + \displaystyle\int_0^t T(t-s)v(s)ds, & \text{if } t \in J_0, \\ T(t-t_k)I_k(y(t_k^-)) + \displaystyle\int_{t_k}^t T(t-s)v(s)ds, & \text{if } t \in J_k. \end{cases} \tag{2.141}$$

For the multivalued map F and for each $y \in C(J_k,E)$ we define $S^1_{F,y}$ by

$$S^1_{F,y} = \{v \in L^1(J_k,E) : v(t) \in F(t,y(t)),\ \text{for a.e. } t \in J_k\}. \tag{2.142}$$

We are now in a position to state and prove our existence result for the IVP (2.140).

Theorem 2.21. *Assume that (2.2.2) holds. In addition suppose the following hypotheses hold.*

(2.21.1) *$F : J \times E \to \mathcal{P}_{b,\mathrm{cp},\mathrm{cv}}(E)$ is an L^1-Carathéodory multivalued map.*

(2.21.2) *There exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ with $\int_0^\infty (du/\psi(u)) = \infty$ and $p \in L^1(J,\mathbb{R}_+)$ such that*

$$\|F(t,y)\| := \sup\{|v| : v \in F(t,y)\} \leq p(t)\psi(|y|) \tag{2.143}$$

for a.e. $t \in J$ and for all $y \in E$.

(2.21.3) *For each bounded set* $B \subseteq C(J_k, E)$ *and for each* $t \in J_k$, *the set*

$$\left\{T(t-t_k)I_k(y(t_k^-)) + \int_0^{t_k} T(t-s)v(s)ds : v \in S^1_{F,B}\right\} \tag{2.144}$$

is relatively compact in E, *where* $S^1_{F,B} = \cup\{S^1_{F,y} : y \in B\}$ *and* $k = 0, \dots, m$.

Then problem (2.140) *has at least one mild solution* $y \in \mathrm{PC}(J, E)$.

Remark 2.22. (i) If $\dim E < \infty$, then for each $y \in C(J_k, E)$, $S^1_{F,y} \neq \emptyset$ (see Lasota and Opial [186]).

(ii) If $\dim E = \infty$ and $y \in C(J_k, E)$, the set $S^1_{F,y}$ is nonempty if and only if the function $Y : J \to \mathbb{R}$ defined by

$$Y(t) := \inf\{|v| : v \in F(t, y)\} \tag{2.145}$$

belongs to $L^1(J, \mathbb{R})$ (see Hu and Papageorgiou [170]).

Proof of Theorem 2.21. The proof is given in several steps.

Step 1. Consider the problem (2.140) on $J_0 := [0, t_1]$,

$$\begin{gathered} y' - Ay \in F(t, y), \quad \text{a.e. } t \in J_0, \\ y(0) = a. \end{gathered} \tag{2.146}$$

We transform this problem into a fixed point problem. A solution to (2.146) is a fixed point of the operator $G : C(J_0, E) \to \mathcal{P}(C(J_0, E))$ defined by

$$G(y) := \left\{h \in C(J_0, E) : h(t) = T(t)a + \int_0^t T(t-s)v(s)ds : v \in S^1_{F,y}\right\}. \tag{2.147}$$

We will show that G satisfies the assumptions of Theorem 1.7.

Claim 1. $G(y)$ is convex for each $y \in C(J_0, E)$.

Indeed, if $h, \overline{h}$ belong to $G(y)$, then there exist $v \in S^1_{F,y}$ and $\overline{v} \in S^1_{F,y}$ such that

$$\begin{aligned} h(t) &= T(t)a + \int_0^t T(t-s)v(s)ds, \quad t \in J_0, \\ \overline{h}(t) &= T(t)a + \int_0^t T(t-s)\overline{v}(s)ds, \quad t \in J_0. \end{aligned} \tag{2.148}$$

Let $0 \le l \le 1$. Then for each $t \in J_0$ we have

$$[lh + (1-l)\overline{h}](t) = T(t)a + \int_0^t T(t-s)[lv(s) + (1-l)\overline{v}(s)]ds. \tag{2.149}$$

Since $S^1_{F,y}$ is convex (because F has convex values), then

$$lh + (1-l)\overline{h} \in G(y). \tag{2.150}$$

Claim 2. G sends bounded sets into bounded sets in $C(J_0, E)$.

Let $B_r := \{y \in C_0(J_0, E) : \|y\|_\infty := \sup\{|y(t)| : t \in J_0\} \le r\}$ be a bounded set in $C_0(J_0, E)$ and $y \in B_r$. Then for each $h \in G(y)$ there exists $v \in S^1_{F,y}$ such that

$$h(t) = T(t)a + \int_0^t T(t-s)v(s)ds, \quad t \in J_0. \tag{2.151}$$

Thus for each $t \in J_0$ we get

$$\begin{aligned} |h(t)| &\le M|a| + M\int_0^t |v(s)|ds \\ &\le M|a| + M\|\phi_r\|_{L^1}. \end{aligned} \tag{2.152}$$

Claim 3. G sends bounded sets in $C(J_0, E)$ into equicontinuous sets.

Let $u_1, u_2 \in J_0$, $u_1 < u_2$, $B_r := \{y \in C(J_0, E) : \|y\|_\infty \le r\}$ be a bounded set in $C_0(J_0, E)$ as in Claim 2 and $y \in B_r$. For each $h \in G(y)$ we have

$$\begin{aligned} |h(u_2) - h(u_1)| \le\ & |T(u_2)a - T(u_1)a| \\ &+ \left|\int_0^{u_2} [T(u_2 - s) - T(u_1 - s)]v(s)ds\right| \\ &+ \left|\int_{u_1}^{u_2} T(u_1 - s)v(s)ds\right| \\ \le\ & |T(u_2)a - T(u_1)a| \\ &+ \left|\int_0^{u_2} [T(u_2 - s) - T(u_1 - s)]v(s)ds\right| \\ &+ M\int_{u_1}^{u_2} |v(s)|ds. \end{aligned} \tag{2.153}$$

As a consequence of Claims 2, 3, and (2.21.3), together with the Arzelá-Ascoli theorem, we can conclude that $G : C(J_0, E) \to \mathcal{P}(C(J_0, E))$ is a compact multivalued map, and therefore, a condensing map.

Claim 4. G has a closed graph.

Let $y_n \to y_*$, $h_n \in G(y_n)$, and $h_n \to h_*$. We will prove that $h_* \in G(y_*)$.

$h_n \in G(y_n)$ means that there exists $v_n \in S_{F,y_n}$ such that

$$h_n(t) = T(t)a + \int_0^t T(t-s)v_n(s)ds, \quad t \in J_0. \tag{2.154}$$

We must prove that there exists $v_* \in S^1_{F,y_*}$ such that

$$h_*(t) = T(t)a + \int_0^t T(t-s)v_*(s)ds, \quad t \in J_0. \tag{2.155}$$

Consider the linear continuous operator $\Gamma : L^1(J_0, E) \to C(J_0, E)$ defined by

$$(\Gamma v)(t) = \int_0^t T(t-s)v(s)ds. \tag{2.156}$$

We have

$$\|(h_n - T(t)a) - (h_* - T(t)a)\|_\infty \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{2.157}$$

From Lemma 1.28, it follows that $\Gamma \circ S_F^1$ is a closed graph operator.

Also from the definition of Γ we have that

$$h_n(t) - T(t)a \in \Gamma(S^1_{F,y_n}). \tag{2.158}$$

Since $y_n \to y_*$, it follows from Lemma 1.28 that

$$h_*(t) = T(t)a + \int_0^t T(t-s)v_*(s)ds, \quad t \in J_0, \tag{2.159}$$

for some $v_* \in S^1_{F,y_*}$.

Claim 5. Now we show that the set

$$\mathcal{M} := \{y \in C(J_0, E) : \lambda y \in G(y) \text{ for some } \lambda > 1\} \tag{2.160}$$

is bounded.

Let $y \in \mathcal{M}$. Then $\lambda y \in G(y)$ for some $\lambda > 1$. Thus there exists $v \in S^1_{F,y}$ such that

$$y(t) = \lambda^{-1}T(t)a + \lambda^{-1}\int_0^t T(t-s)v(s)ds, \quad t \in J_0. \tag{2.161}$$

Thus for each $t \in J_0$ we have

$$\begin{aligned} |y(t)| &\le M|a| + M\int_0^t |v(s)|ds \\ &\le M|a| + M\int_0^t p(s)\psi(|y(s)|)ds. \end{aligned} \tag{2.162}$$

As a consequence of Lemma 1.30, we obtain

$$\|y\|_\infty \le \|z_0\|_\infty, \tag{2.163}$$

where z_0 is the unique solution on J_0 of the integral equation

$$z(t) - M|a| = M\int_0^t p(s)\psi(z(s))ds. \tag{2.164}$$

This shows that $\mathcal{M}$ is bounded. Hence Theorem 1.7 applies and G has a fixed point which is a mild solution to problem (2.146). Denote this solution by y_0.

Step 2. Consider now the following problem on $J_1 := [t_1, t_2]$:

$$\begin{aligned} y' - Ay &\in F(t, y), \quad \text{a.e. } t \in J_1, \\ y(t_1^+) &= I_1(y(t_1^-)). \end{aligned} \tag{2.165}$$

A solution to (2.165) is a fixed point of the operator $G : \mathrm{PC}(J_1, E) \to \mathcal{P}(C(J_1, E))$ defined by

$$\begin{aligned} G(y) := \Big\{ h \in \mathrm{PC}(J_1, E) : h(t) &= T(t - t_1) I_1(y(t_1^-)) \\ &+ \int_{t_1}^{t} T(t - s) v(s) ds : v \in S^1_{F,y} \Big\}. \end{aligned} \tag{2.166}$$

As in Step 1, we can easily show that G has convex values, is condensing and upper semicontinuous. It suffices to show that the set

$$\mathcal{M} := \{ y \in \mathrm{PC}(J_1, E) : \lambda y \in G(y) \text{ for some } \lambda > 1 \} \tag{2.167}$$

is bounded.

Let $y \in \mathcal{M}$. Then $\lambda y \in G(y)$ for some $\lambda > 1$. Thus there exists $v \in S^1_{F,y}$ such that

$$y(t) = \lambda^{-1} T(t - t_1) I_1(y(t_1^-)) + \lambda^{-1} \int_{t_1}^{t} T(t - s) v(s) ds, \quad t \in J_1. \tag{2.168}$$

Thus for each $t \in J_1$ we have

$$\begin{aligned} |y(t)| &\le M \sup_{t \in J_0} |I_1(y_0(t))| + M \int_{t_1}^{t} |v(s)| ds \\ &\le M \sup_{t \in J_0} |I_1(y_0(t))| + M \int_{t_1}^{t} p(s) \psi(|y(s)|) ds. \end{aligned} \tag{2.169}$$

As a consequence of Lemma 1.30, we obtain

$$\|y\|_\infty \le \|z_1\|_\infty, \tag{2.170}$$

where z_1 is the unique solution on J_1 of the integral equation

$$z(t) - M \sup_{t \in J_0} |I_1(y_0(t))| = M \int_{t_1}^{t} p(s) \psi(z(s)) ds. \tag{2.171}$$

This shows that $\mathcal{M}$ is bounded. Hence Theorem 1.7 applies and G has a fixed point which is a mild solution to problem (2.165). Denote this solution by y_1.

Step 3. Continue this process and construct solutions $y_k \in \mathrm{PC}(J_k, E)$, $k = 2, \dots, m$, to

$$\begin{aligned} y'(t) - Ay(t) &\in F(t, y(t)), \quad \text{a.e. } t \in J_k, \\ y(t_k^+) &= I_k(y(t_k^-)). \end{aligned} \tag{2.172}$$

Then

$$y(t) = \begin{cases} y_0(t) & \text{if } t \in [0, t_1], \\ y_1(t) & \text{if } t \in (t_1, t_2], \\ \vdots & \\ y_{m-1}(t) & \text{if } t \in (t_{m-1}, t_m], \\ y_m(t) & \text{if } t \in (t_m, b] \end{cases} \tag{2.173}$$

is a mild solution of (2.140). □

We investigate now the existence of mild solutions for the impulsive semilinear evolution inclusion of the form

$$\begin{aligned} y'(t) - A(t)y(t) &\in F(t, y(t)), \quad t \in J,\ t \neq t_k,\ k = 1, \dots, m, \\ y(t_k^+) &= I_k(y(t_k^-)), \quad k = 1, \dots, m, \\ y(0) &= a, \end{aligned} \tag{2.174}$$

where $F : J \times E \to \mathcal{P}(E)$ is a closed, bounded and convex-valued multivalued map, $a \in E$, $A(t)$, $t \in J$ a linear closed operator from a dense subspace $D(A(t))$ of E into E, E a real "ordered" Banach space with the norm $|\cdot|$, $I_k \in C(E, E)$ $(k = 1, 2, \dots, m)$, and $y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively.

The notions of lower-mild and upper-mild solutions for differential equations in ordered Banach spaces can be found in the book of Heikkilä and Lakshmikantham [163].

In our results we do not assume any type of monotonicity condition on I_k, $k = 1, \dots, m$, which is usually the situation in the literature; see, for instance, [176, 190].

So again, we explain what we mean by a mild solution of problem (2.174).

Definition 2.23. A function $y \in \mathrm{PC}(J, E)$ is said to be a mild solution of (2.174) (see [210]) if there exists a function $v \in L^1(J, E)$ such that $v(t) \in F(t, y(t))$ a.e. on J_k, and

$$y(t) = \begin{cases} T(t,0)a + \displaystyle\int_0^t T(t,s)v(s)ds, & t \in J_0, \\ T(t,t_k)I_k(y(t_k^-)) + \displaystyle\int_{t_k}^t T(t,s)v(s)ds, & t \in J_k. \end{cases} \tag{2.175}$$

For the development, we need the notions of lower-mild and upper-mild solutions for the problem (2.174).

Definition 2.24. A function $\underline{y} \in \mathrm{PC}(J,E)$ is said to be a lower-mild solution of (2.174) if there exists a function $v_1 \in L^1(J,E)$ such that $v_1(t) \in F(t,\underline{y}(t))$ a.e. on J_k, and

$$\underline{y}(t) \leq \begin{cases} T(t,0)a + \displaystyle\int_0^t T(t,s)v_1(s)ds, & t \in J_0, \\ T(t,t_k)I_k\big(\underline{y}(t_k^-)\big) + \displaystyle\int_{t_k}^t T(t,s)v_1(s)ds, & t \in J_k. \end{cases} \tag{2.176}$$

Similarly a function $\overline{y} \in \mathrm{PC}(J,E)$ is said to be an upper-mild solution of (2.174) if there exists a function $v_2 \in L^1(J,E)$ such that $v_2(t) \in F(t,\overline{y}(t))$ a.e. on J_k, and

$$\overline{y}(t) \geq \begin{cases} T(t,0)a + \displaystyle\int_0^t T(t,s)v_2(s)ds, & t \in J_0, \\ T(t,t_k)I_k\big(\overline{y}(t_k^-)\big) + \displaystyle\int_{t_k}^t T(t,s)v_2(s)ds, & t \in J_k. \end{cases} \tag{2.177}$$

For the multivalued map F and for each $y \in C(J_k,E)$ we define $S^1_{F,y}$ by

$$S^1_{F,y} = \{v \in L^1(J_k,E) : v(t) \in F(t,y(t)), \text{ for a.e. } t \in J_k\}. \tag{2.178}$$

We are now in a position to state and prove our first existence result for problem (2.174).

Theorem 2.25. *Assume that $F : J \times E \to \mathcal{P}_{b,\mathrm{cp},\mathrm{cv}}(E)$ and (2.21.1) holds. In addition suppose the following hypotheses hold.*

(2.25.1) *$A(t)$, $t \in J$, is continuous such that*

$$A(t)y = \lim_{h\to 0^+} \frac{T(t+h,t)y - y}{h}, \quad y \in D(A(t)), \tag{2.179}$$

where $T(t,s) \in B(E)$ for each $(t,s) \in \gamma := \{(t,s); 0 \leq s \leq t \leq b\}$, satisfying

(i) *$T(t,t) = I$ (I is the identity operator in E),*

(ii) *$T(t,s)T(s,r) = T(t,r)$ for $0 \leq r \leq s \leq t \leq b$,*

(iii) *the mapping $(t,s) \mapsto T(t,s)y$ is strongly continuous in γ for each $y \in E$,*

(iv) *$|T(t,s)| \leq M$ for $(t,s) \in \gamma$.*

(2.25.2) *There exist $\underline{y}$, $\overline{y}$, respectively, lower-mild and upper-mild solutions for* (2.174) *such that $\underline{y} \leq \overline{y}$.*

(2.25.3) *$\underline{y}(t_k^+) \leq \min_{[\underline{y}(t_k^-),\overline{y}(t_k^-)]} I_k(y) \leq \max_{[\underline{y}(t_k^-),\overline{y}(t_k^-)]} I_k(y) \leq \overline{y}(t_k^+)$, $k = 1,\dots,m$.*

(2.25.4) *$T(t,s)$ is order-preserving for all $(t,s) \in \gamma$.*
(2.25.5) *For each bounded set $B \subseteq C(J_k,E)$ and for each $t \in J_k$, the set*

$$\left\{ \int_0^{t_k} T(t,s)v(s)ds : v \in S^1_{F,B} \right\} \tag{2.180}$$

is relatively compact in E, where $S^1_{F,B} = \cup\{S^1_{F,y} : y \in B\}$ and $k = 0,\dots,m$.

Then problem (2.174) *has at least one mild solution $y \in \mathrm{PC}(J,E)$ with*

$$\underline{y}(t) \le y(t) \le \overline{y}(t), \quad \forall t \in J. \tag{2.181}$$

Remark 2.26. If $T(t,s)$, $(t,s) \in \gamma$, is completely continuous, then (2.25.5) is automatically satisfied.

Proof. The proof is given in several steps.
Step 1. Consider the problem (2.174) on $J_0 := [0,t_1]$,

$$\begin{gathered} y'(t) - A(t)y(t) \in F(t,y(t)), \quad \text{a.e. } t \in J_0, \\ y(0) = a. \end{gathered} \tag{2.182}$$

We transform this problem into a fixed point problem. Let $\tau : C(J_0,E) \to C(J_0,E)$ be the truncation operator defined by

$$(\tau y)(t) = \begin{cases} \underline{y}(t) & \text{if } y < \underline{y}(t), \\ y(t) & \text{if } \underline{y}(t) \le y \le \overline{y}(t), \\ \overline{y}(t) & \text{if } \overline{y}(t) < y. \end{cases} \tag{2.183}$$

Consider the modified problem

$$\begin{gathered} y'(t) - A(t)y(t) \in F(t,(\tau y)(t)), \quad \text{a.e. } t \in J_0, \\ y(0) = a. \end{gathered} \tag{2.184}$$

Set

$$C_0(J_0,E) := \{y \in C(J_0,E) : y(0) = a\}. \tag{2.185}$$

A solution to (2.184) is a fixed point of the operator $G : C_0(J_0,E) \to \mathcal{P}(C_0(J_0,E))$ defined by

$$G(y) := \left\{ h \in C_0(J_0,E) : h(t) = T(t,0)a + \int_0^t T(t,s)v(s)ds : v \in \widetilde{S}^1_{F,\tau y} \right\}, \tag{2.186}$$

where

$$\begin{aligned}\widetilde{S}^1_{F,\tau y} &= \{v \in S^1_{F,\tau y} : v(t) \geq v_1(t) \text{ a.e. on } A_1,\ v(t) \leq v_2(t) \text{ a.e. on } A_2\},\\ S^1_{F,\tau y} &= \{v \in L^1(J_0, E) : v(t) \in F(t,(\tau y)(t)) \text{ for a.e. } t \in J_0\},\\ A_1 = \{t \in J : y(t) < \underline{y}(t) &\leq \overline{y}(t)\}, \qquad A_2 = \{t \in J : \underline{y}(t) \leq \overline{y}(t) < y(t)\}.\end{aligned} \tag{2.187}$$

Remark 2.27. For each $y \in C(J,E)$, the set $\widetilde{S}^1_{F,\tau y}$ is nonempty. Indeed, by (2.21.1), there exists $v \in S^1_{F,y}$. Set

$$w = v_1\chi_{A_1} + v_2\chi_{A_2} + v\chi_{A_3}, \tag{2.188}$$

where

$$A_3 = \{t \in J : \underline{y}(t) \leq y(t) \leq \overline{y}(t)\}. \tag{2.189}$$

Then by decomposability $w \in \widetilde{S}^1_{F,\tau y}$.

We will show that G satisfies the assumptions of Theorem 1.7.

Claim 1. $G(y)$ is convex for each $y \in C_0(J_0,E)$.

This is obvious since $\widetilde{S}^1_{F,\tau y}$ is convex (because F has convex values).

Claim 2. G sends bounded sets into relatively compact sets in $C_0(J_0,E)$.

This is a consequence of the boundedness of $T(t,s)$, $(t,s) \in \gamma$, and the L^1-Carathédory character of F. As a consequence of Claim 2, together with the Arzelá-Ascoli theorem, we can conclude that $G : C_0(J_0,E) \to \mathcal{P}(C_0(J_0,E))$ is a compact multivalued map, and therefore a condensing map.

Claim 3. G has a closed graph.

Let $y_n \to y_*$, $h_n \in G(y_n)$, and $h_n \to h_*$. We will prove that $h_* \in G(y_*)$.

$h_n \in G(y_n)$ means that there exists $v_n \in \widetilde{S}_{F,\tau y_n}$ such that

$$h_n(t) = T(t,0)a + \int_0^t T(t,s)v_n(s)ds, \quad t \in J_0. \tag{2.190}$$

We must prove that there exists $v_* \in \widetilde{S}^1_{F,\tau y_*}$ such that

$$h_*(t) = T(t,0)a + \int_0^t T(t,s)v_*(s)ds, \quad t \in J_0. \tag{2.191}$$

Consider the linear continuous operator $\Gamma : L^1(J_0,E) \to C(J_0,E)$ defined by

$$(\Gamma v)(t) = \int_0^t T(t,s)v(s)ds. \tag{2.192}$$

We have

$$\left|\left|(h_n - T(t,0)a) - (h_* - T(t,0)a)\right|\right|_\infty \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{2.193}$$

From Lemma 1.28, it follows that $\Gamma \circ \widetilde{S}^1_F$ is a closed graph operator.
Also from the definition of Γ we have that

$$h_n(t) - T(t,0)a \in \Gamma\big(\widetilde{S}^1_{F,\tau y_n}\big). \tag{2.194}$$

Since $y_n \to y_*$, it follows from Lemma 1.28 that

$$h_*(t) = T(t,0)a + \int_0^t T(t,s)v_*(s)ds, \quad t \in J_0, \tag{2.195}$$

for some $v_* \in \widetilde{S}^1_{F,\tau y_*}$.
Claim 4. Now we show that the set

$$\mathcal{M} := \{y \in C_0(J_0,E) : \lambda y \in G(y) \text{ for some } \lambda > 1\} \tag{2.196}$$

is bounded.

Let $y \in \mathcal{M}$. Then $\lambda y \in G(y)$ for some $\lambda > 1$. Thus there exists $v \in \widetilde{S}^1_{F,\tau y}$ such that

$$y(t) = \lambda^{-1}T(t,0)a + \lambda^{-1}\int_0^t T(t,s)v(s)ds, \quad t \in J_0. \tag{2.197}$$

Thus

$$|y(t)| \le M|a| + M\int_0^t |v(s)|ds, \quad t \in J_0. \tag{2.198}$$

From the definition of τ there exists $\varphi \in L^1(J,\mathbb{R}^+)$ such that

$$||F(t,(\tau y)(t))|| = \sup\{|v| : v \in F(t,(\tau y)(t))\} \le \varphi(t), \quad \text{for each } y \in C(J,E). \tag{2.199}$$

Thus we obtain

$$\|y\|_\infty \le M|a| + M\|\varphi\|_{L^1}. \tag{2.200}$$

This shows that $\mathcal{M}$ is bounded. Hence Theorem 1.7 applies and G has a fixed point which is a mild solution to problem (2.174).
Claim 5. We will show that the solution y of (2.182) satisfies

$$\underline{y}(t) \le y(t) \le \overline{y}(t), \quad \forall t \in J_0. \tag{2.201}$$

Let y be a solution to (2.182). We prove that

$$\underline{y}(t) \le y(t), \quad \forall t \in J_0. \tag{2.202}$$

Suppose not. Then there exist $e_1, e_2 \in J_0$, $e_1 < e_2$ such that $\underline{y}(e_1) = y(e_1)$ and

$$\underline{y}(t) > y(t), \quad \forall t \in (e_1, e_2). \tag{2.203}$$

In view of the definition of τ, one has

$$y(t) \in T(t, e_1)y(e_1) + \int_{e_1}^{t} T(t,s)F(s, \underline{y}(s))ds, \quad \text{a.e. on } (e_1, e_2). \tag{2.204}$$

Thus there exists $v(t) \in F(t, \underline{y}(t))$ a.e. on (e_1, e_2), with $v(t) \geq v_1(t)$ a.e. on (e_1, e_2), such that

$$y(t) = T(t, e_1)y(e_1) + \int_{e_1}^{t} T(t,s)v(s)ds, \quad t \in (e_1, e_2). \tag{2.205}$$

Since $\underline{y}$ is a lower-mild solution to (2.174), then

$$\underline{y}(t) - T(t, e_1)\underline{y}(e_1) \leq \int_{e_1}^{t} T(t,s)v_1(s)ds, \quad t \in (e_1, e_2). \tag{2.206}$$

Since $y(e_1) = \underline{y}(e_1)$ and $v(t) \geq v_1(t)$, it follows that

$$\underline{y}(t) \leq y(t), \quad \forall t \in (e_1, e_2), \tag{2.207}$$

which is a contradiction since $y(t) < \underline{y}(t)$ for all $t \in (e_1, e_2)$. Consequently

$$\underline{y}(t) \leq y(t), \quad \forall t \in J_0. \tag{2.208}$$

Analogously, we can prove that

$$y(t) \leq \overline{y}(t), \quad \forall t \in J_0. \tag{2.209}$$

This shows that the problem (2.182) has a mild solution in the interval $[\underline{y}, \overline{y}]$. Since $\tau(y) = y$ for all $y \in [\underline{y}, \overline{y}]$, then y is a mild solution to (2.174). Denote this solution by y_0.

Step 2. Consider now the following problem on $J_1 := [t_1, t_2]$:

$$\begin{gathered} y'(t) - A(t)y(t) \in F(t, y(t)), \quad \text{a.e. } t \in J_1, \\ y(t_1^+) = I_1(y_0(t_1^-)), \end{gathered} \tag{2.210}$$

and the modified problem

$$\begin{gathered} y'(t) \in F(t, (\tau y)(t)), \quad \text{a.e. } t \in J_1, \\ y(t_1^+) = I_1(y_0(t_1^-)). \end{gathered} \tag{2.211}$$

Since $y_0(t_1^-) \in [\underline{y}(t_1^-), \overline{y}(t_1^-)]$, then (2.25.3) implies that

$$\underline{y}(t_1^+) \leq I_1(y_0(t_1^-)) \leq \overline{y}(t_1^+); \tag{2.212}$$

that is,

$$\underline{y}(t_1^+) \leq y(t_1^+) \leq \overline{y}(t_1^+). \tag{2.213}$$

Using the same reasoning as that used for problem (2.182), we can conclude the existence of at least one mild solution y to (2.211).

We now show that this solution satisfies

$$\underline{y}(t) \leq y(t) \leq \overline{y}(t), \quad \forall t \in J_1. \tag{2.214}$$

We first show that

$$\underline{y}(t) \leq y(t), \quad \text{on } J_1. \tag{2.215}$$

Assume this is false. Then since $y(t_1^+) \geq \underline{y}(t_1^+)$, there exist $e_3, e_4 \in J_1$ with $e_3 < e_4$ such that $y(e_3) = \underline{y}(e_3)$ and $y(t) < \underline{y}(t)$ on (e_3, e_4).

Consequently,

$$y(t) - T(e_3, t)y(e_3) = \int_{e_3}^{t} T(t,s)v(s)ds, \quad t \in (e_3, e_4), \tag{2.216}$$

where $v(t) \in F(t, \underline{y}(t))$ a.e. on J_1 with $v(t) \geq v_1(t)$ a.e. on (e_3, e_4).

Since $\underline{y}$ is a lower-mild solution to (2.174), then

$$\underline{y}(t) - T(e_3, t)\underline{y}(e_3) \leq \int_{e_3}^{t} v_1(s)ds, \quad t \in (e_3, e_4). \tag{2.217}$$

It follows that

$$\underline{y}(t) \leq y(t), \quad \text{on } (e_3, e_4), \tag{2.218}$$

which is a contradiction. Similarly we can show that $y(t) \leq \overline{y}(t)$ on J_1. Hence y is a solution of (2.174) on J_1. Denote this by y_1.

Step 3. Continue this process and construct solutions $y_k \in C(J_k, E)$, $k = 2, \dots, m$, to

$$\begin{aligned} y'(t) - A(t)y(t) &\in F(t, (\tau y)(t)), \quad \text{a.e. } t \in J_k, \\ y(t_k^+) &= I_k(y(t_k^-)), \end{aligned} \tag{2.219}$$

with $\underline{y}(t) \le y_k(t) \le \overline{y}(t)$, $t \in J_k := [t_k, t_{k+1}]$. Then

$$y(t) = \begin{cases} y_0(t), & t \in [0, t_1], \\ y_1(t), & t \in (t_1, t_2], \\ \vdots & \\ y_{m-1}(t), & t \in (t_{m-1}, t_m], \\ y_m(t), & t \in (t_m, b], \end{cases} \tag{2.220}$$

is a mild solution of (2.174). □

Using the same reasoning as that used in the proof of Theorem 2.25, we can obtain the following result.

Theorem 2.28. *Assume that $F : J \times E \to \mathcal{P}_{b,cp,cv}(E)$, and in addition to (2.21.1), (2.25.1), and (2.25.5), suppose that the following hypotheses hold.*

(2.28.1) *There exist functions $\{r_k\}_{k=0}^{k=m}$ and $\{s_k\}_{k=0}^{k=m}$ with r_k, $s_k \in C(J_k, E)$, $s_0(0) \le a \le r_0(0)$, and $s_k(t) \le r_k(t)$ for $t \in J_k$, $k = 0, \dots, m$, and*

$$\begin{aligned} s_{k+1}(t_{k+1}^+) &\le \min_{[s_k(t_{k+1}^-), r_k(t_{k+1}^-)]} I_{k+1}(y) \le \max_{[s_k(t_{k+1}^-), r_k(t_{k+1}^-)]} I_{k+1}(y) \\ &\le r_{k+1}(t_{k+1}^+), \quad k = 0, \dots, m-1. \end{aligned} \tag{2.221}$$

(2.28.2) *There exist $v_{1,k}, v_{2,k} \in L^1(J_k, E)$, with $v_{1,k}(t) \in F(t, s_k(t))$, $v_{2,k}(t) \in F(t, r_k(t))$ a.e. on J_k such that for each $k = 0, \dots, m$,*

$$\begin{aligned} \int_{z_k}^t T(t,s)v_{1,k}(s)ds &\ge s_k(t) - s_k(z_k), \\ \int_{z_k}^t T(t,s)v_{2,k}(s)ds &\ge r_k(t) - r_k(z_k), \quad \text{with } t, z_k \in J_k. \end{aligned} \tag{2.222}$$

Then the problem (2.174) *has at least one mild solution.*

2.4. Ordinary damped differential inclusions

Again, we let $J = [0, b]$, and $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$ are fixed points of impulse. In this section, we will be concerned with the existence of mild solutions for first- and second-order impulsive semilinear damped differential inclusions in a real Banach space. First, we consider first-order impulsive semilinear differential inclusions of the form

$$\begin{gathered} y'(t) - Ay(t) \in By + F(t, y(t)), \quad \text{a.e. } t \in J,\ t \neq t_k,\ k = 1, \dots, m, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \\ y(0) = y_0, \end{gathered} \tag{2.223}$$

where $F : J \times E \to \mathcal{P}(E)$ is a multivalued map ($\mathcal{P}(E)$ is the family of all nonempty subsets of E), A is the infinitesimal generator of a semigroup $T(t)$, $t \geq 0$, B is a bounded linear operator from E into E, $y_0 \in E$, $I_k \in C(E, E)$ $(k = 1, \dots, m)$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h \to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h \to 0^+} y(t_k - h)$, and E is a real separable Banach space with norm $|\cdot|$.

Later, we study second-order impulsive semilinear evolution inclusions of the form

$$\begin{gathered} y''(t) - Ay \in By'(t) + F(t, y(t)), \quad \text{a.e. } t \in J,\ t \neq t_k,\ k = 1, \dots, m, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \\ \Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \\ y(0) = y_0, \qquad y'(0) = y_1, \end{gathered} \tag{2.224}$$

where F, I_k, B, and y_0 are as in problem (2.223), A is the infinitesimal generator of a family of cosine operators $\{C(t) : t \geq 0\}$, $\overline{I}_k \in C(E, E)$, and $y_1 \in E$.

We study the existence of solutions for problem (2.223) when the right-hand side has convex values. We assume that $F : J \times E \to P(E)$ is a compact and convex valued multivalued map.

Let $\mathrm{PC}(J, E)$ be as given in Section 2.2, and let us start by defining what we mean by a mild solution of problem (2.223).

Definition 2.29. A function $y \in \mathrm{PC}(J, E)$ is said to be a mild solution of (2.223) if there exists a function $v \in L^1(J, E)$ such that $v(t) \in F(t, y(t))$ a.e. on J and

$$\begin{aligned} y(t) = {} & T(t)y_0 + \int_0^t T(t-s)B(y(s))\,ds + \int_0^t T(t-s)v(s)\,ds \\ & + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)). \end{aligned} \tag{2.225}$$

Theorem 2.30. *Assume that hypotheses (2.2.1), (2.21.1) hold. In addition we suppose that the following conditions are satisfied.*

(2.30.1) *$A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t)$, $t \geq 0$, which is compact for $t > 0$, and there exists a constant M such that $\|T(t)\|_{B(E)} \leq M$ for each $t \geq 0$.*

(2.30.2) *There exist a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ and $p \in L^1(J, \mathbb{R}_+)$ such that*

$$||F(t, y)|| \leq p(t)\psi(|y|), \quad \textit{for a.e. } t \in J \textit{ and each } y \in E, \tag{2.226}$$

with

$$\int_0^b m(s)ds < \int_c^\infty \frac{du}{u+\psi(u)}, \tag{2.227}$$

where

$$m(t) = \max\{M\|B\|_{B(E)}, Mp(t)\}, \qquad c = M\left[\,|y_0| + \sum_{k=1}^m c_k\right]. \tag{2.228}$$

Then the IVP (2.223) *has at least one mild solution.*

Proof. Transform the problem (2.223) into a fixed point problem. Consider the multivalued operator $N : \mathrm{PC}(J,E) \to \mathcal{P}(\mathrm{PC}(J,E))$ defined by

$$\begin{aligned} N(y) = \bigg\{ h \in \mathrm{PC}(J,E) : h(t) &= T(t)y_0 + \int_0^t T(t-s)B(y(s))ds \\ &+ \int_0^t T(t-s)g(s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)),\ g \in S_{F(y)} \bigg\}. \end{aligned} \tag{2.229}$$

We will show that N satisfies the assumptions of Theorem 1.9. The proof will be given in several steps. Let

$$K := \{y \in \mathrm{PC}(J,E) : \|y\|_{\mathrm{PC}} \le a(t),\ t \in J\}, \tag{2.230}$$

where

$$\begin{aligned} a(t) &= I^{-1}\left(\int_0^t m(s)ds\right), \\ I(z) &= \int_c^z \frac{du}{u+\psi(u)}. \end{aligned} \tag{2.231}$$

It is clear that K is a closed bounded convex set. Let $k^* = \sup\{\|y\|_{\mathrm{PC}} : y \in K\}$.
Step 1. $N(K) \subset K$.

Indeed, let $y \in K$ and fix $t \in J$. We must show that $N(y) \in K$. There exists $g \in S_{F(y)}$ such that, for each $t \in J$,

$$\begin{aligned} h(t) = T(t)y_0 &+ \int_0^t T(t-s)B(y(s))ds + \int_0^t T(t-s)g(s)ds \\ &+ \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)). \end{aligned} \tag{2.232}$$

Thus

$$
\begin{aligned}
|h(t)| &\le M|y_0| + M\sum_{k=1}^{m} c_k + \int_0^t m(s)(|y(s)| + \psi(|y(s)|))ds \\
&\le M|y_0| + M\sum_{k=1}^{m} c_k + \int_0^t m(s)(a(s) + \psi(a(s)))ds \\
&= M|y_0| + M\sum_{k=1}^{m} c_k + \int_0^t a'(s)ds \\
&= a(t)
\end{aligned}
\tag{2.233}
$$

since

$$
\int_c^{a(s)} \frac{du}{u+\psi(u)} = \int_0^s m(\tau)d\tau. \tag{2.234}
$$

Thus, $N(y) \in K$. So, $N : K \to K$.
Step 2. $N(K)$ is relatively compact.

Since K is bounded and $N(K) \subset K$, it is clear that $N(K)$ is bounded. $N(K)$ is equicontinuous. Indeed, let $\tau_1, \tau_2 \in J'$, $\tau_1 < \tau_2$, and $\epsilon > 0$ with $0 < \epsilon \le \tau_1 < \tau_2$. Let $y \in K$ and $h \in N(y)$. Then there exists $g \in S_{F(y)}$ such that for each $t \in J$ we have

$$
\begin{aligned}
|h(\tau_2) - h(\tau_1)| \le\ & |T(\tau_2)y_0 - T(\tau_1)y_0| \\
&+ \int_0^{\tau_1-\epsilon} ||T(\tau_2 - s) - T(\tau_1 - s)||_{B(E)}|By(s)|ds \\
&+ \int_{\tau_1}^{\tau_1-\epsilon} ||T(\tau_2 - s) - T(\tau_1 - s)||_{B(E)}|By(s)|ds \\
&+ \int_{\tau_1}^{\tau_2} ||T(\tau_2 - s)||_{B(E)}|By(s)|ds \\
&+ \int_0^{\tau_1-\epsilon} ||T(\tau_2 - s) - T(\tau_1 - s)||_{B(E)}|g(s)|ds \\
&+ \int_{\tau_1}^{\tau_1-\epsilon} ||T(\tau_2 - s) - T(\tau_1 - s)||_{B(E)}|g(s)|ds \\
&+ \int_{\tau_1}^{\tau_2} ||T(\tau_2 - s)||_{B(E)}|g(s)|ds + Mc_k(\tau_2 - \tau_1) \\
&+ \sum_{0<t_k<\tau_1} c_k||T(\tau_1 - t_k) - T(\tau_2 - t_k)||_{B(E)}.
\end{aligned}
\tag{2.235}
$$

The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$, and for ϵ sufficiently small, since $T(t)$ is a strongly continuous operator and the compactness of $T(t)$, for $t > 0$, implies the continuity in the uniform operator topology. This proves the equicontinuity for the case where $t \neq t_i$, $i = 1, \dots, m$. It remains to examine the equicontinuity at $t = t_i$. The proof is similar to that given in Theorem 2.2. As a consequence of the Arzelá-Ascoli theorem it suffices to show that the multivalued N maps K into a precompact set in E. Let $0 < t \leq b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in K$, we define

$$\begin{aligned} h_\epsilon(t) = T(t)y_0 + T(\epsilon)\int_0^{t-\epsilon} T(t-s-\epsilon)(By(s))ds \\ + T(\epsilon)\int_0^{t-\epsilon} T(t-s-\epsilon)g(s)ds \\ + T(\epsilon)\sum_{0<t_k<t-\epsilon} T(t-t_k-\epsilon)I_k(y(t_k^-)), \end{aligned} \tag{2.236}$$

where $g \in S_{F(y)}$. Since $T(t)$ is a compact operator, the set $H_\epsilon(t) = \{h_\epsilon(t) : h_\epsilon \in N(y)\}$ is precompact in E for every ϵ, $0 < \epsilon < t$. Moreover, for every $h \in N(y)$, we have

$$\begin{aligned} |h_\epsilon(t) - h(t)| \leq \|B\|_{B(E)}k^* \int_{t-\epsilon}^t \|T(t-s)\|_{B(E)}ds \\ + \int_{t-\epsilon}^t \|T(t-s)\|_{B(E)}|a(s)|ds \\ + \sum_{t-\epsilon\leq t_k<t} c_k\|T(t-t_k)\|_{B(E)}. \end{aligned} \tag{2.237}$$

Therefore there are precompact sets arbitrarily close to the set $\{h(t) : h \in N(y)\}$. Hence the set $\{h(t) : h \in N(y)\}$ is precompact in E.

Step 3. N has a closed graph.

Let $y_n \to y_*$, $h_n \in N(y_n)$, and $h_n \to h_*$. We will prove that $h_* \in N(y_*)$.

$h_n \in N(y_n)$ means that there exists $g_n \in S_{F(y_n)}$ such that, for each $t \in J$,

$$\begin{aligned} h_n(t) = T(t)y_0 + \int^t T(t-s)By_n(s)ds + \int_0^t T(t-s)g_n(s)ds \\ + \sum_{0<t_k<t} T(t-t_k)I_k(y_n(t_k^-)). \end{aligned} \tag{2.238}$$

We must prove that there exists $g_* \in S_{F,y_*}$ such that, for each $t \in J$,

$$\begin{aligned} h_*(t) = T(t)y_0 + \int_0^t T(t-s)By_*(s)ds + \int_0^t T(t-s)g_*(s)ds \\ + \sum_{0<t_k<t} T(t-t_k)I_k(y_*(t_k^-)). \end{aligned} \tag{2.239}$$

Clearly since I_k, $k = 1, \dots, m$, and B are continuous, we have that

$$\begin{aligned}&\Bigg\|\bigg(h_n - T(t)y_0 - \sum_{0<t_k<t} T(t-t_k)I_k(y_n(t_k^-)) - \int_0^t T(t-s)By_n(s)ds\bigg)\\&\quad - \bigg(h_* - T(t)y_0 - \sum_{0<t_k<t} T(t-t_k)I_k(y_*(t_k^-))\\&\qquad - \int_0^t T(t-s)By_*(s)ds\bigg)\Bigg\|_\infty \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{2.240}$$

Consider the linear continuous operator

$$\begin{aligned}\Gamma : L^1(J,E) &\longrightarrow C(J,E),\\ g \longmapsto \Gamma(g)(t) &= \int_0^t T(t-s)g(s)ds.\end{aligned} \tag{2.241}$$

From Lemma 1.28, it follows that $\Gamma \circ S_F$ is a closed graph operator. Moreover, we have that

$$h_n(t) - T(t)y_0 - \sum_{0<t_k<t} T(t-t_k)I_k(y_n(t_k^-)) - \int_0^t T(t-s)By_n(s)ds \in \Gamma(S_{F(y_n)}). \tag{2.242}$$

Since $y_n \to y_*$, it follows from Lemma 1.28 that

$$\begin{aligned}&h_*(t) - T(t)y_0 - \sum_{0<t_k<t} T(t-t_k)I_k(y_*(t_k^-)) - \int_0^t T(t-s)By_*(s)ds\\&\quad = \int_0^t T(t-s)g_*(s)ds\end{aligned} \tag{2.243}$$

for some $g_* \in S_{F(y_*)}$.

As a consequence of Theorem 1.9, we deduce that N has a fixed point which is a mild solution of (2.223). □

We present now a result for the problem (2.223) by using Covitz and Nadler's fixed point theorem.

Theorem 2.31. *Suppose that the following hypotheses hold.*

(2.31.1) $F : J \times E \to P_{\mathrm{cp,cv}}(E)$; $(t,\cdot) \mapsto F(t,y)$ *is measurable for each* $y \in E$.

(2.31.2) *There exists constants* c_k' *such that*

$$|I_k(y) - I_k(\overline{y})| \le c_k'|y - \overline{y}|, \quad \textit{for each } k = 1,\dots,m,\ \forall\, y,\overline{y} \in E. \tag{2.244}$$

(2.31.3) *There exists a function* $l \in L^1(J, \mathbb{R}^+)$ *such that*

$$\begin{gathered} H_d\big(F(t,y),F(t,\overline{y})\big) \leq l(t)|y-\overline{y}|, \quad \textit{for a.e. } t \in J,\ \forall y,\overline{y} \in E, \\ d\big(0,F(t,0)\big) \leq l(t), \quad \textit{for a.e. } t \in J. \end{gathered} \tag{2.245}$$

If

$$\frac{2}{\tau} + M\sum_{k=1}^{m} c_k < 1, \tag{2.246}$$

where $\tau \in \mathbb{R}^+$, *then the IVP* (2.223) *has at least one mild solution.*

Remark 2.32. For each $y \in \mathrm{PC}(J,E)$, the set $S_{F(y)}$ is nonempty since by (2.31.1) F has a measurable selection (see [119, Theorem III.6]).

Proof of Theorem 2.31. Transform the problem (2.223) into a fixed point problem. Let the multivalued operator $N : \mathrm{PC}(J,E) \to \mathcal{P}(\mathrm{PC}(J,E))$ be defined as in Theorem 2.30. We will show that N satisfies the assumptions of Theorem 1.11. The proof will be given in two steps.
Step 1. $N(y) \in P_{\mathrm{cl}}(\mathrm{PC}(J,E))$ for each $y \in \mathrm{PC}(J,E)$.

Indeed, let $(y_n)_{n\geq 0} \in N(y)$ such that $y_n \to \widetilde{y}$ in $\mathrm{PC}(J,E)$. Then $\widetilde{y} \in \mathrm{PC}(J,E)$ and there exists $g_n \in S_{F(y)}$ such that, for each $t \in J$,

$$\begin{aligned} y_n(t) = T(t)y_0 &+ \int_0^t T(t-s)By(s)ds + \int_0^t T(t-s)g_n(s)ds \\ &+ \sum_{0<t_k<t} T(t-t_k)I_k\big(y(t_k^-)\big). \end{aligned} \tag{2.247}$$

Using the fact that F has compact values and from (2.31.3), we may pass to a subsequence if necessary to get that g_n converges to g in $L^1(J,E)$ and hence $g \in S_{F(y)}$. Then, for each $t \in J$,

$$\begin{aligned} y_n(t) \longrightarrow \widetilde{y}(t) = T(t)y_0 &+ \int_0^t T(t-s)By(s)ds \\ &+ \int_0^t T(t-s)g(s)ds + \sum_{0<t_k<t} T(t-t_k)I_k\big(y(t_k^-)\big). \end{aligned} \tag{2.248}$$

So $\widetilde{y} \in N(y)$.
Step 2. There exists $\gamma < 1$ such that

$$H_d\big(N(y),N(\overline{y})\big) \leq \gamma\|y-\overline{y}\|_{\mathrm{PC}}, \quad \text{for each } y,\overline{y} \in \mathrm{PC}(J,E). \tag{2.249}$$

Let $y, \overline{y} \in \mathrm{PC}(J, E)$ and $h \in N(y)$. Then there exists $g(t) \in F(t, y(t))$ such that, for each $t \in J$,

$$\begin{aligned} h(t) = {} & T(t)y_0 + \int_0^t T(t-s)By(s)ds + \int_0^t T(t-s)g(s)ds \\ & + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)). \end{aligned} \tag{2.250}$$

From (2.31.3) it follows that

$$H_d(F(t, y(t)), F(t, \overline{y}(t))) \leq l(t)|y(t) - \overline{y}(t)|. \tag{2.251}$$

Hence there is $w \in F(t, \overline{y}(t))$ such that

$$|g(t) - w| \leq l(t)|y(t) - \overline{y}(t)|, \quad t \in J. \tag{2.252}$$

Consider $U : J \to P(E)$ given by

$$U(t) = \{w \in E : |g(t) - w| \leq l(t)|y(t) - \overline{y}(t)|\}. \tag{2.253}$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \overline{y}(t))$ is measurable (see [119, Proposition III.4]), there exists a function $\overline{g}(t)$, which is a measurable selection for V. So, $\overline{g}(t) \in F(t, \overline{y}(t))$ and

$$|g(t) - \overline{g}(t)| \leq l(t)|y(t) - \overline{y}(t)|, \quad \text{for each } t \in J. \tag{2.254}$$

Let us define, for each $t \in J$,

$$\begin{aligned} \overline{h}(t) = {} & T(t)y_0 + \int_0^t T(t-s)By(s)ds + \int_0^t T(t-s)\overline{g}(s)ds \\ & + \sum_{0<t_k<t} T(t-t_k)I_k(\overline{y}(t_k^-)). \end{aligned} \tag{2.255}$$

We define on $\mathrm{PC}(J, E)$ an equivalent norm to $\|\cdot\|_{\mathrm{PC}}$ by

$$\|y\|_1 = \sup_{t \in J}\{e^{-\tau L(t)}|y(t)|\}, \quad \forall y \in \mathrm{PC}(J, E), \tag{2.256}$$

where $L(t) = \int_0^t \widehat{M}(s)ds$, $\tau \in \mathbb{R}^+$, and $\widehat{M}(t) = \max\{M\|B\|_{B(E)}, Ml(t)\}$.

Then

$$\begin{aligned} |h(t)-\overline{h}(t)| &\le \int_0^t \widehat{M}(s)|y(s)-\overline{y}(s)|ds + \int_0^t \widehat{M}(s)|y(s)-\overline{y}(s)|ds \\ &\quad + M\sum_{k=1}^m c_k'|y(s)-\overline{y}(s)| \\ &\le 2\int_0^t \widehat{M}(s)e^{-\tau L(s)}e^{\tau L(s)}|y(s)-\overline{y}(s)|ds \\ &\quad + M\sum_{k=1}^m c_k' e^{-\tau L(s)}e^{\tau L(s)}|y(s)-\overline{y}(s)| \\ &\le 2\int_0^t \left(e^{\tau L(s)}\right)' ds\|y-\overline{y}\|_1 + M\sum_{k=1}^m c_k' e^{\tau L(s)}\|y-\overline{y}\|_1 \\ &\le \frac{2}{\tau}\|y-\overline{y}\|_1 e^{\tau L(t)} + M\sum_{k=1}^m c_k'\|y-\overline{y}\|_1 e^{\tau L(t)}. \end{aligned} \tag{2.257}$$

Then

$$\|h-\overline{h}\|_1 \le \left(\frac{2}{\tau}+M\sum_{k=1}^m c_k'\right)\|y-\overline{y}\|_1. \tag{2.258}$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H_d(N(y),N(\overline{y})) \le \left(\frac{2}{\tau}+M\sum_{k=1}^m c_k'\right)\|y-\overline{y}\|_1. \tag{2.259}$$

So, N is a contraction and thus, by Theorem 1.11, N has a fixed point y, which is a mild solution to (2.223). □

Now we study the problem (2.224) when the right-hand side has convex values. We give first the definition of mild solution of the problem (2.224).

Definition 2.33. A function $y \in PC^1(J,E)$ is said to be a mild solution of (2.224) if there exists $v \in L^1(J,\mathbb{R}^n)$ such that $v(t) \in F(t,y(t))$ a.e. on J, $y(0)=y_0$, $y'(0)=y_1$, and

$$\begin{aligned} y(t) &= (C(t)-S(t)B)y_0 + S(t)y_1 + \int_0^t C(t-s)By(s)ds + \int_0^t S(t-s)v(s)ds \\ &\quad + \sum_{0<t_k<t}\left[C(t-t_k)I_k(y(t_k^-)) + S(t-t_k)\overline{I}_k(y(t_k^-))\right]. \end{aligned} \tag{2.260}$$

Theorem 2.34. *Assume (2.2.1), (2.21.1), and the following conditions are satisfied:*

(2.34.1) *there exist constants $\overline{d}_k$ such that $|\overline{I}_k(y)| \leq d_k$ for each $y \in E$, $k = 1, \dots, m$;*

(2.34.2) *$A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in J\}$ which is compact for $t > 0$, and there exists a constant $M_1 > 0$ such that $\|C(t)\|_{B(E)} < M_1$ for all $t \in \mathbb{R}$;*

(2.34.3) *there exist a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ and $p \in L^1(J, \mathbb{R}_+)$ such that*

$$\|F(t, y)\| \leq p(t)\psi(|y|), \quad \text{for a.e. } t \in J \text{ and each } y \in E \tag{2.261}$$

with

$$\int_0^b \widehat{m}(s)ds < \int_{\widetilde{c}}^{\infty} \frac{d\tau}{\tau + \psi(\tau)}, \tag{2.262}$$

where

$$\begin{gathered} \widetilde{c} = M_1\left(1 + b\|B\|_{B(E)}\right)|y_0| + bM_1|y_1| + M_1\sum_{k=1}^{m}\left[c_k + bd_k\right], \\ \widehat{m}(t) = \max\left(M_1\|B\|, bM_1 p(t)\right). \end{gathered} \tag{2.263}$$

Then the IVP (2.224) *has at least one mild solution.*

Proof. Transform the problem (2.224) into a fixed point problem. Consider the multivalued operator $\overline{N} : \mathrm{PC}^1(J, E) \to \mathcal{P}(\mathrm{PC}^1(J, E))$ defined by

$$\begin{aligned} \overline{N}(y) = \Bigg\{ h \in \mathrm{PC}^1(J, E) : h(t) = {} & \left(C(t) - S(t)B\right)y_0 + S(t)y_1 \\ & + \int_0^t C(t-s)By(s)ds + \int_0^t S(t-s)v(s)ds \\ & + \sum_{0<t_k<t}\left[C(t-t_k)I_k\left(y(t_k^-)\right)\right. \\ & \left. + S(t-t_k)\overline{I}_k\left(y(t_k^-)\right)\right],\ v \in S_{F(y)} \Bigg\}. \end{aligned} \tag{2.264}$$

As in Theorem 2.30, we will show that $\overline{N}$ satisfies the assumptions of Theorem 1.9. Let

$$K_1 := \left\{y \in \mathrm{PC}^1(J, E) : \|y\|_{\mathrm{PC}} \leq b(t),\ t \in J\right\}, \tag{2.265}$$

where

$$b(t) = I^{-1}\left(\int_0^t \widehat{m}(s)ds\right), \qquad I(z) = \int_{\widetilde{c}}^{z} \frac{du}{u+\psi(u)}. \tag{2.266}$$

It is clear that K is a closed bounded convex set.

Step 1. $\overline{N}(K_1) \subset K_1$.

Indeed, let $y \in K_1$ and fix $t \in J$. We must show that $\overline{N}(y) \subset K_1$. Let $h \in \overline{N}(y)$. Thus there exists $v \in S_{F(y)}$ such that, for each $t \in J$,

$$\begin{aligned} h(t) = {} & (C(t) - S(t)B)y_0 + S(t)y_1 + \int_0^t C(t-s)By(s)ds + \int_0^t S(t-s)v(s)ds \\ & + \sum_{0<t_k<t} \left[C(t-t_k)I_k(y(t_k^-)) + S(t-t_k)\overline{I}_k(y(t_k^-))\right]. \end{aligned} \tag{2.267}$$

This implies that for each $t \in J$ we have

$$\begin{aligned} |h(t)| \le {} & M_1(1+b\|B\|_{B(E)})|y_0| + bM_1|y_1| + M_1\int_0^t |By(s)|ds \\ & + \int_0^t M_1 bp(s)\psi(|y(s)|)ds + M_1\sum_{k=1}^m [c_k + bd_k] \\ \le {} & M_1(1+b\|B\|_{B(E)})|y_0| + bM_1|y_1| + M_1\|B\|_{B(E)}\int_0^t |y(s)|ds \\ & + M_1 b\int_0^t p(s)\psi(|y(s)|)ds + M_1\sum_{k=1}^m [c_k + bd_k] \\ \le {} & M_1(1+b\|B\|_{B(E)})|y_0| + bM_1|y_1| + \int_0^t \widehat{m}(s)(b(s)+\psi(b(s)))ds \\ & + M_1\sum_{k=1}^m [c_k + bd_k] \\ = {} & M_1(1+b\|B\|_{B(E)})|y_0| + bM_1|y_1| + M_1\sum_{k=1}^m [c_k + bd_k] + \int_0^t b'(s)ds \\ = {} & b(t) \end{aligned} \tag{2.268}$$

since

$$\int_{\widetilde{c}}^{b(s)} \frac{du}{u+\psi(u)} = \int_0^s \widehat{m}(\tau)d\tau. \tag{2.269}$$

Thus, $\overline{N}(y) \subset K_1$. So, $\overline{N} : K_1 \to K_1$.

As in Theorem 2.30, we can show that $\overline{N}(K_1)$ is relatively compact and hence by Theorem 1.9 the operator $\overline{N}$ has at least one fixed point which is a mild solution to problem (2.224). □

Theorem 2.35. *Suppose that hypotheses (2.31.1)–(2.31.3) and (2.34.2) hold. In addition, suppose there exist constants $\overline{d}'_k$ such that*

$$\left|\overline{I}_k(y) - \overline{I}_k(\overline{y})\right| \le d'_k|y - \overline{y}|, \quad \text{for each } k = 1,\dots,m, \tag{2.270}$$

and for all $y, \overline{y} \in E$. If

$$\|B\|_{B(E)} + \frac{2}{\tau} + M_1 \sum_{k=1}^{m} \left[c'_k + bd'_k\right] < 1, \tag{2.271}$$

then the IVP (2.224) *has at least one mild solution.*

Proof. Transform the problem (2.224) into a fixed point problem. Consider the multivalued map $\overline{N} : PC^1(J,E) \to \mathcal{P}(PC^1(J,E))$ where $\overline{N}$ is defined as in Theorem 2.34. As in the proof of Theorem 2.31, we can show that $\overline{N}$ has closed values. Here we repeat the proof that $\overline{N}$ is a contraction; that is, there exists $\gamma < 1$ such that

$$H_d\left(\overline{N}(y), \overline{N}(\overline{y})\right) \le \gamma\|y - \overline{y}\|_{PC^1}, \quad \text{for each } y, \overline{y} \in PC^1(J,E). \tag{2.272}$$

Let $y, \overline{y} \in PC^1(J,E)$ and $h \in \overline{N}(y)$. Then there exists $g(t) \in F(t, y(t))$ such that, for each $t \in J$,

$$\begin{aligned} h(t) = {} & \left(C(t) - S(t)B\right)y_0 + S(t)y_1 + \int_0^t C(t-s)By(s)ds + \int_0^t S(t-s)g(s)ds \\ & + \sum_{0<t_k<t} \left[C(t-t_k)I_k(y(t_k^-)) + S(t-t_k)\overline{I}_k(y(t_k^-))\right]. \end{aligned} \tag{2.273}$$

From (2.31.3) it follows that

$$H_d\left(F(t,y(t)), F(t,\overline{y}(t))\right) \le l(t)\left|y(t) - \overline{y}(t)\right|. \tag{2.274}$$

Hence there is $w \in F(t, \overline{y}(t))$ such that

$$\left|g(t) - w\right| \le l(t)\left|y(t) - \overline{y}(t)\right|, \quad t \in J. \tag{2.275}$$

Consider $U : J \to P(E)$, given by

$$U(t) = \left\{w \in E : \left|g(t) - w\right| \le l(t)\left|y(t) - \overline{y}(t)\right|\right\}. \tag{2.276}$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \overline{y}(t))$ is measurable (see [119, Proposition III.4]), there exists a function $\overline{g}(t)$, which is a measurable selection for V. So, $\overline{g}(t) \in F(t, \overline{y}(t))$ and

$$|g(t) - \overline{g}(t)| \leq l(t)|y(t) - \overline{y}(t)|, \quad \text{for each } t \in J. \tag{2.277}$$

Let us define, for each $t \in J$,

$$\begin{aligned} \overline{h}(t) = {} & (C(t) - S(t)B)y_0 + S(t)y_1 + \int_0^t C(t-s)B\overline{y}(s)ds + \int_0^t S(t-s)\overline{g}(s)ds \\ & + \sum_{0<t_k<t} \left[C(t-t_k)I_k(\overline{y}(t_k^-)) + S(t-t_k)\overline{I}_k(y(t_k^-))\right]. \end{aligned} \tag{2.278}$$

We define on $\mathrm{PC}^1(J, E)$ an equivalent norm by

$$\|y\|_2 = \sup_{t\in J} e^{-\tau\widetilde{L}(t)}|y(t)|, \quad \forall y \in \mathrm{PC}^1(J, E), \tag{2.279}$$

where $\widetilde{L}(t) = \int_0^t \widetilde{M}(s)ds$, $\tau \in \mathbb{R}^+$, and $\widetilde{M}(t) = \max\{bM_1\|B\|_{B(E)}\|B\|_{B(E)}, M_1bl(t)\}$. Then we have

$$\begin{aligned} |h(t) - \overline{h}(t)| \leq {} & \int_0^t M_1|By(s) - B\overline{y}(s)|ds + \int_0^t M_1b|g(s) - \overline{g}(s)|ds \\ & + M_1\sum_{k=1}^m |I_k(y(t_k^-)) - I_k(\overline{y}(t_k^-))| \\ & + M_1b\sum_{k=1}^m |\overline{I}_k(y(t_k^-)) - \overline{I}_k(\overline{y}(t_k^-))| \\ \leq {} & \int_0^t M_1\|B\|_{B(E)}|y(s) - \overline{y}(s)|ds + \int_0^t M_1bl(s)|y(s) - \overline{y}(s)|ds \\ & + M_1\sum_{k=1}^m c_k'|y(t_k) - \overline{y}(t_k)| + M_1b\sum_{k=1}^m d_k'|y(t_k) - \overline{y}(t_k)| \\ \leq {} & 2\int_0^t \widetilde{M}(s)e^{\tau\widetilde{L}(t)}e^{-\tau\widetilde{L}(t)}|y(s) - \overline{y}(s)|ds \\ & + M_1e^{\tau\widetilde{L}(t)}\sum_{k=1}^m [c_k' + bd_k']\|y - \overline{y}\|_2 \\ \leq {} & \frac{2}{\tau}e^{\tau\widetilde{L}(t)}\|y - \overline{y}\|_2 + M_1e^{\tau\widetilde{L}(t)}\sum_{k=1}^m [c_k' + bd_k']\|y - \overline{y}\|_2. \end{aligned} \tag{2.280}$$

Similarly we have

$$|h'(t) - \overline{h}'(t)| \leq \left(\|B\|_{B(E)} + \frac{2}{\tau} + M_1 \sum_{k=1}^{m} [c'_k + bd'_k] \right) \|y - \overline{y}\|_2. \tag{2.281}$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H_d(\overline{N}(y), \overline{N}(\overline{y})) \leq \left(\|B\|_{B(E)} + \frac{2}{\tau} + M_1 \sum_{k=1}^{m} [c'_k + bd'_k] \right) \|y - \overline{y}\|_2. \tag{2.282}$$

So, $\overline{N}$ is a contraction and thus, by Theorem 1.11, $\overline{N}$ has a fixed point y, which is a mild solution to (2.224). □

2.5. Notes and remarks

Chapter 2 is devoted to the existence of solutions of ordinary differential inclusions and mild solutions for first- and second-order impulsive semilinear evolution equations and inclusions. In recent years a mixture of classical fixed points theorems, semigroup theory, evolution families, and cosine families has been employed to study these problems. Section 2.2 is based on the work of Benchohra et al. [87]. Section 2.3 uses the method of upper- and lower-solutions combined with a fixed point theorem for condensing maps to investigate some of these problems; see Benchohra and Ntouyas [47, 67, 80, 86]. The techniques in this section have been adapted from [140] where the nonimpulsive case was discussed. In Section 2.4, some results of Section 2.2 are extended to first- and second-order semilinear damped differential inclusions, and are based on the results that were obtained by Benchohra et al. [69]. The second part of Section 2.4 relies on a Covitz and Nadler fixed point theorem for contraction multivalued operators.

3 Impulsive functional differential equations & inclusions

3.1. Introduction

While the previous chapter was devoted to ordinary differential equations and inclusions involving impulses, our attention in this chapter is turned to functional differential equations and inclusions each undergoing impulse effects. These equations and inclusions have played an important role in areas involving hereditary phenomena for which a delay argument arises in the modelling equation or inclusion. There are also a number of applications in which the delayed argument occurs in the derivative of the state variable, which are sometimes modelled by neutral differential equations or neutral differential inclusions.

This chapter presents a theory for the existence of solutions of impulsive functional differential equations and inclusions, including scenarios of neutral equations, as well as semilinear models. The methods used throughout the chapter range over applications of the Leray-Schauder nonlinear alternative, Schaefer's fixed point theorem, a Martelli fixed point theorem for multivalued condensing maps, and a Covitz-Nadler fixed point theorem for multivalued maps.

3.2. Impulsive functional differential equations

In this section, we will establish existence theory for first- and second-order impulsive functional differential equations. The section will be divided into parts. In the first part, by a nonlinear alternative of Leray-Schauder type, we will present an existence result for the first-order initial value problem

$$y'(t) = f(t, y_t), \quad \text{a.e. } t \in J := [0, T], \ t \neq t_k, \ k = 1, \dots, m, \tag{3.1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{3.2}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{3.3}$$

where $f : J \times \mathcal{D} \to E$ is a given function, $\mathcal{D} = \{\psi : [-r, 0] \to E \mid \psi$ is continuous everywhere except for a finite number of points s at which $\psi(s)$ and the right limit $\psi(s^+)$ exist and $\psi(s^-) = \psi(s)\}$, $\phi \in \mathcal{D}$, $(0 < r < \infty)$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $I_k \in C(E, E)$ $(k = 1, 2, \dots, m)$, and E a real separable Banach space with norm $|\cdot|$. Also, throughout, $J' = J \setminus \{t_1, \dots, t_m\}$.

For any continuous function y defined on the interval $[-r,T] \setminus \{t_1,\dots,t_m\}$ and any $t \in J$, we denote by y_t the element of $\mathcal{D}$ defined by

$$y_t(\theta) = y(t+\theta), \quad \theta \in [-r,0]. \tag{3.4}$$

Here $y_t(\cdot)$ represents the history of the state from time $t-r$ up to the present time t. For $\psi \in \mathcal{D}$, the norm of ψ is defined by

$$\|\psi\|_{\mathcal{D}} = \sup\{|\psi(\theta)|,\ \theta \in [-r,0]\}. \tag{3.5}$$

Later, we study the existence of solutions of second-order impulsive differential equations of the form

$$y''(t) = f(t,y_t), \quad t \in J := [0,T],\ t \neq t_k,\ k = 1,\dots,m, \tag{3.6}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m, \tag{3.7}$$

$$\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1,\dots,m, \tag{3.8}$$

$$y(t) = \phi(t), \quad t \in [-r,0], \qquad y'(0) = \eta, \tag{3.9}$$

where f, I_k, and ϕ are as in problem (3.1)–(3.3), $\overline{I}_k \in C(E,E)$ $(k = 1,2,\dots,m)$, and $\eta \in E$.

In order to define the solutions of the above problems, we will consider the spaces $\mathrm{PC}([-r,T],E) = \{y : [-r,T] \to E : y(t)$ is continuous everywhere except for some t_k at which $y(t_k^-)$ and $y(t_k^+)$, $k = 1,\dots,m$, exist and $y(t_k^-) = y(t_k)\}$ and $\mathrm{PC}^1([0,T],E) = \{y : [0,T] \to E : y(t)$ is continuously differentiable everywhere except for some t_k at which $y'(t_k^-)$ and $y'(t_k^+)$, $k = 1,\dots,m$, exist and $y'(t_k^-) = y'(t_k)\}$.

Let

$$Z = \mathrm{PC}([-r,T],E) \cap \mathrm{PC}^1([0,T],E). \tag{3.10}$$

Obviously, for any $t \in [0,T]$ and $y \in Z$, we have $y_t \in \mathcal{D}$, and $\mathrm{PC}([-r,T],E)$ and Z are Banach spaces with the norms

$$\begin{gathered} \|y\| = \sup\{|y(t)| : t \in [-r,T]\}, \\ \|y\|_Z = \|y\| + \|y'\|, \end{gathered} \tag{3.11}$$

where

$$\|y'\| = \sup\{|y'(t)| : t \in [0,T]\}. \tag{3.12}$$

Let us start by defining what we mean by a solution of problem (3.1)–(3.3). In the following, we set for convenience

$$\Omega = \mathrm{PC}([-r,T],E). \tag{3.13}$$

Definition 3.1. A function $y \in \Omega \cap AC((t_k, t_{k+1}), E)$, $k = 1, \ldots, m$, is said to be a solution of (3.1)–(3.3) if y satisfies the equation $y'(t) = f(t, y_t)$ a.e. on J' and the conditions $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $k = 1, \ldots, m$, $y(t) = \phi(t)$, $t \in [-r, 0]$.

The first result of this section concerns a priori estimates on possible solutions of problem (3.1)–(3.3).

Theorem 3.2. *Suppose the following are satisfied.*

(3.2.1) *$f : J \times \mathcal{D} \to E$ is an L^1 Carathéodory function.*

(3.2.2) *There exists a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ and $p \in L^1(J, \mathbb{R}_+)$ such that*

$$|f(t, u)| \leq p(t)\psi(\|u\|_{\mathcal{D}}) \quad \text{for a.e. } t \in J \text{ and each } u \in \mathcal{D} \tag{3.14}$$

with

$$\int_{t_{k-1}}^{t_k} p(s)ds < \int_{N_{k-1}}^{\infty} \frac{du}{\psi(u)}, \quad k = 1, \ldots, m+1, \tag{3.15}$$

where $N_0 = \|\phi\|_{\mathcal{D}}$, and for $k = 2, \ldots, m+1$,

$$\begin{gathered} N_{k-1} = \sup_{y \in [-M_{k-2}, M_{k-2}]} |I_{k-1}(y)| + M_{k-2}, \\ M_{k-2} = \Gamma_{k-1}^{-1}\left(\int_{t_{k-2}}^{t_{k-1}} p(s)ds\right) \end{gathered} \tag{3.16}$$

with

$$\Gamma_l(z) = \int_{N_{l-1}}^{z} \frac{du}{\psi(u)}, \quad z \geq N_{l-1},\ l \in \{1, \ldots, m+1\}. \tag{3.17}$$

Then if $y \in \Omega$ is a solution of (3.1)–(3.3),

$$\sup\{|y(t)| : t \in [t_{k-1}, t_k]\} \leq M_{k-1}, \quad k = 1, \ldots, m+1. \tag{3.18}$$

Consequently, for each possible solution y to (3.1)–(3.3),

$$\|y\| \leq \max\{\|\phi\|_{\mathcal{D}}, M_{k-1} : k = 1, \ldots, m+1\} := b^*. \tag{3.19}$$

Proof. Suppose there exists a solution y to (3.1)–(3.3). Then $y|_{[-r,t_1]}$ is a solution to

$$\begin{gathered} y'(t) = f(t, y_t), \quad \text{for a.e. } t \in (0, t_1), \\ y(t) = \phi(t), \quad t \in [-r, 0]. \end{gathered} \tag{3.20}$$

Then, for each $t \in [0, t_1]$,

$$y(t) - \phi(0) = \int_0^t f(s, y_s)\,ds. \tag{3.21}$$

By (3.2.2), we get

$$|y(t)| \le \|\phi\|_{\mathcal{D}} + \int_0^t p(s)\psi(\|y_s\|_{\mathcal{D}})\,ds, \quad t \in [0, t_1]. \tag{3.22}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \le s \le t\}, \quad 0 \le t \le t_1. \tag{3.23}$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0, t_1]$, then by the previous inequality we have, for $t \in [0, t_1]$,

$$\mu(t) \le \|\phi\|_{\mathcal{D}} + \int_0^t p(s)\psi(\mu(s))\,ds. \tag{3.24}$$

If $t^* \in [-r, 0]$, then $\mu(t) = \|\phi\|_{\mathcal{D}}$ and the previous inequality holds.

Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$\begin{gathered} c = v(0) = \|\phi\|_{\mathcal{D}}, \quad \mu(t) \le v(t), \quad t \in [0, t_1], \\ v'(t) = p(t)\psi(\mu(t)), \quad t \in [0, t_1]. \end{gathered} \tag{3.25}$$

Using the nondecreasing character of ψ, we get

$$v'(t) \le p(t)\psi(v(t)), \quad t \in [0, t_1]. \tag{3.26}$$

This implies, for each $t \in [0, t_1]$, that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \le \int_0^{t_1} p(s)\,ds. \tag{3.27}$$

In view of (3.2.2), we obtain

$$|v(t^*)| \le \Gamma_1^{-1}\left(\int_0^{t_1} p(s)\,ds\right) := M_0. \tag{3.28}$$

Since for every $t \in [0, t_1]$, $\|y_t\|_{\mathcal{D}} \leq \mu(t)$, we have

$$\sup_{t\in[0,t_1]} |y(t)| \leq M_0. \tag{3.29}$$

Now $y|_{[t_1,t_2]}$ is a solution to

$$\begin{gathered} y'(t) = f(t, y_t), \quad \text{for a.e. } t \in (t_1, t_2), \\ y(t_1^+) = I_1(y(t_1)) + y(t_1). \end{gathered} \tag{3.30}$$

Note that

$$\begin{aligned} |y(t_1^+)| &\leq \sup_{r\in[-M_0,+M_0]} |I_1(r)| + \sup_{t\in[0,t_1]} |y(t)| \\ &\leq \sup_{r\in[-M_0,+M_0]} |I_1(r)| + M_0 := N_1. \end{aligned} \tag{3.31}$$

Then, for each $t \in [t_1, t_2]$,

$$y(t) - y(t_1^+) = \int_{t_1}^{t} f(s, y_s)\,ds. \tag{3.32}$$

By (3.2.2), we get

$$|y(t)| \leq N_1 + \int_0^t p(s)\psi(\|y_s\|_{\mathcal{D}})\,ds, \quad t \in [t_1, t_2]. \tag{3.33}$$

We consider the function μ_1 defined by

$$\mu_1(t) = \sup\{|y(s)| : t_1 \leq s \leq t\}, \quad t_1 \leq t \leq t_2. \tag{3.34}$$

Let $t^* \in [t_1, t_2]$ be such that $\mu_1(t) = |y(t^*)|$. By the previous inequality, we have, for $t \in [t_1, t_2]$,

$$\mu_1(t) \leq N_1 + \int_{t_1}^{t} p(s)\psi(\mu_1(s))\,ds. \tag{3.35}$$

Let us take the right-hand side of the above inequality as $v_1(t)$. Then we have

$$\begin{gathered} c = v_1(0) = N_1, \quad \mu_1(t) \leq v_1(t), \quad t \in [t_1, t_2], \\ v_1'(t) = p(t)\psi(\mu_1(t)), \quad t \in [t_1, t_2]. \end{gathered} \tag{3.36}$$

Using the nondecreasing character of ψ, we get

$$v_1'(t) \leq p(t)\psi(v_1(t)), \quad t \in [t_1, t_2]. \tag{3.37}$$

This implies, for each $t \in [t_1, t_2]$, that

$$\int_{v_1(0)}^{v_1(t)} \frac{du}{\psi(u)} \leq \int_{t_1}^{t_2} p(s)ds. \tag{3.38}$$

In view of (3.2.2), we obtain

$$|v_1(t^*)| \leq \Gamma_1^{-1}\left(\int_{t_1}^{t_2} p(s)ds\right) := M_1. \tag{3.39}$$

Since for every $t \in [t_1, t_2]$, $\|y_t\|_{\mathcal{D}} \leq \mu_1(t)$, we have

$$\sup_{t\in[t_1,t_2]} |y(t)| \leq M_1. \tag{3.40}$$

We continue this process taking into account that $y|_{[t_m,T]}$ is a solution to the problem

$$\begin{aligned} y'(t) &= f(t, y_t), \quad \text{for a.e. } t \in (t_m, T), \\ y(t_m^+) &= I_m(y(t_m)) + y(t_m). \end{aligned} \tag{3.41}$$

We obtain that there exists a constant M_m such that

$$\sup_{t\in[t_m,T]} |y(t)| \leq \Gamma_{m+1}^{-1}\left(\int_{t_m}^{T} p(s)ds\right) := M_m. \tag{3.42}$$

But y was an arbitrary solution. Consequently, for each possible solution y to (1)–(3), we have

$$\|y\| \leq \max\{\|\phi\|_{\mathcal{D}}, M_{k-1} : k = 1,\dots,m+1\} := b^*. \tag{3.43}$$

□

Now we are in position to state and prove our main result.

Theorem 3.3. *Let (3.2.1), (3.2.2), and the following hold.*

(3.3.1) *For each bounded $B \subseteq \Omega$ and $t \in J$, the set*

$$\left\{\phi(0) + \int_0^t f(s, y_s)ds + \sum_{0<t_k<t} I_k(y(t_k^-)) : y \in B\right\} \tag{3.44}$$

is relatively compact in E.

Then the IVP (3.1)–(3.3) *has at least one solution.*

Proof. Transform the problem into a fixed point problem. Consider the map $N : \Omega \to \Omega$ defined by

$$N(y)(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \phi(0) + \int_0^t f(s, y_s)\,ds + \sum_{0<t_k<t} I_k(y(t_k^-)), & t \in [0, T]. \end{cases} \tag{3.45}$$

Clearly the fixed points of N are solutions to (3.1)–(3.3).

In order to apply the nonlinear alternative of Leray-Schauder type, we first show that N is completely continuous. The proof will be given in several steps.

Step 1. N maps bounded sets into bounded sets in Ω.

Indeed, it is enough to show that there exists a positive constant ℓ such that, for each $y \in B_q = \{y \in \Omega : \|y\| \le q\}$, one has $\|N(y)\| \le \ell$.

Let $y \in B_q$, then, for each $t \in [0, T]$, we have

$$N(y)(t) = \phi(0) + \int_0^t f(s, y_s)\,ds + \sum_{0<t_k<t} I_k(y(t_k^-)). \tag{3.46}$$

By (3.2.1), we have, for each $t \in J$,

$$\begin{aligned} |N(y)(t)| &\le \|\phi\|_{\mathcal{D}} + \int_0^t |f(s, y_s)|\,ds + \sum_{0<t_k<t} |I_k(y(t_k^-))| \\ &\le \|\phi\|_{\mathcal{D}} + \int_0^t \varphi_q(s)\,ds + \sum_{k=1}^m \sup\{|I_k(y)| : \|y\| \le q\}. \end{aligned} \tag{3.47}$$

Thus

$$||N(y)|| \le \|\phi\|_{\mathcal{D}} + \int_0^T \varphi_q(s)\,ds + \sum_{k=1}^m \sup\{|I_k(y)| : \|y\| \le q\} := \ell. \tag{3.48}$$

Step 2. N maps bounded sets into equicontinuous sets of Ω.

Let $r_1, r_2 \in J'$, $r_1 < r_2$, and let $B_q = \{y \in \Omega : \|y\| \le q\}$ be a bounded set of Ω. Let $y \in B_q$. Then

$$|N(y)(r_2) - N(y)(r_1)| \le \int_{r_1}^{r_2} \varphi_q(s)\,ds + \sum_{0<t_k<r_2-r_1} |I_k(y(t_k^-))|. \tag{3.49}$$

As $r_2 \to r_1$, the right-hand side of the above inequality tends to zero. This proves the equicontinuity for the case where $t \ne t_i$, $i = 1, \dots, m$. It remains to examine the equicontinuity at $t = t_i$. The proof is similar to that given in Theorem 2.2.

The equicontinuity for the cases $r_1 < r_2 \le 0$ and $r_1 \le 0 \le r_2$ is obvious.

Step 3. $N : \Omega \to \Omega$ is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in Ω. Then there is an integer q such that $\|y_n\| \le q$ for all $n \in \mathbb{N}$ and $\|y\| \le q$, so $y_n \in B_q$ and $y \in B_q$.

We have then, by the dominated convergence theorem,

$$\begin{aligned}||N(y_n)-N(y)|| \le \sup_{t\in J}\Bigg[\int_0^t |f(s,y_{ns})-f(s,y_s)|\,ds \\ + \sum_{0<t_k<t} |I_k(y_n(t_k))-I_k(y(t_k^-))|\Bigg] \longrightarrow 0.\end{aligned} \tag{3.50}$$

Thus N is continuous. Set

$$U=\{y\in\Omega : \|y\|<b^*+1\}, \tag{3.51}$$

where b^* is defined in the proof of Theorem 3.2.

As a consequence of Steps 2, 3, and (3.3.3) together with the Ascoli-Arzelá theorem, we can conclude that the map $N:\overline{U}\to\Omega$ is compact.

By the choice of U there is no $y\in\partial U$ such that $y=\lambda N y$ for any $\lambda\in(0,1)$. As a consequence of Theorem 1.8, we deduce that N has a fixed point $y\in\overline{U}$ which is a solution of (3.1)–(3.3). □

In this part we present a result for problem (3.6)–(3.9) in the spirit of Schaefer's fixed point theorem. We begin by giving the definition of the solution of this problem.

Definition 3.4. A function $y\in\Omega\cap AC^1((t_k,t_{k+1}),E)$, $k=0,\dots,m$, is said to be a solution of (3.6)–(3.9) if y satisfies the equation $y''(t)=f(t,y_t)$ a.e. on J' and the conditions $\Delta y|_{t=t_k}=I_k(y(t_k^-))$ and $\Delta y'|_{t=t_k}=\bar{I}_k(y(t_k^-))$, $k=1,\dots,m$ and $y'(0)=\eta$.

Theorem 3.5. *Assume (3.2.1) and the following conditions are satisfied.*

(3.5.1) *There exist constants c_k such that $|I_k(y)|\le c_k$, $k=1,2,\dots,m$, for each $y\in E$.*

(3.5.2) *There exist constants d_k such that $|\bar{I}_k(y)|\le d_k$, $k=1,2,\dots,m$, for each $y\in E$.*

(3.5.3) *$|f(t,u)|\le p(t)\psi(\|u\|_{\mathcal{D}})$ for almost all $t\in J$ and all $u\in\mathcal{D}$, where $p\in L^1(J,\mathbb{R}_+)$ and $\psi:\mathbb{R}_+\to(0,\infty)$ is continuous and increasing with*

$$\int_0^T (T-s)p(s)\,ds<\int_c^\infty \frac{d\tau}{\psi(\tau)}, \tag{3.52}$$

and where $c=\|\phi\|_{\mathcal{D}}+T|\eta|+\sum_{k=1}^m[c_k+(T-t_k)d_k]$.

(3.5.4) *For each bounded $B\subseteq\Omega$ and for each $t\in J$, the set*

$$\left\{\phi(0)+t\eta+\int_0^t (t-s)f(s,y_s)\,ds+\sum_{0<t_k<t} I_k(y(t_k^-))+(t-t_k)\bar{I}_k(y(t_k^-)) : y\in B\right\} \tag{3.53}$$

is relatively compact in E.

Then the IVP (3.6)–(3.9) *has at least one solution.*

Proof. Transform the problem into a fixed point problem. Consider the operator $G:\Omega\to\Omega$ defined by

$$G(y)(t)=\begin{cases}\phi(t) & \text{if } t\in[-r,0],\\ \phi(0)+t\eta+\displaystyle\int_0^t (t-s)f(s,y_s)\,ds & \\ \quad+\displaystyle\sum_{0<t_k<t}\big[I_k(y(t_k^-))+(t-t_k)\bar{I}_k(y(t_k^-))\big] & \text{if } t\in[0,T].\end{cases}\tag{3.54}$$

Step 1. G maps bounded sets into bounded sets in Ω.

Indeed, it is enough to show that there exists a positive constant ℓ such that, for each $y\in B_q=\{y\in\Omega:\|y\|\le q\}$, one has $\|G(y)\|\le\ell$.

Let $y\in B_q$, then, for each $t\in J$, we have

$$\begin{aligned}G(y)(t)&=\phi(0)+ty_0+\int_0^t (t-s)f(s,y_s)\,ds\\&\quad+\sum_{0<t_k<t}\big[I_k(y(t_k^-))+(t-t_k)\bar{I}_k(y(t_k^-))\big].\end{aligned}\tag{3.55}$$

By (3.2.1), we have, for each $t\in J$,

$$\begin{aligned}|G(y)(t)|&\le\|\phi\|_{\mathcal{D}}+t|y_0|+\int_0^t (t-s)|f(s,y_s)|\,ds\\&\quad+\sum_{0<t_k<t}|I_k(y(t_k^-))|+|(t-t_k)||\bar{I}_k(y(t_k^-))|\\&\le\|\phi\|_{\mathcal{D}}+t|y_0|+\int_0^t (t-s)\varphi_q(s)\,ds\\&\quad+\sum_{k=1}^m\big[\sup\{|I_k(|y|)|:\|y\|\le q\}\\&\qquad+(T-t_k)\sup\{|\bar{I}_k(|y|)|:\|y\|\le q\}\big].\end{aligned}\tag{3.56}$$

Then, for each $h\in G(B_q)$, we have

$$\begin{aligned}\|h\|&\le\|\phi\|_{\mathcal{D}}+T|y_0|+\int_0^T (T-s)\varphi_q(s)\,ds\\&\quad+\sum_{k=1}^m\big[\sup\{|I_k(|y|)|:\|y\|\le q\}\\&\qquad+(T-t_k)\sup\{|\bar{I}_k(|y|)|:\|y\|\le q\}\big]:=\ell.\end{aligned}\tag{3.57}$$

Step 2. G maps bounded sets into equicontinuous sets of Ω.

Let $\tau_1, \tau_2 \in J'$, $\tau_1 < \tau_2$, and let $B_q = \{y \in \Omega : \|y\| \leq q\}$ be a bounded set of Ω. Let $y \in B_q$. Then

$$\begin{aligned}
|G(y)(\tau_2) - G(y)(\tau_1)| \leq (\tau_2 - \tau_1)|y_0| &+ \int_{\tau_1}^{\tau_2} \varphi_q(s)ds \\
&+ \int_0^{\tau_2} (\tau_2 - \tau_1)\varphi_q(s)ds + \int_{\tau_1}^{\tau_2} |\tau_1 - s|\varphi_q(s)ds \\
&+ \sum_{0<t_k<\tau_2-\tau_1} |I_k(y(t_k^-))| \\
&+ \sum_{0<t_k<\tau_2-\tau_1} |\tau_1 - t_k|\,|\bar{I}_k(y(t_k^-))| \\
&+ \sum_{0<t_k<\tau_2} (\tau_2 - \tau_1)|\bar{I}_k(y(t_k^-))|.
\end{aligned} \tag{3.58}$$

As $\tau_2 \to \tau_1$, the right-hand side of the above inequality tends to zero. This proves the equicontinuity for the case where $t \neq t_i$, $i = 1, \ldots, m$. It remains to examine the equicontinuity at $t = t_i$. The proof is similar to that given in Theorem 2.2.

The equicontinuity for the cases $\tau_1 < \tau_2 \leq 0$ and $\tau_1 \leq 0 \leq \tau_2$ is obvious.

Step 3. $G : \Omega \to \Omega$ is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in Ω. Then there is an integer q such that $\|y_n\| \leq q$ for all $n \in \mathbb{N}$ and $\|y\| \leq q$, so $y_n \in B_q$ and $y \in B_q$.

We have then by the dominated convergence theorem

$$\begin{aligned}
&||G(y_n) - G(y)|| \\
&\quad \leq \sup_{t\in J}\Bigg[\int_0^t (t-s)|f(s, y_{ns}) - f(s, y_s)|ds \\
&\qquad\quad + \sum_{0<t_k<t} \big[|I_k(y_n(t_k)) - I_k(y(t_k^-))| \\
&\qquad\qquad\quad + |t - t_k|\,|\bar{I}_k(y_n(t_k)) - \bar{I}_k(y(t_k^-))|\big]\Bigg] \longrightarrow 0.
\end{aligned} \tag{3.59}$$

Thus G is continuous.

As a consequence of Steps 1, 2, 3, and (3.5.3) together with the Ascoli-Arzelá theorem, we can conclude that $G : \Omega \to \Omega$ is completely continuous.

Step 4. Now it remains to show that the set

$$\mathcal{E}(G) := \{y \in \Omega : y = \lambda G(y) \text{ for some } 0 < \lambda < 1\} \tag{3.60}$$

is bounded.

Let $y \in \mathcal{E}(G)$. Then $y = \lambda G(y)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$,

$$\begin{aligned} y(t) = \lambda\phi(0) + \lambda t y_0 + \lambda \int_0^t (t-s) f(s, y_s)\,ds \\ + \lambda \sum_{0<t_k<t} \left[I_k(y(t_k^-)) + (t - t_k)\bar{I}_k(y(t_k^-)) \right]. \end{aligned} \tag{3.61}$$

This implies that, for each $t \in J$, we have

$$|y(t)| \le \|\phi\|_{\mathcal{D}} + T|\eta| + \int_0^t (T-s)p(s)\psi(\|y_s\|_{\mathcal{D}})\,ds + \sum_{k=1}^m \left[c_k + (T-t_k)d_k\right]. \tag{3.62}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \le s \le t\}, \quad 0 \le t \le T. \tag{3.63}$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0, T]$, by the previous inequality, we have, for $t \in [0, T]$,

$$\mu(t) \le \|\phi\|_{\mathcal{D}} + T|\eta| + \int_0^t (T-s)p(s)\psi(\mu(s))\,ds + \sum_{k=1}^m \left[c_k + (T-t_k)d_k\right]. \tag{3.64}$$

If $t^* \in [-r, 0]$, then $\mu(t) = \|\phi\|_{\mathcal{D}}$ and the previous inequality holds.

Let us take the right-hand side of the above inequality as $v(t)$, then we have

$$\begin{gathered} c = v(0) = \|\phi\|_{\mathcal{D}} + T|\eta| + \sum_{k=1}^m \left[c_k + (T-t_k)d_k\right], \quad \mu(t) \le v(t), \quad t \in [0, T], \\ v'(t) = (T-t)p(t)\psi(\mu(t)), \quad t \in [0, T]. \end{gathered} \tag{3.65}$$

Using the nondecreasing character of ψ, we get

$$v'(t) \le (T-t)p(t)\psi(v(t)), \quad t \in [0, T]. \tag{3.66}$$

This implies, for each $t \in J$, that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \le \int_0^T (T-s)p(s)\,ds < \int_{v(0)}^{\infty} \frac{du}{\psi(u)}. \tag{3.67}$$

This inequality implies that there exists a constant $b = b(T, p, \psi)$ such that $v(t) \le b$, $t \in J$, and hence $\mu(t) \le b$, $t \in J$. Since for every $t \in [0, T]$, $\|y_t\|_{\mathcal{D}} \le \mu(t)$, we have

$$\|y\| := \sup\{|y(t)| : -r \le t \le T\} \le b, \tag{3.68}$$

where b depends only on T and on the functions p and ψ. This shows that $\mathcal{E}(G)$ is bounded.

Set $X := \Omega$. As a consequence of Theorem 1.6, we deduce that G has a fixed point which is a solution of (3.6)–(3.9). □

3.3. Impulsive neutral differential equations

This section is concerned with the existence of solutions for initial value problems for first- and second-order neutral functional differential equations with impulsive effects. In the first part, we consider the first-order initial value problem (IVP for short)

$$\begin{gathered}\frac{d}{dt}[y(t)-g(t,y_t)] = f(t,y_t), \quad \text{a.e. } t\in[0,T],\ t\neq t_k,\ k=1,\dots,m,\\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k=1,\dots,m,\\ y(t)=\phi(t), \quad t\in[-r,0],\end{gathered} \tag{3.69}$$

where f, I_k, ϕ are in problem (3.1)–(3.3) and $g : J \times \mathcal{D} \to E$ is a given function.

In the second part, we consider the second-order IVP

$$\begin{gathered}\frac{d}{dt}[y'(t)-g(t,y_t)] = f(t,y_t), \quad \text{a.e. } t\in[0,T],\ t\neq t_k,\ k=1,\dots,m,\\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k=1,\dots,m,\\ \Delta y'|_{t=t_k} = \overline{I}_k(y(t)), \quad k=1,\dots,m,\\ y(t)=\phi(t), \quad t\in[-r,0], \quad y'(0)=\eta,\end{gathered} \tag{3.70}$$

where f, I_k, and ϕ are as in problem (3.1)–(3.3), $\overline{I}_k, \eta$ are as in problem (3.6)–(3.9) and g as in (3.69).

Let us start by defining what we mean by a solution of problem (3.69).

Definition 3.6. A function $y \in \Omega \cap AC((t_k, t_{k+1}), E)$, $k = 1,\dots,m$, is said to be a solution of (3.69) if y satisfies the equation $(d/dt)[y(t)-g(t,y_t)] = f(t,y_t)$ a.e. on J, $t \neq t_k$, $k = 1,\dots,m$, and the conditions $\Delta y|_{t=t_k} = I_k(y(t))$, $t = t_k$, $k = 1,\dots,m$, and $y(t) = \phi(t)$ on $[-r,0]$.

We are now in a position to state and prove our existence result for problem (3.69).

Theorem 3.7. *Assume (3.2.1), (3.5.1), and the following conditions are satisfied.*

(3.7.1) (i) *The function g is completely continuous.*

(ii) *For any bounded set B in $C([-r,T],E)$, the set $\{t \to g(t,y_t) : y \in B\}$ is equicontinuous in Ω.*

(iii) *There exist constants $0 \le c_1^* < 1$ and $c_2^* \ge 0$ such that*

$$|g(t,u)| \le c_1^*\|u\| + c_2^*, \quad t \in J,\ u \in \mathcal{D}. \tag{3.71}$$

(3.7.2) *There exist a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$, *and* $p \in L^1([0,T],\mathbb{R}_+)$ *such that*

$$\begin{gathered} |f(t,u)| \le p(t)\psi(\|u\|_{\mathcal{D}}) \quad \text{for a.e. } t \in [0,T], \text{ and each } u \in \mathcal{D}, \\ \frac{1}{1-c_1^*}\int_0^T p(s)ds < \int_c^\infty \frac{d\tau}{\psi(\tau)}, \end{gathered} \tag{3.72}$$

where

$$c = \frac{1}{1-c_1^*}\left[(1+c_1^*)\|\phi\| + 2c_2^* + \sum_{k=1}^m c_k\right]. \tag{3.73}$$

Then the IVP (3.69) *has at least one solution.*

Proof. Consider the operator $\overline{N} : \Omega \to \Omega$ defined by

$$\overline{N}(y)(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ \phi(0) - g(0,\phi(0)) + g(t,y_t) & \\ \quad + \displaystyle\int_0^t f(s,y_s)ds + \sum_{0<t_k<t} I_k(y(t_k^-)), & t \in [0,T]. \end{cases} \tag{3.74}$$

We will show that $\overline{N}$ satisfies the assumptions of Schaefer's fixed point theorem. Using (3.7.1), it suffices to show that the operator $N_1 : \Omega \to \Omega$ defined by

$$N_1(y)(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ \phi(0) + \displaystyle\int_0^t f(s,y_s)ds + \sum_{0<t_k<t} I_k(y(t_k^-)), & t \in [0,T], \end{cases} \tag{3.75}$$

is completely continuous. As in Theorem 3.3, we can prove that N_1 is a completely continuous operator. We omit the details. Here we repeat only the proof that the set

$$\mathcal{E}(\overline{N}) := \{y \in \Omega : y = \lambda\overline{N}(y) \text{ for some } 0 < \lambda < 1\} \tag{3.76}$$

is bounded. Let $y \in \mathcal{E}(\overline{N})$. Then $y = \lambda\overline{N}(y)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$,

$$y(t) = \lambda\left[\phi(0) - g(0,\phi(0)) + g(t,y_t) + \int_0^t f(s,y_s)ds + \sum_{0<t_k<t} I_k(y(t_k^-))\right]. \tag{3.77}$$

This implies by our assumptions that, for each $t \in J$, we have

$$|y(t)| \leq \|\phi\|_{\mathcal{D}} + c_1^*\|\phi\|_{\mathcal{D}} + 2c_2^* + c_1^*\|y_t\|_{\mathcal{D}} + \int_0^t p(s)\psi(\|y_s\|_{\mathcal{D}})ds + \sum_{k=1}^m c_k. \tag{3.78}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq T. \tag{3.79}$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0, T]$, by the previous inequality, we have, for $t \in [0, T]$,

$$\begin{aligned} \mu(t) &\leq \|\phi\|_{\mathcal{D}} + c_1^*\|\phi\|_{\mathcal{D}} + 2c_2^* + c_1^*\|y_t\|_{\mathcal{D}} + \int_0^t p(s)\psi(\|y_s\|_{\mathcal{D}})ds \\ &\quad + \sum_{0<t_k<t} |I_k(y(t_k^-))| \\ &\leq \|\phi\|_{\mathcal{D}} + c_1^*\|\phi\|_{\mathcal{D}} + 2c_2^* + c_1^*\mu(t) + \int_0^t p(s)\psi(\mu(s))ds + \sum_{k=1}^m c_k. \end{aligned} \tag{3.80}$$

Thus

$$\mu(t) \leq \frac{1}{1-c_1^*}\left\{(1+c_1^*)\|\phi\|_{\mathcal{D}} + 2c_2^* + \int_0^t p(s)\psi(\mu(s))ds + \sum_{k=1}^m c_k\right\}, \quad t \in J. \tag{3.81}$$

If $t^* \in [-r, 0]$, then $\mu(t) = \|\phi\|_{\mathcal{D}}$ and the previous inequality holds.

Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$\begin{gathered} c = v(0) = \frac{1}{1-c_1^*}\left\{(1+c_1^*)\|\phi\|_{\mathcal{D}} + 2c_2^* + \sum_{k=1}^m c_k\right\}, \quad \mu(t) \leq v(t), \quad t \in J, \\ v'(t) = \frac{1}{1-c_1^*}p(t)\psi(\mu(t)), \quad t \in J. \end{gathered} \tag{3.82}$$

Using the nondecreasing character of ψ, we get

$$v'(t) \leq \frac{1}{1-c_1^*}p(t)\psi(v(t)), \quad t \in J. \tag{3.83}$$

This implies, for each $t \in J$, that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq \frac{1}{1-c_1^*}\int_0^T p(s)ds < \int_{v(0)}^{\infty} \frac{du}{\psi(u)}. \tag{3.84}$$

This inequality implies that there exists a constant $\overline{b}$ such that $v(t) \le \overline{b}$, $t \in J$, and hence $\mu(t) \le \overline{b}$, $t \in J$. Since for every $t \in [0,T]$, $\|y_t\|_{\mathcal{D}} \le \mu(t)$, we have

$$\|y\| \le b' = \max\{\|\phi\|_{\mathcal{D}}, \overline{b}\}, \tag{3.85}$$

where b' depends only on T and on the functions p and ψ. Thus $\mathcal{E}(\overline{N})$ is bounded.

Set $X := \Omega$. As a consequence of Schaefer's fixed point theorem (Theorem 1.6), we deduce that $\overline{N}$ has a fixed point which is a solution of (3.69). □

In this next part we study problem (3.70). We give first the definition of solution of problem (3.70).

Definition 3.8. A function $y \in \Omega \cap AC^1((t_k, t_{k+1}), E)$, $k = 0, \ldots, m$, is said to be a solution of (3.70) if y satisfies the equation $(d/dt)[y'(t) - g(t, y_t)] = f(t, y_t)$ a.e. on J, $t \ne t_k$, $k = 1, \ldots, m$, and the conditions $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $t = t_k$, $k = 1, \ldots, m$, $\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-))$, $k = 1, \ldots, m$, $y(t) = \phi(t)$, on $[-r, 0]$ and $y'(0) = \eta$.

Theorem 3.9. *Assume (3.2.1), (3.7.1), (3.5.1), and (3.5.2) hold. In addition assume the following conditions are satisfied.*

(3.9.1) *There exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and $p \in L^1([0,T], \mathbb{R}_+)$ such that*

$$|f(t,u)| \le p(t)\psi(\|u\|_{\mathcal{D}}) \quad \text{for a.e. } t \in [0,T] \text{ and each } u \in \mathcal{D}, \tag{3.86}$$

where $p \in L^1(J, \mathbb{R}_+)$ and

$$\int_0^T M(s)\,ds < \int_{\widetilde{c}}^{\infty} \frac{ds}{s + \psi(s)}, \tag{3.87}$$

where

$$\widetilde{c} = \|\phi\|_{\mathcal{D}} + [|\eta| + c_1^*\|\phi\|_{\mathcal{D}} + 2c_2^*]T + \sum_{k=1}^{m} [c_k + (T - t_k)d_k] \tag{3.88}$$

and $M(t) = \max\{1, c_1, p(t)\}$.

(3.9.2) *For each bounded $B \subseteq \Omega$ and $t \in J$, the set*

$$\left\{\phi(0) + t\eta + \int_0^t \int_0^s f(u, y_u)\,du\,ds + \sum_{0<t_k<t} [I_k(y(t_k^-)) + (t - t_k)\overline{I}_k(y(t_k^-))] : y \in B\right\} \tag{3.89}$$

is relatively compact in E.

Then the IVP (3.70) *has at least one solution.*

Proof. Transform the problem into a fixed point problem. Consider the operator $N_2 : \Omega \to \Omega$ defined by

$$N_2(y)(t) = \begin{cases} \phi(t) & \text{if } t \in [-r, 0], \\ \phi(0) + [\eta - g(0, \phi(0))]t + \int_0^t g(s, y_s)\,ds & \\ \quad + \int_0^t \int_0^s f(u, y_u)\,du\,ds & \\ \quad + \sum_{0<t_k<t} [I_k(y(t_k^-)) + (t - t_k)\overline{I}_k] & \text{if } t \in [0, T]. \end{cases} \tag{3.90}$$

As in Theorem 3.3, we can show that N_2 is completely continuous.

Now we prove only that the set

$$\mathcal{E}(N_2) := \{y \in \Omega : y = \lambda N_2(y) \text{ for some } 0 < \lambda < 1\} \tag{3.91}$$

is bounded.

Let $y \in \mathcal{E}(N_2)$. Then $y = \lambda N_2(y)$ for some $0 < \lambda < 1$. Thus

$$\begin{aligned} y(t) = \lambda\phi(0) + \lambda[\eta - g(0, \phi(0))]t + \lambda \int_0^t g(s, y_s)\,ds + \lambda \int_0^t \int_0^s f(u, y_u)\,du\,ds \\ + \lambda \sum_{0<t_k<t} [I_k(y(t_k^-)) + (t - t_k)\overline{I}_k(y(t_k^-))]. \end{aligned} \tag{3.92}$$

This implies that, for each $t \in [0, T]$, we have

$$\begin{aligned} |y(t)| &\le \|\phi\|_{\mathcal{D}} + [|\eta| + c_1^*\|\phi\|_{\mathcal{D}} + 2c_2^*]T + c_1^* \int_0^t \|y_s\|_{\mathcal{D}}\,ds \\ &\quad + \int_0^t \int_0^s p(u)\psi(\|y_u\|_{\mathcal{D}})\,du\,ds + \sum_{k=1}^m [c_k + (T - t_k)d_k] \\ &\le \|\phi\|_{\mathcal{D}} + [|\eta| + c_1^*\|\phi\|_{\mathcal{D}} + 2c_2^*]T + \int_0^t M(s)\|y_s\|_{\mathcal{D}}\,ds \\ &\quad + \int_0^t M(s) \int_0^s \psi(\|y_u\|_{\mathcal{D}})\,du\,ds + \sum_{k=1}^m [c_k + (T - t_k)d_k]. \end{aligned} \tag{3.93}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \le s \le t\}, \quad 0 \le t \le T. \tag{3.94}$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0,T]$, by the previous inequality, we have, for $t \in [0,T]$,

$$\begin{aligned} \mu(t) \le \|\phi\|_{\mathcal{D}} + [|\eta| + c_1^*\|\phi\|_{\mathcal{D}} + 2c_2^*]T + \int_0^t M(s)\mu(s)ds \\ + \int_0^t M(s)\int_0^s \psi(\mu(u))du\,ds + \sum_{k=1}^m [c_k + (T-t_k)d_k]. \end{aligned} \tag{3.95}$$

If $t^* \in [-r,0]$, then $\mu(t) = \|\phi\|_{\mathcal{D}}$ and the previous inequality holds.

Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$\begin{gathered} \tilde{c} = v(0) = \|\phi\|_{\mathcal{D}} + [|\eta| + c_1^*\|\phi\|_{\mathcal{D}} + 2c_2^*]T + \sum_{k=1}^m [c_k + (T-t_k)d_k], \\ \mu(t) \le v(t), \quad t \in J, \\ v'(t) = M(t)\mu(t) + M(t)\int_0^t \psi(\mu(s))ds \le M(t)\left[v(t) + \int_0^t \psi(v(s))ds\right], \quad t \in J. \end{gathered} \tag{3.96}$$

Put

$$u(t) = v(t) + \int_0^t \psi(v(s))ds, \quad t \in J. \tag{3.97}$$

Then

$$\begin{gathered} u(0) = v(0) = \tilde{c}, \quad v(t) \le u(t), \quad t \in J, \\ u'(t) = v'(t) + \psi(v(t)) \le M(t)[u(t) + \psi(u(t))], \quad t \in J. \end{gathered} \tag{3.98}$$

This implies, for each $t \in J$, that

$$\int_{u(0)}^{u(t)} \frac{du}{u+\psi(u)} \le \int_0^T M(s)ds < \int_{u(0)}^{\infty} \frac{du}{u+\psi(u)}. \tag{3.99}$$

This inequality implies that there exists a constant b_* such that $u(t) \le b_*$, $t \in J$, and hence $\mu(t) \le b_*$, $t \in J$. Since for every $t \in [0,T]$, $\|y_t\|_{\mathcal{D}} \le \mu(t)$, we have

$$\|y\| \le b'' = \max\{\|\phi\|_{\mathcal{D}}, b_*\}, \tag{3.100}$$

where b'' depends only on T and on the functions p and ψ. This shows that $\mathcal{E}(N_2)$ is bounded.

Hence, by Theorem 1.6, we have the result. $\square$

3.4. Impulsive functional differential inclusions

In this section, we will present existence results for impulsive functional differential inclusions. These results constitute, in some sense, extensions of Section 3.2 to differential inclusions. Initially, we will consider first-order impulsive functional differential inclusions,

$$\begin{gathered} y'(t) \in F(t, y_t), \quad \text{a.e. } t \in J := [0, T],\ t \neq t_k,\ k = 1, \dots, m, \\ \Delta y|_{t=t_k} = I_k\big(y(t_k^-)\big), \quad k = 1, \dots, m, \\ y(t) = \phi(t), \quad t \in [-r, 0] \end{gathered} \tag{3.101}$$

with $\mathcal{D}$ as in problem (3.1)–(3.3), $F : J \times \mathcal{D} \to \mathcal{P}(E)$ is a multivalued map, $\phi \in \mathcal{D}$, and $\mathcal{P}(E)$ is the family of all subsets of E.

Later we study second-order initial value problems for impulsive functional differential inclusions,

$$\begin{gathered} y''(t) \in F(t, y_t), \quad t \in J = [0, T],\ t \neq t_k,\ k = 1, \dots, m, \\ \Delta y|_{t=t_k} = I_k\big(y(t_k^-)\big), \quad k = 1, \dots, m, \\ \Delta y'|_{t=t_k} = \bar{I}_k\big(y(t_k^-)\big), \quad k = 1, \dots, m, \\ y(t) = \phi(t), \quad t \in [-r, 0], \qquad y'(0) = \eta, \end{gathered} \tag{3.102}$$

where F, I_k, and ϕ are as in problem (3.101), $\bar{I}_k \in C(E, E)$ and $\eta \in E$.

In our consideration of problem (3.101), a fixed point theorem for condensing maps is used to investigate the existence of solutions for first-order impulsive functional differential inclusions. So, let us start by defining what we mean by a solution of problem (3.101).

Definition 3.10. A function $y \in \Omega \cap \mathrm{AC}((t_k, t_{k+1}), E)$ is said to be a solution of (3.101) if y satisfies the differential inclusion $y'(t) \in F(t, y_t)$ a.e. on J', the conditions $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $k = 1, \dots, m$, and $y(t) = \phi(t)$, $t \in [-r, 0]$.

Theorem 3.11. *Assume that (3.5.1) holds. Moreover assume the following are satisfied.*

(3.11.1) $F : J \times \mathcal{D} \to \mathcal{P}_{b,cp,cv}(E)$; $(t, u) \mapsto F(t, u)$ *is measurable with respect to* t, *for each* $u \in \mathcal{D}$, *u.s.c. with respect to* u, *for each* $t \in J$, *and for each fixed* $u \in \mathcal{D}$, *the set*

$$S_{F,u} = \{g \in L^1(J, E) : g(t) \in F(t, u) \text{ for a.e. } t \in J\} \tag{3.103}$$

is nonempty.

(3.11.2) $\|F(t, u)\| := \sup\{|v| : v \in F(t, u)\} \leq p(t)\psi(\|u\|_{\mathcal{D}})$ *for almost all* $t \in J$ *and all* $u \in \mathcal{D}$, *where* $p \in L^1(J, \mathbb{R}_+)$ *and* $\psi : \mathbb{R}_+ \to (0, \infty)$ *is* *continuous and increasing with*

$$\int_0^T p(s)ds < \int_c^\infty \frac{d\tau}{\psi(\tau)}, \tag{3.104}$$

where $c = \|\phi\|_{\mathcal{D}} + \sum_{k=1}^m c_k$.

(3.11.3) *For each bounded* $B \subseteq \Omega$ *and* $t \in J$, *the set*

$$\left\{\phi(0) + \int_0^t g(s)ds + \sum_{0<t_k<t} I_k\big(y(t_k^-)\big) : g \in S_{F,B}\right\} \tag{3.105}$$

is relatively compact in E *where* $S_{F,B} = \cup\{S_{F,y} : y \in B\}$.

Then the IVP (3.101) *has at least one solution on* $[-r, T]$.

Proof. Consider the multivalued map $N : \Omega \to \mathcal{P}(\Omega)$, defined by

$$N(y) := \left\{ h \in \Omega : h(t) = \begin{cases} \phi(t), & t \in [-r, 0] \\ \phi(0) + \displaystyle\int_0^t g(s)ds & \\ \quad + \displaystyle\sum_{0<t_k<t} I_k\big(y(t_k^-)\big), & t \in [0, T] \end{cases} \right\}, \tag{3.106}$$

where $g \in S_{F,y}$. We will show that N is a completely continuous multivalued map, u.s.c. with convex closed values. The proof will be given in several steps.

Step 1. $N(y)$ is convex, for each $y \in \Omega$.

Indeed, if h_1, h_2 belong to $N(y)$, then there exist $g_1, g_2 \in S_{F,y}$ such that, for each $t \in J$, we have

$$\begin{aligned} h_1(t) &= \phi(0) + \int_0^t g_1(s)ds + \sum_{0<t_k<t} I_k\big(y(t_k^-)\big), \\ h_2(t) &= \phi(0) + \int_0^t g_2(s)ds + \sum_{0<t_k<t} I_k\big(y(t_k^-)\big). \end{aligned} \tag{3.107}$$

Let $0 \le l \le 1$. Then, for each $t \in J$, we have

$$(lh_1 + (1-l)h_2)(t) = \phi(0) + \int_0^t [lg_1(s) + (1-l)g_2(s)]ds + \sum_{0<t_k<t} I_k\big(y(t_k^-)\big). \tag{3.108}$$

Since $S_{F,y}$ is convex (because F has convex values), then

$$lh_1 + (1-l)h_2 \in N(y). \tag{3.109}$$

Step 2. N maps bounded sets into bounded sets in Ω.

Indeed, it is enough to show that there exists a positive constant ℓ such that, for each $h \in N(y)$, $y \in B_q = \{y \in \Omega : \|y\| \le q\}$, one has $\|h\| \le \ell$. If $h \in N(y)$, then there exists $g \in S_{F,y}$ such that, for each $t \in J$, we have

$$h(t) = \phi(0) + \int_0^t g(s)ds + \sum_{0<t_k<t} I_k(y(t_k^-)). \tag{3.110}$$

By (3.11.2), we have, for each $t \in J$,

$$\begin{aligned} |h(t)| &\le \|\phi\|_{\mathcal{D}} + \int_0^t |g(s)|ds + \sum_{0<t_k<t} |I_k(y(t_k^-))| \\ &\le \|\phi\|_{\mathcal{D}} + \sup_{y\in[0,q]} \psi(y)\left(\int_0^t p(s)ds\right) + \sum_{k=1}^m \sup\{|I_k(|y|)| : \|y\| \le q\}. \end{aligned} \tag{3.111}$$

Then, for each $h \in N(B_q)$, we have

$$\|h\| \le \|\phi\|_{\mathcal{D}} + \sup_{y\in[0,q]} \psi(y) \sup_{t\in J}\left(\int_0^t p(s)ds\right) + \sum_{k=1}^m \sup\{|I_k(|y|)| : \|y\| \le q\} := \ell. \tag{3.112}$$

Step 3. N maps bounded sets into equicontinuous sets of Ω.

Let $\tau_1, \tau_2 \in J'$, $\tau_1 < \tau_2$, and let $B_q = \{y \in \Omega : \|y\| \le q\}$ be a bounded set of Ω. For each $y \in B_r$ and $h \in N(y)$, there exists $g \in S_{F,y}$ such that

$$h(t) = \phi(0) + \int_0^t g(s)ds + \sum_{0<t_k<t} I_k(y(t_k^-)), \quad t \in J. \tag{3.113}$$

Thus

$$|h(\tau_2) - h(\tau_1)| \le \int_{\tau_1}^{\tau_2} |g(s)|ds + \sum_{0<t_k<\tau_2-\tau_1} |I_k(y(t_k^-))|. \tag{3.114}$$

As $\tau_2 \to \tau_1$, the right-hand side of the above inequality tends to zero. This proves the equicontinuity for the case where $t \ne t_i$, $i = 1,\dots,m$. It remains to examine the equicontinuity at $t = t_i$. The proof is similar to that given in Theorem 2.2. The equicontinuity for the cases $\tau_1 < \tau_2 \le 0$ and $\tau_1 \le 0 \le \tau_2$ are obvious.

As a consequence of Steps 2, 3, (3.11.4) together with the Ascoli-Arzelá theorem we can conclude that $N : \Omega \to \mathcal{P}(\Omega)$ is a compact multivalued map, and therefore, a condensing map.

Step 4. N has a closed graph.

Let $y_n \to y_*$, $h_n \in N(y_n)$, and $h_n \to h_*$. We will prove that $h_* \in N(y_*)$.
$h_n \in N(y_n)$ means that there exists $g_n \in S_{F,y_n}$ such that

$$h_n(t) = \phi(0) + \int_0^t g_n(s)ds + \sum_{0<t_k<t} I_k(y_n(t_k^-)), \quad t \in J. \tag{3.115}$$

We must prove that there exists $g_* \in S_{F,y_*}$ such that

$$h_*(t) = \phi(0) + \int_0^t g_*(s)ds + \sum_{0<t_k<t} I_k(y_*(t_k^-)), \quad t \in J. \tag{3.116}$$

Clearly since I_k, $k = 1,\dots,m$, are continuous, we have

$$\left\| \left(h_n - \phi(0) - \sum_{0<t_k<t} I_k(y_n(t_k^-)) \right) - \left(h_* - \phi(0) - \sum_{0<t_k<t} I_k(y_*(t_k^-)) \right) \right\| \longrightarrow 0, \tag{3.117}$$

as $n \to \infty$.

Consider the linear continuous operator

$$\begin{gathered} \Gamma : L^1(J,E) \longrightarrow C(J,E), \\ g \longmapsto \Gamma(g)(t) = \int_0^t g(s)ds. \end{gathered} \tag{3.118}$$

From Lemma 1.28, it follows that $\Gamma \circ S_F$ is a closed graph operator. Moreover, we have

$$\left(h_n(t) - \phi(0) - \sum_{0<t_k<t} I_k(y_n(t_k^-)) \right) \in \Gamma(S_{F,y_n}). \tag{3.119}$$

Since $y_n \to y_*$, it follows from Lemma 1.28 that

$$\left(h_*(t) - \phi(0) - \sum_{0<t_k<t} I_k(y_*(t_k^-)) \right) = \int_0^t g_*(s)ds \tag{3.120}$$

for some $g_* \in S_{F,y_*}$.

Step 5. The set

$$M := \{ y \in \Omega : \lambda y \in N(y) \text{ for some } \lambda > 1 \} \tag{3.121}$$

is bounded.

Let $y \in M$. Then $\lambda y \in N(y)$ for some $\lambda > 1$. Thus there exists $g \in S_{F,y}$ such that

$$y(t) = \lambda^{-1}\phi(0) + \lambda^{-1} \int_0^t g(s)ds + \lambda^{-1} \sum_{0<t_k<t} I_k(y(t_k^-)), \quad t \in J. \tag{3.122}$$

This implies by our assumptions that, for each $t \in J$, we have

$$|y(t)| \leq \|\phi\|_{\mathcal{D}} + \int_0^t p(s)\psi(\|y_s\|_{\mathcal{D}})\,ds + \sum_{k=1}^{m} c_k. \tag{3.123}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq T. \tag{3.124}$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0, T]$, by the previous inequality, we have, for $t \in [0, T]$,

$$\mu(t) \leq \|\phi\|_{\mathcal{D}} + \int_0^t p(s)\psi(\mu(s))\,ds + \sum_{k=1}^{m} c_k. \tag{3.125}$$

If $t^* \in [-r, 0]$, then $\mu(t) = \|\phi\|_{\mathcal{D}}$ and the previous inequality holds. Let us take the right-hand side of the above inequality as $v(t)$, then we have

$$\begin{gathered} c = v(0) = \|\phi\|_{\mathcal{D}} + \sum_{k=1}^{m} c_k, \quad \mu(t) \leq v(t), \quad t \in J, \\ v'(t) = p(t)\psi(\mu(t)), \quad t \in J. \end{gathered} \tag{3.126}$$

Using the nondecreasing character of ψ, we get

$$v'(t) \leq p(t)\psi(v(t)), \quad t \in J. \tag{3.127}$$

This implies, for each $t \in J$, that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq \int_0^T p(s)\,ds < \int_{v(0)}^{\infty} \frac{du}{\psi(u)}. \tag{3.128}$$

This inequality implies that there exists a constant b such that $v(t) \leq b$, $t \in J$, and hence $\mu(t) \leq b$, $t \in J$. Since for every $t \in [0, T]$, $\|y_t\|_{\mathcal{D}} \leq \mu(t)$, we have

$$\|y\| = \sup\{|y(t)| : -r \leq t \leq T\} \leq b, \tag{3.129}$$

where b depends only T and on the functions p and ψ. This shows that M is bounded.

Set $X := \Omega$. As a consequence of Theorem 1.7, we deduce that N has a fixed point which is a solution of (3.101). □

For the next part, we study the case where F is not necessarily convex-valued. Our approach here is based on Schaefer's fixed point theorem combined with a selection theorem due to Bressan and Colombo [105] for lower semicontinuous multivalued operators.

Theorem 3.12. *Suppose that (3.3.1), (3.5.1), (3.11.2), and the following conditions are satisfied.*

(3.12.1) $F : [0,T] \times \mathcal{D} \to \mathcal{P}(E)$ *is a nonempty, compact-valued, multivalued map such that*
  (a) $(t,u) \mapsto F(t,u)$ *is* $\mathcal{L} \otimes \mathcal{B}$ *measurable;*
  (b) $u \mapsto F(t,u)$ *is lower semicontinuous for a.e.* $t \in [0,T]$.

(3.12.2) *For each* $\rho > 0$, *there exists a function* $h_\rho \in L^1([0,T],\mathbb{R}^+)$ *such that for* $u \in \mathcal{D}$ *with* $\|u\|_{\mathcal{D}} \le \rho$,

$$||F(t,u)|| = \sup\{|v| : v \in F(t,u)\} \le h_\rho(t) \quad \text{for a.e. } t \in [0,T]. \tag{3.130}$$

Then the impulsive initial value problem (3.101) *has at least one solution.*

Proof. Assumptions (3.12.1) and (3.12.2) imply that F is of lower semicontinuous type. Then, from Theorem 1.5, there exists a continuous function $f : \Omega \to L^1([0,T],\mathbb{R}^n)$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in \Omega$.

Consider the problem

$$\begin{aligned} y'(t) &= f(y_t), \quad t \in [0,T],\ t \neq t_k,\ k = 1,\dots,m, \\ \Delta y|_{t=t_k} &= I_k(y(t_k^-)), \quad k = 1,\dots,m, \\ y(t) &= \phi(t), \quad t \in [-r,0]. \end{aligned} \tag{3.131}$$

It is obvious that if $y \in \Omega$ is a solution of problem (3.131), then y is a solution to problem (3.101).

Transform problem (3.131) into a fixed point problem. Consider the operator $N : \Omega \to \Omega$ defined by

$$N(y)(t) := \begin{cases} \phi(t) & \text{if } t \in [-r,0], \\ \phi(0) + \displaystyle\int_0^t f(y_s)\,ds + \sum_{0<t_k<t} I_k(y(t_k^-)) & \text{if } t \in [0,T]. \end{cases} \tag{3.132}$$

As in Theorem 3.3, we can show that N is completely continuous, and the set

$$\mathcal{E}(N) := \{y \in \Omega : y = \lambda N(y) \text{ for some } 0 < \lambda < 1\} \tag{3.133}$$

is bounded. Set $X := \Omega$. As a consequence of Schaefer's fixed point theorem, we deduce that N has a fixed point y which is a solution to problem (3.131) and hence a solution to problem (3.101). □

Now by using a fixed point theorem for contraction multivalued operators given by Covitz and Nadler [123] we present a result for problem (3.101).

Theorem 3.13. *Assume the following are satisfied.*

(3.13.1) $F : [0,T] \times \mathcal{D} \to \mathcal{P}_{\mathrm{cp,cv}}(E)$ *has the property that* $F(\cdot,u) : [0,T] \to \mathcal{P}_{\mathrm{cp}}(E)$ *is measurable, for each* $u \in \mathcal{D}$.

(3.13.2) *$H_d(F(t,u),F(t,\overline{u})) \le l(t)\|u-\overline{u}\|_{\mathcal{D}}$, for each $t \in [0,T]$ and $u,\overline{u} \in \mathcal{D}$, where $l \in L^1([0,T],\mathbb{R})$; and*

$$d\big(0,F(t,0)\big) \le l(t) \quad \text{for a.e. } t \in J. \tag{3.134}$$

(3.13.3) *$|I_k(y) - I_k(\overline{y})| \le c_k|y-\overline{y}|$, for each $y,\overline{y} \in E$, $k = 1,\dots,m$, where c_k are nonnegative constants.*

If

$$\max\left\{\int_0^T l(s)ds + c_k : k = 1,\dots,m\right\} < 1, \tag{3.135}$$

then the IVP (3.101) *has at least one solution on* $[-r,T]$.

Proof. Transform problem (3.101) into a fixed point problem. Consider first problem (3.101) on the interval $[-r,t_1]$, that is, the problem

$$\begin{aligned} y'(t) &\in F(t,y_t), \quad \text{a.e. } t \in (0,t_1), \\ y(t) &= \phi(t), \quad t \in [-r,0]. \end{aligned} \tag{3.136}$$

It is clear that the solutions of problem (3.136) are fixed points of the multivalued operator $N : \mathrm{PC}([-r,t_1]) \to \mathcal{P}(\mathrm{PC}[-r,t_1])$ defined by

$$N(y) := \left\{h \in \mathrm{PC}\big([-r,t_1]\big) : h(t) = \begin{cases}\phi(t) & \text{if } t \in [-r,0],\\ \phi(0) + \int_0^t g(s)ds & \text{if } t \in [0,t_1],\end{cases}\right\}, \tag{3.137}$$

where

$$g \in S_{F,y} = \{g \in L^1([0,t_1],E) : g(t) \in F(t,y_t) \text{ for a.e. } t \in [0,t_1]\}. \tag{3.138}$$

We will show that N satisfies the assumptions of Theorem 1.11. The proof will be given in two steps.

Step 1. $N(y) \in P_{\mathrm{cl}}(\mathrm{PC}([-r,t_1]))$, for each $y \in \mathrm{PC}([-r,t_1])$.

Indeed, let $(y_n)_{n\ge 0} \in N(y)$ such that $y_n \to \widetilde{y}$ in $\mathrm{PC}([-r,t_1])$. Then $\widetilde{y} \in \mathrm{PC}([-r,t_1])$ and, for each $t \in [0,t_1]$,

$$y_n(t) \in \phi(0) + \int_0^t F(s,y_s)ds. \tag{3.139}$$

Using the fact that F has compact values and from (3.13.2), we may pass to a subsequence if necessary to get that g_n converges to g in $L^1(J,E)$, and hence $g \in S_{F(y)}$. Then, for each $t \in J$,

$$y_n(t) \longrightarrow \widetilde{y}(t) \in \phi(0) + \int_0^t F(s,y_s)ds. \tag{3.140}$$

So $\widetilde{y} \in N(y)$.

Step 2. There exists $\gamma < 1$ such that $H(N(y), N(\overline{y})) \le \gamma \|y - \overline{y}\|_{[-r,t_1]}$, for each $y, \overline{y} \in PC([-r, t_1])$.

Let $y, \overline{y} \in PC([-r, t_1])$ and $h_1 \in N(y)$. Then there exists $g_1(t) \in F(t, y_t)$ such that, for each $t \in [0, t_1]$,

$$h_1(t) = \phi(0) + \int_0^t g_1(s)ds. \tag{3.141}$$

From (3.13.2), it follows that

$$H(F(t, y_t), F(t, \overline{y}_t)) \le l(t)\|y_t - \overline{y}_t\|_{\mathcal{D}}. \tag{3.142}$$

Hence there is $w \in F(t, \overline{y}_t)$ such that

$$|g_1(t) - w| \le l(t)\|y_t - \overline{y}_t\|_{\mathcal{D}}, \quad t \in [0, t_1]. \tag{3.143}$$

Consider $U : [0, t_1] \to \mathcal{P}(E)$, given by

$$U(t) = \{w \in E : |g_1(t) - w| \le l(t)\|y_t - \overline{y}_t\|_{\mathcal{D}}\}. \tag{3.144}$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \overline{y}_t)$ is measurable (see [119, Proposition III.4]), there exists a function $g_2(t)$, which is a measurable selection for V. So, $g_2(t) \in F(t, \overline{y}_t)$ and

$$|g_1(t) - g_2(t)| \le l(t)\|y - \overline{y}\|_{\mathcal{D}}, \quad \text{for each } t \in [0, t_1]. \tag{3.145}$$

Let us define, for each $t \in [0, t_1]$,

$$h_2(t) = \phi(0) + \int_0^t g_2(s)ds. \tag{3.146}$$

Then we have

$$\begin{aligned}
|h_1(t) - h_2(t)| &\le \int_0^t |g_1(s) - g_2(s)|ds \le \int_0^t l(s)\|y_{1s} - y_{2s}\|_{\mathcal{D}}ds \\
&= \int_0^t l(s)\Big(\sup_{-r\le\theta\le 0} |y_{1s}(\theta) - y_{2s}(\theta)|\Big)ds \\
&= \int_0^t l(s)\Big(\sup_{-r\le\theta\le 0} |y_1(s+\theta) - y_2(s+\theta)|\Big)ds \\
&= \int_0^t l(s)\Big(\sup_{s-r\le z\le s} |y_1(z) - y_2(z)|\Big)ds \\
&\le \int_0^t l(s)\Big(\sup_{-r\le z\le t_1} |y_1(z) - y_2(z)|\Big)ds \\
&\le \Big(\int_0^T l(s)ds\Big)\|y_1 - y_2\|_{[-r,t_1]}.
\end{aligned} \tag{3.147}$$

Then

$$||h_1 - h_2||_{[-r,t_1]} \le \left(\int_0^T l(s)ds \right) ||y_1 - y_2||_{[-r,t_1]}. \tag{3.148}$$

By the analogous relation, obtained by interchanging the roles of y_1 and y_2, it follows that

$$H_d(N(y_1), N(y_2)) \le \left(\int_0^T l(s)ds \right) ||y_1 - y_2||_{[-r,t_1]}. \tag{3.149}$$

So, N is a contraction and thus, by Theorem 1.11, N has a fixed point y_1, which is a solution to (3.136).

Now let $y_2 := y|_{[t_1,t_2]}$ be a solution to the problem

$$\begin{gathered} y'(t) \in F(t, y_t), \quad \text{a.e. } t \in (t_1, t_2), \\ \Delta y|_{t=t_1} = I_1(y(t_1^-)). \end{gathered} \tag{3.150}$$

Then y_2 is a fixed point of the multivalued operator $N : \mathrm{PC}([t_1, t_2]) \to \mathcal{P}(\mathrm{PC}([t_1, t_2]))$ defined by

$$N(y) := \left\{ h \in \mathrm{PC}([t_1, t_2]) : h(t) = \int_0^t g(s)ds + I_1(y(t_1)),\ t \in ([t_1, t_2]) \right\}, \tag{3.151}$$

where

$$g \in S_{F,y} = \{ g \in L^1([t_1, t_2], E) : g(t) \in F(t, y_t) \text{ for a.e. } t \in [t_1, t_2] \}. \tag{3.152}$$

We will show that N satisfies the assumptions of Theorem 1.11. Clearly, $N(y) \in \mathcal{P}_{\mathrm{cl}}(\mathrm{PC}([t_1, t_2]))$, for each $y \in \mathrm{PC}([t_1, t_2])$. It remains to show

$$H(N(y), N(\overline{y})) \le \gamma \| y - \overline{y} \|_{[t_1,t_2]}, \quad \text{for each } y, \overline{y} \in \mathrm{PC}([t_1, t_2]) \text{ (where } \gamma < 1). \tag{3.153}$$

To that end, let $y, \overline{y} \in \mathrm{PC}([t_1, t_2])$ and $h_1 \in N(y)$. Then there exists $g_1(t) \in F(t, y_t)$ such that, for each $t \in [t_1, t_2]$,

$$h_1(t) = \int_0^t g_1(s)ds + I_1(y(t_1)). \tag{3.154}$$

From (3.13.2), it follows that

$$H(F(t,y_t),F(t,\overline{y}_t)) \leq l(t)\|y_t-\overline{y}_t\|_{\mathcal{D}}. \tag{3.155}$$

Hence there is a $w \in F(t,\overline{y}_t)$ such that

$$|g_1(t)-w| \leq l(t)\|y_t-\overline{y}_t\|_{\mathcal{D}}, \quad t\in[t_1,t_2]. \tag{3.156}$$

Consider $U:[t_1,t_2]\to\mathcal{P}(E)$, given by

$$U(t)=\{w\in E: |g_1(t)-w| \leq l(t)\|y_t-\overline{y}_t\|_{\mathcal{D}}\}. \tag{3.157}$$

Since the multivalued operator $V(t) = U(t)\cap F(t,\overline{y}_t)$ is measurable (see [119, Proposition III.4]), there exists $g_2(t)$, which is a measurable selection for V. So, $g_2(t)\in F(t,\overline{y}_t)$ and

$$|g_1(t)-g_2(t)| \leq l(t)\|y_t-\overline{y}_t\|_{\mathcal{D}}, \quad \text{for each } t\in[t_1,t_2]. \tag{3.158}$$

Let us define, for each $t\in[t_1,t_2]$,

$$h_2(t)=\int_0^t g_2(s)ds+I_1(\overline{y}(t_1)). \tag{3.159}$$

Then we have

$$\begin{aligned}|h_1(t)-h_2(t)| &\leq \int_0^t |g_1(s)-g_2(s)|ds+|I_1(y(t_1))-I_1(\overline{y}(t_1))|\\ &\leq \int_0^t l(s)\|y_{1s}-y_{2s}\|_{\mathcal{D}}ds+c_1|y(t_1)-\overline{y}(t_1)|\\ &\leq \int_0^t l(s)\Big(\sup_{-r\leq\theta\leq 0}|y_{1s}(\theta)-y_{2s}(\theta)|\Big)ds+c_1|y(t_1)-\overline{y}(t_1)|\\ &\leq \Big(\int_0^T l(s)ds+c_1\Big)\|y-\overline{y}\|_{[t_1,t_2]}.\end{aligned} \tag{3.160}$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H(N(y),N(\overline{y})) \leq \Big(\int_0^T l(s)ds+c_1\Big)\|y-\overline{y}\|_{[t_1,t_2]}. \tag{3.161}$$

So, N is a contraction and thus, by Theorem 1.11, N has a fixed point y_2, which is solution to (3.150).

We continue this process taking into account that $y_m := y|_{[t_m,T]}$ is a solution to the problem

$$\begin{aligned} y'(t) &\in F(t, y_t), \quad \text{a.e. } t \in (t_m, T], \\ \Delta y|_{t=t_m} &= I_m(y(t_m^-)). \end{aligned} \tag{3.162}$$

The solution y of problem (3.101) is then defined by

$$y(t) = \begin{cases} y_1(t) & \text{if } t \in [-r, t_1], \\ y_2(t) & \text{if } t \in (t_1, t_2], \\ \vdots & \\ y_m(t) & \text{if } t \in (t_m, T]. \end{cases} \tag{3.163}$$

□

In this last part of Section 3.4, we establish existence results for problem (3.102).

Definition 3.14. A function $y \in \Omega \cap \mathrm{AC}^1((t_k, t_{k+1}), E)$, $k = 1, \dots, m$, is said to be a solution of (3.102) if y satisfies the differential inclusion $y''(t) \in F(t, y_t)$ a.e. on J' and the conditions $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $\Delta y'|_{t=t_k} = \bar{I}_k(y(t_k^-))$, $k = 1, \dots, m$.

Theorem 3.15. *Let (3.5.1), (3.5.2), and (3.11.1) hold. Suppose also the following are satisfied.*

(3.15.1) $\|F(t,u)\| := \sup\{|v| : v \in F(t,u)\} \le p(t)\psi(\|u\|_{\mathcal{D}})$ *for almost all* $t \in J$ *and all* $u \in \mathcal{D}$, *where* $p \in L^1(J, \mathbb{R}_+)$ *and* $\psi : \mathbb{R}_+ \to (0, \infty)$ *is continuous and increasing with*

$$\int_0^T (T-s)p(s)ds < \int_c^\infty \frac{d\tau}{\psi(\tau)}, \tag{3.164}$$

where $c = \|\phi\|_{\mathcal{D}} + T|\eta| + \sum_{k=1}^m [c_k + (T - t_k)d_k]$.

(3.15.2) *For each bounded* $B \subseteq \Omega$ *and for each* $t \in J$, *the set*

$$\begin{aligned} \Big\{ \phi(0) + t\eta + \int_0^t (t-s)g(s)ds \\ + \sum_{0<t_k<t} \big[I_k(y(t_k^-)) + (t-t_k)\bar{I}_k(y(t_k^-))\big] : g \in S_{F,B} \Big\} \end{aligned} \tag{3.165}$$

is relatively compact in E, where $S_{F,B} = \cup\{S_{F,y} : y \in B\}$.

Then the impulsive initial value problem (3.102) *has at least one solution on* $[-r, T]$.

Proof. Transform the problem into a fixed point problem. Consider the multivalued map $G : \Omega \to \mathcal{P}(\Omega)$ defined by

$$G(y) = \left\{ h \in \Omega : h(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \phi(0) + t\eta + \displaystyle\int_0^t (t-s)g(s)ds & \\ \quad + \displaystyle\sum_{0<t_k<t} \big[I_k(y(t_k^-)) & \\ \qquad\qquad + (t-t_k)\bar{I}_k(y(t_k^-))\big], & t \in [0, T], \end{cases} \right\}, \tag{3.166}$$

where $g \in S_{F,y}$.

We will show that G satisfies the assumptions of Theorem 1.7. As in Theorem 3.11, we can show that G is completely continuous. We will show now that the set

$$\mathcal{M} := \{ y \in \Omega : \lambda y \in G(y) \text{ for some } \lambda > 1 \} \tag{3.167}$$

is bounded.

Let $y \in \mathcal{M}$. Then $\lambda y \in G(y)$ for some $\lambda > 1$. Thus there exists $g \in S_{F,y}$ such that

$$\begin{aligned} y(t) = {} & \lambda^{-1}\phi(0) + \lambda^{-1}t\eta + \lambda^{-1}\int_0^t (t-s)g(s)ds \\ & + \lambda^{-1}\sum_{0<t_k<t} \big[I_k(y(t_k^-)) + (t-t_k)\bar{I}_k(y(t_k^-))\big], \quad t \in J. \end{aligned} \tag{3.168}$$

This implies that, for each $t \in J$, we have

$$|y(t)| \le \|\phi\|_{\mathcal{D}} + T|\eta| + \int_0^t (T-s)p(s)\psi(\|y_s\|_{\mathcal{D}})ds + \sum_{k=1}^m \big[c_k + (T-t_k)d_k\big]. \tag{3.169}$$

We consider the function μ defined by

$$\mu(t) = \sup\{ |y(s)| : -r \le s \le t \}, \quad 0 \le t \le T. \tag{3.170}$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0, T]$, by the previous inequality, we have for $t \in [0, T]$,

$$\mu(t) \le \|\phi\|_{\mathcal{D}} + T|\eta| + \int_0^t (T-s)p(s)\psi(\mu(s))ds + \sum_{k=1}^k \big[c_k + (T-t_k)d_k\big]. \tag{3.171}$$

If $t^* \in [-r, 0]$, then $\mu(t) = \|\phi\|_{\mathcal{D}}$ and the previous inequality holds.

Let us take the right-hand side of the above inequality as $v(t)$. Then, we have

$$
\begin{gathered}
c = v(0) = \|\phi\|_{\mathcal{D}} + T|\eta| + \sum_{k=1}^{m}\left[c_k + (T - t_k)d_k\right], \quad \mu(t) \leq v(t), \quad t \in [0, T], \\
v'(t) = (T - t)p(t)\psi(\mu(t)), \quad t \in [0, T], \\
v'(t) = (T - t)p(t)\psi(\mu(t)), \quad t \in J.
\end{gathered}
\tag{3.172}
$$

Using the nondecreasing character of ψ, we get

$$v'(t) \leq (T - t)p(t)\psi(v(t)), \quad t \in [0, T]. \tag{3.173}$$

This implies, for each $t \in J$, that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq \int_0^T (T - s)p(s)ds < \int_{v(0)}^{\infty} \frac{du}{\psi(u)}. \tag{3.174}$$

This inequality implies that there exists a constant b such that $v(t) \leq b$, $t \in J$, and hence $\mu(t) \leq b$, $t \in J$. Since for every $t \in [0, T]$, $\|y_t\|_{\mathcal{D}} \leq \mu(t)$, we have

$$\|y\|_{\Omega} \leq b, \tag{3.175}$$

where b depends only on T and on the functions p and ψ. This shows that $\mathcal{M}$ is bounded.

Set $X := \Omega$. As a consequence of Theorem 1.7, we deduce that G has a fixed point y which is a solution of problem (3.102). □

Theorem 3.16. *Assume hypotheses (3.5.1), (3.5.2), (3.12.1), (3.12.2), and (3.15.1) are satisfied. Then the IVP* (3.102) *has at least one solution.*

Proof. First, (3.12.1) and (3.12.2) imply that F is of lower semicontinuous type. Then from Theorem 1.5 there exists a continuous function $f : \Omega \to L^1([0, T], \mathbb{R}^n)$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in \Omega$.

Consider the problem

$$y''(t) = f(y_t), \quad t \in [0, T],\ t \neq t_k,\ k = 1, \dots, m, \tag{3.176}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{3.177}$$

$$\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{3.178}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \qquad y'(0) = \eta. \tag{3.179}$$

Transform problem (3.177)–(3.179) into a fixed point problem. Consider the operator $\overline{N} : \Omega \to \Omega$ defined by

$$\overline{N}(y)(t) := \begin{cases} \phi(t) & \text{if } t \in [-r, 0], \\ \phi(0) + t\eta + \int_0^t (t-s) f(y_s)\,ds & \\ \quad + \sum_{0<t_k<t} \left[I_k(y(t_k^-)) + (t - t_k)\overline{I}_k(y(t_k^-)) \right] & \text{if } t \in [0, T]. \end{cases} \tag{3.180}$$

As in Theorem 3.5, we can show that $\overline{N}$ is completely continuous and that the set

$$\mathcal{E}(\overline{N}) := \{ y \in \Omega : y = \lambda \overline{N}(y) \text{ for some } 0 < \lambda < 1 \} \tag{3.181}$$

is bounded.

Set $X := \Omega$. As a consequence of Schaefer's fixed point theorem, we deduce that $\overline{N}$ has a fixed point y which is a solution to problem (3.176)–(3.179) and hence a solution to problem (3.102). □

Theorem 3.17. *Assume that (3.13.1)–(3.13.3) and the following condition hold.*

(3.17.1) $|\overline{I}_k(y) - \overline{I}_k(\overline{y})| \le d_k' |y - \overline{y}|$, *for each* $y, \overline{y} \in E$, $k = 1, \ldots, m$, *where* d_k' *are nonnegative constants.*

If

$$T \int_0^T l(s)\,ds + \sum_{k=1}^m c_k + \sum_{k=1}^m (T - t_k) d_k' < 1, \tag{3.182}$$

then the IVP (3.102) *has at least one solution on* $[-r, T]$.

Proof. Transform problem (3.102) into a fixed point problem. Consider the operator $N : \Omega \to \mathcal{P}(\Omega)$ defined by

$$N(y) := \left\{ h \in \Omega : h(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \phi(0) + t\eta + \int_0^t (t-s) g(s)\,ds & \\ \quad + \sum_{0<t_k<t} \left[I_k(y(t_k^-)) \right. & \\ \qquad \left. + (t - t_k)\overline{I}_k(y(t_k^-)) \right], & t \in J, \end{cases} \right\}, \tag{3.183}$$

where $g \in S_{F,y}$.

We can easily show that $N(y) \in P_{\text{cl}}(\Omega)$, for each $y \in \Omega$.

There remains to show that N is a contraction multivalued operator. Indeed, let $y, \overline{y} \in \Omega$, and $h_1 \in N(y)$. Then there exists $g_1(t) \in F(t, y_t)$ such that, for $t \in J$,

$$h_1(t) = \phi(0) + t\eta + \int_0^t (t-s)g_1(s)ds + \sum_{0<t_k<t} \left[I_k\left(y(t_k^-)\right) + (t-t_k)\bar{I}_k\left(y(t_k^-)\right)\right]. \tag{3.184}$$

From (3.13.2), it follows that

$$H\left(F(t,y_t), F(t,\overline{y}_t)\right) \le l(t)\|y_t - \overline{y}_t\|_{\mathcal{D}}. \tag{3.185}$$

Hence there is $w \in F(t, \overline{y}_t)$ such that

$$\left|g_1(t) - w\right| \le l(t)\|y_t - \overline{y}_t\|_{\mathcal{D}}, \quad t \in J. \tag{3.186}$$

Consider $U : J \to \mathcal{P}(E)$, given by

$$U(t) = \left\{w \in E : \left|g_1(t) - w\right| \le l(t)\|y_t - \overline{y}_t\|_{\mathcal{D}}\right\}. \tag{3.187}$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \overline{y}_t)$ is measurable (see [119, Proposition III.4]), there exists $g_2(t)$, a measurable selection for V. So, $g_2(t) \in F(t, \overline{y}_t)$ and

$$\left|g_1(t) - g_2(t)\right| \le l(t)\|y - \overline{y}\|_{\mathcal{D}}, \quad \text{for each } t \in J. \tag{3.188}$$

Let us define, for each $t \in J$,

$$h_2(t) = \phi(0) + t\eta + \int_0^t (t-s)g_2(s)ds + \sum_{0<t_k<t} \left[I_k\left(y(t_k^-)\right) + (t-t_k)\bar{I}_k\left(y(t_k^-)\right)\right]. \tag{3.189}$$

Then, we have

$$\begin{aligned}
\left|h_1(t) - h_2(t)\right| &\le \int_0^t (t-s)\left|g_1(s) - g_2(s)\right| ds + \sum_{0<t_k<t} \left|I_k\left(y(t_k^-)\right) - I_k\left(\overline{y}(t_k^-)\right)\right| \\
&\quad + \sum_{0<t_k<t} (T-t_k)\left|\bar{I}_k\left(y(t_k^-)\right) - \bar{I}_k\left(\overline{y}(t_k^-)\right)\right| \\
&\le \left(T\int_0^T l(s)ds\right)\|y - \overline{y}\| + \sum_{k=1}^m c_k\|y - \overline{y}\| + \sum_{k=1}^m (T-t_k)d_k'\|y - \overline{y}\|.
\end{aligned} \tag{3.190}$$

Then

$$\|h_1 - h_2\|_\Omega \le \left[T\int_0^T l(s)ds + \sum_{k=1}^m \left(c_k + (T-t_k)d_k'\right)\right]\|y - \overline{y}\|. \tag{3.191}$$

Again, by an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H(N_1(y), N_1(\overline{y})) \leq \left[T \int_0^T l(s)ds + \sum_{k=1}^{m} (c_k + (T - t_k)d'_k)\right] \|y - \overline{y}\|. \tag{3.192}$$

So, N is a contraction, and thus, by Theorem 1.11, N has a fixed point y, which is a solution to (3.102). □

3.5. Impulsive neutral functional DIs

In this section, we are concerned with the existence of solutions for first- and second-order initial value problems for neutral functional differential inclusions with impulsive effects,

$$\begin{gathered}\frac{d}{dt}[y(t) - g(t, y_t)] \in F(t, y_t), \quad t \in J := [0, T],\ t \neq t_k,\ k = 1, \dots, m, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \\ y(t) = \phi(t), \quad t \in [-r, 0],\end{gathered} \tag{3.193}$$

where F, I_k, ϕ are as in problem (3.101) and $g : J \times \mathcal{D} \to E$ and

$$\frac{d}{dt}[y'(t) - g(t, y_t)] \in F(t, y_t), \quad t \in J := [0, T],\ t \neq t_k,\ k = 1, \dots, m, \tag{3.194}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{3.195}$$

$$\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{3.196}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \qquad y'(0) = \eta, \tag{3.197}$$

where F, I_k, ϕ are as in problem (3.101), g as in problem (3.193), and $\overline{I}_k$, η as in (3.102).

Definition 3.18. A function $y \in \Omega \cap AC((t_k, t_{k+1}), E)$, $k = 0, \dots, m$, is said to be a solution of (3.193) if $y(t) - g(t, y_t)$ is absolutely continuous on J' and (3.193) are satisfied.

Theorem 3.19. *Assume that (3.2.1), (3.5.1), (3.7.1), and the following conditions hold.*

(3.19.1) $\|F(t, u)\| = \sup\{|v| : v \in F(t, u)\} \leq p(t)\psi(\|u\|_{\mathcal{D}})$ *for almost all* $t \in J$ *and all* $u \in \mathcal{D}$, *where* $p \in L^1(J, \mathbb{R}_+)$ *and* $\psi : \mathbb{R}_+ \to (0, \infty)$ *is continuous and increasing with*

$$\frac{1}{1 - c_1^*} \int_0^T p(s)ds < \int_c^\infty \frac{d\tau}{\psi(\tau)}, \tag{3.198}$$

where $c = (1/(1 - c_1^*))\{(1 + c_1^*)\|\phi\|_{\mathcal{D}} + 2c_2^* + \sum_{k=1}^m c_k\}$.

(3.19.2) *For each bounded $B \subseteq \Omega$ and $t \in J$, the set*

$$\left\{\phi(0) + \int_0^t v(s)ds + \sum_{0<t_k<t} I_k\big(y(t_k^-)\big) : v \in S_{F,B}\right\} \tag{3.199}$$

is relatively compact in E, where $S_{F,B} = \cup\{S_{F,y} : y \in B\}$.
Then the IVP (3.193) *has at least one solution on* $[-r, T]$.

Proof. Consider the operator $N : \Omega \to \mathcal{P}(\Omega)$ defined by

$$N(y) = \left\{h \in \Omega : h(t)\begin{cases}\phi(t), & t \in [-r,0],\\ \phi(0) - g(0,\phi(0)) + g(t,y_t) & \\ \quad + \displaystyle\int_0^t v(s)ds + \sum_{0<t_k<t} I_k\big(y(t_k^-)\big), & t \in [0,T],\end{cases}\right\}, \tag{3.200}$$

where $v \in S_{F,y}$.

We will show that N satisfies the assumptions of Theorem 1.7. Using (3.7.1), it suffices to show that the operator $N_1 : \Omega \to \mathcal{P}(\Omega)$ defined by

$$N_1(y) = \left\{h \in \Omega : h(t) = \begin{cases}\phi(t), & t \in [-r,0],\\ \phi(0) + \displaystyle\int_0^t v(s)ds & \\ \quad + \displaystyle\sum_{0<t_k<t} I_k\big(y(t_k^-)\big), & t \in [0,T],\end{cases}\right\}, \tag{3.201}$$

where $v \in S_{F,y}$, is u.s.c. and condensing with bounded, closed, and convex values. The proof will be given in several steps.

Step 1. $N_1(y)$ is convex, for each $y \in \Omega$.

This is obvious since $S_{F,y}$ is convex (because F has convex values).

Step 2. N_1 maps bounded sets into relatively compact sets in Ω.

This is a consequence of the L^1-Carathéodory character of F. As a consequence of Steps 1 and 2 and (3.19.2) together with the Arzelá-Ascoli theorem, we can conclude that $N : \Omega \to \mathcal{P}(\Omega)$ is a completely continuous multivalued map and therefore a condensing map.

Step 3. N has a closed graph.

Let $y_n \to y_*$, $h_n \in N(y_n)$, and $h_n \to h_*$. We will prove that $h_* \in N(y_*)$.

$h_n \in N(y_n)$ means that there exists $v_n \in S_{F,y_n}$ such that, for each $t \in J$,

$$h_n(t) = \phi(0) - g(0,\phi(0)) + g(t,y_{nt}) + \int_0^t v_n(s)ds + \sum_{0<t_k<t} I_k\big(y_n(t_k^-)\big). \tag{3.202}$$

We must prove that there exists $v_* \in S_{F,y_*}$ such that, for each $t \in J$,

$$h_*(t) = \phi(0) - g(0,\phi(0)) + g(t,y_{*t}) + \int_0^t v_*(s)ds + \sum_{0<t_k<t} I_k(y_*(t_k^-)). \quad (3.203)$$

Since the functions $g(t,\cdot)$, $t \in J$, I_k, $k = 1,\dots,m$, are continuous, we have

$$\begin{aligned} &\Bigg\|\Bigg(h_n - \phi(0) + g(0,\phi(0)) - g(t,y_{nt}) - \sum_{0<t_k<t} I_k(y_n(t_k^-))\Bigg) \\ &\quad - \Bigg(h_* - \phi(0) + g(0,\phi(0)) - g(t,y_{*t}) - \sum_{0<t_k<t} I_k(y_*(t_k^-))\Bigg)\Bigg\|_\Omega \longrightarrow 0, \end{aligned} \quad (3.204)$$

as $n \to \infty$.

Consider the linear continuous operator

$$\begin{aligned} \Gamma : L^1(J,E) &\longrightarrow C(J,E), \\ v \longmapsto \Gamma(v)(t) &= \int_0^t v(s)ds. \end{aligned} \quad (3.205)$$

By Lemma 1.28, it follows that $\Gamma \circ S_F$ is a closed graph operator.

Moreover, we have

$$\Bigg(h_n(t) - \phi(0) + g(0,\phi(0)) - g(t,y_{nt}) - \sum_{0<t_k<t} I_k(y_n(t_k^-))\Bigg) \in \Gamma(S_{F,y_n}). \quad (3.206)$$

Since $y_n \to y_*$, it follows from Lemma 1.28 that

$$\Bigg(h_*(t) - \phi(0) + g(0,\phi(0)) - g(t,y_{*t}) - \sum_{0<t_k<t} I_k(y_*(t_k^-))\Bigg) = \int_0^t v_*(s)ds \quad (3.207)$$

for some $g_* \in S_{F,y_*}$.

Step 4. Now it remains to show that the set

$$\mathcal{M} := \{y \in \Omega : \lambda y \in N(y) \text{ for some } \lambda > 1\} \quad (3.208)$$

is bounded.

Let $y \in \mathcal{M}$. Then $y \in \lambda N(y)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$,

$$\begin{aligned} y(t) &= \lambda^{-1}\phi(0) - \lambda^{-1}g(0,\phi(0)) + \lambda^{-1}g(t,y_t) \\ &\quad + \lambda^{-1}\int_0^t v(s)ds + \lambda^{-1}\sum_{0<t_k<t} I_k(y(t_k^-)). \end{aligned} \quad (3.209)$$

This implies by our assumptions that, for each $t \in J$, we have

$$|y(t)| \leq \|\phi\|_{\mathcal{D}} + c_1^*\|\phi\|_{\mathcal{D}} + 2c_2^* + c_1^*\|y_t\|_{\mathcal{D}} + \int_0^t p(s)\psi(\|y_s\|_{\mathcal{D}})ds + \sum_{k=1}^m c_k. \tag{3.210}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq T. \tag{3.211}$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0,T]$, by the previous inequality, we have, for $t \in [0,T]$,

$$\begin{aligned}
\mu(t) \leq{}& \|\phi\|_{\mathcal{D}} + c_1^*\|\phi\|_{\mathcal{D}} + 2c_2^* + c_1^*\|y_t\|_{\mathcal{D}} + \int_0^t p(s)\psi(\|y_s\|_{\mathcal{D}})ds \\
&+ \sum_{0<t_k<t} |I_k(y(t_k^-))| \\
\leq{}& \|\phi\|_{\mathcal{D}} + c_1^*\|\phi\|_{\mathcal{D}} + 2c_2^* + c_1^*\mu(t) + \int_0^t p(s)\psi(\mu(s))ds + \sum_{k=1}^m c_k.
\end{aligned} \tag{3.212}$$

Thus

$$\mu(t) \leq \frac{1}{1-c_1^*}\left\{(1+c_1^*)\|\phi\|_{\mathcal{D}} + 2c_2^* + \int_0^t p(s)\psi(\mu(s))ds + \sum_{k=1}^m c_k\right\}. \tag{3.213}$$

If $t^* \in [-r,0]$, then $\mu(t) = \|\phi\|_{\mathcal{D}}$ and the previous inequality holds.

Let us take the right-hand side of the above inequality as $v(t)$. Then, we have

$$\begin{gathered}
c = v(0) = \frac{1}{1-c_1^*}\left\{(1+c_1^*)\|\phi\|_{\mathcal{D}} + 2c_2^* + \sum_{k=1}^m c_k\right\}, \quad \mu(t) \leq v(t), \quad t \in J, \\
v'(t) = \frac{1}{1-c_1^*}p(t)\psi(\mu(t)), \quad t \in J.
\end{gathered} \tag{3.214}$$

Using the nondecreasing character of ψ, we get

$$v'(t) \leq \frac{1}{1-c_1^*}p(t)\psi(v(t)), \quad t \in J. \tag{3.215}$$

This implies, for each $t \in J$, that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq \frac{1}{1-c_1^*}\int_0^T p(s)ds < \int_{v(0)}^{\infty} \frac{du}{\psi(u)}. \tag{3.216}$$

This inequality implies that there exists a constant b such that $v(t) \leq b$, $t \in J$, and hence $\mu(t) \leq b$, $t \in J$. Since for every $t \in [0,T]$, $\|y_t\|_{\mathcal{D}} \leq \mu(t)$, we have

$$\|y\|_{\Omega} \leq b' = \max\{\|\phi\|_{\mathcal{D}}, b\}, \tag{3.217}$$

where b' depends only T and on the functions p and ψ. This shows that $\mathcal{M}$ is bounded.

Set $X := \Omega$. As a consequence of Theorem 1.7, we deduce that N has a fixed point which is a solution of (3.193). □

Theorem 3.20. *Assume that hypotheses (3.5.1), (3.7.1), (3.12.1), (3.12.2), and (3.19.1) hold. Then problem* (3.193) *has at least one solution.*

Proof. (3.12.1) and (3.12.2) imply by Lemma 1.29 that F is of lower semicontinuous type. Then from Theorem 1.5, there exists a continuous function $f : \Omega \to L^1([0,T],E)$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in \Omega$. Consider the problem

$$\begin{gathered}\frac{d}{dt}[y(t) - g(t,y_t)] = f(y_t), \quad t \in J,\ t \neq t_k,\ k = 1,\dots,m, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m, \\ y(t) = \phi(t), \quad t \in [-r,0].\end{gathered} \tag{3.218}$$

Transform the problem into a fixed point problem. Consider the operator $\overline{N}_1 : \Omega \to \Omega$ defined by

$$\overline{N}_1(y)(t) := \begin{cases}\phi(t) & \text{if } t \in [-r,0], \\ \phi(0) - g(0,\phi(0)) + g(t,y_t) + \displaystyle\int_0^t f(y_s)\,ds & \\ \quad + \displaystyle\sum_{0<t_k<t} I_k(y(t_k^-)) & \text{if } t \in [0,T].\end{cases} \tag{3.219}$$

We will show that $\overline{N}_1$ is a completely continuous multivalued operator. Using (3.7.1), it suffices to show that the operator $\widetilde{N}_1 : \Omega \to \Omega$ defined by

$$\widetilde{N}_1(y)(t) := \begin{cases}\phi(t) & \text{if } t \in [-r,0], \\ \phi(0) + \displaystyle\int_0^t f(y_s)\,ds + \sum_{0<t_k<t} I_k(y(t_k^-)) & \text{if } t \in [0,T],\end{cases} \tag{3.220}$$

is completely continuous. This was proved in Theorem 3.12. Also, as in Theorem 3.19, we can prove that the set

$$\mathcal{E}(\widetilde{N}_1) := \{y \in \Omega : y = \lambda \widetilde{N}_1(y) \text{ for some } 0 < \lambda < 1\} \tag{3.221}$$

is bounded.

Set $X := \Omega$. As a consequence of Schaefer's fixed point theorem, we deduce that $\overline{N}$ has a fixed point y which is a solution to problem (3.218) and hence a solution to problem (3.193). □

Theorem 3.21. *Assume (3.13.1)–(3.13.3) and the following condition holds.*

(3.21.1) $|g(t,u) - g(t,\overline{u})| \le p\|u - \overline{u}\|_{\mathcal{D}}$, *for each* $u, \overline{u} \in \mathcal{D}$, *where* p *is a nonnegative constant.*

If

$$\int_0^T l(s)ds + p + \sum_{k=1}^m c_k < 1, \tag{3.222}$$

then the IVP (3.193) *has at least one solution on* $[-r, T]$.

Proof. Transform problem (3.193) into a fixed point problem. It is clear that the solutions of problem (3.193) are fixed points of the multivalued operator $N : \Omega \to \mathcal{P}(\Omega)$ defined by

$$N(y) := \left\{ h \in \Omega : h(t) = \begin{cases} \phi(t) & \text{if } t \in [-r, 0], \\ \phi(0) - g(0, \phi(0)) + g(t, y_t) & \\ \quad + \displaystyle\int_0^t v(s)ds + \sum_{0<t_k<t} I_k(y(t_k^-)) & \text{if } t \in J, \end{cases} \right\}, \tag{3.223}$$

where $v \in S_{F,y}$.

We will show that N satisfies the assumptions of Theorem 1.11.

The proof will be given in two steps.

Step 1. $N(y) \in P_{cl}(\Omega)$, for each $y \in \Omega$.

Indeed, let $(y_n)_{n\ge 0} \in N(y)$ such that $y_n \to \tilde{y}$ in Ω. Then $\tilde{y} \in \Omega$ and, for each $t \in J$,

$$y_n(t) \in \phi(0) - g(0, \phi(0)) + g(t, y_t) + \int_0^t F(s, y_s)ds + \sum_{0<t_k<t} I_k(y(t_k^-)). \tag{3.224}$$

Using the fact that F has compact values and from (3.13.2), we may pass to a subsequence if necessary to get that g_n converges to g in $L^1(J, E)$ and hence

$g \in S_{F(y)}$. Then, for each $t \in J$,

$$y_n(t) \longrightarrow \widetilde{y}(t) \in \phi(0) - g(0, \phi(0)) + g(t, y_t) + \int_0^t F(s, y_s)\,ds + \sum_{0<t_k<t} I_k(y(t_k^-)). \tag{3.225}$$

So $\widetilde{y} \in N(y)$.

Step 2. $H(N(y), N(\overline{y})) \leq \gamma \|y - \overline{y}\|_\Omega$, for each $y, \overline{y} \in \Omega$ (where $\gamma < 1$).

Let $y, \overline{y} \in \Omega$, and $h_1 \in N(y)$. Then there exists $v_1(t) \in F(t, y_t)$ such that, for each $t \in J$,

$$h_1(t) = \phi(0) - g(0, \phi(0)) + g(t, y_t) + \int_0^t v_1(s)\,ds + \sum_{0<t_k<t} I_k(y(t_k^-)). \tag{3.226}$$

From (3.13.2), it follows that

$$H(F(t, y_t), F(t, \overline{y}_t)) \leq l(t)\|y_t - \overline{y}_t\|_{\mathcal{D}}, \quad t \in J. \tag{3.227}$$

Hence there is $w \in F(t, \overline{y}_t)$ such that

$$|v_1(t) - w| \leq l(t)\|y_t - \overline{y}_t\|_{\mathcal{D}}, \quad t \in J. \tag{3.228}$$

Consider $U : J \to \mathcal{P}(E)$, given by

$$U(t) = \{w \in E : |v_1(t) - w| \leq l(t)\|y_t - \overline{y}_t\|_{\mathcal{D}}\}. \tag{3.229}$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \overline{y}_t)$ is measurable (see [119, Proposition III.4]), there exists $v_2(t)$, which is a measurable selection for V. So, $v_2(t) \in F(t, \overline{y}_t)$ and

$$|v_1(t) - v_2(t)| \leq l(t)\|y - \overline{y}\|_{\mathcal{D}}, \quad \text{for each } t \in J. \tag{3.230}$$

Let us define, for each $t \in J$,

$$h_2(t) = \phi(0) - g(0, \phi(0)) + g(t, \overline{y}_t) + \int_0^t v_2(s)\,ds + \sum_{0<t_k<t} I_k(\overline{y}(t_k^-)). \tag{3.231}$$

Then we have

$$\begin{aligned} |h_1(t)-h_2(t)| &\le \int_0^t |v_1(s)-v_2(s)|\,ds + |g(t,y_t)-g(t,\overline{y}_t)| \\ &\quad + \sum_{0<t_k<t} |I_k(y(t_k^-)) - I_k(\overline{y}(t_k^-))| \\ &\le \int_0^t l(s)\|y_s-\overline{y}_s\|_{\mathcal{D}}ds + p\|y_t-\overline{y}_t\|_{\mathcal{D}} + \sum_{k=1}^m c_k\|y-\overline{y}\| \\ &\le \left(\int_0^T l(s)ds + p + \sum_{k=1}^m c_k\right)\|y-\overline{y}\|. \end{aligned} \tag{3.232}$$

Then

$$\|h_1-h_2\| \le \left(\int_0^T l(s)ds + p + \sum_{k=1}^m c_k\right)\|y-\overline{y}\|. \tag{3.233}$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H(N(y),N(\overline{y})) \le \left(\int_0^T l(s)ds + p + \sum_{k=1}^m c_k\right)\|y-\overline{y}\|. \tag{3.234}$$

So, N is a contraction and thus, by Theorem 1.11, N has a fixed point y, which is a solution to (3.193). □

In this last part, we present results concerning problem (3.194)–(3.197).

Definition 3.22. A function $y \in \Omega \cap AC^1((t_k,t_{k+1}),E)$, $k=0,\dots,m$, is said to be a solution of (3.194)–(3.197) if y and $y'(t)-g(t,y_t)$ are absolutely continuous on J' and (3.194) to (3.197) are satisfied.

Theorem 3.23. *Assume (3.2.1), (3.5.1), (3.5.2), (3.7.1) (with $c_1 \ge 0$ in (iii)), and the following conditions hold.*

(3.23.1) $\|F(t,u)\| \le p(t)\psi(\|u\|_{\mathcal{D}})$ *for almost all $t\in J$ and all $u\in\mathcal{D}$, where $p\in L^1(J,\mathbb{R}_+)$ and $\psi:\mathbb{R}_+\to(0,\infty)$ is continuous and increasing with*

$$\int_0^T M(s)ds < \int_{\overline{c}}^{\infty} \frac{ds}{s+\psi(s)}, \tag{3.235}$$

where $\overline{c} = \|\phi\|_{\mathcal{D}} + [\|\eta\| + c_1^*\|\phi\|_{\mathcal{D}} + 2c_2^*]T + \sum_{k=1}^m [c_k + (T-t_k)d_k]$, *and* $M(t) = \max\{1, c_1^*, p(t)\}$.

(3.23.2) *For each bounded $B \subseteq \Omega$ and $t \in J$, the set*

$$\begin{aligned}\Big\{&\phi(0) + t\eta + \int_0^t \int_0^s v(u)du\,ds \\ &+ \sum_{0<t_k<t} \big[I_k\big(y(t_k^-)\big) + (t-t_k)\bar{I}_k\big(y(t_k^-)\big)\big] : v \in S_{F,B}\Big\}\end{aligned} \tag{3.236}$$

is relatively compact in E, where $S_{F,B} = \cup\{S_{F,y} : y \in B\}$.

Then the IVP (3.194)–(3.197) *has at least one solution on* $[-r, T]$.

Proof. Transform the problem into a fixed point problem. Consider the operator $N^* : \Omega \to \Omega$ defined by

$$N^*(y) = \left\{ h \in \Omega : h(t) \begin{cases} \phi(t), & t \in [-r,0], \\ \phi(0) + [\eta - g(0,\phi(0))]t & \\ \quad + \displaystyle\int_0^t g(s,y_s)ds + \int_0^t \int_0^u v(u)du\,ds & \\ \quad + \displaystyle\sum_{0<t_k<t} \big[I_k\big(y(t_k^-)\big) & \\ \qquad + (t-t_k)\bar{I}_k\big(y(t_k^-)\big)\big], & t \in J, \end{cases} \right\}, \tag{3.237}$$

where $v \in S_{F,y}$. As in Theorem 3.11, we can prove that N^* is a bounded-, closed-, and convex-valued multivalued map and is u.s.c. and that the set

$$\mathcal{E}(N^*) := \{y \in \Omega : y \in \lambda N^*(y) \text{ for some } 0 < \lambda < 1\} \tag{3.238}$$

is bounded. We omit the details.

Set $X := \Omega$. As a consequence of Theorem 1.7, we deduce that N^* has a fixed point y which is a solution to problem (3.194)–(3.197). □

Theorem 3.24. *Assume that (3.5.1), (3.5.2), [(3.7.1)(i), (iii)], (3.12.1), (3.12.2), and (3.23.1) are satisfied. Then the IVP* (3.194)–(3.197) *has a least one solution.*

Proof. Conditions (3.12.1) and (3.12.2) imply by Lemma 1.29 that F is of lower semicontinuous type. Then from Theorem 1.5, there exists a continuous function $f : \Omega \to L^1([0,T],E)$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in \Omega$. Consider the problem

$$\begin{gathered} \frac{d}{dt}[y'(t) - g(t,y_t)] = f(y_t), \quad t \in [0,T],\ t \neq t_k,\ k = 1,\dots,m, \\ \Delta y|_{t=t_k} = I_k\big(y(t_k^-)\big), \quad k = 1,\dots,m, \\ \Delta y'|_{t=t_k} = \bar{I}_k\big(y(t_k^-)\big), \quad k = 1,\dots,m, \\ y(t) = \phi(t), \quad t \in [-r,0], \quad y'(0) = \eta. \end{gathered} \tag{3.239}$$

Transform problem (3.239) into a fixed point problem. Consider the operator $\overline{N}_2 : \Omega \to \Omega$ defined by

$$\overline{N}_2(y)(t) := \begin{cases} \phi(t) & \text{if } t \in [-r, 0], \\ \phi(0) + [\eta - g(0, \phi(0))]t + \int_0^t g(s, y_s)\,ds & \\ \quad + \int_0^t (t-s) f(y_s)\,ds & \\ \quad + \sum_{0<t_k<t} [I_k(y(t_k^-)) + (t - t_k)\overline{I}_k] & \text{if } t \in [0, T]. \end{cases} \tag{3.240}$$

As in Theorem 3.7, we can show that $\overline{N}_2$ is completely continuous.

Now we prove only that the set

$$\mathcal{E}(\overline{N}_2) := \{y \in \Omega : y = \lambda \overline{N}_2(y) \text{ for some } 0 < \lambda < 1\} \tag{3.241}$$

is bounded.

Let $y \in \mathcal{E}(\overline{N}_2)$. Then $y = \lambda \overline{N}_2(y)$ for some $0 < \lambda < 1$. Thus

$$\begin{aligned} y(t) = {} & \lambda\phi(0) + \lambda[\eta - g(0, \phi(0))]t \\ & + \lambda \int_0^t g(s, y_s)\,ds + \lambda \int_0^t (t-s) f(y_s)\,ds \\ & + \lambda \sum_{0<t_k<t} [I_k(y(t_k^-)) + (t - t_k)\overline{I}_k]. \end{aligned} \tag{3.242}$$

This implies that, for each $t \in [0, T]$, we have

$$\begin{aligned} |y(t)| \le {} & \|\phi\|_{\mathcal{D}} + T(|\eta| + c_1^* \|\phi\|_{\mathcal{D}} + 2c_2^*) + \int_0^t c_1^* \|y_s\|_{\mathcal{D}}\,ds \\ & + \int_0^t (T-s) p(s) \psi(\|y_s\|_{\mathcal{D}})\,ds + \sum_{k=1}^m [c_k + (T - t_k) d_k]. \end{aligned} \tag{3.243}$$

We consider the function μ defined by

$$\mu(t) := \sup\{|y(s)| : -r \le s \le t\}, \quad 0 \le t \le T. \tag{3.244}$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by inequality (3.243), we have, for $t \in [0, T]$,

$$\begin{aligned} \mu(t) \le {} & \|\phi\|_{\mathcal{D}} + T(|\eta| + c_1^* \|\phi\|_{\mathcal{D}} + 2c_2^*) + \int_0^t M(s)\mu(s)\,ds \\ & + \int_0^t M(s) \psi(\mu(s))\,ds + \sum_{k=1}^m [c_k + (T - t_k) d_k]. \end{aligned} \tag{3.245}$$

If $t^* \in [-r,0]$, then $\mu(t) = \|\phi\|_{\mathcal{D}}$ and inequality (3.245) holds. Let us take the right-hand side of inequality (3.245) as $v(t)$. Then, we have

$$\begin{aligned} v(0) &= \|\phi\|_{\mathcal{D}} + T\big(|\eta| + c_1^*\|\phi\|_{\mathcal{D}} + 2c_2^*\big) + \sum_{k=1}^{m}\big(c_k + (T-s)d_k\big), \\ v'(t) &= M(t)\mu(t) + M(t)\psi\big(\mu(t)\big), \quad t \in [0,T]. \end{aligned} \tag{3.246}$$

Using the nondecreasing character of ψ, we get

$$v'(t) \le M(t)\big[\mu(t) + \psi\big(v(t)\big)\big], \quad t \in [0,T]. \tag{3.247}$$

This inequality implies, for each $t \in [0,T]$, that

$$\int_{v(0)}^{v(t)} \frac{d\tau}{\tau + \psi(\tau)} \le \int_0^T M(s)ds < \int_{v(0)}^{\infty} \frac{d\tau}{\tau + \psi(\tau)}. \tag{3.248}$$

This inequality implies that there exists a constant b such that $v(t) \le b$, $t \in [0,T]$, and hence $\mu(t) \le b$, $t \in [0,T]$. Since for every $t \in [0,T]$, $\|y_t\|_{\mathcal{D}} \le \mu(t)$, we have

$$\|y\| \le \max\big\{\|\phi\|_{\mathcal{D}}, b\big\}, \tag{3.249}$$

where b depends only on T and on the functions p and ψ. This shows that $\mathcal{E}(\overline{N}_2)$ is bounded.

Set $X := \Omega$. As a consequence of Schaefer's theorem, we deduce that $\overline{N}_2$ has a fixed point y which is a solution to problem (3.239). Then y is a solution to problem (3.194)–(3.197). □

Theorem 3.25. *Assume (3.13.1)–(3.13.3), (3.17.1), and (3.21.1) hold. If*

$$T\int_0^T l(s)ds + pT + \sum_{k=1}^{m}\big(c_k + (T-t_k)d_k\big) < 1, \tag{3.250}$$

then the IVP (3.194)–(3.197) *has at least one solution on* $[-r,T]$.

Proof. We transform problem (3.194)–(3.197) into a fixed point problem. Consider the operator $\overline{N} : \Omega \to \Omega$ defined by

$$\overline{N}(y) = \left\{ h \in \Omega : h(t) \begin{cases} \phi(t), & t \in [-r,0], \\ \phi(0) + [\eta - g(0,\phi(0))]t & \\ \quad + \displaystyle\int_0^t g(s,y_s)ds + \int_0^t\int_0^s v(\mu)d\mu\, ds & \\ \quad + \displaystyle\sum_{0<t_k<t} \big[I_k(y(t_k)) + (t-t_k)\overline{I}_k(y(t_k))\big], & t \in J, \end{cases} \right\}, \tag{3.251}$$

where $v \in S_{F,y}$. It is clear that the fixed points of $\overline{N}$ are solutions to problem (3.194)–(3.197). As in Theorem 3.21, we can easily prove that $\overline{N}$ has closed values.

We prove now that $H(\overline{N}(y),\overline{N}(\overline{y})) \leq \gamma\|y-\overline{y}\|$, *for each* $y,\overline{y} \in \Omega$ (*where* $\gamma < 1$).

Let $y,\overline{y} \in \Omega$ and $h_1 \in \overline{N}(y)$. Then there exists $v_1(t) \in F(t,y_t)$ such that, for each $t \in J$,

$$\begin{aligned} h_1(t) = \phi(0) - [\eta - g(0,\phi(0))]t + \int_0^t g(s,y_s)\,ds + \int_0^t\int_0^s v_1(\mu)\,d\mu\,ds \\ + \sum_{0<t_k<t}\left[I_k(y(t_k^-)) - (t-t_k)\overline{I}_k(y(t_k^-))\right]. \end{aligned} \tag{3.252}$$

From (3.13.2), it follows that

$$H(F(t,y_t),F(t,\overline{y}_t)) \leq l(t)\|y_t-\overline{y}_t\|_{\mathcal{D}}, \quad t \in J. \tag{3.253}$$

Hence there is $w \in F(t,\overline{y}_t)$ such that

$$|v_1(t)-w| \leq l(t)\|y_t-\overline{y}_t\|_{\mathcal{D}}, \quad t \in J. \tag{3.254}$$

Consider $U : J \to \mathcal{P}(E)$, given by

$$U(t) = \{w \in E : |v_1(t)-w| \leq l(t)\|y_t-\overline{y}_t\|_{\mathcal{D}}\}. \tag{3.255}$$

Since the multivalued operator $V(t) = U(t) \cap F(t,\overline{y}_t)$ is measurable (see [119, Proposition III.4]), there exists $v_2(t)$ a measurable selection for V. So, $v_2(t) \in F(t,\overline{y}_t)$ and

$$|v_1(t)-v_2(t)| \leq l(t)\|y-\overline{y}\|_{\mathcal{D}}, \quad \text{for each } t \in J. \tag{3.256}$$

Let us define, for each $t \in J$,

$$\begin{aligned} h_2(t) = \phi(0) - [\eta - g(0,\phi(0))]t + \int_0^t g(s,\overline{y}_s)\,ds + \int_0^t\int_0^s v_2(\mu)\,d\mu\,ds \\ + \sum_{0<t_k<t}\left[I_k(\overline{y}(t_k^-)) - (t-t_k)\overline{I}_k(\overline{y}(t_k^-))\right]. \end{aligned} \tag{3.257}$$

Then we have

$$\begin{aligned}
|h_1(t)-h_2(t)| &\le \int_0^t |g(s,y_s)-g(s,\overline{y}_s)|\,ds+\int_0^t\int_0^s |v_1(\mu)-v_2(\mu)|\,d\mu\,ds\\
&\quad+\sum_{0<t_k<t}|I_k(y(t_k^-))-I_k(\overline{y}(t_k^-))|\\
&\quad+\sum_{0<t_k<t}(T-t_k)|\overline{I}_k(y(t_k^-))-\overline{I}_k(\overline{y}(t_k^-))|\\
&\le p\int_0^t ||y_s-\overline{y}_s||_{\mathcal{D}}ds+T\int_0^t l(s)||y_s-\overline{y}_s||_{\mathcal{D}}ds\\
&\quad+\sum_{k=1}^m c_k\|y-\overline{y}\|+\sum_{k=1}^m (T-t_k)d_k\|y-\overline{y}\|\\
&\le \left[T\int_0^T l(s)ds+p+\sum_{k=1}^m (c_k+(T-t_k)d_k)\right]\|y-\overline{y}\|.
\end{aligned}\tag{3.258}$$

Then

$$||h_1-h_2||_{\Omega}\le\left[T\int_0^T l(s)ds+p+\sum_{k=1}^m (c_k+(T-t_k)d_k)\right]\|y-\overline{y}\|.\tag{3.259}$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H(\overline{N}(y),\overline{N}(\overline{y}))\le\left[T\int_0^T l(s)ds+p+\sum_{k=1}^m (c_k+(T-t_k)d_k)\right]\|y-\overline{y}\|.\tag{3.260}$$

So, $\overline{N}$ is a contraction and thus, by Theorem 1.11, $\overline{N}$ has a fixed point y, which is a solution to (3.194)–(3.197). □

3.6. Impulsive semilinear functional DIs

This section is concerned with the existence of mild solutions for first-order impulsive semilinear functional differential inclusions of the form

$$\begin{gathered}
y'(t)-Ay\in F(t,y_t),\quad t\in J=[0,b],\ t\neq t_k,\ k=1,\dots,m,\\
y(t_k^+)=I_k(y(t_k^-)),\quad k=1,\dots,m,\\
y(t)=\phi(t),\quad t\in[-r,0],
\end{gathered}\tag{3.261}$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in E, $F:J\times\mathcal{D}\to\mathcal{P}(E)$ is a bounded-, closed-, and convex-valued multivalued map, $\phi \in \mathcal{D}$, $(0 < r < \infty)$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, $I_k \in C(E,E)$ $(k = 1,2,\ldots,m)$, are bounded functions, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^-)$, and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively, and E a real separable Banach space with norm $|\cdot|$.

Definition 3.26. A function $y \in \Omega$ is said to be a mild solution of (3.261) if there exists a function $v \in L^1(J,E)$ such that $v(t) \in F(t,y(t))$ a.e. on J and

$$y(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ T(t)\phi(0) + \int_0^t T(t-s)v(s)ds, & t \in [0,t_1], \\ T(t-t_k)I_k\big(y(t_k^-)\big) + \int_{t_k}^t T(t-s)v(s)ds, & t \in J_k,\ k = 1,\ldots,m. \end{cases} \tag{3.262}$$

We are now in a position to state and prove our existence result for the IVP (3.261).

Theorem 3.27. *Suppose (3.11.1) holds and in addition assume that the following conditions are satisfied.*

(3.27.1) *A is the infinitesimal generator of a linear bounded semigroup $T(t)$, $t \geq 0$, which is compact for $t > 0$, and there exists $M > 1$ such that $\|T(t)\|_{B(E)} \leq M$.*

(3.27.2) *There exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and $p \in L^1(J,\mathbb{R}_+)$ such that*

$$||F(t,u)|| := \sup\{|v| : v \in F(t,u)\} \leq p(t)\psi(\|u\|_{\mathcal{D}}) \tag{3.263}$$

for a.e. $t \in J$ and each $u \in \mathcal{D}$ with

$$\int_{t_{k-1}}^{t_k} p(s)ds < \int_{N_{k-1}}^{\infty} \frac{d\tau}{\psi(\tau)}, \quad k = 1,\ldots,m+1, \tag{3.264}$$

where $N_0 = M\|\phi\|_{\mathcal{D}}$, and for $k = 2,\ldots,m+1$,

$$\begin{aligned} N_{k-1} &= \sup_{y \in [-M_{k-2},M_{k-2}]} M\big|I_{k-1}(y)\big|, \\ M_{k-2} &= \Gamma_{k-1}^{-1}\Big(M\int_{t_{k-2}}^{t_{k-1}} p(s)ds\Big), \end{aligned} \tag{3.265}$$

with

$$\Gamma_l(z) = \int_{N_{l-1}}^{z} \frac{d\tau}{\psi(\tau)}, \quad z \geq N_{l-1},\ l \in \{1,\ldots,m+1\}. \tag{3.266}$$

Then problem (3.261) *has at least one mild solution $y \in \Omega$.*

Proof. The proof is given in several steps.
Step 1. Consider problem (3.261) on $[-r, t_1]$,

$$\begin{aligned} y' - Ay \in F(t, y_t), \quad t \in J_0 = [0, t_1], \\ y(t) = \phi(t), \quad t \in [-r, 0]. \end{aligned} \tag{3.267}$$

We will show that the possible mild solutions of (3.267) are a priori bounded, that is, there exists a constant b_0 such that, if $y \in \Omega$ is a mild solution of (3.267), then

$$\sup\{|y(t)| : t \in [-r, 0] \cup (0, t_1]\} \le b_0. \tag{3.268}$$

So assume that there exists a mild solution y to (3.267). Then, for each $t \in [0, t_1]$,

$$y(t) - T(t)\phi(0) \in \int_0^t T(t-s)F(s, y_s)\,ds. \tag{3.269}$$

By (3.27.2), we get

$$|y(t)| \le M\|\phi\|_{\mathcal{D}} + M\int_0^t p(s)\psi(\|y_s\|_{\mathcal{D}})\,ds, \quad t \in [0, t_1]. \tag{3.270}$$

We consider the function μ_0 defined by

$$\mu_0(t) = \sup\{|y(s)| : -r \le s \le t\}, \quad 0 \le t \le t_1. \tag{3.271}$$

Let $t^* \in [-r, t]$ be such that $\mu_0(t) = |y(t^*)|$. If $t^* \in [0, t_1]$, by the previous inequality, we have, for $t \in [0, t_1]$,

$$\mu_0(t) \le M\|\phi\|_{\mathcal{D}} + M\int_0^t p(s)\psi(\mu_0(s))\,ds. \tag{3.272}$$

If $t^* \in [-r, 0]$, then $\mu_0(t) = \|\phi\|_{\mathcal{D}}$ and the previous inequality holds since $M \ge 1$.

Let us take the right-hand side of the above inequality as $v_0(t)$. Then we have

$$\begin{aligned} v_0(0) = M\|\phi\|_{\mathcal{D}} = N_0, \quad \mu_0(t) \le v_0(t), \quad t \in [0, t_1], \\ v_0'(t) = Mp(t)\psi(\mu_0(t)), \quad t \in [0, t_1]. \end{aligned} \tag{3.273}$$

Using the nondecreasing character of ψ, we get

$$v_0'(t) \le Mp(t)\psi(v_0(t)), \quad t \in [0, t_1]. \tag{3.274}$$

This implies, for each $t \in [0, t_1]$, that

$$\int_{N_0}^{v_0(t)} \frac{d\tau}{\psi(\tau)} \le M\int_0^{t_1} p(s)\,ds. \tag{3.275}$$

In view of (3.27.2), we obtain

$$|v_0(t^*)| \le \Gamma_1^{-1}\left(M\int_0^{t_1} p(s)ds\right) := M_0. \tag{3.276}$$

Since for every $t \in [0,t_1]$, $\|y_t\| \le \mu_0(t)$, we have

$$\sup_{t\in[-r,t_1]} |y(t)| \le \max\left(\|\phi\|_{\mathcal{D}}, M_0\right) = b_0. \tag{3.277}$$

We transform this problem into a fixed point problem. A mild solution to (3.267) is a fixed point of the operator $G : \mathrm{PC}([-r,t_1],E) \to \mathcal{P}(\mathrm{PC}([-r,t_1],E))$ defined by

$$G(y) := \left\{ h \in \mathrm{PC}\left([-r,t_1],E\right) : h(t) = \begin{cases} \phi(t) & \text{if } t \in [-r,0] \\ T(t)\phi(0) & \\ \quad + \displaystyle\int_0^t T(t-s)v(s)ds & \text{if } t \in [0,t_1], \end{cases} \right\}, \tag{3.278}$$

where $v \in S^1_{F,y}$. We will show that G satisfies the assumptions of Theorem 1.11.

Claim 1. $G(y)$ is convex, for each $y \in \mathrm{PC}([-r,t_1],E)$.

Indeed, if $h, \overline{h}$ belong to $G(y)$, then there exist $v \in S^1_{F,y}$ and $\overline{v} \in S^1_{F,y}$ such that

$$\begin{aligned} h(t) &= T(t)\phi(0) + \int_0^t T(t-s)v(s)ds, \quad t \in J_0, \\ \overline{h}(t) &= T(t)\phi(0) + \int_0^t T(t-s)\overline{v}(s)ds, \quad t \in J_0. \end{aligned} \tag{3.279}$$

Let $0 \le l \le 1$. Then, for each $t \in [0,t_1]$, we have

$$[lh + (1-l)\overline{h}](t) = T(t)\phi(0) + \int_0^t T(t-s)[lv(s) + (1-l)\overline{v}(s)]ds. \tag{3.280}$$

Since $S^1_{F,y}$ is convex (because F has convex values), then

$$lh + (1-l)\overline{h} \in G(y). \tag{3.281}$$

Claim 2. G sends bounded sets into bounded sets in $\mathrm{PC}([-r,t_1],E)$.

Let $B_q := \{y \in \mathrm{PC}([-r,t_1],E) : \|y\| = \sup_{t\in[-r,t_1]}|y(t)| \le q\}$ be a bounded set in $\mathrm{PC}([-r,t_1],E)$ and $y \in B_q$. Then, for each $h \in G(y)$, there exists $v \in S^1_{F,y}$ such that

$$h(t) = T(t)\phi(0) + \int_0^t T(t-s)v(s)ds, \quad t \in J_0. \tag{3.282}$$

Thus, for each $t \in [-r, t_1]$, we get

$$|h(t)| \leq M\|\phi\|_{\mathcal{D}} + M\int_0^t |v(s)|\,ds \leq M\|\phi\|_{\mathcal{D}} + M\|\varphi_q\|_{L^1}. \tag{3.283}$$

Claim 3. G sends bounded sets in $\mathrm{PC}([-r, t_1], E)$ into equicontinuous sets.

Let $r_1, r_2 \in [-r, t_1]$, $r_1 < r_2$, and let $B_q := \{y \in \mathrm{PC}([-r, t_1], E) : \|y\| \leq q\}$ be a bounded set in $\mathrm{PC}([-r, t_1], E)$ as in Claim 2 and $y \in B_q$. For each $h \in G(y)$, there exists $v \in S^1_{F,y}$ such that

$$h(t) = T(t)\phi(0) + \int_0^t T(t-s)v(s)ds, \quad t \in J_0. \tag{3.284}$$

Hence

$$\begin{aligned}
&|h(r_2) - h(r_1)| \\
&\quad\leq |T(r_2)\phi(0) - T(r_1)\phi(0)| + \left|\int_0^{r_2}[T(r_2-s) - T(r_1-s)]v(s)ds\right| \\
&\qquad + \left|\int_{r_1}^{r_2} T(r_1-s)v(s)ds\right| \leq |T(r_2)\phi(0) - T(r_1)\phi(0)| \\
&\qquad + \left|\int_0^{r_2}[T(r_2-s) - T(r_1-s)]v(s)ds\right| + M\int_{r_1}^{r_2}|v(s)|\,ds \\
&\quad\leq |T(r_2)\phi(0) - T(r_1)\phi(0)| \\
&\qquad + \left|\int_0^{r_2}[T(r_2-s) - T(r_1-s)]\varphi_r(s)ds\right| + M\int_{r_1}^{r_2}\varphi_r(s)ds.
\end{aligned} \tag{3.285}$$

The right-hand side of the above inequality tends to zero, as $r_1 \to r_2$, since $T(t)$ is a strongly continuous operator, and the compactness of $T(t)$ for $t > 0$ implies the continuity in the uniform operator topology. The equicontinuity for the cases $r_1 < r_2 \leq 0$ and $r_1 \leq 0 \leq r_2$ follows from the uniform continuity of ϕ on the interval $[-r, 0]$. As a consequence of Claims 1 to 3, together with the Arzelá-Ascoli theorem, it suffices to show that multivalued G maps B_q into a precompact set in E. Let $0 < t \leq b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B_q$, we define

$$h_\epsilon(t) = T(t)\phi(0) + T(\epsilon)\int_0^{t-\epsilon} T(t-s-\epsilon)v(s)ds, \tag{3.286}$$

where $v \in S^1_{F,y}$. Then we have, since $T(t)$ is a compact operator, the set $H_\epsilon(t) = \{h_\epsilon(t) : h_\epsilon \in G(y)\}$ is a precompact in E for every ϵ, $0 < \epsilon < t$. Moreover, for every $h \in G(y)$, we have

$$|h(t) - h_\epsilon(t)| \leq \int_{t-\epsilon}^{t} |T(t-s)|\varphi_q(s)ds. \tag{3.287}$$

Therefore there are precompact sets arbitrarily close to the set $H(t) = \{h_\epsilon(t) : h \in G(y)\}$. Hence the set $H = \{h_\epsilon(t) : h \in G(y)\}$ is precompact in E. We can conclude that $G : \mathrm{PC}([-r, t_1], E) \to \mathcal{P}(\mathrm{PC}([-r, t_1], E))$ is completely continuous. Set

$$U = \{y \in \mathrm{PC}([-r, t_1], E) : \|y\|_\Omega < b_0 + 1\}. \tag{3.288}$$

As a consequence of Claims 2 and 3 together with the Arzelá-Ascoli theorem, we can conclude that $G : \overline{U} \to \mathcal{P}(\mathrm{PC}([-r, t_1], E))$ is a compact multivalued map.
Claim 4. G has a closed graph.

Let $y_n \to y_*$, $h_n \in G(y_n)$, and $h_n \to h_*$. We will prove that $h_* \in G(y_*)$.
$h_n \in G(y_n)$ means that there exists $v_n \in S_{F,y_n}$ such that

$$h_n(t) = T(t)\phi(0) + \int_0^t T(t-s)v_n(s)ds, \quad t \in [-r, t_1]. \tag{3.289}$$

We must prove that there exists $v_* \in S^1_{F,y_*}$ such that

$$h_*(t) = T(t)\phi(0) + \int_0^t T(t-s)v_*(s)ds, \quad t \in [-r, t_1]. \tag{3.290}$$

Consider the linear continuous operator $\Gamma : L^1([0, t_1], E) \to C([0, t_1], E)$ defined by

$$(\Gamma v)(t) = \int_0^t T(t-s)v(s)ds. \tag{3.291}$$

We have

$$\|(h_n - T(t)\phi(0)) - (h_* - T(t)\phi(0))\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.292}$$

By Lemma 1.28, it follows that $\Gamma \circ S^1_F$ is a closed graph operator.

Also from the definition of Γ, we have

$$h_n(t) - T(t)\phi(0) \in \Gamma(S^1_{F,y_n}). \tag{3.293}$$

Since $y_n \to y_*$, it follows from Lemma 1.28 that

$$h_*(t) = T(t)\phi(0) + \int_0^t T(t-s)v_*(s)ds, \quad t \in J_0 \tag{3.294}$$

for some $v_* \in S^1_{F,y_*}$.

By the choice of U, there is no $y \in \partial U$ such that $y \in \lambda G(y)$ for any $\lambda \in (0,1)$.

As a consequence of Theorem 1.8, we deduce that G has a fixed point $y_0 \in \overline{U}$ which is a mild solution of (3.267).

Step 2. Consider now the following problem on $J_1 := [t_1, t_2]$:

$$\begin{gathered} y' - Ay \in F(t, y_t), \quad t \in J_1, \\ y(t_1^+) = I_1(y(t_1^-)). \end{gathered} \tag{3.295}$$

Let y be a (possible) mild solution to (3.295). Then, for each $t \in [t_1, t_2]$,

$$y(t) - T(t - t_1)I_1(y(t_1^-)) \in \int_{t_1}^{t} T(t - s)F(s, y_s)\,ds. \tag{3.296}$$

By (3.27.2), we get

$$|y(t)| \le M \sup_{t\in[-r,t_1]} |I_1(y_0(t^-))| + M\int_{t_1}^{t} p(s)\psi(\|y_s\|_{\mathcal{D}})\,ds, \quad t \in [t_1, t_2]. \tag{3.297}$$

We consider the function μ_1 defined by

$$\mu_1(t) = \sup\{|y(s)| : t_1 \le s \le t\}, \quad t_1 \le t \le t_2. \tag{3.298}$$

Let $t^* \in [t_1, t]$ be such that $\mu_1(t) = |y(t^*)|$. Then we have, for $t \in [t_1, t_2]$,

$$\mu_1(t) \le N_1 + M\int_{t_1}^{t} p(s)\psi(\mu_1(s))\,ds. \tag{3.299}$$

Let us take the right-hand side of the above inequality as $v_1(t)$. Then we have

$$\begin{gathered} v_1(t_1) = N_1, \quad \mu_1(t) \le v_1(t), \quad t \in [t_1, t_2], \\ v_1'(t) = Mp(t)\psi(\mu_1(t)), \quad t \in [t_1, t_2]. \end{gathered} \tag{3.300}$$

Using the nondecreasing character of ψ, we get

$$v_1'(t) \le Mp(t)\psi(v_1(t)), \quad t \in [t_1, t_2]. \tag{3.301}$$

This implies, for each $t \in [t_1, t_2]$, that

$$\int_{N_1}^{v_1(t)} \frac{d\tau}{\psi(\tau)} \le M\int_{t_1}^{t_2} p(s)\,ds. \tag{3.302}$$

In view of (3.27.2), we obtain

$$|v_1(t^*)| \le \Gamma_2^{-1}\left(M\int_{t_1}^{t_2} p(s)\,ds\right) := M_1. \tag{3.303}$$

Since for every $t \in [t_1, t_2]$, $\|y_t\|_{\mathcal{D}} \leq \mu_1(t)$, we have

$$\sup_{t\in[t_1,t_2]} |y(t)| \leq M_1. \tag{3.304}$$

A mild solution to (3.3)-(3.6) is a fixed point of the operator $G : C(J_1, E) \to \mathcal{P}(C(J_1, E))$ defined by

$$G(y) := \left\{ h \in \mathrm{PC}(J_1, E) : h(t) = \left\{ \begin{aligned} &T(t-t_1)I_1(y(t_1^-)) \\ &\quad + \int_{t_1}^{t} T(t-s)v(s)ds : v \in S^1_{F,y} \end{aligned} \right\} \right\}. \tag{3.305}$$

Set

$$U = \{y \in \mathrm{PC}([t_1, t_2], E) : \|y\| < M_1 + 1\}. \tag{3.306}$$

As in Step 1, we can show that $G : \overline{U} \to \mathcal{P}(\Omega)$ is a compact multivalued map and u.s.c. By the choice of U, there is no $y \in \partial U$ such that $y \in \lambda G(y)$ for any $\lambda \in (0, 1)$.

As a consequence of Theorem 1.8, we deduce that G has a fixed point $y_1 \in \overline{U}$ which is a mild solution of (3.295).

Step 3. Continue this process and construct solutions $y_k \in \mathrm{PC}(J_k, E)$, $k = 2, \dots, m$, to

$$\begin{gathered} y'(t) - Ay \in F(t, y_t), \quad \text{a.e. } t \in J_k, \\ y(t_k^+) = I_k(y(t_k^-)). \end{gathered} \tag{3.307}$$

Then

$$y(t) = \begin{cases} y_0(t), & t \in [-r, t_1], \\ y_1(t), & t \in (t_1, t_2], \\ \vdots & \\ y_{m-1}(t), & t \in (t_{m-1}, t_m], \\ y_m(t), & t \in (t_m, b], \end{cases} \tag{3.308}$$

is a mild solution of (3.261). □

In the second part, a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued operators with nonempty closed decomposable values combined with Schaefer's fixed point theorem is used to investigate the existence of mild solution for first-order impulsive semilinear functional differential inclusions with nonconvex-valued right-hand side.

Theorem 3.28. *Suppose that (3.5.1), (3.12.1), (3.12.2), and (3.27.1) are satisfied. In addition we assume that the following condition holds.*

(3.28.1) *There exists a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *and* $p \in L^1([0,b],\mathbb{R}_+)$ *such that*

$$\|F(t,u)\| \le p(t)\psi(\|u\|_{\mathcal{D}}) \quad \text{for a.e. } t \in [0,b] \text{ and each } u \in D, \tag{3.309}$$

with

$$M\int_0^b p(s)ds < \int_c^{\infty} \frac{d\tau}{\psi(\tau)}, \quad c = M\|\phi\|_{\mathcal{D}} + \sum_{k=1}^m c_k. \tag{3.310}$$

Then the impulsive initial value problem (3.261) *has at least one solution.*

Proof. First, (3.12.1) and (3.12.2) imply by Lemma 1.29 that F is of lower semicontinuous type. Then from Theorem 1.5, there exists a continuous function $f : \Omega \to L^1([0,b],E)$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in \Omega$. Consider the problem

$$\begin{gathered} y'(t) - Ay(t) = f(y_t), \quad t \in [0,b],\ t \neq t_k,\ k = 1,\dots,m, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m, \\ y(t) = \phi(t), \quad t \in [-r,0]. \end{gathered} \tag{3.311}$$

Clearly, if $y \in \Omega$ is a solution of the problem (3.311), then y is a solution to problem (3.261).

Transform problem (3.311) into a fixed point problem. Consider the operator $N : \Omega \to \Omega$ defined by

$$N(y)(t) = \begin{cases} \phi(t) & \text{if } t \in [-r,0], \\ T(t)\phi(0) + \displaystyle\int_0^t T(t-s)f(y_s)ds & \\ \quad + \displaystyle\sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)) & \text{if } t \in [0,b]. \end{cases} \tag{3.312}$$

We will show that N is completely continuous. We show first that N *is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in Ω. Then

$$\begin{aligned} &|N(y_n(t)) - N(y(t))| \\ &\quad \le M\int_0^t |f(y_{n,s}) - f(y_s)|ds + M\sum_{0<t_k<t} |I_k(y_n(t_k)) - I_k(y(t_k^-))| \\ &\quad \le M\int_0^b |f(y_{n,s}) - f(y_s)|ds + M\sum_{0<t_k<t} |I_k(y_n(t_k)) - I_k(y(t_k^-))|. \end{aligned} \tag{3.313}$$

Since the functions f and I_k, $k = 1, \dots, m$, are continuous, then

$$\begin{aligned} \|N(y_n) - N(y)\| &\le M\|f(y_n) - f(y)\|_{L^1} \\ &\quad + M\sum_{k=1}^{m} \left| I_k(y_n(t_k^-)) - I_k(y(t_k^-)) \right| \longrightarrow 0 \end{aligned} \tag{3.314}$$

as $n \to \infty$.

As in Theorem 3.27, we can prove that $N : \Omega \to \Omega$ is completely continuous. *Now it remains to show that the set*

$$\mathcal{E}(N) := \{y \in \Omega : y = \lambda N(y) \text{ for some } 0 < \lambda < 1\} \tag{3.315}$$

is bounded.

Let $y \in \mathcal{E}(N)$. Then $y = \lambda N(y)$ for some $0 < \lambda < 1$. Thus, for each $t \in [0, b]$,

$$y(t) = \lambda\left[T(t)\phi(0) + \int_0^t T(t-s)f(y_s)\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-))\right]. \tag{3.316}$$

This implies that, for each $t \in [0, b]$, we have

$$|y(t)| \le M\|\phi\|_{\mathcal{D}} + M\int_0^t p(s)\psi(\|y_s\|_{\mathcal{D}})\,ds + M\sum_{k=1}^{m} c_k. \tag{3.317}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \le s \le t\}, \quad 0 \le t \le b. \tag{3.318}$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, then by inequality (3.317) we have, for $t \in [0, b]$,

$$\mu(t) \le M\|\phi\|_{\mathcal{D}} + M\int_0^t p(s)\psi(\mu(s))\,ds + M\sum_{k=1}^{m} c_k. \tag{3.319}$$

If $t^* \in [-r, 0]$, then $\mu(t) = \|\phi\|_{\mathcal{D}}$ and inequality (3.319) holds. Let us take the right-hand side of inequality (3.319) as $v(t)$. Then we have

$$\begin{gathered} c = v(0) = M\|\phi\|_{\mathcal{D}} + M\sum_{k=1}^{m} c_k, \quad \mu(t) \le v(t), \quad t \in [0, b], \\ v'(t) = Mp(t)\psi(\mu(t)), \quad t \in [0, b]. \end{gathered} \tag{3.320}$$

Using the nondecreasing character of ψ, we get

$$v'(t) \le Mp(t)\psi(v(t)), \quad t \in [0, b]. \tag{3.321}$$

This implies, for each $t \in [0,b]$, that

$$\int_{v(0)}^{v(t)} \frac{d\tau}{\psi(\tau)} \leq M \int_0^b p(s)ds < \int_{v(0)}^{\infty} \frac{d\tau}{\psi(\tau)}. \tag{3.322}$$

(3.28.1) implies that there exists a constant K such that $v(t) \leq K$, $t \in [0,b]$, and hence $\mu(t) \leq K$, $t \in [0,b]$. Since for every $t \in [0,b]$, $\|y_t\|_{\mathcal{D}} \leq \mu(t)$, we have

$$\|y\| \leq \max\{\|\phi\|_{\mathcal{D}}, K\} := K', \tag{3.323}$$

where K' depends only on b, M and on the functions p and ψ. This shows that $\mathcal{E}(N)$ is bounded.

Set $X := \Omega$. As a consequence of Schaefer's fixed point theorem (Theorem 1.6), we deduce that N has a fixed point y which is a mild solution to problem (3.311). Then y is a mild solution to problem (3.261). □

For second-order impulsive functional differential inclusions, we have the following theorem, which we state without proof, since it follows the same steps as the previous theorem.

Theorem 3.29. *Assume (3.5.1), (3.5.2), (3.12.1), (3.12.2), and the following conditions hold.*

(3.29.1) $C(t)$, $t > 0$ *is compact, and there exists a constant* $M_1 \geq 1$ *such that* $\|C(t)\|_{B(E)} \leq M_1$ *for all* $t \in \mathbb{R}$.

(3.29.2) *There exist a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *and* $p \in L^1([0,b],\mathbb{R}_+)$ *such that*

$$\|F(t,u)\| \leq p(t)\psi(\|u\|_{\mathcal{D}}) \quad \text{for a.e. } t \in J \text{ and each } u \in D \tag{3.324}$$

with

$$\begin{gathered} bM_1 \int_0^b p(s)ds < \int_c^{\infty} \frac{d\tau}{\psi(\tau)}, \\ c = M_1\|\phi\|_{\mathcal{D}} + bM_1|\eta| + \sum_{k=1}^{m} [M_1 c_k + M_1(b-t_k)d_k]. \end{gathered} \tag{3.325}$$

Then the IVP

$$\begin{gathered} y''(t) - Ay(t) \in F(t,y_t), \quad \text{a.e. } t \in J = [0,b],\ t \neq t_k,\ k = 1,\dots,m, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m, \\ \Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1,\dots,m, \\ y(t) = \phi(t), \quad t \in [-r,0], \qquad y'(0) = \eta, \end{gathered} \tag{3.326}$$

has at least one mild solution.

3.7. Notes and remarks

The techniques in this chapter have been adapted from [138, 162, 164, 202], where the nonimpulsive case was discussed. The arguments of Section 3.2 are dependent upon the nonlinear alternative of Leray-Schauder. Theorems 3.2, 3.3, 3.5 are taken from Benchohra et al. [46] and Benchohra and Ntouyas [85]. The results of Section 3.3 are adapted from Benchohra et al. [49] and extend those of Section 3.2. Section 3.4 is taken from Benchohra and Ntouyas [82] and Benchohra et al. [53, 54, 60, 66], with the major tools based on Martelli's fixed point theorem for multivalued condensing maps, Schaefer's fixed point theorem combined with a selection theorem due to Bressan and Colombo, and the Covitz-Nadler fixed point theorem for contraction multivalued maps. The material of Section 3.5 is based on the results given by Benchohra et al. [56, 57], and this section extends some results given in Section 3.4. The results of last section of Chapter 3 are taken from Benchohra et al. [64].

4 Impulsive differential inclusions with nonlocal conditions

4.1. Introduction

In this chapter, we will prove existence results for impulsive semilinear ordinary and functional differential inclusions, with nonlocal conditions. Often, nonlocal conditions are motivated by physical problems. For the importance of nonlocal conditions in different fields we refer to [112]. As indicated in [112, 113, 126] and the references therein, the nonlocal condition $y(0) + g(y) = y_0$ can be more descriptive in physics with better effect than the classical initial condition $y(0) = y_0$. For example, in [126], the author used

$$g(y) = \sum_{k=1}^{p} c_i y(t_i), \tag{4.1}$$

where c_i, $i = 1, \dots, p$ are given constants and $0 < t_1 < t_2 < \cdots < t_p \leq b$, to describe the diffusion phenomenon of a small amount of gas in a transparent tube. In this case, (4.1) allows the additional measurements at t_i, $i = 1, \dots, p$.

Nonlocal Cauchy problems for ordinary differential equations have been investigated by several authors, (see, e.g., [103, 113, 114, 202–204, 206, 207]). Nonlocal Cauchy problems, in the case where F is a multivalued map, were studied by Benchohra and Ntouyas [77–79], and Boucherif [103]. Akça et al. [14] initiated the study of a class of first-order semilinear functional differential equations for which the nonlocal conditions and the impulse effects are combined. Again, in this chapter, we will invoke some of our fixed point theorems in establishing solutions for these nonlocal impulsive differential inclusions.

4.2. Nonlocal impulsive semilinear differential inclusions

In this section, we begin the study of nonlocal impulsive initial value problems by proving existence results for the problem

$$y'(t) \in Ay(t) + F(t, y(t)), \quad t \in J := [0, b],\ t \neq t_k,\ k = 1, 2, \dots, m, \tag{4.2}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{4.3}$$

$$y(0) + \sum_{k=1}^{m+1} c_k y(\eta_k) = y_0, \tag{4.4}$$

where A is the infinitesimal generator of a strongly continuous semigroup, $T(t)$, $t \geq 0$, $F : J \times E \to \mathcal{P}(E)$ is a multivalued map, $y_0 \in E$, $\mathcal{P}(E)$ is the family of all subsets of E, $0 \leq \eta_1 < t_1 < \eta_2 < t_2 < \eta_3 < \cdots < t_m < \eta_{m+1} \leq b$, $c_k \neq 0$, $k = 1, 2, \dots, m+1$, are real numbers, $I_k \in C(E, E)$ $(k = 1, \dots, m)$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h \to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h \to 0^+} y(t_k - h)$, and E is a real separable Banach space with norm $|\cdot|$. We are concerned with the existence of solutions for problem (4.2)–(4.4) when $F : J \times E \to \mathcal{P}(E)$ is a compact and convex-valued multivalued map.

We recall that $\mathrm{PC}(J, E) = \{y : J \to E$ such that $y(t)$ is continuous everywhere except for some t_k at which $y(t_k^-)$ and $y(t_k^+)$ exist, and $y(t_k^-) = y(t_k)$, $k = 1, 2, \dots, m\}$. Evidently, $\mathrm{PC}(J, E)$ is a Banach space with norm

$$\|y\|_{\mathrm{PC}} = \sup\{|y(t)| : t \in J\}. \tag{4.5}$$

Let us define what we mean by a mild solution of problem (4.2)–(4.4).

Definition 4.1. A function $y \in \mathrm{PC}(J, E) \cap \mathrm{AC}((t_k, t_{k+1}), E)$ is said to be a mild solution of (4.2)–(4.4) if $y(0) + \sum_{k=1}^{m+1} c_k y(\eta_k) = y_0$, $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $k = 1, \dots, m$, and there exists a function $f \in L^1(J, E)$ such that $f(t) \in F(t, y(t))$ a.e. on $t \in J$, and $y'(t) = Ay(t) + f(t)$.

Lemma 4.2. *Assume*

(4.2.1) *there exists a bounded operator $B : E \to E$ such that*

$$B = \left(I + \sum_{k=1}^{m+1} c_k T(\eta_k)\right)^{-1}. \tag{4.6}$$

If y is a solution of (4.2)–(4.4)*, then it is given by*

$$\begin{aligned} y(t) = {} & T(t)By_0 - \sum_{k=1}^{m+1} c_k T(t)B \int_0^{\eta_k} T(\eta_k - s) f(s)\,ds + \int_0^t T(t-s) f(s)\,ds \\ & - T(t)B \sum_{k=1}^{m+1} c_k T(\eta_k - t_{k-1}) I_{k-1}(y(t_{k-1}^-)) \\ & + \sum_{0<t_k<t} T(t - t_k) I_k(y(t_k^-)), \quad f \in S_{F,y}. \end{aligned} \tag{4.7}$$

Proof. Let y be a solution of problem (4.2)–(4.4). Then there exists $f \in S_{F,y}$ such that $y'(t) = Ay(t) + f(t)$. We put $w(s) = T(t-s)y(s)$. Then

$$\begin{aligned} w'(s) &= -T'(t-s)y(s) + T(t-s)y'(s) \\ &= -AT(t-s)y(s) + T(t-s)y'(s) \\ &= T(t-s)[y'(s) - Ay(s)] \\ &= T(t-s)f(s). \end{aligned} \tag{4.8}$$

Let $t < t_1$. Integrating the above equation, we have

$$\begin{aligned} \int_0^t w'(s)ds &= \int_0^t T(t-s)f(s)ds, \\ w(t) - w(0) &= \int_0^t T(t-s)f(s)ds, \\ y(t) &= T(t)y(0) + \int_0^t T(t-s)f(s)ds. \end{aligned} \tag{4.9}$$

Consider $t_k < t$, $k = 1,\dots,m$. By integrating (4.8) for $k = 1,2,\dots,m$, we have

$$\int_0^{t_1} w'(s)ds + \int_{t_1}^{t_2} w'(s)ds + \cdots + \int_{t_k}^{t} w'(s)ds = \int_0^t T(t-s)f(s)ds \tag{4.10}$$

or

$$w(t_1^-) - w(0) + w(t_2^-) - w(t_1^+) + \cdots + w(t_k^+) - w(t) = \int_0^t T(t-s)f(s)ds, \tag{4.11}$$

and consequently

$$\begin{aligned} w(t) &= w(0) + \sum_{0<t_k<t} [w(t_k^+) - w(t_k^-)] + \int_0^t T(t-s)f(s)ds, \\ y(t) &= w(0) + \sum_{0<t_k<t} T(t-t_k)I(y(t_k^-)) + \int_0^t T(t-s)f(s)ds, \end{aligned} \tag{4.12}$$

where $w(0) = T(t)y(0) = T(t)[y_0 - \sum_{k=1}^{m+1} c_k y(\eta_k)]$.

It remains to find $y(\eta_k)$. For that reason we use (4.8) and integrate it from 0 to η_k, $k = 1,\dots,m+1$.

For $k = 1$,

$$\begin{aligned} \int_0^{\eta_1} w'(s)ds &= \int_0^{\eta_1} T(t-s)f(s)ds - \int_0^{\eta_1} y(s)ds \Longleftrightarrow w(\eta_1) - w(0) \\ &= \int_0^{\eta_1} S(t-s)f(s)ds - \int_0^{\eta_1} y(s)ds \Longleftrightarrow T(t-\eta_1)y(\eta_1) \\ &= T(t)y(0) + \int_0^{\eta_1} T(t-s)f(s)ds - \int_0^{\eta_1} y(s)ds. \end{aligned} \tag{4.13}$$

For $k = 2,\dots,m+1$,

$$\begin{aligned}
\int_0^{\eta_k} w'(s)ds = \int_0^{\eta_k} T(t-s)f(s)ds - \int_0^{\eta_k} y(s)ds \Longleftrightarrow \int_0^{t_1} w'(s)ds \\
+ \int_{t_1}^{t_2} w'(s)ds + \cdots + \int_{t_{k-1}}^{\eta_k} w'(s)ds \\
= \int_0^{\eta_k} T(t-s)f(s)ds - \int_0^{\eta_k} y(s)ds \Longleftrightarrow w(t_1^-) \\
- w(0) + w(t_2^-) - w(t_1^+) + \cdots + w(\eta_k) - w(t_{k-1}^+) \\
= \int_0^{\eta_k} T(t-s)f(s)ds,
\end{aligned} \tag{4.14}$$

and thus

$$\begin{aligned}
&T(t-t_1)y(t_1^-) - T(t)y(0) + T(t-t_2)y(t_2^-) \\
&\quad - T(t-t_1)y(t_1^+) + \cdots + T(t-t_k)y(\eta_k) - T(t-t_{k-1})y(t_{k-1}^+) \\
&= \int_0^{\eta_k} T(t-s)f(s)ds.
\end{aligned} \tag{4.15}$$

Hence

$$T(t-\eta_k)y(\eta_k) = T(t)y(0) + \sum_{0<t_j<\eta_k} T(t-t_j)I_j(y(t_j^-)) + \int_0^{\eta_k} T(t-s)f(s)ds, \tag{4.16}$$

$$y(\eta_k) = T(\eta_k)y(0) + \sum_{0<t_j<\eta_k} T(\eta_k-t_j)I_j(y(t_j^-)) + \int_0^{\eta_k} T(\eta_k-s)f(s)ds. \tag{4.17}$$

The nonlocal condition, with the help of (4.17), becomes

$$\begin{aligned}
&y(0)+\sum_{k=1}^{m+1} c_k\left[T(\eta_k)y(0)+\sum_{0<t_j<\eta_k} T(\eta_k-t_j)I_j(y(t_j^-))+\int_0^{\eta_k} T(\eta_k-s)f(s)ds\right] = y_0, \\
&y(0)\left(I+\sum_{k=1}^{m+1} c_kT(\eta_k)\right) \\
&\quad = y_0 - \sum_{k=2}^{m+1} c_k \sum_{\mu=1}^{k-1} T(\eta_k-t_\mu)I_\mu(y(t_\mu^-)) - \sum_{k=1}^{m+1} c_k \int_0^{\eta_k} T(\eta_k-s)f(s)ds.
\end{aligned} \tag{4.18}$$

Hence

$$y(0) = By_0 - B\sum_{k=2}^{m+1} c_k \sum_{\mu=1}^{k-1} T(\eta_k-t_\mu)I_\mu(y(t_\mu^-)) - B\sum_{k=1}^{m+1} c_k \int_0^{\eta_k} T(\eta_k-s)f(s)ds. \tag{4.19}$$

Equation (4.12), with the help of (4.19), becomes

$$y(t) = T(t)\left[By_0 - B\sum_{k=2}^{m+1} c_k \sum_{\mu=1}^{k-1} T(\eta_k - t_\mu) I_\mu(y(t_\mu^-)) - B\sum_{k=1}^{m+1} c_k \int_0^{\eta_k} T(\eta_k - s) f(s)ds\right]$$
$$+ \sum_{0<t_k<t} T(t-t_k) I_k(y(t_k^-)) + \int_0^t T(t-s) f(s)ds, \tag{4.20}$$

which completes the proof. □

Now we are able to state and prove our main theorem.

Theorem 4.3. *Assume (3.11.1), (3.27.1), (4.2.1), and the following conditions are satisfied:*

(4.3.1) *there exist constants θ_k such that*

$$|I_k(x)| \leq \theta_k, \quad k = 1,\dots,m, \ \forall x \in E; \tag{4.21}$$

(4.3.2) *there exist a continuous nondecreasing function $\psi : \mathbb{R}_+ \to (0,\infty)$, a function $p \in L^1(J,\mathbb{R}_+)$, and a constant $M > 0$ such that*

$$||F(t,y)|| := \sup\{|v| : v \in F(t,y)\} \leq p(t)\psi(|y|) \tag{4.22}$$

for almost all $t \in J$ and all $y \in E$, and

$$\frac{M}{\alpha + M^2\|B\|_{B(E)}\sum_{k=1}^{m+1}|c_k|\psi(M)\int_0^{\eta_k} p(t)dt + M\int_0^b p(s)\psi(M)ds} > 1, \tag{4.23}$$

where

$$\alpha = M\|B\|_{B(E)}|y_0| + M^2\|B\|_{B(E)}\sum_{k=1}^{m+1}|c_k|\theta_k + M\sum_{k=1}^{m}\theta_k; \tag{4.24}$$

(4.3.3) *the set $\{y_0 - \sum_{k=1}^{m+1} c_k y(\eta_k),\ y \in \mathrm{PC}(J,E),\ \|y\|_{\mathrm{PC}} \leq r,\ r > 0\}$ is relatively compact.*

Then the IVP (4.2)–(4.4) *has at least one mild solution on J.*

Proof. We transform problem (4.2)–(4.4) into a fixed point problem. Consider the multivalued map $N : \mathrm{PC}(J,E) \to \mathcal{P}(\mathrm{PC}(J,E))$ defined by

$$\begin{aligned} N(y) := \Bigg\{ h \in \mathrm{PC}(J,E) : h(t) &= T(t)By_0 - \sum_{k=1}^{m+1} c_k T(t)B \int_0^{\eta_k} T(\eta_k - s)g(s)ds \\ &+ \int_0^t T(t-s)g(s)ds - T(t)B\sum_{k=1}^{m+1} c_k T(\eta_k - t_{k-1})I_{k-1}\big(y(t_{k-1}^-)\big) \\ &+ \sum_{0<t_k<t} T(t-t_k)I_k\big(y(t_k^-)\big)ds : g \in S_{F,y} \Bigg\}. \end{aligned} \tag{4.25}$$

It is clear that the fixed points of N are mild solutions to (4.2)-(4.3).

We will show that N has a fixed point. The proof will be given in several steps. We first will show that N is a completely continuous multivalued map, upper semicontinuous (u.s.c.), with convex closed values.

Step 1. $N(y)$ is convex, for each $y \in \mathrm{PC}(J,E)$.

Indeed, if h_1, h_2 belong to $N(y)$, then there exist $g_1, g_2 \in S_{F,y}$ such that, for each $t \in J$, we have

$$\begin{aligned} h_i(t) &= T(t)By_0 - \sum_{k=1}^{m+1} c_k T(t)B \int_0^{\eta_k} T(\eta_k - s)g_i(s)ds \\ &\quad + \int_0^t T(t-s)g_i(s)ds - T(t)B\sum_{k=1}^{m+1} c_k T(\eta_k - t_{k-1})I_{k-1}\big(y(t_{k-1}^-)\big) \\ &\quad + \sum_{0<t_k<t} T(t-t_k)I_k\big(y(t_k^-)\big), \quad i = 1,2. \end{aligned} \tag{4.26}$$

Let $0 \le k \le 1$. Then, for each $t \in J$, we have

$$\begin{aligned} &(kh_1 + (1-k)h_2)(t) \\ &\quad = T(t)By_0 - \sum_{k=1}^{m+1} c_k T(t)B \int_0^{\eta_k} T(\eta_k - s)\big[kg_1(s) + (1-k)g_2(s)\big]ds \\ &\qquad - T(t)B\sum_{k=1}^{m+1} c_k T(\eta_k - t_{k-1})I_{k-1}\big(y(t_{k-1}^-)\big) \\ &\qquad + \int_0^t T(t-s)\big[kg_1(s) + (1-k)g_2(s)\big]ds \\ &\qquad + \sum_{0<t_k<t} T(t-t_k)I_k\big(y(t_k^-)\big). \end{aligned} \tag{4.27}$$

Since $S_{F,y}$ is convex (because F has convex values), then

$$kh_1 + (1-k)h_2 \in N(y). \tag{4.28}$$

Step 2. N is bounded on bounded sets of $\mathrm{PC}(J,E)$.

Indeed, it is enough to show that for any $r>0$, there exists a positive constant ℓ such that, for each $h \in N(y)$, $y \in B_r = \{y \in \mathrm{PC}(J,E) : \|y\|_{\mathrm{PC}} \leq r\}$, one has $\|N(y)\| := \{\|h\|_{\mathrm{PC}} : h \in N(y)\} \leq \ell$. By (3.27.1), (4.3.2), and (4.3.3), we have, for each $t \in J$, that

$$\begin{aligned}
|h(t)| &\leq M\|B\|_{B(E)}|y_0| + M^2\|B\|_{B(E)}\sum_{k=1}^{m+1}|c_k|\int_0^{\eta_k} p(t)\psi(|y(t)|)dt \\
&\quad + M\int_0^t p(s)\psi(|y(s)|)ds + M^2\|B\|_{B(E)}\sum_{k=1}^{m+1}|c_k|\theta_k + M\sum_{k=1}^{m}\theta_k \\
&\leq M\|B\|_{B(E)}|y_0| + M^2\|B\|_{B(E)}\sum_{k=1}^{m+1}|c_k|\psi(\|y\|_{\mathrm{PC}})\|p\|_{L^1} \\
&\quad + M\psi(\|y\|_{\mathrm{PC}})\|p\|_{L^1} + M^2\|B\|_{B(E)}\sum_{k=1}^{m+1}|c_k|\theta_k + M\sum_{k=1}^{m}\theta_k.
\end{aligned} \tag{4.29}$$

Then, for each $h \in N(B_r)$, we have

$$\begin{aligned}
\|h\|_{\mathrm{PC}} &\leq M\|B\|_{B(E)}|y_0| + M^2\|B\|_{B(E)}\sum_{k=1}^{m+1}|c_k|\|p\|_{L^1}\psi(r) \\
&\quad + M\|p\|_{L^1}\psi(r) + M^2\|B\|_{B(E)}\sum_{k=1}^{m+1}|c_k|\theta_k + M\sum_{k=1}^{m}\theta_k := \ell.
\end{aligned} \tag{4.30}$$

Step 3. N sends bounded sets into equicontinuous sets of $\mathrm{PC}(J,E)$.

Let $\tau_1,\tau_2 \in J \setminus \{t_1,\dots,t_m\}$, $\tau_1<\tau_2$, and B_r be a bounded set in $\mathrm{PC}(J,E)$. Then we have

$$\begin{aligned}
|h(\tau_2)-h(\tau_1)| &\leq |[T(\tau_2)B - T(\tau_1)B]y_0| \\
&\quad + M\|B\|_{B(E)}\sum_{k=1}^{m+1}|c_k|\int_0^{\eta_k}\|T(\tau_2)-T(\tau_1)\|_{B(E)}|g(s)|ds \\
&\quad + \int_0^{\tau_1}\|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)}|g(s)|ds \\
&\quad + \int_{\tau_1}^{\tau_2}\|T(\tau_2-s)\|_{B(E)}|g(s)|ds \\
&\quad + \|T(\tau_2)-T(\tau_1)\|_{B(E)}\|B\|_{B(E)}\sum_{k=1}^{m+1}|c_k|M\theta_k \\
&\quad + \sum_{0<t_k<\tau_1}\|T(\tau_2-t_k)-T(\tau_1-t_k)\|_{B(E)}\theta_k + \sum_{\tau_1<t_k<\tau_2}M\theta_k.
\end{aligned} \tag{4.31}$$

As $\tau_2 \to \tau_1$, the right-hand side of the above inequality tends to zero, since $T(t)$ is a strongly continuous operator, and the compactness of $T(t)$ for $t > 0$ implies the continuity in the uniform operator topology.

This proves the equicontinuity for the case where $t \neq t_i$, $i = 1, \ldots, m$. It remains to examine the equicontinuity at $t = t_i$.

First we prove equicontinuity at $t = t_i^+$. Fix $\delta_1 > 0$ such that $\{t_k : k \neq i\} \cap [t_i - \delta_1, t_i + \delta_1] = \emptyset$.

For $0 < h < \delta_1$, we have

$$\begin{aligned}
&|h(t_i+h) - h(t_i)| \\
&\quad \leq |[T(t_i+h) - T(t_i)]By_0| \\
&\qquad + M^2\|B\|_{B(X)}\sum_{k=1}^{m+1}|c_k|\int_0^{\eta_k}|[T(t_i+h) - T(t_i)]g(s)|\,ds \\
&\qquad + \int_0^{t_i}|[T(t_i+h-s) - T(t_i-s)]g(s)|\,ds + \int_{t_i}^{t_i+h}M|g(s)|\,ds \\
&\qquad + ||T(t_i+h) - T(t_i)||_{B(X)}\|B\|_{B(X)}M\sum_{k=2}^{m+1}|c_k|\sum_{\lambda=1}^{k-1}\theta_\lambda \\
&\qquad + \sum_{0<t_k\leq t_i}|[T(t_i+h-t_k) - T(t_i-t_k)]I_k(y(t_k^-))| + \sum_{t_i<t_k<t_i+h}M\theta_k.
\end{aligned} \tag{4.32}$$

The right-hand side tends to zero as $h \to 0$.

Next we prove the equicontinuity at $t = t_i^-$. Fix $\delta_1 > 0$ such that $\{t_k : k \neq i\} \cap [t_i - \delta_1, t_i + \delta_1] = \emptyset$.

For $0 < h < \delta_1$, we have

$$\begin{aligned}
|h(t_i) - h(t_i-h)| \leq{} & |[T(t_i) - T(t_i-h)]By_0| \\
& + M^2\|B\|_{B(X)}\sum_{k=1}^{m+1}|c_k|\int_0^{\eta_k}|[T(t_i) - T(t_i-h)]g(s)|\,ds \\
& + \int_0^{t_i-h}|[T(t_i-s) - T(t_i-h-s)]g(s)|\,ds \\
& + ||T(t_i) - T(t_i-h)||_{B(X)}\|B\|_{B(X)}M\sum_{k=2}^{m+1}|c_k|\sum_{\lambda=1}^{k-1}\theta_\lambda \\
& + \sum_{k=1}^{i-1}|[T(t_i-t_k) - T(t_i-h-t_k)](I_k(y(t_k^-)))| \\
& + M\int_{t_i-h}^{t_i}p(s)\psi(r)\,ds + \sum_{t_i-h<t<t_i}M\theta_k.
\end{aligned} \tag{4.33}$$

The right-hand side tends to zero as $h \to 0$.

As a consequence of Steps 1 to 3 and (4.3.3), together with the Arzelá-Ascoli theorem, it suffices to show that N maps B_r into a precompact set in E.

Let $Y = \{h \in N(y) : y \in B_r,\ y(0) + \sum_{k=1}^{m+1} c_k y(\eta_k) = y_0\}$. We show that N maps Y into relatively compact sets $N(Y)$ of Y. For this reason we will prove that $Y(t) = \{h(t) : h \in Y\}$, $t \in J$ is precompact in $\mathrm{PC}(J,E)$.

From assumption (4.3.3), we have that $Y(0)$ is relatively compact.

Let $0 < t \leq b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B_r$ we define

$$\begin{aligned} h_\epsilon(t) = {} & T(t)By_0 - \sum_{k=1}^{m+1} c_k T(t)B \int_0^{\eta_k} T(\eta_k - s)g(s)ds \\ & + \int_0^{t-\epsilon} T(t-\epsilon-s)g(s)ds \\ & - T(t)B \sum_{k=2}^{m+1} c_k \sum_{\lambda=1}^{k-1} T(\eta_k - t_\lambda)I_\lambda(y(t_\lambda^-)) \\ & + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)). \end{aligned} \tag{4.34}$$

Since $T(t)$ is a compact operator for $t > 0$, the set $Y_\varepsilon(t) = \{h_\varepsilon(t) : h_\varepsilon \in N(y)\}$ is relatively compact in $\mathrm{PC}(J,E)$, for every ε, $0 < \varepsilon < t$. Moreover, for every $h \in N(y)$, we have

$$\left| h(t) - h_\varepsilon(t) \right| \leq M \int_{t-\varepsilon}^{t} p(s)\psi(r)ds. \tag{4.35}$$

Therefore there are precompact sets arbitrarily close to the set $Y(t)$. Hence the set $Y(t)$ is precompact.

Step 4. N has a closed graph.

Let $y_n \to y^*$, $h_n \in N(y_n)$, and $h_n \to h^*$. We will prove that $h^* \in N(y^*)$. $h_n \in N(y_n)$ means that there exists $g_n \in S_{F,y_n}$ such that

$$\begin{aligned} h_n(t) = {} & T(t)By_0 - \sum_{k=1}^{m+1} c_k T(t)B \int_0^{\eta_k} T(\eta_k - s)g_n(s)ds \\ & + \int_0^{t} T(t-s)g_n(s)ds \\ & - T(t)B \sum_{k=1}^{m+1} c_k T(\eta_k - t_{k-1})I_{k-1}(y_n(t_{k-1}^-)) \\ & + \sum_{0<t_k<t} T(t-t_k)I_k(y_n(t_k^-))ds. \end{aligned} \tag{4.36}$$

We must prove that there exists $g^* \in S_{F,y^*}$ such that

$$h^*(t) = T(t)By_0 - \sum_{k=1}^{m+1} c_k T(t)B \int_0^{\eta_k} T(\eta_k - s)g^*(s)ds + \int_0^t T(t-s)g^*(s)ds - T(t)B\sum_{k=1}^{m+1} c_k T(\eta_k - t_{k-1})I_{k-1}(y^*(t_{k-1}^-)) + \sum_{0<t_k<t} T(t-t_k)I_k(y^*(t_k^-)), \quad t \in J. \tag{4.37}$$

Consider the operator

$$\Gamma : L^1(J,E) \longrightarrow C(J,E),$$
$$g \longmapsto \Gamma(g)(t) = \int_0^t T(t-s)g(s)ds - \sum_{k=1}^{m+1} c_k T(t)B\int_0^{\eta_k} T(\eta_k - s)g(s)ds. \tag{4.38}$$

We can see that the operator Γ is linear and continuous. Indeed, one has

$$\big\|(\Gamma g)\big\|_\infty \le \overline{M}\|g\|_{L^1}, \tag{4.39}$$

where $\overline{M}$ is given by

$$\overline{M} = M + M^2\|B\|_{B(E)} \sum_{k=1}^{m+1} |c_k|. \tag{4.40}$$

Clearly, we have

$$\Bigg\|\Big(h_n - T(t)By_0 - \sum_{0<t_k<t} T(t-t_k)I_k(y_n(t_k^-)) + T(t)B\sum_{k=1}^{m+1} c_k T(\eta_k - t_{k-1})I_{k-1}(y_n(t_{k-1}^-))\Big) - \Big(h^* - T(t)By_0 - \sum_{0<t_k<t} T(t-t_k)I_k(y^*(t_k^-)) + T(t)B\sum_{k=1}^{m+1} c_k T(\eta_k - t_{k-1})I_{k-1}(y^*(t_{k-1}^-))\Big)\Bigg\|_{PC} \longrightarrow 0, \tag{4.41}$$

as $n \to \infty$. From Lemma 1.28, it follows that $\Gamma \circ S_F$ is a closed graph operator. Moreover, we have

$$h_n(t) - T(t)By_0 - \sum_{0<t_k<t} T(t-t_k)I_k(y_n(t_k^-)) + T(t)B\sum_{k=1}^{m+1} c_k T(\eta_k - t_{k-1})I_{k-1}(y_n(t_{k-1}^-)) \in \Gamma(S_{F,y_n}). \tag{4.42}$$

Since $y_n \to y^*$, it follows from Lemma 1.28 that

$$\begin{aligned}
h^*(t) &- T(t)By_0 - \sum_{0<t_k<t} T(t-t_k)I_k(y^*(t_k^-)) \\
&= T(t)B\sum_{k=1}^{m+1} c_k T(\eta_k - t_{k-1})I_{k-1}(y^*(t_{k-1}^-)) \\
&= \int_0^t T(t-s)g^*(s)ds - \sum_{k=1}^{m+1} c_k T(t)B\int_0^{\eta_k} T(\eta_k - s)g^*(s)ds
\end{aligned} \tag{4.43}$$

for some $g^* \in S_{F,y^*}$.

Therefore N is a completely continuous multivalued map, u.s.c., with convex closed values.

Step 5. A priori bounds on solutions.

Let y be such that $y \in \lambda N(y)$, for some $\lambda \in (0,1)$. Then

$$\begin{aligned}
y(t) = \lambda T(t)By_0 &- \lambda\sum_{k=1}^{m+1} c_k T(t)B\int_0^{\eta_k} T(\eta_k - s)g(s)ds \\
&- \lambda T(t)B\sum_{k=1}^{m+1} c_k T(\eta_k - t_{k-1})I_{k-1}(y_n(t_{k-1}^-)) \\
&+ \lambda\int_0^t T(t-s)g(s)ds + \lambda^{-1}\sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)), \quad t \in J.
\end{aligned} \tag{4.44}$$

This implies by (4.3.1), (4.3.2), and (4.3.3) that, for each $t \in J$, we have

$$\begin{aligned}
|y(t)| &\le M\|B\|_{B(E)}|y_0| + M^2\|B\|_{B(E)}\sum_{k=1}^{m+1}|c_k|\psi(\|y\|_{\mathrm{PC}})\int_0^{\eta_k} p(t)dt \\
&\quad + M\int_0^t p(s)\psi(|y(t)|)ds + M^2\|B\|_{B(E)}\sum_{k=1}^{m+1}|c_k|\theta_k + M\sum_{k=1}^{m}\theta_k \\
&\le \alpha + M^2\|B\|_{B(E)}\sum_{k=1}^{m+1}|c_k|\psi(\|y\|_{\mathrm{PC}})\int_0^{\eta_k} p(t)dt \\
&\quad + M\int_0^b p(s)\psi(\|y\|_{\mathrm{PC}})ds.
\end{aligned} \tag{4.45}$$

Consequently,

$$\frac{\|y\|_{\mathrm{PC}}}{\alpha + M^2\|B\|_{B(E)}\sum_{k=1}^{m+1}|c_k|\psi(\|y\|_{\mathrm{PC}})\int_0^{\eta_k} p(t)dt + M\int_0^b p(s)\psi(\|y\|_{\mathrm{PC}})ds} \le 1. \tag{4.46}$$

Then by (4.3.3), there exists K such that $\|y\|_{\mathrm{PC}} \ne K$. Set

$$U = \{y \in \mathrm{PC}(J,E) : \|y\|_{\mathrm{PC}} < K+1\}. \tag{4.47}$$

The operator N is continuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y = \lambda N(y)$, for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type (Theorem 1.8), we deduce that N has a fixed point y in $\overline{U}$ which is a solution of (4.2)–(4.4). □

Theorem 4.4. *Assume that hypotheses (3.27.1), (4.3.1), and (4.3.2) are satisfied. In addition we suppose that the following conditions hold:*

(4.4.1) $F : J \times E \to P_{\mathrm{cp,cv}}(E)$ *has the property that* $F(\cdot, y) : J \to P_{\mathrm{cp}}(E)$ *is measurable, for each* $y \in E$;

(4.4.2) *there exists* $l \in L^1(J, \mathbb{R}^+)$ *such that* $H_d(F(t,y), F(t,\overline{y})) \le l(t)|y - \overline{y}|$, *for almost each* $t \in J$ *and* $y, \overline{y} \in E$, *and*

$$d(0, F(t,0)) \le \ell(t), \quad \text{for almost each } t \in J; \tag{4.48}$$

(4.4.3) *there exists constant* d_k *such that*

$$|I_k(y_2) - I_k(y_1)| \le d_k |y_2 - y_1|, \quad \forall y_1, y_2 \in E; \tag{4.49}$$

(4.4.4) *assume that*

$$M\left(M\|B\|_{B(E)} \sum_{k=1}^{m+1} |c_k| L(\eta_k) + L(b) + M\|B\|_{B(E)} \sum_{k=1}^{m+1} |c_k| d_{k-1} + \sum_{k=1}^{m} d_k\right) < 1, \tag{4.50}$$

where $L(t) = \int_0^t \ell(s)ds$.

Then the IVP (4.2)–(4.4) *has at least one mild solution on* J.

Proof. Set

$$\Omega_0 = \left\{ y \in \mathrm{PC}(J,E) : y(0) + \sum_{k=1}^{m+1} c_k y(\eta_k) = y_0 \right\}. \tag{4.51}$$

Transform problem (4.2)–(4.4) into a fixed point problem. Consider the multivalued operator $N : \Omega_0 \to \mathcal{P}(\Omega_0)$ defined in Theorem 4.3; that is,

$$\begin{aligned} N(y) := \Bigg\{ h \in \Omega_0 : h(t) &= T(t)By_0 - \sum_{k=1}^{m+1} c_k T(t)B \int_0^{\eta_k} T(\eta_k - s)g(s)ds \\ &\quad - T(t)B \sum_{k=1}^{m+1} c_k T(\eta_k - t_{k-1}) I_{k-1}(y(t_{k-1}^-)) \\ &\quad + \int_0^t T(t-s)g(s)ds : g \in S_{F,y} \Bigg\}. \end{aligned} \tag{4.52}$$

We will show that N satisfies the assumptions of Theorem 1.11. The proof will be given in two steps.

Step 1. $N(y) \in P_{\text{cl}}(\Omega_0)$, for each $y \in \Omega_0$.

Indeed, let $(y_n)_{n\geq 0} \in N(y)$ such that $y_n \to \tilde{y}$ in Ω_0. Then $\tilde{y} \in \Omega_0$ and there exists $g_n \in S_{F,y}$ such that, for every $t \in J$,

$$\begin{aligned} y_n(t) = {} & T(t)By_0 - \sum_{k=1}^{m+1} c_k T(t)B \int_0^{\eta_k} T(\eta_k - s)g_n(s)ds \\ & + \int_0^t T(t-s)g_n(s)ds - T(t)B\sum_{k=1}^{m+1} c_k T(\eta_k - t_{k-1})I_{k-1}(y(t_{k-1}^-)) \\ & + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)). \end{aligned} \tag{4.53}$$

Using the fact that F has compact values and from (4.4.2), we may pass to a subsequence if necessary to get that g_n converges to g in $L^1(J,E)$, and hence $g \in S_{F,y}$. Then, for each $t \in J$,

$$\begin{aligned} y_n(t) \longrightarrow \tilde{y}(t) = {} & T(t)By_0 - \sum_{k=1}^{m+1} c_k T(t)B \int_0^{\eta_k} T(\eta_k - s)g(s)ds \\ & + \int_0^t T(t-s)g(s)ds \\ & - T(t)B\sum_{k=1}^{m+1} c_k T(\eta_k - t_{k-1})I_{k-1}(y(t_{k-1}^-)) \\ & + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)), \quad t \in J. \end{aligned} \tag{4.54}$$

So, $\tilde{y} \in N(y)$.

Step 2. $H_d(N(y_1), N(y_2)) \leq \gamma \|y_1 - y_2\|_{\text{PC}}$, for each $y_1, y_2 \in \text{PC}(J,E)$ (where $\gamma < 1$).

Let $y_1, y_2 \in \text{PC}(J,E)$ and $h_1 \in N(y_1)$. Then there exists $g_1(t) \in F(t, y_1(t))$ such that

$$\begin{aligned} h_1(t) = {} & T(t)By_0 - \sum_{k=1}^{m+1} c_k T(t)B \int_0^{\eta_k} T(\eta_k - s)g_1(s)ds \\ & + \int_0^t T(t-s)g_1(s)ds - T(t)B\sum_{k=1}^{m+1} c_k T(\eta_k - t_{k-1})I_{k-1}(y_1(t_{k-1}^-)) \\ & + \sum_{0<t_k<t} T(t-t_k)I_k(y_1(t_k^-)), \quad t \in J. \end{aligned} \tag{4.55}$$

From (4.4.2) it follows that

$$H_d(F(t, y_1(t)), F(t, y_2(t))) \leq l(t)|y_1(t) - y_2(t)|, \quad t \in J. \tag{4.56}$$

Hence there is $w \in F(t, y_2(t))$ such that

$$|g_1(t) - w| \leq l(t)|y_1(t) - y_2(t)|, \quad t \in J. \tag{4.57}$$

Consider $U : J \to \mathcal{P}(E)$, given by

$$U(t) = \{w \in E : |g_1(t) - w| \leq l(t)|y_1(t) - y_2(t)|\}. \tag{4.58}$$

Since the multivalued operator $V(t) = U(t) \cap F(t, y_2(t))$ is measurable (see [119, Proposition III.4]), there exists $g_2(t)$ a measurable selection for V. So, $g_2(t) \in F(t, y_2(t))$ and

$$|g_1(t) - g_2(t)| \leq l(t)|y_1(t) - y_2(t)|, \quad \text{for each } t \in J. \tag{4.59}$$

Let us define, for each $t \in J$,

$$\begin{aligned} h_2(t) = {} & T(t)By_0 - \sum_{k=1}^{m+1} c_k T(t)B \int_0^{\eta_k} T(\eta_k - s) g_2(s)\,ds \\ & + \int_0^t T(t-s) g_2(s)\,ds - T(t)B \sum_{k=1}^{m+1} c_k T(\eta_k - t_{k-1}) I_{k-1}(y_2(t_{k-1}^-)) \\ & + \sum_{0<t_k<t} T(t-t_k) I_k(y_2(t_k^-)). \end{aligned} \tag{4.60}$$

Then we have

$$\begin{aligned} |h_1(t) - h_2(t)| \leq {} & \bigg| \sum_{k=1}^{m+1} c_k T(t)B \int_0^{\eta_k} T(\eta_k - s)[g_1(s) - g_2(s)]\,ds \\ & + \int_0^t T(t-s)[g_1(s) - g_2(s)]\,ds \\ & - T(t)B \sum_{k=1}^{m+1} c_k T(\eta_k - t_{k-1}) \big(I_{k-1}(y_2(t_{k-1}^-)) - I_{k-1}(y_1(t_{k-1}^-))\big) \\ & + \sum_{0<t_k<t} T(t-t_k)\big(I_k(y_2(t_k^-)) - I_k(y_1(t_k^-))\big) \bigg| \\ \leq {} & \sum_{k=1}^{m+1} |c_k| M^2 \|B\|_{B(E)} \|y_1 - y_2\|_{PC} \int_0^{\eta_k} \ell(s)\,ds \end{aligned}$$

$$
\begin{aligned}
&+ M^2\|B\|_{B(E)} \sum_{k=1}^{m+1} |c_k|\, d_{k-1} \,|\, y_2(t_{k-1}^-) - y_1(t_{k-1}^-)\,| \\
&+ M\|y_1 - y_2\|_{\mathrm{PC}} \int_0^t \ell(s)ds + M\sum_{k=1}^{m} d_k \,|\, y_2(t_k^-) - y_1(t_k^-)\,| \\
\le{}& \sum_{k=1}^{m+1} |c_k|\, L(\eta_k) M^2 \|B\|_{B(E)} \|y_1 - y_2\|_{\mathrm{PC}} \\
&+ M^2\|B\|_{B(E)} \sum_{k=1}^{m+1} |c_k|\, d_{k-1} \|y_2 - y_1\|_{\mathrm{PC}} \\
&+ L(b)M\|y_1 - y_2\|_{\mathrm{PC}} + M\sum_{k=1}^{m} d_k \|y_2 - y_1\|_{\mathrm{PC}}.
\end{aligned}
\tag{4.61}
$$

Then

$$
\begin{aligned}
\|h_1 - h_2\|_{\mathrm{PC}} \le M\Bigg(& M\|B\|_{B(E)} \sum_{k=1}^{m+1} |c_k|\, L(\eta_k) + L(b) \\
& + M\|B\|_{B(E)} \sum_{k=1}^{m+1} |c_k|\, d_{k-1} + \sum_{k=1}^{m} d_k \Bigg) \|y_1 - y_2\|_{\mathrm{PC}}.
\end{aligned}
\tag{4.62}
$$

By the analogous relation, obtained by interchanging the roles of y_1 and y_2, it follows that

$$
\begin{aligned}
H_d(N(y_1), N(y_2))M \le \Bigg(& M\|B\|_{B(E)} \sum_{k=1}^{m+1} |c_k|\, L(\eta_k) + L(b) \\
& + M\|B\|_{B(E)} \sum_{k=1}^{m+1} |c_k|\, d_{k-1} + \sum_{k=1}^{m} d_k \Bigg) \|y_1 - y_2\|_{\mathrm{PC}}.
\end{aligned}
\tag{4.63}
$$

From (4.4.4), we have

$$
\gamma := M\left(M\|B\|_{B(E)} \sum_{k=1}^{m+1} |c_k|\, L(\eta_k) + L(b) + M\|B\|_{B(E)} \sum_{k=1}^{m+1} |c_k|\, d_{k-1} + \sum_{k=1}^{m} d_k \right) < 1.
\tag{4.64}
$$

Then N is a contraction and thus, by Theorem 1.11, it has a fixed point y, which is a mild solution to (4.2)–(4.4). □

By the help of the nonlinear alternative of Leray-Schauder type, combined with the selection theorem of Bressan and Colombo for lower semicontinuous maps with decomposable values, we will present a second existence result for problem (4.2)–(4.4), with a nonconvex-valued right-hand side.

Theorem 4.5. *Suppose, in addition to hypotheses (3.27.1), (4.3.1)–(4.3.3), the following also hold:*

(4.5.1) $F : J\times E \to \mathcal{P}(E)$ *is a nonempty compact-valued multivalued map such that*
 (a) $(t, y) \mapsto F(t, y)$ *is* $\mathcal{L} \otimes \mathcal{B}$ *measurable,*
 (b) $y \mapsto F(t, y)$ *is lower semicontinuous for a.e.* $t \in J$*;*

(4.5.2) *for each* $r > 0$*, there exists a function* $h_r \in L^1(J,\mathbb{R}^+)$ *such that*

$$\|F(t, y)\| := \sup\{|v| : v \in F(t, y)\} \le h_r(t), \quad \text{for a.e. } t \in J,\ y \in E \text{ with } |y| \le r. \tag{4.65}$$

Then the initial value problem (4.2)–(4.4) *has at least one solution on* J.

Proof. Conditions (4.5.1) and (4.5.2) imply that F is of lower semicontinuous type. Then from Theorem 1.5 there exists a continuous function $f : \mathrm{PC}(J,E) \to L^1(J,E)$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in \mathrm{PC}(J,E)$.

We consider the problem

$$\begin{gathered} y'(t) = Ay(t) + f(y)(t), \quad t \in J,\ t \neq t_k,\ k = 1,2,\dots,m, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m, \\ y(0) + \sum_{k=1}^{m+1} c_k y(\eta_k) = y_0. \end{gathered} \tag{4.66}$$

We remark that if $y \in \mathrm{PC}(J,E)$ is a solution of problem (4.66), then y is a solution to problem (4.2)–(4.4).

Transform problem (4.66) into a fixed point problem by considering the operator $N_1 : \mathrm{PC}(J,E) \to \mathrm{PC}(J,E)$ defined by

$$\begin{aligned} N_1(y) := {} & T(t)By_0 - \sum_{k=1}^{m+1} c_k T(t)B\int_0^{\eta_k} T(\eta_k - s)f(y)(s)ds \\ & + \int_0^t T(t-s)f(y)(s)ds \\ & - T(t)B\sum_{k=1}^{m+1} c_k T(\eta_k - t_{k-1})I_{k-1}(y(t_{k-1}^-)) \\ & + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)). \end{aligned} \tag{4.67}$$

We will show that N_1 is a completely continuous operator.

First we prove that N_1 *is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J,E)$. Then

$$\begin{aligned}|N_1(y_n)(t) - N_1(y)(t)| \leq {} & M^2 \sum_{k=1}^{m+1} |c_k| \, \|B\|_{B(E)} \int_0^{\eta_k} |f(y_n)(s) - f(y)(s)| \, ds \\ & + M \int_0^t |f(y_n)(s) - f(y)(s)| \, ds \\ & + M^2 \|B\|_{B(E)} \sum_{k=1}^{m+1} |c_k| \, |I_{k-1}(y_n(t_{k-1}^-)) - I_{k-1}(y(t_{k-1}^-))| \\ & + M \sum_{0<t_k<t} |I_k(y_n(t_k^-)) - I_k(y(t_k^-))|.\end{aligned} \tag{4.68}$$

Since the function f is continuous, then

$$||N_1(y_n) - N_1(y)||_{\mathrm{PC}} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{4.69}$$

The proof that N_1 is completely continuous is similar to that given in Theorem 4.3. Finally we establish a priori bounds on the solutions. Let $y \in \mathcal{E}(N_1)$. Then $y = \lambda N_1(y)$, for some $0 < \lambda < 1$ and

$$\begin{aligned}y(t) = {} & \lambda T(t) B y_0 - \lambda \sum_{k=1}^{m+1} c_k T(t) B \int_0^{\eta_k} T(\eta_k - s) f(y)(s) ds \\ & + \lambda \int_0^t T(t-s) f(y)(s) ds \\ & - \lambda T(t) B \sum_{k=1}^{m+1} c_k T(\eta_k - t_{k-1}) I_{k-1}(y(t_{k-1}^-)) \\ & + \lambda \sum_{0<t_k<t} T(t-t_k) I_k(y(t_k^-)), \quad t \in J.\end{aligned} \tag{4.70}$$

This implies by (3.27.1), (4.3.2), and (4.3.3) that, for each $t \in J$, we have

$$\begin{aligned}|y(t)| \leq {} & M\|B\|_{B(E)} \left(|y_0| + M \sum_{k=1}^{m+1} |c_k| \int_0^{\eta_k} p(t) \psi(|y(t)|) dt \right) \\ & + M \int_0^t p(s) \psi(|y(t)|) ds + M^2 \|B\|_{B(E)} \sum_{k=1}^{m+1} |c_k| \theta_k + M \sum_{k=1}^{m} \theta_k \\ \leq {} & M\|B\|_{B(E)} \left(|y_0| + M \sum_{k=1}^{m+1} |c_k| \psi(\|y\|_{\mathrm{PC}}) \int_0^{\eta_k} p(t) dt \right) \\ & + M \int_0^b p(s) \psi(\|y\|_{\mathrm{PC}}) ds + M^2 \|B\|_{B(E)} \sum_{k=1}^{m+1} |c_k| \theta_k + M \sum_{k=1}^{m} \theta_k.\end{aligned} \tag{4.71}$$

We continue as in Theorem 4.3. □

4.3. Existence results for impulsive functional semilinear differential inclusions with nonlocal conditions

In this section, we will be concerned with the existence of mild solutions for the first-order impulsive functional semilinear differential inclusions with nonlocal conditions in a Banach space of the form

$$\begin{gathered} y'(t) - Ay(t) \in F(t, y_t), \quad \text{a.e. } t \in J := [0,b],\ t \neq t_k,\ k = 1,2,\dots,m, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m, \\ y(t) + (g(y_{\eta_1},\dots,y_{\eta_p}))(t) = \phi(t), \quad t \in [-r,0], \end{gathered} \tag{4.72}$$

where $F : J \times \mathcal{D} \to \mathcal{P}(E)$ is a bounded-, closed-, convex-valued multivalued map, $\mathcal{D} = \{\psi : [-r,0] \to E \mid \psi$ is continuous everywhere except for a finite number of points $\bar{t}$ at which $\psi(\bar{t})$ and $\psi(\bar{t}^+)$ exist and $\psi(\bar{t}^-) = \psi(\bar{t})\}$, $\phi \in \mathcal{D}$ $(0 < r < \infty)$, ($\mathcal{P}(E)$ is the family of all nonempty subsets of E), $\eta_1 < \cdots < \eta_p \leq b$, $p \in \mathbb{N}$, $g : \mathcal{D}^p \to D$, ($\mathcal{D}^p = \mathcal{D} \times \mathcal{D} \times \cdots \times \mathcal{D}$, p-times), A is the infinitesimal generator of a family of semigroup $\{T(t) : t \geq 0\}$, $0 < t_1 < \cdots < t_m < t_{m+1} = b$, $I_k \in C(E,E)$ $(k = 1,\dots,m)$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$, and $y(t_k^-) = \lim_{h\to 0^-} y(t_k - h)$, and E is a real separable Banach space with norm $|\cdot|$.

Recall that $\Omega = \mathrm{PC}([-r,b],E)$ and that the spaces $\mathrm{PC}([-r,b],E)$ and $\mathrm{PC}^1([0,b],E)$ are defined in Section 3.2.

Definition 4.6. A function $y \in \Omega \cap \mathrm{AC}((t_k,t_{k+1}),E)$ is said to be a mild solution of (4.72) if there exists a function $v \in L^1(J,E)$ such that $v(t) \in F(t,y(t))$ a.e. on J and

$$\begin{aligned} y(t) = {} & T(t)[\phi(0) - (g(y_{\eta_1},\dots,y_{\eta_p}))(0)] + \int_0^t T(t-s)v(s)ds \\ & + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)), \end{aligned} \tag{4.73}$$

and $y(t) + (g(y_{\eta_1},\dots,y_{\eta_p}))(t) = \phi(t)$, $t \in [-r,0]$.

Theorem 4.7. *Assume that (3.2.1), (3.11.1), [(3.7.1)(i), (ii)], (3.27.1), and the following conditions hold:*

(4.7.1) *g is completely continuous and there exists a constant Q such that*

$$|g(u_1,\dots,u_p)(t)| \leq Q, \quad \text{for } (u_1,\dots,u_p) \in \mathcal{D}^p, \quad t \in [-r,0]; \tag{4.74}$$

(4.7.2) *there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and $p \in L^1(J,\mathbb{R}_+)$ such that*

$$\|F(t,u)\| \leq p(t)\psi(\|u\|_{\mathcal{D}}), \quad \text{for a.e. } t \in J \text{ and each } u \in D, \tag{4.75}$$

with

$$\int_1^\infty \frac{d\tau}{\psi(\tau)} = \infty. \tag{4.76}$$

Then the IVP (4.72) *has at least one mild solution.*

Proof. Transform problem (4.72) into a fixed point problem. Consider the multivalued operator $N : \Omega \to \mathcal{P}(\Omega)$ defined by

$$N(y) = \left\{ h \in \Omega : h(t) = \begin{cases} \phi(t) - (g(y_{\eta_1}, \dots, y_{\eta_p}))(t), & t \in [-r, 0], \\ T(t)[\phi(0) - (g(y_{\eta_1}, \dots, y_{\eta_p}))(0)] & \\ \quad + \int_0^t T(t-s)v(s)ds & \\ \quad + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)), & t \in J, \end{cases} \right\}, \tag{4.77}$$

where $v \in S_{F,y}$.

We will show that N satisfies the assumptions of Theorem 1.7. The proof will be given in several steps.

Step 1. $N(y)$ is convex, for each $y \in \Omega$.

Indeed, if h_1, h_2 belong to $N(y)$, then there exist $v_1, v_2 \in S_{F(y)}$ such that, for each $t \in J$, we have

$$\begin{aligned} h_i(t) = T(t)[\phi(0) - (g(y_{\eta_1}, \dots, y_{\eta_p}))(0)] + \int_0^t T(t-s)v_i(s)ds \\ + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)), \quad i = 1, 2. \end{aligned} \tag{4.78}$$

Let $0 \le d \le 1$. Then, for each $t \in J$, we have

$$\begin{aligned} (dh_1 + (1-d)h_2)(t) = T(t)&[\phi(0) - (g(y_{\eta_1}, \dots, y_{\eta_p}))(0)] \\ &+ \int_0^t T(t-s)[dv_1(s) + (1-d)v_2(s)]ds \\ &+ \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)). \end{aligned} \tag{4.79}$$

Since $S_{F(y)}$ is convex (because F has convex values), then

$$dh_1 + (1-d)h_2 \in N(y). \tag{4.80}$$

Step 2. N maps bounded sets into bounded sets in Ω.

Indeed, it is enough to show that there exists a positive constant ℓ such that, for each $y \in \mathcal{B}_q = \{y \in \Omega : \|y\| \le q\}$, one has $\|N(y)\| := \sup\{\|h\| : h \in N(y)\} \le \ell$.

Let $y \in \mathcal{B}_q$ and $h \in N(y)$. Then there exists $v \in S_{F(y)}$ such that, for each $t \in J$, we have

$$\begin{aligned} h(t) = {} & T(t)[\phi(0) - (g(y_{\eta_1},\dots,y_{\eta_p}))(0)] + \int_0^t T(t-s)v(s)ds \\ & + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)). \end{aligned} \tag{4.81}$$

We have, for each $t \in J$,

$$\begin{aligned} |h(t)| & \le M[\|\phi\|_{\mathcal{D}} + Q] + M\int_0^b \varphi_q(s)ds + M\sum_{k=1}^m c_k \\ & \le M[\|\phi\|_{\mathcal{D}} + Q] + M\|\varphi_q\|_{L^1} + M\sum_{k=1}^m c_k, \end{aligned} \tag{4.82}$$

where ϕ_q is defined in the definition of a Carathéodory function. Then, for each $h \in N(\mathcal{B}_q)$, we obtain

$$\|N(y)\| \le M[\|\phi\|_{\mathcal{D}} + Q] + M\|\varphi_q\|_{L^1} + M\sum_{k=1}^m c_k := \ell. \tag{4.83}$$

Step 3. N maps bounded sets into equicontinuous sets of Ω.

Let $\tau_1, \tau_2 \in J \setminus \{t_1,\dots,t_m\}$, $\tau_1 < \tau_2$, and $\delta > 0$ such that $\{t_1,\dots,t_m\} \cap [t-\delta, t+\delta] = \emptyset$, and let $\mathcal{B}_q$ be a bounded set of Ω as in Step 2. Let $y \in \mathcal{B}_q$ and $h \in N(y)$. Then there exists $v \in S_{F,y}$ such that, for each $t \in J$, we have

$$\begin{aligned} |h(\tau_2) - h(\tau_1)| \le {} & |[T(\tau_2) - T(\tau_1)]\phi(0)| \\ & + |[T(\tau_2) - T(\tau_1)]g(y_{\eta_1},\dots,y_{\eta_p})(0)| \\ & + \int_0^{\tau_1-\epsilon} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)}\varphi_q(s)ds \\ & + \int_{\tau_1-\epsilon}^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)}\varphi_q(s)ds \\ & + \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)}\varphi_q(s)ds + \sum_{0<t<\tau_2-\tau_1} Mc_k \\ & + \sum_{0<t<\tau_2} \|T(\tau_2 - t_k) - T(\tau_1 - t_k)\|_{B(E)}c_k. \end{aligned} \tag{4.84}$$

As $\tau_2 \to \tau_1$, and for ϵ sufficiently small, the right-hand side of the above inequality tends to zero, since $T(t)$ is a strongly continuous operator, and the compactness of $T(t)$, for $t > 0$, implies the continuity in the uniform operator topology.

This proves the equicontinuity for the case where $t \ne t_i$, $i = 1,\dots,m$. It remains to examine the equicontinuity at $t = t_i$.

Set

$$\begin{aligned} h_1(t) &= T(t)[\phi(0) - g(y_{\eta_1},\dots,y_{\eta_p})(0)] + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k)),\\ h_2(t) &= \int_0^t T(t-s)v(s)ds. \end{aligned} \tag{4.85}$$

First we prove equicontinuity at $t = t_i^-$. Fix $\delta_1 > 0$ such that $\{t_k : k \neq i\} \cap [t_i - \delta_1, t_i + \delta_1] = \emptyset$,

$$\begin{aligned} h_1(t_i) &= T(t_i)[\phi(0) - g(y_{\eta_1},\dots,y_{\eta_p})(0)] + \sum_{0<t_k<t_i} T(t-t_k)I_k(y(t_k))\\ &= T(t_i)[\phi(0) - g(y_{\eta_1},\dots,y_{\eta_p})(0)] + \sum_{k=1}^{i-1} T(t_i-t_k)I_k(y(t_k)). \end{aligned} \tag{4.86}$$

For $0 < h < \delta_1$, we have

$$\begin{aligned} |h_1(t_i-h) - h_1(t_i)| \le{}& |(T(t_i-h) - T(t_i))[\phi(0) - g(y_{\eta_1},\dots,y_{\eta_p})(0)]|\\ &+ \sum_{k=1}^{i-1} |[T(t_i-h-t_k) - T(t_i-t_k)]I(y(t_k^-))|. \end{aligned} \tag{4.87}$$

The right-hand side tends to zero as $h \to 0$.

Moreover,

$$\begin{aligned} |h_2(t_i-h) - h_2(t_i)| \le{}& \int_0^{t_i-h} |[T(t_i-h-s) - T(t_i-s)]v(s)|ds\\ &+ \int_{t_i-h}^{t_i} M\phi_q(s)ds, \end{aligned} \tag{4.88}$$

which tends to zero as $h \to 0$.

Define

$$\begin{aligned} \widehat{h}_0(t) &= h(t), \quad t \in [0,t_1],\\ \widehat{h}_i(t) &= \begin{cases} h(t), & t \in (t_i, t_{i+1}],\\ h(t_i^+), & t = t_i. \end{cases} \end{aligned} \tag{4.89}$$

Next we prove equicontinuity at $t = t_i^+$. Fix $\delta_2 > 0$ such that $\{t_k : k \neq i\} \cap [t_i - \delta_2, t_i + \delta_2] = \emptyset$. Then

$$\begin{aligned} \widehat{h}(t_i) = {}& T(t_i)[\phi(0) - g(y_{\eta_1},\dots,y_{\eta_p})(0)] + \int_0^{t_i} T(t_i-s)v(s)ds\\ &+ \sum_{k=1}^{i} T(t_i-t_k)I_k(y(t_k^-)). \end{aligned} \tag{4.90}$$

For $0 < h < \delta_2$, we have

$$
\begin{aligned}
|\widehat{h}(t_i+h) - \widehat{h}(t_i)| \leq & \ |(T(t_i+h) - T(t_i))[\phi(0) - g(y_{\eta_1},\dots,y_{\eta_p})(0)]| \\
& + \int_0^{t_i} |[T(t_i+h-s) - T(t_i-s)]v(s)|\,ds \\
& + \int_{t_i}^{t_i+h} M\varphi_q(s)\,ds \\
& + \sum_{k=1}^{i} |[T(t_i+h-t_k) - T(t_i-t_k)]I(y(t_k^-))|.
\end{aligned}
\tag{4.91}
$$

The right-hand side tends to zero as $h \to 0$.

The equicontinuity for the cases $\tau_1 < \tau_2 \leq 0$ and $\tau_1 \leq 0 \leq \tau_2$ follows from the uniform continuity of ϕ on the interval $[-r, 0]$ and the complete continuity of g. As a consequence of Steps 1 to 3 and (4.7.1), together with the Arzelá-Ascoli theorem, it suffices to show that N maps B_q into a precompact set in E.

Let $0 < t \leq b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B_q$ we define

$$
\begin{aligned}
h_\epsilon(t) = & \ T(t)[\phi(0) - (g(y_{\eta_1},\dots,y_{\eta_p}))(0)] + T(\epsilon)\int_0^{t-\epsilon} T(t-s-\epsilon)v_1(s)\,ds \\
& + T(\epsilon)\sum_{0<t_k<t-\epsilon} T(t-t_k-\epsilon)I_k(y(t_k^-)),
\end{aligned}
\tag{4.92}
$$

where $v_1 \in S_{F(y)}$. Since $T(t)$ is a compact operator, the set $H_\epsilon(t) = \{h_\epsilon(t) : h_\epsilon \in N(y)\}$ is precompact in E, for every ϵ, $0 < \epsilon < t$. Moreover, for every $h \in N(y)$, we have

$$
|h(t) - h_\epsilon(t)| \leq \int_{t-\epsilon}^{t} \|T(t-s)\|_{B(E)}\varphi_q(s)\,ds + \sum_{t-\epsilon<t_k<t} \|T(t-t_k)\|_{B(E)}c_k.
\tag{4.93}
$$

Therefore there are precompact sets arbitrarily close to the set $H(t) = \{h_\epsilon(t) : h \in N(y)\}$. Hence the set $H(t) = \{h(t) : h \in N(B_q)\}$ is precompact in E. Hence the operator $N : \Omega \to \mathcal{P}(\Omega)$ is completely continuous, and therefore a condensing operator.

Step 4. N has a closed graph.

Let $y_n \to y_*$, $h_n \in N(y_n)$, and $h_n \to h_*$. We will prove that $h_* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $v_n \in S_{F(y_n)}$ such that, for each $t \in J$,

$$
\begin{aligned}
h_n(t) = & \ T(t)[\phi(0) - (g((y_n)_{\eta_1},\dots,(y_n)_{\eta_p}))(0)] + \int_0^t T(t-s)v_n(s)\,ds \\
& + \sum_{0<t_k<t} T(t-t_k)I_k(y_n(t_k^-)).
\end{aligned}
\tag{4.94}
$$

We have to prove that there exists $v_* \in S_{F(y_*)}$ such that, for each $t \in J$,

$$\begin{aligned} h_*(t) = {} & T(t)[\phi(0) - (g((y_*)_{\eta_1},\ldots,(y_*)_{\eta_p}))(0)] + \int_0^t T(t-s)v_*(s)ds \\ & + \sum_{0<t_k<t} T(t-t_k)I_k(y_*(t_k^-)). \end{aligned} \tag{4.95}$$

Clearly, since I_k, $k = 1,\ldots,m$, are continuous and g is completely continuous, we obtain that

$$\begin{aligned} & \Bigg\| \bigg(h_n - T(t)[\phi(0) - (g((y_n)_{\eta_1},\ldots,(y_n)_{\eta_p}))(0)] - \sum_{0<t_k<t} T(t-t_k)I_k(y_n(t_k^-)) \bigg) \\ & \quad - \bigg(h_* - T(t)[\phi(0) - (g((y_*)_{\eta_1},\ldots,(y_*)_{\eta_p}))(0)] \\ & \qquad - \sum_{0<t_k<t} T(t-t_k)I_k(y_*(t_k^-)) \bigg) \Bigg\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{4.96}$$

Consider the linear continuous operator

$$\begin{gathered} \Gamma : L^1(J,E) \longrightarrow C(J,E), \\ g \longmapsto \Gamma(g)(t) = \int_0^t T(t-s)v(s)ds. \end{gathered} \tag{4.97}$$

From Lemma 1.28, it follows that $\Gamma \circ S_F$ is a closed graph operator. Moreover, we have

$$\begin{aligned} & h_n(t) - T(t)[\phi(0) - (g((y_n)_{\eta_1},\ldots,(y_n)_{\eta_p}))(0)] \\ & \quad - \sum_{0<t_k<t} T(t-t_k)I_k(y_n(t_k)) \in \Gamma(S_{F(y_n)}). \end{aligned} \tag{4.98}$$

Since $y_n \to y_*$, it follows, from Lemma 1.28, that

$$\begin{aligned} & h_*(t) - T(t)[\phi(0) - (g((y_*)_{\eta_1},\ldots,(y_*)_{\eta_p}))(0)] - \sum_{0<t_k<t} T(t-t_k)I_k(y_*(t_k^-)) \\ & \quad = \int_0^t T(t-s)v_*(s)ds \end{aligned} \tag{4.99}$$

for some $v_* \in S_{F(y_*)}$.

Step 5. Now it remains to show that the set

$$\mathcal{M} := \{y \in \Omega : \lambda y \in N(y) \text{ for some } \lambda > 1\} \tag{4.100}$$

is bounded. Let $y \in \mathcal{M}$. Then $\lambda y \in N(y)$, for some $\lambda > 1$. Thus, for each $t \in J$,

$$y(t) = \lambda^{-1}\Big[T(t)[\phi(0) - (g(y_{\eta_1},\dots,y_{\eta_p}))(0)] + \int_0^t T(t-s)v(s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-))\Big] \tag{4.101}$$

for some $v \in S_{F(y)}$. This implies that, for each $t \in J$, we have

$$|y(t)| \leq M[\|\phi\|_{\mathcal{D}} + Q] + \int_0^t Mp(s)\psi(\|y_s\|_{\mathcal{D}})ds + M\sum_{k=1}^m c_k. \tag{4.102}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq b. \tag{4.103}$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by the previous inequality, we have, for $t \in J$,

$$\mu(t) \leq M[\|\phi\|_{\mathcal{D}} + Q] + M\int_0^t p(s)\psi(\mu(s))ds + M\sum_{k=1}^m c_k. \tag{4.104}$$

If $t^* \in [-r,0]$, then $\mu(t) \leq \|\phi\|_{\mathcal{D}} + Q$ and the previous inequality holds.

Let us denote the right-hand side of the above inequality as $\overline{v}(t)$. Then, we have

$$\begin{gathered}\mu(t) \leq \overline{v}(t), \quad t \in J,\\ \overline{v}(0) = M[\|\phi\|_{\mathcal{D}} + Q] + M\sum_{k=1}^m c_k, \qquad \overline{v}'(t) = Mp(t)\psi(\mu(t)), \quad t \in J.\end{gathered} \tag{4.105}$$

Using the increasing character of ψ, we get

$$\overline{v}'(t) \leq Mp(t)\psi(\overline{v}(t)), \quad \text{a.e. } t \in J. \tag{4.106}$$

Then, for each $t \in J$, we have

$$\int_{\overline{v}(0)}^{\overline{v}(t)} \frac{du}{\psi(u)} \leq M\int_0^b p(s)ds < \infty. \tag{4.107}$$

Assumption (4.7.2) shows that there exists a constant K such that $\overline{v}(t) \leq K$, $t \in J$, and hence $\mu(t) \leq KZ$, $t \in J$. Since, for every $t \in J$, $\|y_t\| \leq \mu(t)$, we have

$$\|y\| \leq K' = \max\{\|\phi\|_{\mathcal{D}} + Q, K\}, \tag{4.108}$$

where K' depends on b, ϕ, Q, and on the functions p and ψ. This shows that $\mathcal{M}$ is bounded. As a consequence of Theorem 1.7, we deduce that N has a fixed point which is a mild solution of (4.72). $\square$

Theorem 4.8. *Suppose that hypotheses (3.13.1)–(3.13.3) and the following condition are satisfied:*

(4.8.1) *there exist constants $\overline{c}_k$ such that*

$$|g(u_1,\dots,u_p)(0)-g(\overline{u}_1,\dots,\overline{u}_p)(0)| \leq \sum_{k=1}^{p}\overline{c}_k\,|u_k(0)-\overline{u}_k(0)|, \tag{4.109}$$

for each $(u_1,\dots,u_2)$, $(\overline{u}_1,\dots,\overline{u}_p)$ in D^p.

If

$$M\left(l^*+\sum_{k=1}^{p}\overline{c}_k+\sum_{k=1}^{m}c_k\right)<1, \quad l^*=\int_0^b l(s)ds, \tag{4.110}$$

then the IVP (4.72) *has at least one mild solution.*

Proof. Set

$$\Omega_0=\{y\in\Omega: y(t)=\phi(t)-(g(y_{\eta_1},\dots,y_{\eta_p}))(t),\ \forall t\in[-r,0]\}. \tag{4.111}$$

Transform problem (4.72) into a fixed point problem. Let the multivalued operator $N:\Omega_0\to\mathcal{P}(\Omega_0)$ be defined as in the proof of Theorem 4.7. We will show that N satisfies the assumptions of Theorem 1.11. The proof will be given in two steps.

Step 1. $N(y)\in\mathcal{P}_{\text{cl}}(\Omega_0)$, for each $y\in\Omega$.

Indeed, let $(y_n)_{n\geq 0}\in N(y)$ be such that $y_n\to\widetilde{y}$ in Ω. Then $\widetilde{y}\in\Omega$ and there exists $v_n\in S_{F(y)}$ such that, for each $t\in J$,

$$\begin{aligned} y_n(t)=T(t)[\phi(0)-(g(y_{\eta_1},\dots,y_{\eta_p}))(0)]&+\int_0^t T(t-s)v_n(s)ds\\ &+\sum_{0<t_k<t}T(t-t_k)I_k(y(t_k^-)).\end{aligned} \tag{4.112}$$

Using the fact that F has compact values and from (3.13.2) we may pass to a subsequence if necessary to get that v_n converges to v in $L^1(J,E)$ and hence $v\in S_{F(y)}$. Then, for each $t\in J$,

$$\begin{aligned} y_n(t)\longrightarrow\widetilde{y}(t)=T(t)[\phi(0)-(g(y_{\eta_1},\dots,y_{\eta_p}))(0)]&+\int_0^t T(t-s)v(s)ds\\ &+\sum_{0<t_k<t}T(t-t_k)I_k(y(t_k^-)).\end{aligned} \tag{4.113}$$

So, $\widetilde{y}\in N(y)$.

Step 2. There exists $\gamma < 1$ such that

$$H_d(N(y),N(\overline{y})) \leq \gamma\|y-\overline{y}\|, \quad \text{for each } y,\overline{y} \in \Omega. \tag{4.114}$$

Let $y,\overline{y} \in \Omega$ and $h \in N(y)$. Then there exists $v(t) \in F(t,y_t)$ such that, for each $t \in J$,

$$\begin{aligned} h(t) = {} & T(t)\big[\phi(0) - \big(g\big(y_{\eta_1},\dots,y_{\eta_p}\big)\big)(0)\big] + \int_0^t T(t-s)v(s)ds \\ & + \sum_{0<t_k<t} T(t-t_k)I_k\big(y\big(t_k^-\big)\big). \end{aligned} \tag{4.115}$$

From (3.13.2), it follows that

$$H_d\big(F(t,y_t),F\big(t,\overline{y}_t\big)\big) \leq l(t)\big|\big|y_t - \overline{y}_t\big|\big|_{\mathcal{D}}. \tag{4.116}$$

Hence there is $w \in F(t,\overline{y}_t)$ such that

$$\big|v(t) - w\big| \leq l(t)\big|\big|y_t - \overline{y}_t\big|\big|_{\mathcal{D}}, \quad t \in J. \tag{4.117}$$

Consider $U : J \to \mathcal{P}(E)$, given by

$$U(t) = \big\{w \in E : \big|v(t) - w\big| \leq l(t)\big|\big|y_t - \overline{y}_t\big|\big|_{\mathcal{D}}\big\}. \tag{4.118}$$

Since the multivalued operator $V(t) = U(t) \cap F(t,\overline{y}_t)$ is measurable (see [119, Proposition III.4]), there exists a function $t \to \overline{v}(t)$, which is a measurable selection for V. So, $\overline{v}(t) \in F(t,\overline{y}_t)$ and

$$\big|v(t) - \overline{v}(t)\big| \leq l(t)\big|\big|y_t - \overline{y}_t\big|\big|_{\mathcal{D}}, \quad \text{for each } t \in J. \tag{4.119}$$

Let us define, for each $t \in J$,

$$\begin{aligned} \overline{h}(t) = {} & T(t)\big[\phi(0) - \big(g\big(\overline{y}_{\eta_1},\dots,\overline{y}_{\eta_p}\big)\big)(0)\big] + \int_0^t T(t-s)\overline{v}(s)ds \\ & + \sum_{0<t_k<t} T(t-t_k)I_k\big(\overline{y}\big(t_k^-\big)\big). \end{aligned} \tag{4.120}$$

Then we have

$$
\begin{aligned}
|h(t)-\overline{h}(t)| &\leq M\left|\left(g\left(y_{\eta_1},\ldots,y_{\eta_p}\right)\right)(0)-g\left(\overline{y}_{\eta_1},\ldots,\overline{y}_{\eta_p}\right)(0)\right| \\
&\quad + M\int_0^t |v(s)-\overline{v}(s)|\,ds \\
&\quad + \sum_{0<t_k<t} |T(t-t_k)|\,\left|I_k(y(t_k))-I_k\left(\overline{y}\left(t_k^-\right)\right)\right| \\
&\leq M\sum_{k=1}^{p}\overline{c}_k\left|\left(y_{\eta_k}-\overline{y}_{\eta_k}\right)(0)\right| \\
&\quad + M\int_0^t l(s)\left\|y_s-\overline{y}_s\right\|_{\mathcal{D}}ds + M\sum_{k=1}^{m}c_k\|y-\overline{y}\| \\
&= M\sum_{k=1}^{p}\overline{c}_k\|y-\overline{y}\| + M\int_0^b l(s)\|y-\overline{y}\|ds + M\sum_{k=1}^{m}c_k\|y-\overline{y}\| \\
&\leq \left[M\sum_{k=1}^{p}\overline{c}_k + Ml^* + M\sum_{k=1}^{m}c_k\right]\|y-\overline{y}\|.
\end{aligned}
\tag{4.121}
$$

Consequently,

$$
\|h-\overline{h}\| \leq M\left[\sum_{k=1}^{p}\overline{c}_k + l^* + \sum_{k=1}^{m}c_k\right]\|y-\overline{y}\|. \tag{4.122}
$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$
H_d(N(y),N(\overline{y})) \leq M\left[\sum_{k=1}^{p}\overline{c}_k + l^* + \sum_{k=1}^{m}c_k\right]\|y-\overline{y}\|. \tag{4.123}
$$

So, N is a contraction and thus, by Theorem 1.11, N has a fixed point y, which is a mild solution to (4.72). □

4.4. Notes and remarks

The results of Section 4.2 are adapted from Benchohra et al. [40], while the results of Section 4.3 come from Benchohra et al. [87]. The techniques in this chapter have been adapted from [112] where the nonimpulsive case was discussed.

5 Positive solutions for impulsive differential equations

5.1. Introduction

Positive solutions and multiple positive solutions of differential equations have received a tremendous amount of attention. Studies have involved initial value problems, as well as boundary value problems, for both ordinary and functional differential equations. In some cases, impulse effects have also been present. The methods that have been used include multiple applications of the Guo-Krasnosel'skii fixed point theorem [158], the Leggett-Williams multiple fixed point theorem [187], and extensions such as the Avery-Henderson double fixed point theorem [26]. Many such multiple-solution works can be found in the papers [6, 8–10, 19, 52, 94, 95, 137, 159, 194].

This chapter is devoted to positive solutions and multiple positive solutions of impulsive differential equations.

5.2. Positive solutions for impulsive functional differential equations

Throughout this section, let $J = [0,b]$, and the points $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$ are fixed. This section is concerned with the existence of three nonnegative solutions for initial value problems for first- and second-order functional differential equations with impulsive effects. In Section 5.2.1, we consider the first-order IVP

$$\begin{aligned} y'(t) &= f(t, y_t), \quad t \in J = [0,b],\ t \neq t_k, \quad k = 1,\dots,m, \\ \Delta y|_{t=t_k} &= I_k\big(y(t_k^-)\big), \quad k = 1,\dots,m, \\ y(t) &= \phi(t), \quad t \in [-r,0], \end{aligned} \tag{5.1}$$

where $f : J\times\mathcal{D} \to \mathbb{R}$ is a given function, $\mathcal{D} = \{\psi : [-r,0] \to \mathbb{R}_+ \mid \psi$ is continuous everywhere except for a finite number of points s at which $\psi(s)$ and the right limit $\psi(s^+)$ exist and $\psi(s^-) = \psi(s)\}$, $\phi \in \mathcal{D}$, $0 < r < \infty$, $I_k : \mathbb{R} \to \mathbb{R}_+$ $(k = 1,2,\dots,m)$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, and $J' = J \setminus \{t_1,\dots,t_m\}$.

In Section 5.2.2, we study the second-order impulsive functional differential equations of the form

$$\begin{gathered} y''(t) = f(t, y_t), \quad t \in J = [0,b],\ t \neq t_k,\ k = 1,\dots,m, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m, \\ \Delta y'|_{t=t_k} = \bar{I}_k(y(t_k^-)), \quad k = 1,\dots,m, \\ y(t) = \phi(t), \quad t \in [-r,0], \qquad y'(0) = \eta, \end{gathered} \tag{5.2}$$

where f, I_k, and ϕ are as in problem (5.1), $\bar{I}_k \in C(\mathbb{R},\mathbb{R}_+)$, and $\eta \in \mathbb{R}$.

5.2.1. First-order impulsive FDEs

In what follows we will assume that f is an L^1-Carathéodory function. We seek a solution of (5.1) via the Leggett-Williams fixed theorem, which employs the concept of concave continuous functionals.

By a concave nonnegative continuous functional ψ on a space C we mean a continuous mapping $\psi : C \to [0,\infty)$ with

$$\psi(\lambda x + (1-\lambda)y) \geq \lambda\psi(x) + (1-\lambda)\psi(y), \quad \forall x, y \in C,\ \lambda \in [0,1]. \tag{5.3}$$

Let us start by defining what we mean by a solution of problem (5.1). We recall here that $\Omega = \mathrm{PC}([-r,b],\mathbb{R})$.

Definition 5.1. A function $y \in \Omega \cap \mathrm{AC}((t_k,t_{k+1}),\mathbb{R})$ is said to be a solution of (5.1) if y satisfies the equation $y'(t) = f(t,y_t)$ a.e. on J', the conditions $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $k = 1,\dots,m$, and $y(t) = \phi(t)$, $t \in [-r,0]$.

Theorem 5.2. *Assume that the following assumptions are satisfied:*

(5.2.1) *there exist constants c_k such that*

$$|I_k(y)| \leq c_k, \quad k = 1,\dots,m, \text{ for each } y \in \mathbb{R}; \tag{5.4}$$

(5.2.2) *there exist a function $p \in L^1(J,\mathbb{R}_+)$, $\rho > 0$, and $0 < M < 1$ such that*

$$\begin{gathered} |f(t,u)| \leq Mp(t) \quad \text{for a.e. } t \in J \text{ and each } u \in D, \\ \|\phi\|_{\mathcal{D}} + \sum_{k=1}^{m} c_k + M\int_0^b p(t)dt < \rho; \end{gathered} \tag{5.5}$$

(5.2.3) *there exist $L > \rho$, $M \leq M_1 < 1$, and an interval $[c,d] \subset (0,b)$ such that*

$$\begin{aligned} &\min_{t\in[c,d]} \left(\phi(0) + \sum_{0<t_k<t} I_k(y(t_k)) + \int_0^t f(s,y_s)ds\right) \\ &\quad \geq M_1\left(\phi(0) + \sum_{k=1}^{m} I_k(y(t_k)) + \int_0^b f(s,y_s)ds\right) > L; \end{aligned} \tag{5.6}$$

(5.2.4) *there exist $R > L$ and M_2 with $M_1 \leq M_2 < 1$ such that*

$$\|\phi\|_{\mathcal{D}} + \sum_{k=1}^{m} c_k + M_2 \int_0^b p(t)dt \leq R. \tag{5.7}$$

Then problem (5.1) *has three nonnegative solutions y_1, y_2, y_3 with*

$$\begin{gathered} \|y_1\| < \rho, \qquad y_2(t) > L \quad \text{for } t \in [0,b], \\ \|y_3\| > \rho \quad \text{with} \min_{t\in[c,d]} y_3(t) < L. \end{gathered} \tag{5.8}$$

Proof. Transform problem (5.1) into a fixed point problem. Consider the operator $N : \Omega \to \Omega$ defined by

$$N(y)(t) = \begin{cases} \phi(t) & \text{if } t \in [-r,0], \\ \phi(0) + \displaystyle\sum_{0<t_k<t} I_k(y(t_k^-)) + \int_0^t f(s,y_s)ds & \text{if } t \in [0,b]. \end{cases} \tag{5.9}$$

We will show that N is a completely continuous operator.

Step 1. N is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in Ω. Then

$$\begin{aligned} &|N(y_n(t)) - N(y(t))| \\ &\quad \leq \int_0^t |f(s,y_{n_s}) - f(s,y_s)|ds + \sum_{0<t_k<t} |I_k(y_n(t_k^-)) - I_k(y(t_k^-))| \\ &\quad \leq \int_0^b |f(s,y_{n_s}) - f(s,y_s)|ds + \sum_{0<t_k<t} |I_k(y_n(t_k^-)) - I_k(y(t_k^-))|. \end{aligned} \tag{5.10}$$

Since f is an L^1-Carathéodory function and I_k, $k = 1,\dots,m$, are continuous, then

$$\begin{aligned} &\|N(y_n) - N(y)\| \\ &\quad \leq \|f(\cdot,y_{n_{(\cdot)}}) - f(\cdot,y_{(\cdot)})\|_{L^1} + \sum_{k=1}^{m} |I_k(y_n(t_k^-)) - I_k(y(t_k^-))| \longrightarrow 0 \end{aligned} \tag{5.11}$$

as $n \to \infty$.

Step 2. N maps bounded sets into bounded sets in Ω.

Indeed, it is enough to show that, for any $q > 0$, there exists a positive constant ℓ such that, for each $y \in B_q = \{y \in \Omega : \|y\| \leq q\}$, we have $\|N(y)\| \leq \ell$.

By (5.2.1) we have, for each $t \in J$,

$$\begin{aligned} |N(y)(t)| &\leq \|\phi\|_{\mathcal{D}} + \int_0^t |f(s,y_s)|ds + \sum_{0<t_k<t} |I_k(y(t_k^-))| \\ &\leq \|\phi\|_{\mathcal{D}} + \|h_q\|_{L^1} + \sum_{k=1}^{m} c_k := \ell. \end{aligned} \tag{5.12}$$

Then

$$||N(y)|| \leq \ell. \tag{5.13}$$

Step 3. N maps bounded sets into equicontinuous sets of Ω.

Let $\tau_1, \tau_2 \in [0, b]$, $\tau_1 < \tau_2$, and let B_q be a bounded set of Ω. Let $y \in B_q$. Then

$$|N(y)(\tau_2) - N(y)(\tau_1)| \leq \int_{\tau_1}^{\tau_2} h_q(s)ds + \sum_{0<t_k<\tau_1-\tau_2} c_k. \tag{5.14}$$

As $\tau_2 \to \tau_1$ the right-hand side of the above inequality tends to zero. This proves the equicontinuity for the case where $t \neq t_i$, $i = 1, \dots, m$. It remains to examine the equicontinuity at $t = t_i$. The proof is similar to that given in Theorem 4.3. The equicontinuity for the cases $\tau_1 < \tau_2 \leq 0$ and $\tau_1 \leq 0 \leq \tau_2$ is obvious.

As a consequence of Steps 1 to 3, together with the Arzelá-Ascoli theorem, we can conclude that $N : \Omega \to \Omega$ is completely continuous.

Let

$$C = \{y \in \Omega : y(t) \geq 0 \text{ for } t \in [-r, b]\} \tag{5.15}$$

be a cone in Ω. Since f and I_k, $k = 1, \dots, m$, are all positive functions, $N(C) \subset C$ and $N : \overline{C}_R \to \overline{C}_R$ is a completely continuous operator. In addition, by (5.2.4), we can show that if $y \in \overline{C}_R$, then $N(y) \in \overline{C}_R$. Next, let $\psi : C \to [0, \infty)$ be defined by

$$\psi(y) = \min_{t\in[c,d]} y(t). \tag{5.16}$$

It is clear that ψ is a nonnegative concave continuous functional and $\psi(y) \leq \|y\|_\Omega$ for $y \in \overline{C}_R$. Now it remains to show that the hypotheses of Theorem 1.14 are satisfied.

Claim 1. $\{y\in C(\psi, L, K) : \psi(y) > L\} \neq \varnothing$ and $\psi(N(y)) > L$ for all $y \in C(\psi, L, K)$.

Let K be such that $LM^{-1} \leq K \leq R$ and $y(t) = (L+K)/2$ for $t \in [-r, b]$. By the definition of $C(\psi, L, K)$, y belongs to $C(\psi, L, K)$. Then y belongs to $\{y \in C(\psi, L, K) : \psi(y) > L\}$ and hence it is nonempty. Also if $y \in C(\psi, L, K)$, then

$$\begin{aligned}\psi(N(y)) &= \min_{t\in[c,d]} \left(\phi(0) + \sum_{0<t_k<t} I_k(y(t_k^-)) + \int_0^t f(s, y_s)ds\right) \\ &\geq M_1\left(\phi(0) + \sum_{k=1}^{m} I_k(y(t_k^-)) + \int_0^b f(s, y_s)ds\right) > L,\end{aligned} \tag{5.17}$$

by using (5.2.3).

Claim 2. $\|N(y)\|_{\Omega} \leq \rho$ for all $y \in \overline{C}_{\rho}$.

For $y \in \overline{C}_{\rho}$, we have, from (5.2.1) and (5.2.2),

$$\begin{aligned} |N(y)(t)| &\leq |\phi(0)| + \int_0^t |f(s, y_s)|\,ds + \sum_{0<t_k<t} |I_k(y(t_k^-))| \\ &\leq \|\phi\|_{\mathcal{D}} + M\|p\|_{L^1} + \sum_{k=1}^{m} c_k < \rho. \end{aligned} \tag{5.18}$$

Claim 3. $\psi(N(y)) > L$ for each $y \in C(\psi, L, R)$ with $\|N(y)\| \geq K$. Let $y \in C(\psi, L, R)$ with $\|N(y)\| > K$. From (5.2.3) we have

$$\begin{aligned} \psi(N(y)) &= \min_{t\in[c,d]} \left(\phi(0) + \sum_{0<t_k<t} I_k(y(t_k^-)) + \int_0^t f(s, y_s)\,ds\right) \\ &\geq M_1\left(\phi(0) + \sum_{k=1}^{m} I_k(y(t_k)) + \int_0^b f(s, y_s)\,ds\right) = M_1\|N(y)\| > M_1 K \geq L. \end{aligned} \tag{5.19}$$

Then the Leggett-Williams fixed point theorem implies that N has at least three fixed points y_1, y_2, y_3 which are solutions to problem (5.1). Furthermore, we have

$$\begin{gathered} y_1 \in C_{\rho}, \qquad y_2 \in \{y \in C(\psi, L, R) : \psi(y) > L\}, \\ y_3 \in C_R - \{C(\psi, L, R) \cup C_{\rho}\}. \end{gathered} \tag{5.20}$$

□

5.2.2. Second-order impulsive FDEs

In this section, we study the existence of three solutions for the second-order IVP (5.2). Again, these solutions will arise from the Leggett-Williams fixed point theorem.

We adopt the following definition.

Definition 5.3. A function $y \in \Omega \cap AC^1((t_k, t_{k+1}), \mathbb{R})$ is said to be a solution of (5.2) if y satisfies the equation $y''(t) = f(t, y_t)$ a.e. on J, $t \neq t_k$, $k = 1, \ldots, m$, and the conditions $\Delta y|_{t=t_k} = I_k(y(t))$, $\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k))$, $k = 1, \ldots, m$, $y'(0) = \eta$, and $y(t) = \phi(t)$ on $[-r, 0]$.

We are now in a position to state and prove our existence result for problem (5.2).

Theorem 5.4. *Suppose that hypothesis (5.2.1) holds. In addition assume that the following conditions are satisfied:*

(5.4.1) *there exist constants d_k such that*

$$|\overline{I}_k(y)| \leq d_k, \quad k = 1, \ldots, m, \text{ for each } y \in \mathbb{R}; \tag{5.21}$$

(5.4.2) *there exist a function* $h \in L^1(J, \mathbb{R}_+)$, $r^* > 0$, *and* $0 < M^* < 1$ *such that*

$$\begin{gathered} |f(t,u)| \leq M^* h(t) \quad \text{for a.e. } t \in J \text{ and each } u \in D, \\ \|\phi\|_{\mathcal{D}} + b|\eta| + M^* \int_0^b (b-s)h(s)ds + \sum_{k=1}^m [c_k + (b - t_k)d_k] < r^*; \end{gathered} \tag{5.22}$$

(5.4.3) *there exist* $L^* > r^*$, $M^* \leq M_1^* < 1$, *and an interval* $[c,d] \subset (0,b)$ *such that*

$$\begin{aligned} &\min_{t\in[c,d]} \left(\phi(0) + t\eta + \int_0^t (t-s)f(s,y_s)ds + \sum_{0<t_k<t} [I_k(y(t_k^-)) + (t-t_k)\overline{I}_k(y(t_k^-))] \right) \\ &\quad \geq M_1^* \left(\phi(0) + b\eta + \int_0^b bf(s,y_s)ds + \sum_{k=1}^m [I_k(y(t_k^-)) + (b-t_k)\overline{I}_k(y(t_k^-))] \right) > L^*; \end{aligned} \tag{5.23}$$

(5.4.4) *there exist* $R^* > L^*$ *and* M_2^* *with* $M_1^* \leq M_2^* < 1$ *such that*

$$\|\phi\|_{\mathcal{D}} + b|\eta| + M_2^* \int_0^b (b-s)h(s)ds + \sum_{k=1}^m [c_k + (b-t_k)d_k] < R^*. \tag{5.24}$$

Then problem (5.2) *has three nonnegative solutions* y_1, y_2, y_3 *with*

$$\begin{gathered} \|y_1\| < r^*, \qquad y_2(t) > L^* \quad \text{for } t \in [0,b], \\ \|y_3\| > r^* \quad \text{with } \min_{t\in[c,d]} y_3(t) < L^*. \end{gathered} \tag{5.25}$$

Proof. Transform problem (5.2) into a fixed point problem. Consider the operator $N_1 : \Omega \to \Omega$ defined by

$$N_1(y)(t) = \begin{cases} \phi(t) & \text{if } t \in [-r,0], \\ \phi(0) + \eta t + \displaystyle\int_0^t (t-s)f(s,y_s)ds & \\ \quad + \displaystyle\sum_{0<t_k<t} [I_k(y(t_k^-)) + (t-t_k)\overline{I}_k(y(t_k^-))] & \text{if } t \in [0,b]. \end{cases} \tag{5.26}$$

As in Theorem 5.2 we can show that N_1 is completely continuous. Now we prove only that the hypotheses of Theorem 1.14 are satisfied.

Let

$$C = \{y \in \Omega : y(t) \geq 0 \text{ for } t \in [-r, b]\} \tag{5.27}$$

be a cone in Ω. Since $f, I_k, \bar{I}_k, k = 1, \dots, m$, are all positive functions, then $N_1(C) \subset C$ and $N_1 : \overline{C}_{R^*} \to \overline{C}_{R^*}$ is completely continuous. Moreover, by (5.4.4), we can show that if $y \in \overline{C}_{R^*}$, then $N_1(y) \in \overline{C}_{R^*}$. Next, let $\psi : C \to [0, \infty)$ be defined by

$$\psi(y) = \min_{t \in [c,d]} y(t). \tag{5.28}$$

It is clear that ψ is a nonnegative concave continuous functional and $\psi(y) \leq \|y\|$ for $y \in \overline{C}_R$.

Now it remains to show that the hypotheses of Theorem 1.14 are satisfied. First notice that condition (A2) of Theorem 1.14 holds since for $y \in \overline{C}_{r^*}$ we have, from (5.4.1) and (5.4.2),

$$\begin{aligned} |N_1(y)(t)| &\leq |\phi(0)| + b|\eta| + \int_0^t (b-s)|f(s, y_s)|\,ds \\ &\quad + \sum_{0<t_k<t} \left[|I_k(y(t_k^-))| + (b - t_k)\bar{I}_k(y(t_k^-)) \right] \\ &\leq \|\phi\|_{\mathcal{D}} + b|\eta| + M^* \int_0^b (b-s)h(s)\,ds + \sum_{k=1}^m \left[c_k + (b - t_k)d_k \right] < r^*. \end{aligned} \tag{5.29}$$

Let K^* be such that $L^* M^{*-1} \leq K^* \leq R^*$ and $y(t) = (L^* + K^*)/2$ for $t \in [-r, b]$. By the definition of $C(\psi, L^*, K^*)$, y belongs to $C(\psi, L^*, K^*)$. Then $y \in \{y \in C(\psi, L^*, K^*) : \psi(y) > L^*\}$. Also if $y \in C(\psi, L^*, K^*)$, then

$$\begin{aligned} &\psi(N_1(y)) \\ &\quad = \min_{t \in [c,d]} \left(\phi(0) + t\eta + \sum_{0<t_k<t} \left[I_k(y(t_k^-)) + (t - t_k)\bar{I}_k(y(t_k^-)) \right] + \int_0^t (t-s) f(s, y_s)\,ds \right). \end{aligned} \tag{5.30}$$

Then from (5.4.3) we have

$$\begin{aligned} &\psi(N_1(y)) \\ &\quad \geq \min_{t \in [c,d]} \left(\phi(0) + t\eta + \sum_{0<t_k<t} \left[I_k(y(t)) + (t - t_k)\bar{I}_k(y(t_k)) \right] + \int_0^t (t-s) f(s, y_s)\,ds \right) \\ &\quad \geq M_1^* \left(\phi(0) + b\eta + \int_0^b b f(s, y_s)\,ds + \sum_{k=1}^m \left[I_k(y(t_k^-)) + (b - t_k)\bar{I}_k(y(t_k^-)) \right] \right) > L^*. \end{aligned} \tag{5.31}$$

So conditions (A1) and (A2) of Theorem 1.14 are satisfied. Finally, to see that Theorem 1.14(A3) holds, let $y \in C(\psi, L^*, R^*)$ with $\|N_1(y)\| > K^*$. From (5.4.3) we have

$$\begin{aligned}
&\psi(N_1(y)) \\
&\quad= \min_{t\in[c,d]}\left(\phi(0)+t\eta+\sum_{0<t_k<t}\left[I_k(y(t_k^-))+(t-t_k)\bar{I}_k(y(t_k^-))\right]+\int_0^t(t-s)f(s,y_s)\,ds\right) \\
&\quad\geq M_1^*\left(\phi(0)+b\eta+\sum_{k=1}^{m}\left[I_k(y(t_k))+(b-t_k)\bar{I}_k(yt_k)\right]+\int_0^b bf(s,y_s)\,ds\right) \\
&\quad\geq M_1^*||N_1(y)|| > M_1^*K^* \geq L^*.
\end{aligned} \tag{5.32}$$

Then the Leggett-Williams fixed point theorem implies that N_1 has at least three fixed points y_1, y_2, y_3 which are solutions to problem (5.2). Furthermore, we have

$$\begin{gathered}
y_1 \in C_{r^*}, \qquad y_2 \in \{y \in C(\psi, L^*, R^*) : \psi(y) > L^*\}, \\
y_3 \in C_{R^*} - \{C(\psi, L^*, R^*) \cup C_{r^*}\}.
\end{gathered} \tag{5.33}$$

□

5.3. Positive solutions for impulsive boundary value problems

In this section, we will be concerned with the existence of positive solutions of the second-order boundary value problem for the impulsive functional differential equation,

$$\begin{gathered}
y'' = f(t, y_t), \quad t \in J = [0, T],\ t \neq t_k,\ k = 1, \dots, m, \\
\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \\
\Delta y'|_{t=t_k} = \bar{I}_k(y(t_k^-)), \quad k = 1, \dots, m,
\end{gathered} \tag{5.34}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \qquad y(T) = y_T, \tag{5.35}$$

where $f : J \times \mathcal{D} \to \mathbb{R}$ is a given function, $\mathcal{D} = \{\psi : [-r, 0] \to \mathbb{R}_+ \mid \psi$ is continuous everywhere except for a finite number of points s at which $\psi(s)$ and the right limit $\psi(s^+)$ exist and $\psi(s^-) = \psi(s)\}$, $\phi \in \mathcal{D}$, $0 < r < \infty$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $I_k, \bar{I}_k \in C(\mathbb{R}, \mathbb{R})$ $(k = 1, 2, \dots, m)$ are bounded, $y_T \in \mathbb{R}$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $\Delta y'|_{t=t_k} = y'(t_k^+) - y'(t_k^-)$, and $y(t_k^-)$, $y(t_k^+)$ and $y'(t_k^-)$, $y'(t_k^+)$ represent the left and right limits of $y(t)$ and $y'(t)$, respectively, at $t = t_k$.

Definition 5.5. A function $y \in \Omega \cap \mathrm{AC}^1((t_k, t_{k+1}), \mathbb{R})$ is said to be a solution of (5.34)–(5.35) if y satisfies the differential equation $y''(t) = f(t, y_t)$ a.e. on J', the conditions $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $\Delta y'|_{t=t_k} = \bar{I}_k(y(t_k^-))$, $k = 1, \dots, m$, and the conditions (5.35).

In what follows we will use the notation $\sum_{0<t_k<t}[y(t_k^+)-y(t_k)]$ to mean 0 when $k=0$ and $0<t<t_1$, and to mean $\sum_{i=1}^{k}[y(t_k^+)-y(t_k)]$ when $k\geq 1$ and $t_k<t\leq t_{k+1}$. We establish solutions of (5.34)–(5.35) by an application of a Schaefer fixed point theorem.

Theorem 5.6. *Suppose that the following assumptions are satisfied:*
(5.6.1) $\phi\in\mathcal{D}$ *and* $y_T\geq 0$*;*
(5.6.2) $f:J\times\mathcal{D}\to(-\infty,0]$ *is an* L^1*-Carathéodory map;*
(5.6.3) $I_k(v)+(t-t_k)\bar{I}_k(v)\geq 0$ *for each* $v\in\mathbb{R}$, $t\geq t_k$, *and* $k=1,\dots,m$*;*
(5.6.4) $I_k(v)+(T-t_k)\bar{I}_k(v)\leq 0$ *for each* $v\in\mathbb{R}$ *and* $k=1,\dots,m$*;*
(5.6.5) *there exist constants* c_k, d_k *such that* $|I_k(y)|\leq c_k$, $|\bar{I}_k(y)|\leq d_k$, $k=1,\dots,m$, *for each* $y\in\mathbb{R}$*;*
(5.6.6) *there exists a function* $m\in L^1(J,\mathbb{R}^+)$ *such that*

$$|f(t,u)|\leq m(t)\quad \text{for almost all } t\in J,\ \forall u\in\mathcal{D}. \tag{5.36}$$

Then the impulsive boundary value problem (5.34)–(5.35) *has at least one positive solution on* $[-r,T]$.

Proof. Transform problem (5.34)–(5.35) into a fixed point problem. Consider the multivalued map $G:\Omega\to\Omega$ defined by

$$G(y)(t)=\begin{cases}\phi(t), & t\in[-r,0],\\ \dfrac{T-t}{T}\phi(0)+\dfrac{t}{T}y_T+\displaystyle\int_0^T H(t,s)f(s,y_s)\,ds & \\ \quad+\displaystyle\sum_{0<t_k<t}\left[I_k(y(t_k))+(t-t_k)\bar{I}_k(y(t_k))\right] & \\ \quad-\dfrac{t}{T}\displaystyle\sum_{k=1}^{m}\left[I_k(y(t_k))+(T-t_k)\bar{I}_k(y(t_k))\right], & t\in J,\end{cases} \tag{5.37}$$

where

$$H(t,s)=\begin{cases}\dfrac{t}{T}(s-T), & 0\leq s\leq t\leq T,\\ \dfrac{s}{T}(t-T), & 0\leq t<s\leq T.\end{cases} \tag{5.38}$$

Remark 5.7. We first show that the fixed points of G are positive solutions to (5.34)–(5.35).

Indeed, assume that $y\in\Omega$ is a fixed point of G. It is clear that $y(t)=\phi(t)$ for each $t\in[-r,0]$, $y(T)=y_T$, and $\Delta y|_{t=t_k}=I_k(y(t_k^-))$, $k=1,\dots,m$. By performing direct differentiation twice, we find

$$\begin{aligned}
y'(t) &= \frac{-1}{T}\phi(0) + \frac{1}{T}y_T + \int_0^T \frac{\partial H}{\partial t}(t,s)f(s,y_s)\,ds + \sum_{0<t_k<t} \bar{I}_k(y(t_k)) \\
&\quad - \frac{1}{T}\sum_{k=1}^{m}\left[I_k(y(t_k)) + (T-t_k)\bar{I}_k(y(t_k))\right], \quad t \neq t_k, \\
&\qquad \Delta y'|_{t=t_k} = \bar{I}_k(y(t_k^-)), \quad k = 1,\dots,m, \\
&\qquad\quad y''(t) = f(t,y_t), \quad t \neq t_k,
\end{aligned} \tag{5.39}$$

which imply that y is a solution of the BVP (5.34)–(5.35).

If y is a fixed point of G, then (5.6.1) through (5.6.4) imply that $y(t) \geq 0$ for each $t \in [-r, T]$. We will now show that G satisfies the assumptions of Schaefer's fixed point theorem. The proof will be given in several steps.

Step 1. G maps bounded sets into bounded sets in Ω.

Indeed, it is enough to show that there exists a positive constant ℓ such that, for each $y \in B_q = \{y \in \Omega : \|y\| \leq q\}$, one has $\|G(y)\| \leq \ell$. For each $t \in J$, we have

$$\begin{aligned}
G(y)(t) &= \frac{T-t}{T}\phi(0) + \frac{t}{T}y_T + \int_0^T H(t,s)f(s,y_s)\,ds \\
&\quad + \sum_{0<t_k<t}\left[I_k(y(t_k)) + (t-t_k)\bar{I}_k(y(t_k))\right] \\
&\quad - \frac{t}{T}\sum_{k=1}^{m}\left[I_k(y(t_k)) + (T-t_k)\bar{I}_k(y(t_k))\right].
\end{aligned} \tag{5.40}$$

By (5.6.2) we have, for each $t \in J$,

$$\begin{aligned}
|G(y)(t)| &\leq \|\phi\|_{\mathcal{D}} + |y_T| + \sup_{(t,s)\in J\times J}|H(t,s)| \int_0^T |f(s,y_s)|\,ds \\
&\quad + \sum_{0<t_k<t}\left[|I_k(y(t_k))| + |(t-t_k)|\,|\bar{I}_k(y(t_k))|\right] \\
&\quad + \sum_{k=1}^{m}\left[I_k(y(t_k)) + (T-t_k)\bar{I}_k(y(t_k))\right] \leq \|\phi\|_{\mathcal{D}} + |y_T| \\
&\quad + \sup_{(t,s)\in J\times J}|H(t,s)| \int_0^T g_q(s)\,ds \\
&\quad + \sum_{k=1}^{m}\Big[2\sup\{|I_k(|y|)| : \|y\| \leq q\} \\
&\qquad + 2(T-t_k)\sup\{|\bar{I}_k(|y|)| : \|y\| \leq q\}\Big] = \ell.
\end{aligned} \tag{5.41}$$

Step 2. G maps bounded sets into equicontinuous sets of Ω.

Let $r_1, r_2 \in J'$, $0 < r_1 < r_2$, and let $B_q = \{y \in \Omega : \|y\| \le q\}$ be a bounded set of Ω.

For each $y \in B_q$ and $t \in J$, we have

$$\begin{aligned}
|G(y)(r_2) - G(y)(r_1)| \le\ & (r_2 - r_1)|\phi(0)| + (r_2 - r_1)\frac{|y_T|}{T} \\
& + \int_0^T |H(r_2, s) - H(r_1, s)| g_q(s)ds \\
& + \sum_{0<t_k<r_2-r_1} \left[I_k(y(t_k)) + (r_2 - r_1)\bar{I}_k(y(t_k))\right] \\
& - \frac{r_2 - r_1}{T}\sum_{k=1}^{m}\left[I_k(y(t_k)) + (T - t_k)\bar{I}_k(y(t_k))\right].
\end{aligned} \tag{5.42}$$

As $r_2 \to r_1$ the right-hand side of the above inequality tends to zero. This proves the equicontinuity for the case where $t \neq t_i$, $i = 1, \dots, m$. It remains to examine the equicontinuity at $t = t_i$. The proof is similar to that given in Theorem 4.3.

The equicontinuity for the cases $r_1 < r_2 \le 0$ and $r_1 \le 0 \le r_2$ is similar.

Step 3. G is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in Ω. Then there is an integer q such that $\|y_n\| \le q$, for all $n \in \mathbb{N}$ and $\|y\| \le q$. So $y_n \in B_q$ and $y \in B_q$.

We have then by the dominated convergence theorem

$$\begin{aligned}
&\|G(y_n) - G(y)\| \\
&\quad \le \sup_{t\in J}\Bigg[\int_0^T H(t, s)|f(s, y_{ns}) - f(s, y_s)|ds \\
&\qquad + \sum_{0<t_k<t}\left[|I_k(y_n(t_k)) - I_k(y(t_k))| + |t - t_k||\bar{I}_k(y_n(t_k)) - \bar{I}_k(y(t_k))|\right] \\
&\qquad + \frac{t}{T}\sum_{1}^{m}\big[|I_k(y_n(t_k)) - I_k(y(t_k))| \\
&\qquad\qquad + |T - t_k||\bar{I}_k(y_n(t_k)) - \bar{I}_k(y(t_k))|\big]\Bigg] \longrightarrow 0.
\end{aligned} \tag{5.43}$$

Thus G is continuous. As a consequence of Steps 1, 2, and 3 together with the Ascoli-Arzelá theorem we can conclude that $G : \Omega \to \Omega$ is completely continuous.

Step 4. Now it remains to show that the set

$$\mathcal{M} := \{y \in \Omega : \lambda y = G(y), \text{ for some } \lambda > 1\} \tag{5.44}$$

is bounded.

Let $y \in \mathcal{M}$. Then $\lambda y \in G(y)$ for some $\lambda > 1$. Thus

$$
\begin{aligned}
y(t) = \lambda^{-1}\frac{T-t}{T}\phi(0) + \lambda^{-1}\frac{t}{T}y_T + \lambda^{-1}\int_0^T H(t,s)f(s,y_s)ds \\
+ \lambda^{-1}\sum_{0<t_k<t}\left[I_k(y(t_k)) + (t-t_k)\bar{I}_k(y(t_k))\right] \\
- \lambda^{-1}\frac{t}{T}\sum_{k=1}^{m}\left[I_k(y(t_k)) + (T-t_k)\bar{I}_k(y(t_k))\right].
\end{aligned}
\tag{5.45}
$$

This implies by (5.6.5) and (5.6.6) that for each $t \in J$ we have

$$
\begin{aligned}
|y(t)| \le \|\phi\|_{\mathcal{D}} + |y_T| + \sup_{(t,s)\in J\times J}|H(t,s)|\int_0^T m(s)ds \\
+ \sum_{k=1}^{m}\left[2c_k + 2(T-t_k)d_k\right] := b.
\end{aligned}
\tag{5.46}
$$

This inequality implies that there exists a constant b depending only on T and on the function m such that

$$
|y(t)| \le b \quad \text{for each } t \in J. \tag{5.47}
$$

Hence

$$
\|y\| := \sup\{|y(t)| : -r \le t \le T\} \le \max(\|\phi\|_{\mathcal{D}}, b). \tag{5.48}
$$

This shows that $\mathcal{M}$ is bounded.

Set $X := \Omega$. As a consequence of Theorem 1.6 we deduce that G has a fixed point y which is a positive solution of (5.34)–(5.35). □

Remark 5.8. We can analogously (with obvious modifications) study the existence of positive solutions for the following BVP:

$$
\begin{gathered}
y'' = f(t,y_t), \quad t \in J = [0,T],\ t \ne t_k,\ k = 1,\dots,m, \\
\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m, \\
\Delta y'|_{t=t_k} = \bar{I}_k(y(t_k^-)), \quad k = 1,\dots,m, \\
y(t) = \phi(t), \quad t \in [-r,0], \qquad y'(T) = y_T.
\end{gathered}
\tag{5.49}
$$

We omit the details.

5.4. Double positive solutions for impulsive boundary value problems

Let $0 < \tau < 1$ be fixed. We apply an Avery-Henderson fixed point theorem to obtain multiple positive solutions of the nonlinear impulsive differential equation

$$y'' = f(y), \quad t \in [0,1] \setminus \{\tau\}, \tag{5.50}$$

subject to the underdetermined impulse condition

$$\Delta y(\tau) = I(y(\tau)), \tag{5.51}$$

and satisfying the right focal boundary conditions

$$y(0) = y'(1) = 0, \tag{5.52}$$

where $\Delta y(\tau) = y(\tau^+) - y(\tau^-)$, $f : \mathbb{R} \to [0,\infty)$ is continuous, and $I : [0,\infty) \to [0,\infty)$ is continuous. By a positive solution we will mean *positive with respect to a suitable cone*.

Definition 5.9. Let $(\mathcal{B}, \|\cdot\|)$ be a real Banach space. A nonempty, closed, convex set $\mathcal{P} \subset \mathcal{B}$ is said to be a *cone* provided the following are satisfied:

(a) if $y \in \mathcal{P}$ and $\lambda \geq 0$, then $\lambda y \in \mathcal{P}$;
(b) if $y \in \mathcal{P}$ and $-y \in \mathcal{P}$, then $y = 0$.

Every cone $\mathcal{P} \subset \mathcal{B}$ induces a partial ordering, $\leq$, on $\mathcal{B}$ defined by

$$x \leq y \quad \text{iff } y - x \in \mathcal{P}. \tag{5.53}$$

Definition 5.10. Given a cone $\mathcal{P}$ in a real Banach space $\mathcal{B}$, a functional $\psi : \mathcal{P} \to R$ is said to be *increasing* on $\mathcal{P}$, provided $\psi(x) \leq \psi(y)$, for all $x, y \in \mathcal{P}$ with $x \leq y$.

Given a nonnegative continuous functional γ on a cone $\mathcal{P}$ of a real Banach space $\mathcal{B}$ (i.e., $\gamma : \mathcal{P} \to [0,\infty)$ continuous), we define, for each $d > 0$, the convex set

$$\mathcal{P}(\gamma, d) = \{x \in \mathcal{P} \mid \gamma(x) < d\}. \tag{5.54}$$

In this section, we impose growth conditions on f and I and then apply Theorem 1.16 to establish the existence of double positive solutions of (5.50)–(5.52). We note that, from the nonnegativity of f and I, a solution y of (5.50)–(5.52) is nonnegative and concave on each of $[0,\tau]$ and $(\tau,1]$. We will apply Theorem 1.16 to a completely continuous operator whose kernel $G(t,s)$ is the Green's function for

$$-y'' = 0, \tag{5.55}$$

satisfying (5.52). In this instance,

$$G(t,s) = \begin{cases} t, & 0 \le t \le s \le 1, \\ s, & 0 \le s \le t \le 1. \end{cases} \tag{5.56}$$

Properties of $G(t,s)$ for which we will make use are

$$G(t,s) \le G(s,s) = s, \quad 0 \le t,\ s \le 1, \tag{5.57}$$

and for each $0 < p < 1$,

$$G(t,s) \ge pG(s,s) = ps, \quad p \le t \le 1,\ 0 \le s \le 1. \tag{5.58}$$

In particular, from (5.58), we have

$$\min_{t\in[p,1]} G(t,s) \ge ps, \quad 0 \le s \le 1. \tag{5.59}$$

To apply Theorem 1.16, we must define an appropriate Banach space $\mathcal{B}$, a cone $\mathcal{P}$, and an operator A. To that end, let

$$\mathcal{B} = \{y : [0,1] \longrightarrow \mathbb{R} \mid y \in C[0,\tau],\ y \in C(\tau,1],\ \text{and } y(\tau^+) \in \mathbb{R}\}, \tag{5.60}$$

equipped with the norm

$$\|y\| = \max\left\{ \sup_{0\le t\le\tau} |y(t)|,\ \sup_{\tau<t\le 1} |y(t)| \right\}. \tag{5.61}$$

Naturally, for $y \in \mathcal{B}$, we will consider in a piecewise manner that $y \in C[0,\tau]$ and $y \in C[\tau,1]$. We also note that if $y \in \mathcal{B}$, then $y(\tau^-) = y(\tau)$. Next, let the cone $\mathcal{P} \subset \mathcal{B}$ be defined by

$$\begin{aligned} \mathcal{P} = \{y \in \mathcal{B} \mid\ & y \text{ is concave, nondecreasing, and nonnegative on} \\ & \text{each of } [0,\tau] \text{ and } [\tau,1],\ \text{and } \Delta y(\tau) \ge 0\}. \end{aligned} \tag{5.62}$$

We note that, for each $y \in \mathcal{P}$, $I(y(\tau)) \ge 0$ so that $y(\tau^+) \ge y(\tau) \ge 0$. It follows that, for $y \in \mathcal{P}$,

$$\|y\| = \max\{y(\tau),\ y(1)\} = y(1). \tag{5.63}$$

Moreover, if $y \in \mathcal{P}$,

$$\begin{aligned} y(t) &\geq \frac{1}{2} \sup_{s\in[\tau/2,\tau]} y(s) = \frac{1}{2} y(\tau), \quad \frac{\tau}{2} \leq t \leq \tau, \\ y(t) &\geq \frac{1}{2} \sup_{s\in[(\tau+1)/2,1]} y(s) = \frac{1}{2} y(1), \quad \frac{\tau+1}{2} \leq t \leq 1; \end{aligned} \tag{5.64}$$

see [19].

For the remainder of this section, fix

$$\frac{\tau+1}{2} < r < 1, \tag{5.65}$$

and define the nonnegative, increasing, continuous functionals γ, θ, and α on $\mathcal{P}$ by

$$\begin{aligned} \gamma(y) &= \min_{(\tau+1)/2\leq t\leq r} y(t) = y\left(\frac{\tau+1}{2}\right), \\ \theta(y) &= \max_{\tau\leq t\leq(\tau+1)/2} y(t) = y\left(\frac{\tau+1}{2}\right), \\ \alpha(y) &= \max_{\tau\leq t\leq r} y(t) = y(r). \end{aligned} \tag{5.66}$$

We observe that, for each $y \in \mathcal{P}$,

$$\gamma(y) = \theta(y) \leq \alpha(y). \tag{5.67}$$

Furthermore, for each $y \in \mathcal{P}$, $\gamma(y) = y((\tau+1)/2) \geq (1/2)y(1) = (1/2)\|y\|$. So

$$\|y\| \leq 2\gamma(y), \quad \forall y \in \mathcal{P}. \tag{5.68}$$

Finally, we also note that

$$\theta(\lambda y) = \lambda\theta(y), \quad 0 \leq \lambda \leq 1, \ y \in \partial\mathcal{P}(\theta, b). \tag{5.69}$$

We now state growth conditions on f and I so that problem (5.50)–(5.52) has at least two positive solutions.

Theorem 5.11. *Let* $0 < a < (r^2/2)b < (r^2/4)c$, *and suppose that* f *and* I *satisfy the following conditions:*

(A) $f(w) > 4c/(1-\tau^2)$ *if* $c \leq w \leq 2c$,
(B) $f(w) < b$ *if* $0 \leq w \leq 2b$,
(C) $f(w) > 2a/r^2$ *if* $0 \leq w \leq a$,
(D) $I(w) \leq b/2$ *if* $0 \leq w \leq b$.

Then the impulsive boundary value problem (5.50)–(5.52) *has at least two positive solutions* x_1 *and* x_2 *such that*

$$\begin{aligned} a < \max_{\tau \le t \le r} x_1(t), \quad &\textit{with} \max_{\tau \le t \le (\tau+1)/2} x_1(t) < b, \\ b < \max_{\tau \le t \le (\tau+1)/2} x_2(t), \quad &\textit{with} \min_{(\tau+1)/2 \le t \le r} x_2(t) < c. \end{aligned} \tag{5.70}$$

Proof. We begin by defining the completely continuous integral operator $A : \mathcal{B} \to \mathcal{B}$ by

$$Ax(t) = I(x(\tau))\chi_{(\tau,1]}(t) + \int_0^1 G(t,s) f(x(s))\,ds, \quad x \in \mathcal{B},\ 0 \le t \le 1, \tag{5.71}$$

where $\chi_{(\tau,1]}(t)$ is the characteristic function. It is immediate that solutions of (5.50)–(5.52) are fixed points of A and conversely. We proceed to show that the conditions of Theorem 1.16 are satisfied.

First, let $x \in \overline{\mathcal{P}(\gamma, c)}$. By the nonnegativity of I, f, and G, for $0 \le t \le 1$,

$$Ax(t) = I(x(\tau))\chi_{(\tau,1]}(t) + \int_0^1 G(t,s) f(x(s))\,ds \ge 0. \tag{5.72}$$

Moreover, $(Ax)''(t) = -f(x(t)) \le 0$ on $[0,1] \setminus \{\tau\}$, which implies $(Ax)(t)$ is concave on each of $[0,\tau]$ and $[\tau,1]$. In addition,

$$(Ax)'(t) = \int_0^1 \frac{\partial}{\partial t} G(t,s) f(x(s))\,ds \ge 0 \quad \text{on } [0,1] \setminus \{\tau\}, \tag{5.73}$$

so that $(Ax)(t)$ is nondecreasing on each of $[0,\tau]$ and $[\tau,1]$. From $(Ax)(0) = 0$, we have $(Ax)(t) \ge 0$ on $[0,\tau]$. Also, since $x \in \overline{\mathcal{P}(\gamma, c)}$,

$$\Delta(Ax)(\tau) = (Ax)(\tau^+) - (Ax)(\tau) = I(x(\tau)) \ge 0. \tag{5.74}$$

This yields $(Ax)(\tau^+) \ge (Ax)(\tau) \ge 0$. Consequently, $(Ax)(t) \ge 0$, $\tau \le t \le 1$, as well. Ultimately, we have $Ax \in \mathcal{P}$ and, in particular, $A : \overline{\mathcal{P}(\gamma, c)} \to \mathcal{P}$.

We now turn to property (i) of Theorem 1.16. We choose $x \in \partial\mathcal{P}(\gamma, c)$. Then $\gamma(x) = \min_{(\tau+1)/2 \le t \le r} x(t) = x((\tau+1)/2) = c$. Since $x \in \mathcal{P}$, $x(t) \ge c$, $(\tau+1)/2 \le t \le 1$. If we recall that $\|x\| \le 2\gamma(x) = 2c$, we have

$$c \le x(t) \le 2c, \quad \frac{\tau+1}{2} \le t \le 1. \tag{5.75}$$

By hypothesis (A),

$$f(x(s)) > \frac{4c}{1-\tau^2}, \quad \frac{\tau+1}{2} \le s \le r. \tag{5.76}$$

By above $Ax \in \mathcal{P}$, and so

$$\begin{aligned}
\gamma(Ax) &= (Ax)\left(\frac{\tau+1}{2}\right) = I(x(\tau))\chi_{(\tau,1]}\left(\frac{\tau+1}{2}\right) + \int_0^1 G\left(\frac{\tau+1}{2}, s\right) f(x(s))\,ds \\
&= \int_0^1 G\left(\frac{\tau+1}{2}, s\right) f(x(s))\,ds \geq \int_{(\tau+1)/2}^1 G\left(\frac{\tau+1}{2}, s\right) f(x(s))\,ds \\
&= \left(\frac{\tau+1}{2}\right)\int_{(\tau+1)/2}^1 f(x(s))\,ds > \left(\frac{\tau+1}{2}\right)\left(\frac{4c}{1-\tau^2}\right)\int_{(\tau+1)/2}^1 ds = c.
\end{aligned} \tag{5.77}$$

We conclude that Theorem 1.16(i) is satisfied.

We next address Theorem 1.16(ii). This time, we choose $x \in \partial\mathcal{P}(\theta, b)$. Then $\theta(x) = \max_{\tau \leq t \leq (\tau+1)/2} x(t) = x((\tau+1)/2) = b$. Thus, $0 \leq x(t) \leq b$, $\tau^+ \leq t \leq (\tau+1)/2$. Yet $x \in \mathcal{P}$ implies $x(\tau) \leq x(\tau^+)$, and also $x(t)$ is nondecreasing on $[0, \tau]$. Thus

$$x(t) \leq b, \quad 0 \leq t \leq \frac{\tau+1}{2}. \tag{5.78}$$

By hypothesis (D), we have

$$I(x(\tau)) \leq \frac{b}{2}. \tag{5.79}$$

If we recall that $\|x\| \leq 2\gamma(x) \leq 2\theta(x) = 2b$, then we have

$$0 \leq x(t) \leq 2b, \quad 0 \leq t \leq 1, \tag{5.80}$$

and by (B),

$$f(x(s)) < b, \quad 0 \leq s \leq 1. \tag{5.81}$$

Then

$$\begin{aligned}
\theta(Ax) &= (Ax)\left(\frac{\tau+1}{2}\right) = I(x(\tau))\chi_{(\tau,1]}\left(\frac{\tau+1}{2}\right) + \int_0^1 G\left(\frac{\tau+1}{2}, s\right) f(x(s))\,ds \\
&\leq \frac{b}{2} + \int_0^1 s f(x(s))\,ds < \frac{b}{2} + b\int_0^1 s\,ds = b.
\end{aligned} \tag{5.82}$$

In particular, Theorem 1.16(ii) holds.

For the final part, we consider Theorem 1.16(iii). If we define $y(t) = a/2$, for all $0 \leq t \leq 1$, then $\alpha(y) = a/2 < a$ and $\mathcal{P}(\alpha, a) \neq \varnothing$.

Now choose $x \in \partial\mathcal{P}(\alpha, a)$. Then $\alpha(x) = \max_{\tau \le t \le r} x(t) = x(r) = a$. This implies $0 \le x(t) \le a$, $\tau^+ \le t \le r$. Since x is nondecreasing and $x(\tau^+) \ge x(\tau)$,

$$0 \le x(t) \le a, \quad 0 \le t \le r. \tag{5.83}$$

By assumption (C),

$$f(x(s)) > \frac{2a}{r^2}, \quad 0 \le s \le r. \tag{5.84}$$

As before $Ax \in \mathcal{P}$, and so

$$\begin{aligned}\alpha(Ax) &= (Ax)(r) = I(x(\tau))\chi_{(\tau,1]}(r) + \int_0^1 G(r,s)f(x(s))\,ds \ge \int_0^1 G(r,s)f(x(s))\,ds \\ &\ge \int_0^r G(r,s)f(x(s))\,ds \ge \int_0^r sf(x(s))\,ds > \left(\frac{2a}{r^2}\right)\int_0^r s\,ds = a.\end{aligned} \tag{5.85}$$

Thus Theorem 1.16(iii) is satisfied. Hence there exist at least two fixed points of A which are solutions x_1 and x_2, belonging to $\overline{\mathcal{P}(\gamma, c)}$, of the impulsive boundary value problem (5.50)–(5.52) such that

$$\begin{aligned} a < \alpha(x_1) &\quad \text{with } \theta(x_1) < b, \\ b < \theta(x_2) &\quad \text{with } \gamma(x_2) < c. \end{aligned} \tag{5.86}$$

The proof is complete. □

Example 5.12. For $(\tau + 1)/2 < r < 1$ fixed and for $0 < a < (r^2/2)b < (r^2/4)c$, if $f : \mathbb{R} \to [0, \infty)$ and $I : [0, \infty) \to [0, \infty)$ are defined by

$$\begin{aligned} f(w) &= \begin{cases} \dfrac{br^2 + 2a}{2r^2}, & w \le 2b, \\ \ell(w), & 2b \le w \le c, \\ \dfrac{4c+1}{1-\tau^2}, & c \le w, \end{cases} \\ I(w) &= \begin{cases} \dfrac{b}{2}, & 0 \le w \le b, \\ w - \dfrac{b}{2}, & b \le w, \end{cases} \end{aligned} \tag{5.87}$$

where $\ell(w)$ satisfies $\ell'' = 0$, $\ell(2b) = (br^2 + 2a)/2r^2$, and $\ell(c) = (4c + 1)/(1 - \tau^2)$, then by Theorem 5.11 the impulsive boundary value problem (5.50)–(5.52) has at least two solutions belonging to $\overline{\mathcal{P}(\gamma, c)}$.

5.5. Notes and remarks

There is much current interest in multiple fixed point theorems and their applications to impulsive functional differential equations. The techniques in this chapter have been adapted from [8, 9, 19] where the nonimpulsive case was discussed. Section 5.2 deals with the existence of multiple positive solutions for first- and second-order impulsive functional differential equations by applying the Leggett-Williams fixed point theorem. The material of Section 5.2 is based on the results given by Benchohra et al. [95]. The Krasnoselskii twin fixed point theorem is used in Section 5.3 to obtain two positive solutions for initial value problems for first- and second-order impulsive semilinear functional differential equations in Hilbert space. The results of Section 5.3 are adapted from Benchohra et al. [75]. Positive solutions for impulsive boundary value problems are studied in Section 5.4 and the results are adapted from Benchohra et al. [52]. The results of Section 5.5 are taken from Benchohra et al. [25] and concern double positive solutions for impulsive boundary value problems. A new fixed point theorem of Avery and Henderson [26] is applied in Section 5.5.

6 Boundary value problems for impulsive differential inclusions

6.1. Introduction

The method of upper and lower solutions has been successfully applied to study the existence of solutions for first-order impulsive initial value problems and boundary value problems. This method generates solutions of such problems, with the solutions located in an order interval with the upper and lower solutions serving as bounds. Moreover, this method, coupled with some monotonicity-type hypotheses, leads to monotone iterative techniques which generate in a constructive way (amenable to numerical treatment) the extremal solutions within the order interval determined by the upper and lower solutions.

This method has been used only in the context of single-valued impulsive differential equations. We refer to the monographs of Lakshmikantham et al. [180], Samoĭlenko and Perestyuk [217], the papers of Cabada and Liz [117], Frigon and O'Regan [151], Heikkilä and Lakshmikantham [163], Liu [188], Liz [192, 193], Liz and Nieto [194], and Pierson-Gorez [212]. However, this method has been used recently by Benchohra and Boucherif [35] for the study of first-order initial value problems for impulsive differential inclusions.

Let us mention that other methods like the nonlinear alternative, such as in the papers of Benchohra and Boucherif [34, 35], Frigon and O'Regan [150], and the topological transversality theorem Erbe and Krawcewicz [140], have been used to analyze first- and second-order impulsive differential inclusions. The first part of this chapter presents existence results using upper- and lower-solutions methods to obtain solutions of first-order impulsive differential inclusions with periodic boundary conditions and nonlinear boundary conditions. The last section of the chapter deals with boundary value problems for second-order impulsive differential inclusions.

6.2. First-order impulsive differential inclusions with periodic boundary conditions

This section is devoted to the existence of solutions for the impulsive periodic multivalued problem

$$\begin{aligned} y'(t) &\in F(t, y(t)), \quad t \in J = [0, T],\ t \neq t_k,\ k = 1, \ldots, m, \\ \Delta y|_{t=t_k} &= I_k(y(t_k^-)), \quad k = 1, \ldots, m, \\ y(0) &= y(T), \end{aligned} \tag{6.1}$$

where $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a compact and convex-valued multivalued map, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $I_k \in C(\mathbb{R}, \mathbb{R})$ $(k = 1, 2, \ldots, m)$ $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^-)$, and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively. Also, throughout, $J' = J \setminus \{t_1, \ldots, t_m\}$.

$\mathrm{AC}(J, \mathbb{R})$ is the space of all absolutely continuous functions $y : J \to \mathbb{R}$.

Condition

$$y \leq \overline{y} \quad \text{iff } y(t) \leq \overline{y}(t),\ \forall t \in J \tag{6.2}$$

defines a partial ordering in $\mathrm{AC}(J, \mathbb{R})$. If $\alpha, \beta \in \mathrm{AC}(J, \mathbb{R})$ and $\alpha \leq \beta$, we denote

$$[\alpha, \beta] = \{y \in \mathrm{AC}(J, \mathbb{R}) : \alpha \leq y \leq \beta\}. \tag{6.3}$$

Here $\mathrm{PC}(J, \mathbb{R}) = \{y \mid y : J \to \mathbb{R}$ such that $y(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$, and $y(t_k^+)$ exist, $k = 1, 2, \ldots, m\}$, which is a Banach space with norm

$$\|y\|_{\mathrm{PC}} = \sup\{|y(t)| : t \in J\}. \tag{6.4}$$

In our results, we do not assume any type of monotonicity condition on I_k, $k = 1, \ldots, m$, which is usually the situation in the literature.

Now we introduce concepts of lower and upper solutions for (6.1). These will be the basic tools in the approach that follows.

Definition 6.1. A function $\alpha \in \mathrm{PC}(J, \mathbb{R}) \cap \mathrm{AC}(J', \mathbb{R})$ is said to be a lower solution of (6.1) if there exists $v_1 \in L^1(J, \mathbb{R})$ such that $v_1(t) \in F(t, \alpha(t))$ a.e. on J, $\alpha'(t) \leq v_1(t)$ a.e. on J', $\Delta\alpha|_{t=t_k} \leq I_k(\alpha(t_k^-))$, $k = 1, \ldots, m$, and $\alpha(0) \leq \alpha(T)$.

Similarly a function $\beta \in \mathrm{PC}(J, \mathbb{R}) \cap \mathrm{AC}(J', \mathbb{R})$ is said to be an upper solution of (6.1) if there exists $v_2 \in L^1(J, \mathbb{R})$ such that $v_2(t) \in F(t, \beta(t))$ a.e. on J, $\beta'(t) \geq v_2(t)$ a.e. on J', $\Delta\beta|_{t=t_k} \geq I_k(\beta(t_k^-))$, $k = 1, \ldots, m$, and $\beta(0) \geq \beta(T)$.

So let us begin by defining what we mean by a solution of problem (6.1).

Definition 6.2. A function $y \in \mathrm{PC}(J, \mathbb{R}) \cap \mathrm{AC}(J', \mathbb{R})$ is said to be a solution of (6.1) if there exists a function $v \in L^1(J, \mathbb{R})$ such that $v(t) \in F(t, y(t))$ a.e. on J, $y'(t) = v(t)$ a.e. on J', $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $k = 1, \ldots, m$, and $y(0) = y(T)$.

We need the following auxiliary result.

Lemma 6.3. *Let $g \in L^1(J,\mathbb{R})$. $y \in \mathrm{PC}(J,\mathbb{R}) \cap \mathrm{AC}(J',\mathbb{R})$ be a solution to the periodic problem*

$$\begin{gathered} y'(t) + y(t) = g(t), \quad t \in J,\ t \neq t_k,\ k = 1,\dots,m, \\ \Delta y|_{t=t_k} = I_k\big(y(t_k^-)\big), \quad k = 1,\dots,m, \\ y(0) = y(T), \end{gathered} \tag{6.5}$$

if and only if $y \in \mathrm{PC}(J,\mathbb{R})$ is a solution of the impulsive integral equation

$$y(t) = \int_0^T H(t,s)g(s)ds + \sum_{k=1}^{m} H(t,t_k)I_k\big(y(t_k)\big), \tag{6.6}$$

where

$$H(t,s) = \big(e^T - 1\big)^{-1} \begin{cases} e^{T+s-t}, & 0 \leq s < t \leq T, \\ e^{s-t}, & 0 \leq t \leq s < T. \end{cases} \tag{6.7}$$

Proof. The proof appears as [194, Lemma 2.1]. □

We are now in a position to state and prove our existence result for problem (6.1).

Theorem 6.4. *Let $t_0 = 0$, $t_{m+1} = T$, and assume that $F : J \times \mathbb{R} \to \mathcal{P}_{\mathrm{cp,cv}}(\mathbb{R})$ is an L^1-Carathéodory multivalued map. In addition suppose that the following hold.*

(6.4.1) *There exist α and β in $\mathrm{PC}(J,\mathbb{R}) \cap \mathrm{AC}(J',\mathbb{R})$ lower and upper solutions, respectively, for the problem (6.1) such that $\alpha \leq \beta$.*

(6.4.2) $\Delta\alpha|_{t=t_k} \leq \min_{[\alpha(t_k^-),\, \beta(t_k^-)]} I_k(y) \leq \max_{[\alpha(t_k^-),\, \beta(t_k^-)]} I_k(y) \leq \Delta\beta|_{t=t_k}$, $k = 1,\dots,m$.

Then the problem (6.1) has at least one solution such that

$$\alpha(t) \leq y(t) \leq \beta(t), \quad \forall t \in J. \tag{6.8}$$

Proof. Transform the problem (6.1) into a fixed point problem. Consider the modified problem

$$\begin{gathered} y'(t) + y(t) \in F_1\big(t, y(t)\big), \quad t \in J,\ t \neq t_k,\ k = 1,\dots,m, \\ \Delta y|_{t=t_k} = I_k\big((\tau y)(t_k^-)\big), \quad k = 1,\dots,m, \\ y(0) = y(T), \end{gathered} \tag{6.9}$$

where $F_1(t,y) = F(t,(\tau y)(t)) + (\tau y)(t)$ and $\tau : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ is the truncation operator defined by

$$(\tau y)(t) = \begin{cases} \alpha(t) & \text{if } y(t) < \alpha(t), \\ y(t) & \text{if } \alpha(t) \leq y \leq \beta(t), \\ \beta(t) & \text{if } \beta(t) < y(t). \end{cases} \tag{6.10}$$

Remark 6.5. (i) $\Delta\alpha|_{t=t_k} \leq I_k((\tau y)(t_k^-)) \leq \Delta\beta|_{t=t_k}$ for all $y \in \mathbb{R}$, $k = 1, \dots, m$.

(ii) F_1 is an L^1-Carathéodory multivalued map with compact convex values and there exists $\phi \in L^1(J, \mathbb{R}_+)$ such that

$$||F_1(t, y(t))|| \leq \phi(t) + \max\left(\sup_{t\in J} |\alpha(t)|, \sup_{t\in J} |\beta(t)|\right) \tag{6.11}$$

for a.e. $t \in J$ and all $y \in C(J, \mathbb{R})$.

Set

$$C_0(J, \mathbb{R}) := \{y \in \mathrm{PC}(J, \mathbb{R}) : y(0) = y(T)\}. \tag{6.12}$$

From Lemma 6.3, it follows that a solution to (6.9) is a fixed point of the operator $N : C_0(J, \mathbb{R}) \to \mathcal{P}(C_0(J, \mathbb{R}))$ defined by

$$\begin{aligned} N(y)(t) := \Big\{ h \in C_0(J, \mathbb{R}) : h(t) = &\int_0^T H(t,s)[v(s) + (\tau y)(s)]ds \\ &+ \sum_{k=1}^m H(t, t_k) I_k((\tau y)(t_k)) : v \in \widetilde{S}^1_{F,y} \Big\}, \end{aligned} \tag{6.13}$$

where

$$\begin{gathered} \widetilde{S}_{F,y} = \{v \in S_{F,\tau y} : v(t) \geq v_1(t) \text{ a.e. on } A_1,\ v(t) \leq v_2(t) \text{ a.e. on } A_2\}, \\ S_{F,\tau y} = \{v \in L^1(J, \mathbb{R}) : v(t) \in F(t, (\tau y)(t)) \text{ for a.e. } t \in J\}, \\ A_1 = \{t \in J : y(t) < \alpha(t) \leq \beta(t)\}, \qquad A_2 = \{t \in J : \alpha(t) \leq \beta(t) < y(t)\}. \end{gathered} \tag{6.14}$$

Remark 6.6. (i) For each $y \in C(J, \mathbb{R})$, the set $S_{F,y}$ is nonempty (see Lasota and Opial [186]).

(ii) For each $y \in C(J, \mathbb{R})$, the set $\widetilde{S}_{F,y}$ is nonempty. Indeed, by (i) there exists $v \in S_{F,y}$. Set

$$w = v_1\chi_{A_1} + v_2\chi_{A_2} + v\chi_{A_3}, \tag{6.15}$$

where

$$A_3 = \{t \in J : \alpha(t) \leq y(t) \leq \beta(t)\}. \tag{6.16}$$

Then by decomposability, $w \in \widetilde{S}_{F,y}$.

We will show that N has a fixed point, by applying Theorem 1.7. The proof will be given in several steps. We first will show that N is a completely continuous multivalued map, upper semicontinuous with convex closed values.

Step 1. $N(y)$ is convex for each $y \in C_0(J,\mathbb{R})$.

Indeed, if $h, \overline{h}$ belong to $N(y)$, then there exist $v \in \widetilde{S}_{F,y}$ and $\overline{v} \in \widetilde{S}_{F,y}$ such that

$$\begin{aligned} h(t) &= \int_0^T H(t,s)[v(s)+(\tau y)(s)]ds + \sum_{k=1}^m H(t,t_k)I_k((\tau y)(t_k)), \quad t \in J, \\ \overline{h}(t) &= \int_0^T H(t,s)[\overline{v}(s)+(\tau y)(s)]ds + \sum_{k=1}^m H(t,t_k)I_k((\tau y)(t_k)), \quad t \in J. \end{aligned} \tag{6.17}$$

Let $0 \le l \le 1$. Then, for each $t \in J$, we have

$$\begin{aligned} [lh+(1-l)\overline{h}](t) = & \int_0^T H(t,s)[lv(s)+(1-l)\overline{v}(s)+(\tau y)(s)]ds \\ & + \sum_{k=1}^m H(t,t_k)I_k((\tau y)(t_k)). \end{aligned} \tag{6.18}$$

Since $\widetilde{S}_{F,y}$ is convex (because F has convex values), then

$$lh+(1-l)\overline{h} \in N(y). \tag{6.19}$$

Step 2. N is completely continuous.

Let $B_r := \{y \in C_0(J,\mathbb{R}) : \|y\|_{\mathrm{PC}} \le r\}$ be a bounded set in $C_0(J,\mathbb{R})$ and let $y \in B_r$. Then for each $h \in N(y)$, there exists $v \in \widetilde{S}_{F,y}$ such that

$$h(t) = \int_0^T H(t,s)[v(s)+(\tau y)(s)]ds + \sum_{k=1}^m H(t,t_k)I_k((\tau y)(t_k)), \quad t \in J. \tag{6.20}$$

Thus, for each $t \in J$, we get

$$\begin{aligned} |h(t)| \le & \int_0^T \|H(t,s)\|\,|v(s)+(\tau y)(s)|\,ds + \sum_{k=1}^m \|H(t,t_k)\|\,|I_k((\tau y)(t_k))| \\ \le & \max_{(t,s)\in J\times J} |H(t,s)| \left[\|\phi_R\|_{L^1} + T\max\left(r, \sup_{t\in J}|\alpha(t)|, \sup_{t\in J}|\beta(t)|\right)\right] \\ & + \sum_{k=1}^m \sup_{t\in J} |H(t,t_k)| \max\left(|\Delta\alpha|_{t=t_k}|, |\Delta\beta|_{t=t_k}|\right) := K. \end{aligned} \tag{6.21}$$

Furthermore,

$$
\begin{aligned}
|h'(t)| &\leq \int_0^T \|H_t'(t,s)\| \, |v(s)+(\tau y)(s)| \, ds + \sum_{k=1}^m \|H_t'(t,t_k)\| \, |I_k((\tau y)(t_k))| \\
&\leq \max_{(t,s)\in J\times J} |H_t'(t,s)| \left[\|\phi_R\|_{L^1} + T \max\left(r, \sup_{t\in J} |\alpha(t)|, \sup_{t\in J} |\beta(t)| \right) \right] \\
&\quad + \sum_{k=1}^m \sup_{t\in J} |H_t'(t,t_k)| \max\left(|\Delta\alpha|_{t=t_k}|, |\Delta\beta|_{t=t_k}|\right) =: K_1.
\end{aligned}
\tag{6.22}
$$

We note that $H(t,s)$ and $H_t'(t,s)$ are continuous on $J\times J$. Thus N maps bounded sets of $C_0(J,\mathbb{R})$ into uniformly bounded and equicontinuous sets of $C_0(J,\mathbb{R})$.

Step 3. N has a closed graph.

Let $y_n \to y_*$, $h_n \in N(y_n)$, and $h_n \to h_*$. We will prove that $h_* \in N(y_*)$.

$h_n \in N(y_n)$ means that there exists $v_n \in \widetilde{S}_{F,y_n}$ such that

$$
h_n(t) = \int_0^T H(t,s)[v_n(s)+(\tau y_n)(s)]ds + \sum_{k=1}^m H(t,t_k) I_k((\tau y_n)(t_k)), \quad t\in J.
\tag{6.23}
$$

We must prove that there exists $v_* \in \widetilde{S}_{F,y_*}$ such that

$$
h_*(t) = \int_0^T H(t,s)[v_*(s)+(\tau y_*)(s)]ds + \sum_{k=1}^m H(t,t_k) I_k((\tau y_*)(t_k)), \quad t\in J.
\tag{6.24}
$$

Since $y_n \to y_*$, $h_n \to h_*$, τ and I_k, $k=1,\dots,m$, are continuous functions, we have that

$$
\begin{aligned}
&\Bigg\| \left(h_n - \int_0^T H(t,s)(\tau y_n)(s)ds - \sum_{k=1}^m H(t,t_k) I_k((\tau y_n)(t_k)) \right) \\
&\quad - \left(h_* - \int_0^T H(t,s)(\tau y_*)(s)ds - \sum_{k=1}^m H(t,t_k) I_k((\tau y_*)(t_k)) \right) \Bigg\|_{PC} \longrightarrow 0,
\end{aligned}
\tag{6.25}
$$

as $n\to\infty$.

Now we consider the linear continuous operator

$$
\begin{aligned}
\Gamma : L^1(J,\mathbb{R}) &\longrightarrow C(J,\mathbb{R}), \\
v &\longmapsto \Gamma(v)(t) = \int_0^T H(t,s)v(s)ds.
\end{aligned}
\tag{6.26}
$$

From Lemma 1.28, it follows that $\Gamma\circ \widetilde{S}_F$ is a closed graph operator.

Moreover, we have that

$$\left(h_n(t) - \int_0^T H(t,s)(\tau y_n)(s)ds - \sum_{k=1}^m H(t,t_k)I_k((\tau y_n)(t_k))\right) \in \Gamma(\widetilde{S}_{F,y_n}). \quad (6.27)$$

Since $y_n \to y_*$, it follows from Lemma 1.28 that

$$h_*(t) - \int_0^T H(t,s)(\tau y_*)(s)ds - \sum_{k=1}^m H(t,t_k)I_k((\tau y_*)(t_k)) = \int_0^T H(t,s)v_*(s)ds \quad (6.28)$$

for some $v_* \in \widetilde{S}_{F,y_*}$.

Therefore N is a completely continuous multivalued map, upper semicontinuous, with convex closed values.

Step 4. The set

$$\mathcal{M} := \{y \in C_0(J,\mathbb{R}) : \lambda y \in N(y) \text{ for some } \lambda > 1\} \quad (6.29)$$

is bounded.

Let $\lambda y \in N(y)$, $\lambda > 1$. Then there exists $v \in \widetilde{S}_{F,y}$ such that

$$y(t) = \lambda^{-1}\int_0^T H(t,s)[v(s) + (\tau y)(s)]ds + \lambda^{-1}\sum_{k=1}^m H(t,t_k)I_k((\tau y)(t_k)), \quad t \in J. \quad (6.30)$$

Thus, for each $t \in J$,

$$\begin{aligned} |y(t)| \le |H(t,s)| &\int_0^T |v(s) + (\tau y)(s)|\,ds \\ &+ \sum_{k=1}^m \sup_{t\in J} |H(t,t_k)| \max\left(|\Delta\alpha|_{t=t_k}|, |\Delta\beta|_{t=t_k}|\right). \end{aligned} \quad (6.31)$$

Thus we obtain

$$\begin{aligned} \|y\|_{\mathrm{PC}} \le \frac{1}{1-e^{-T}} &\left[\|\varphi\|_{L^1} + T\max\left(\sup_{t\in J}|\alpha(t)|, \sup_{t\in J}|\beta(t)|\right)\right] \\ &+ \sum_{k=1}^m \sup_{t\in J}|H(t,t_k)| \max\left(|\Delta\alpha|_{t=t_k}|, |\Delta\beta|_{t=t_k}|\right). \end{aligned} \quad (6.32)$$

This shows that $\mathcal{M}$ is bounded. Hence Theorem 1.7 applies and N has a fixed point which is a solution to problem (6.9).

Step 5. The solution y of (6.9) satisfies

$$\alpha(t) \le y(t) \le \beta(t), \quad \forall t \in J. \quad (6.33)$$

Let y be a solution to (6.9). We prove that

$$y(t) \leq \beta(t), \quad \forall t \in J. \tag{6.34}$$

Assume that $y - \beta$ attains a positive maximum on $[t_k^+, t_{k+1}^-]$ at $\bar{t}_k \in [t_k^+, t_{k+1}^-]$ for some $k = 0, \dots, m$, that is,

$$(y - \beta)(\bar{t}_k) = \max\left\{y(t) - \beta(t) : t \in [t_k^+, t_{k+1}^-],\ k = 0, \dots, m\right\} > 0. \tag{6.35}$$

We distinguish the following cases.
Case 1. If $\bar{t}_k \in (t_k^+, t_{k+1}^-]$, there exists $t_k^* \in [t_k^+, \bar{t}_k)$ such that

$$0 < y(t) - \beta(t) \leq y(\bar{t}_k) - \beta(\bar{t}_k), \quad \forall t \in [t_k^*, \bar{t}_k]. \tag{6.36}$$

By the definition of τ, there exist $v \in L^1(J, \mathbb{R})$ with $v(t) \leq v_2(t)$ a.e. on $[t_k^*, \bar{t}_k]$ and $v(t) \in F(t, \beta(t))$ a.e. on $[t_k^*, \bar{t}_k]$ such that

$$\begin{aligned} y(\bar{t}_k) - y(t_k^*) &= \int_{t_k^*}^{\bar{t}_k} \left(v(s) - y(s) + \beta(s)\right) ds \\ &\leq \int_{t_k^*}^{\bar{t}_k} \left(v_2(s) - \left(y(s) - \beta(s)\right)\right) ds. \end{aligned} \tag{6.37}$$

Using the fact that β is an upper solution to (6.1), the above inequality yields

$$\begin{aligned} y(\bar{t}_k) - y(t_k^*) &\leq \beta(\bar{t}_k) - \beta(t_k^*) - \int_{t_k^*}^{\bar{t}_k} \left(y(s) - \beta(s)\right) ds \\ &< \beta(\bar{t}_k) - \beta(t_k^*). \end{aligned} \tag{6.38}$$

Thus we obtain the contradiction

$$\beta(\bar{t}_k) - \beta(t_k^*) < \beta(\bar{t}_k) - \beta(t_k^*). \tag{6.39}$$

Case 2. $\bar{t}_k = t_k^+$, $k = 1, \dots, m$.
Then

$$\Delta\beta|_{t=t_k} < \Delta y|_{t=t_k} = I_k^*\left(y(t_k^-)\right) \leq \Delta\beta|_{t=t_k}, \tag{6.40}$$

which is a contradiction. Thus

$$y(t) \leq \beta(t), \quad \forall t \in [t_1^+, T]. \tag{6.41}$$

Case 3. $\bar{t}_0 = 0$.

Then

$$\beta(T) \leq \beta(0) < y(0) = y(T), \tag{6.42}$$

which is also a contradiction. Consequently,

$$y(t) \leq \beta(t), \quad \forall t \in J. \tag{6.43}$$

Analogously, we can prove that

$$y(t) \geq \alpha(t), \quad \forall t \in J. \tag{6.44}$$

This shows that the problem (6.9) has a solution in the interval $[\alpha, \beta]$. Since $\tau(y) = y$ for all $y \in [\alpha, \beta]$, then y is a solution to (6.1). □

Now we will be concerned with the existence of solutions of the following first-order impulsive periodic multivalued problem:

$$\begin{gathered} y'(t) - a(t, y(t))y(t) \in F(t, y(t)), \quad t \in J,\ t \neq t_k,\ k = 1, \dots, m, \\ y(0) = y(T), \\ \Delta y|_{t=t_k} = I_k(y(t_k)), \quad k = 1, \dots, m, \end{gathered} \tag{6.45}$$

where $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a convex compact-valued multivalued map, $a : J \times \mathbb{R} \to \mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $I_k \in C(\mathbb{R}, \mathbb{R})$ $(k = 1, 2, \dots, m)$ are bounded, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$ and $y(t_k^-)$, $y(t_k^+)$ represent the left and right limits of $y(t)$, respectively, at $t = t_k$. Without loss of generality, we assume that $a(t, y) > 0$ for each $(t, y) \in J \times \mathbb{R}$.

We will provide sufficient conditions on F and I_k, $k = 1, \dots, m$, in order to insure the existence of solutions of the problem (6.45).

For short, we will refer to (6.45) as (NP).

Definition 6.7. A function $y \in \mathrm{PC}(J, \mathbb{R}) \cap \mathrm{AC}(J', \mathbb{R})$ is said to be a solution of (NP) if y satisfies the differential inclusion $y'(t) \in F(t, y(t))$ a.e. on J' and the conditions $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $k = 1, \dots, m$, and $y(0) = y(T)$.

We now consider for each $u \in \mathrm{PC}(J, \mathbb{R}) \cap \mathrm{AC}(J', \mathbb{R})$ the following "linear problem":

$$y'(t) - a(t, u(t))y(t) = g(t), \quad t \neq t_k,\ k = 1, \dots, m, \tag{6.46}$$

$$y(0) = y(T), \tag{6.47}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k)), \quad k = 1, \dots, m, \tag{6.48}$$

where $g \in \mathrm{PC}(J, \mathbb{R})$ and $I_k \in C(\mathbb{R}, \mathbb{R})$, $k = 1, \dots, m$.

For short, we will refer to (6.46)–(6.48) as $(\mathrm{LP})_u$. Note that $(\mathrm{LP})_u$ is not really a linear problem since the impulsive functions are not necessarily linear. However, if I_k, $k = 1, \dots, m$, are linear, then $(\mathrm{LP})_u$ is a linear impulsive problem.

We have the following auxiliary result.

Lemma 6.8. *$y \in \mathrm{PC}(J, \mathbb{R}) \cap \mathrm{AC}(J', \mathbb{R})$ is a solution of $(\mathrm{LP})_u$ if and only if $y \in \mathrm{PC}(J, \mathbb{R})$ is a solution of the impulsive integral equation*

$$y(t) = \int_0^T H(t,s)g(s)ds + \sum_{k=1}^{m} H(t,t_k)I_k(y(t_k)), \tag{6.49}$$

where

$$H(t,s) = (A(T)-1)^{-1}\begin{cases} \dfrac{A(T)A(s)}{A(t)}, & 0 \le s \le t \le T, \\ \dfrac{A(s)}{A(t)}, & 0 \le t < s \le T, \end{cases} \tag{6.50}$$
$$A(t) = \exp\left(-\int_0^t a(s,u(s))ds\right).$$

Proof. First, suppose that $y \in \mathrm{PC}(J, \mathbb{R}) \cap \mathrm{AC}(J', \mathbb{R})$ is a solution of $(\mathrm{LP})_u$. Then

$$y' - a(t,u(t))y = g(t), \quad t \neq t_k, \tag{6.51}$$

that is,

$$(A(t)y(t))' = A(t)g(t), \quad t \neq t_k. \tag{6.52}$$

Assume that $t_k < t \le t_{k+1}$, $k = 0, \dots, m$. By integration of (6.52), we obtain

$$\begin{aligned} A(t_1)y(t_1) - A(0)y(0) &= \int_0^{t_1} A(s)g(s)ds, \\ A(t_2)y(t_2) - A(t_1)y(t_1^+) &= \int_{t_1}^{t_2} A(s)g(s)ds, \\ &\vdots \\ A(t_k)y(t_k) - A(t_{k-1})y(t_{k-1}^+) &= \int_{t_{k-1}}^{t_k} A(s)g(s)ds, \\ A(t)y(t) - A(t_k)y(t_k^+) &= \int_{t_k}^{t} A(s)g(s)ds. \end{aligned} \tag{6.53}$$

Adding these together, we get

$$A(t)y(t) - y(0) = \sum_{0<t_k<t} A(t_k)y(t_k^+) - \sum_{0<t_k<t} A(t_k)y(t_k) + \int_0^t A(s)g(s)ds, \tag{6.54}$$

that is,

$$A(t)y(t) = y(0) + \sum_{0<t_k<t} A(t_k)I_k(y(t_k)) + \int_0^t A(s)g(s)ds. \tag{6.55}$$

In view of (6.55) with $y(0) = y(T)$, we get

$$A(T)y(0) = y(0) + \sum_{k=1}^{m} A(t_k)I_k(y(t_k)) + \int_0^T A(s)g(s)ds. \tag{6.56}$$

Hence

$$y(0) = (A(T)-1)^{-1}\left[\sum_{k=1}^{m} A(t_k)I_k(y(t_k)) + \int_0^T A(s)g(s)ds\right]. \tag{6.57}$$

Substituting (6.57) into (6.55), we obtain

$$\begin{aligned} A(t)y(t) = {} & (A(T)-1)^{-1}\left[\sum_{k=1}^{m} A(t_k)I_k(y(t_k)) + \int_0^T A(s)g(s)ds\right] \\ & + \sum_{0<t_k<t} A(t_k)I_k(y(t_k)) + \int_0^t A(s)g(s)ds. \end{aligned} \tag{6.58}$$

Using (6.58) and the fact that

$$\sum_{k=1}^{m} I_k(y(t_k)) = \sum_{0<t_k<T} I_k(y(t_k)) = \sum_{0<t_k<t} I_k(y(t_k)) + \sum_{t\le t_k<T} I_k(y(t_k)), \tag{6.59}$$

we get

$$\begin{aligned} A(t)y(t) &= (A(T)-1)^{-1}\Bigg[\sum_{0<t_k<t} A(t_k)I_k(y(t_k)) + \sum_{t\le t_k<T} A(t_k)I_k(y(t_k)) \\ &\qquad + \int_0^t A(s)g(s)ds + \int_t^T A(s)g(s)ds \\ &\qquad + (A(T)-1)\sum_{0<t_k<t} A(t_k)I_k(y(t_k)) \\ &\qquad + (A(T)-1)\int_0^t A(s)g(s)ds\Bigg] \\ &= (A(T)-1)^{-1}\Bigg[A(T)\sum_{0<t_k<t} A(t_k)I_k(y(t_k)) + \sum_{t\le t_k<T} A(t_k)I_k(y(t_k)) \\ &\qquad + A(T)\int_0^t A(s)g(s)ds + \int_t^T A(s)g(s)ds\Bigg]. \end{aligned} \tag{6.60}$$

Thus

$$\begin{aligned} y(t) = (A(T)-1)^{-1}\Bigg[\int_0^t \frac{A(T)A(s)}{A(t)}g(s)ds + \int_t^T \frac{A(s)}{A(t)}g(s)ds \\ + \sum_{0<t_k<t} \frac{A(T)A(t_k)}{A(t)} I_k(y(t_k)) + \sum_{t\le t_k<T} \frac{A(t_k)}{A(t)} I_k(y(t_k))\Bigg] \\ = \int_0^T H(t,s)g(s)ds + \sum_{k=1}^m H(t,t_k)I_k(y(t_k)). \end{aligned} \tag{6.61}$$

In particular, y is a solution of (6.49).

Conversely, assume that $y \in \mathrm{PC}(J,\mathbb{R}) \cap \mathrm{AC}(J',\mathbb{R})$ is a solution of (6.49).

Direct differentiation on (6.49) implies that for $t \neq t_k$,

$$\begin{aligned} y'(t) &= \int_0^T \frac{\partial H(t,s)}{\partial t} g(s)ds + \sum_{k=1}^m \left[\frac{\partial H(t,t_k)}{\partial t} I_k(y(t_k))\right] \\ &= g(t) + \int_0^T [a(t,u(t))]H(t,s)g(s)ds + \sum_{k=1}^m [a(t,u(t))]H(t,t_k)I_k(y(t_k)) \\ &= g(t) + a(t,u(t))\left[\int_0^T H(t,s)g(s)ds + \sum_{k=1}^m H(t,t_k)I_k(y(t_k))\right], \\ &= g(t) + a(t,u(t))\,y(t). \end{aligned} \tag{6.62}$$

It is easy to see that

$$\Delta\left[\sum_{k=1}^m H(t,t_k)I_k\right]\Bigg|_{t=t_k} = I_k. \tag{6.63}$$

Moreover, we have

$$\Delta y|_{t=t_k} = I_k(y(t_k)). \tag{6.64}$$

Making use of the fact that $H(0,s) = H(T,s)$ for $s \in J$, we obtain that $y(0) = y(T)$.

Hence $y \in \mathrm{PC}(J,\mathbb{R}) \cap \mathrm{AC}(J',\mathbb{R})$ is a solution of impulsive periodic problem $(\mathrm{LP})_u$. □

Although the linear differential problem (6.46)-(6.47) possesses a unique solution $y \in \mathrm{PC}(J,\mathbb{R}) \cap \mathrm{AC}(J',\mathbb{R})$ for any $g \in \mathrm{PC}(J,\mathbb{R})$, the impulse problem $(\mathrm{LP})_u$ is not always solvable even for $g \equiv 0$.

Example 6.9. Consider the problem $(\mathrm{LP})_u$ with $a(t,u(t)) \equiv 1$, $g \equiv 0$, and $I_1(x) = (e^{-T}-1)x + 1$.

The general solution of the equation $y' - y = 0$ subject to the impulse $\Delta y|_{t=t_1} = I_1(y(t_1))$ is

$$y(t) = \begin{cases} y(0)e^t, & t \in [0, t_1], \\ [y(0)e^{t_1} + I_1(y(0)e^{t_1})]e^{t-t_1}, & t \in (t_1, T]. \end{cases} \tag{6.65}$$

This solution satisfies the periodic boundary condition (6.47) if and only if

$$y(0) = [y(0)e^{t_1} + I_1(y(0)e^{t_1})]e^{(T-t_1)}, \tag{6.66}$$

that is,

$$y(0)e^{t_1}(e^{-T} - 1) = I_1(y(0)e^{t_1}). \tag{6.67}$$

By the definition of I_1, there is no initial condition $y(0)$ satisfying this last equality. Hence the problem has no solution. In this example, the impulse function is not linear.

We now present another example with linear impulse so that $(\mathrm{LP})_u$ is indeed a linear problem, but with no solution.

Example 6.10. We now inspect problem $(\mathrm{LP})_u$ with $a(t, u(t)) \equiv 1$, $k = 1$, and $I_1(x) = (e^{-T} - 1)x$, and $g \in \Omega$, $e^{-T}d_1 + d_2 \neq 0$, where

$$d_1 = \int_0^{t_1} e^{T-s}g(s)ds, \qquad d_2 = \int_{t_1}^{T} e^{T-s}g(s)ds. \tag{6.68}$$

In this case, the general solution of (6.46) and (6.48) is

$$y(t) = \begin{cases} y(0)e^t + \displaystyle\int_0^t e^{t-s}g(s)ds, & t \in [0, t_1], \\ y(t_1^+)e^{t-t_1} + \displaystyle\int_{t_1}^{T} e^{t-s}g(s)ds, & t \in (t_1, T], \end{cases} \tag{6.69}$$

where

$$\begin{aligned} y(t_1^+) &= y(t_1^-) + I_1(y(t_1)), \\ y(t_1^-) = y(t_1) &= y(0)e^{t_1} + \int_0^{t_1} e^{t_1-s}g(s)ds. \end{aligned} \tag{6.70}$$

Thus y satisfies the periodic boundary condition (6.47) if and only if

$$\begin{aligned}
y(0) &= e^{T-t_1}\Big[y(0)e^{t_1} + \int_0^{t_1} e^{t_1-s}g(s)ds + I_1\Big(y(0)e^{t_1} + \int_0^{t_1} e^{t_1-s}g(s)ds\Big)\Big] \\
&\quad + \int_{t_1}^{T} e^{T-s}g(s)ds \\
&= e^{T-t_1}\Big[y(0)e^{t_1} + \int_0^{t_1} e^{t_1-s}g(s)ds + (e^{-T}-1)\Big(y(0)e^{t_1} + \int_0^{t_1} e^{t_1-s}g(s)ds\Big)\Big] \\
&\quad + \int_{t_1}^{T} e^{T-s}g(s)ds \\
&= y(0) + \int_0^{t_1} e^{-s}g(s)ds + \int_{t_1}^{T} e^{T-s}g(s)ds.
\end{aligned} \tag{6.71}$$

Thus

$$\int_0^{t_1} e^{-s}g(s)ds + \int_{t_1}^{T} e^{T-s}g(s)ds = 0. \tag{6.72}$$

But

$$\int_0^{t_1} e^{-s}g(s)ds + \int_{t_1}^{T} e^{T-s}g(s)ds = e^{-T}d_1 + d_2, \tag{6.73}$$

which is a contradiction.

Example 6.11. Consider now a simple example of a periodic problem $y'(t) = f(t)$, $t \in [0,T]$, $y(0) = y(T)$. It is clear that without impulses, this problem does not have a solution if $f(t) > 0$. If we consider the corresponding impulsive problem with the impulsive conditions $y(t_i) = \beta_i y(t_i - 0)$, $i = 1,2,\dots,m$, where $\beta_1\beta_2\cdots\beta_m \neq 1$, this problem has a solution for each $f(t)$. In this case, impulses "improve" existence.

As a consequence of Lemma 6.8, we have that y is a solution of (NP) if and only if y satisfies the impulsive integral inclusion

$$y(t) \in \int_0^T H(t,s)F(s,y(s))ds + \sum_{k=1}^{m} H(t,t_k)I_k(y(t_k)). \tag{6.74}$$

We now give the existence result for the nonlinear problem (NP).

Theorem 6.12. *Assume that the following hold.*

(6.9.1) $F : J \times \mathbb{R} \to \mathcal{P}_{\mathrm{cp,cv}}(\mathbb{R})$ *is an* L^1*- Carathéodory multivalued map.*

(6.9.2) *There exist constants* c_k *such that* $|I_k(y)| \leq c_k$, $k = 1,\dots,m$, *for each* $y \in \mathbb{R}$.

(6.9.3) *There exists a function* $m \in L^1(J, \mathbb{R}^+)$ *such that*

$$||F(t,y)|| := \sup\{|v| : v \in F(t,y)\} \le m(t) \tag{6.75}$$

for almost all $t \in J$ *and for all* $y \in \mathbb{R}$.
Then the nonlinear impulsive problem (NP) has at least one solution.

Proof. Transform the problem (NP) into a fixed point problem. Consider the multivalued map $G : \mathrm{PC}(J,\mathbb{R}) \to \mathcal{P}(\mathrm{PC}(J,\mathbb{R}))$ defined by

$$G(y)(t) = \left\{h \in \mathrm{PC}(J,\mathbb{R}) : h(t) = \int_0^T H(t,s)g(s)ds + \sum_{k=1}^m H(t,t_k)I_k(y(t_k))\right\}, \tag{6.76}$$

where $g \in S_{F,y}$.

We will show that G satisfies the assumptions of Theorem 1.7. The proof will be given in several steps.

Step 1. $G(y)$ is convex for each $y \in \mathrm{PC}(J,\mathbb{R})$.

Indeed, if h_1, h_2 belong to $G(y)$, then there exist $g_1, g_2 \in S_{F,y}$ such that, for each $t \in J$, we have

$$h_i(t) = \int_0^T H(t,s)g_i(s)ds + \sum_{k=1}^m H(t,t_k)I_k(y(t_k)), \quad i = 1,2. \tag{6.77}$$

Let $0 \le d \le 1$. Then, for each $t \in J$, we have

$$(dh_1 + (1-d)h_2)(t) = \int_0^T H(t,s)[dg_1(s) + (1-d)g_2(s)]ds + \sum_{k=1}^m H(t,t_k)I_k(y(t_k)). \tag{6.78}$$

Since $S_{F,y}$ is convex (because F has convex values), then

$$dh_1 + (1-d)h_2 \in G(y). \tag{6.79}$$

Step 2. G maps bounded sets into bounded sets in $\mathrm{PC}(J,\mathbb{R})$.

Indeed, it is enough to show that there exists a positive constant ℓ such that for each $h \in G(y)$ with $y \in B_q = \{y \in \mathrm{PC}(J,\mathbb{R}) : \|y\|_{\mathrm{PC}} \le q\}$, one has $\|h\|_{\mathrm{PC}} \le \ell$. If $h \in G(y)$, then there exists $g \in S_{F,y}$ such that for each $t \in J$, we have

$$h(t) = \int_0^T H(t,s)g(s)ds + \sum_{k=1}^m H(t,t_k)I_k(y(t_k)). \tag{6.80}$$

By (6.9.1) and (6.9.2), we have for each $t \in J$,

$$\begin{aligned} |h(t)| &\leq \int_0^T |H(t,s)||g(s)|ds + \sum_{k=1}^m |H(t,t_k)||I_k(y(t_k))| \\ &\leq \sup_{(t,s)\in J\times J} |H(t,s)| \int_0^T g_q(s)ds \\ &\quad + \sum_{k=1}^m \sup_{t\in J} |H(t,t_k)| \sup\{|I_k(|y|)| : \|y\|_{\mathrm{PC}} \leq q\} = \ell. \end{aligned} \tag{6.81}$$

Step 3. G maps bounded sets into equicontinuous sets of $\mathrm{PC}(J,\mathbb{R})$.

Let $r_1, r_2 \in J'$, $r_1 < r_2$, and let $B_q = \{y \in \mathrm{PC}(J,\mathbb{R}) : \|y\|_{\mathrm{PC}} \leq q\}$ be a bounded set of $\mathrm{PC}(J,\mathbb{R})$.

For each $y \in B_q$ and $h \in G(y)$, there exists $g \in S_{F,y}$ such that

$$h(t) = \int_0^T H(t,s)g(s)ds + \sum_{k=1}^m H(t,t_k)I_k(y(t_k)). \tag{6.82}$$

Thus

$$\begin{aligned} |h(r_2) - h(r_1)| &\leq \int_0^T |H(r_2,s) - H(r_1,s)| g_q(s)ds \\ &\quad + \sum_{k=1}^m |H(r_2,s) - H(r_1,s)| I_k(y(t_k)). \end{aligned} \tag{6.83}$$

As $r_2 \to r_1$, the right-hand side of the above inequality tends to zero. This proves the equicontinuity for the case where $t \neq t_i$, $i = 1,\dots,m$. The proof of the equicontinuity at $t = t_i$ is similar to that given in Theorem 4.3.

Step 4. G has a closed graph.

Let $y_n \to y_*$, $h_n \in G(y_n)$, and $h_n \to h_*$. We will prove that $h_* \in G(y_*)$.

$h_n \in G(y_n)$ means that there exists $g_n \in S_{F,y_n}$ such that, for each $t \in J$,

$$h_n(t) = \int_0^T H(t,s)g_n(s)ds + \sum_{k=1}^m H(t,t_k)I_k(y_n(t_k)). \tag{6.84}$$

We must prove that there exists $g_* \in S_{F,y_*}$ such that, for each $t \in J$,

$$h_*(t) = \int_0^T H(t,s)g_*(s)ds + \sum_{k=1}^m H(t,t_k)I_k(y_*(t_k)). \tag{6.85}$$

Clearly since I_k, $k = 1,\dots,m$, are continuous, we have that

$$\left\|\left(h_n - \sum_{k=1}^{m} H(t,t_k)I_k(y_n(t_k))\right) - \left(h_* - \sum_{k=1}^{m} H(t,t_k)I_k(y_*(t_k))\right)\right\|_{\mathrm{PC}} \longrightarrow 0, \tag{6.86}$$

as $n \to \infty$.

Consider the linear continuous operator

$$\begin{aligned} \Gamma : L^1(J,\mathbb{R}) &\longrightarrow C(J,\mathbb{R}), \\ g \longmapsto \Gamma(g)(t) &= \int_0^T H(t,s)g(s)ds. \end{aligned} \tag{6.87}$$

From Lemma 1.28, it follows that $\Gamma \circ S_F$ is a closed graph operator.

Moreover, we have that

$$\left(h_n(t) - \sum_{k=1}^{m} H(t,t_k)I_k(y_n(t_k))\right) \in \Gamma(S_{F,y_n}). \tag{6.88}$$

Since $y_n \to y_*$, it follows from Lemma 1.28 that

$$\left(h_*(t) - \sum_{k=1}^{m} H(t,t_k)I_k(y_*(t_k))\right) = \int_0^T H(t,s)g_*(s)ds \tag{6.89}$$

for some $g_* \in S_{F,y_*}$.

Step 5. Now it remains to show that the set

$$\mathcal{M} := \{y \in \mathrm{PC}(J,\mathbb{R}) : \lambda y \in G(y),\ \text{for some } \lambda > 1\}. \tag{6.90}$$

is bounded.

Let $y \in \mathcal{M}$. Then $\lambda y \in G(y)$ for some $\lambda > 1$. Thus there exists $g \in S_{F,y}$ such that

$$y(t) = \lambda^{-1}\int_0^T H(t,s)g(s)ds + \lambda^{-1}\sum_{k=1}^{m} H(t,t_k)I_k(y(t_k)). \tag{6.91}$$

This implies by (6.9.2)-(6.9.3) that, for each $t \in J$, we have

$$|y(t)| \le \sup_{(t,s)\in J\times J} |H(t,s)| \int_0^T m(s)ds + \sum_{k=1}^{m} \sup_{t\in J} |H(t,t_k)| c_k = b. \tag{6.92}$$

This inequality implies that there exists a constant b such that $|y(t)| \le b$, $t \in J$. This shows that $\mathcal{M}$ is bounded.

Set $X := \mathrm{PC}(J,\mathbb{R})$. As a consequence of Theorem 1.7, we deduce that G has a fixed point y which is a solution of (6.45). $\square$

6.3. Upper- and lower-solutions method for impulsive differential inclusions with nonlinear boundary conditions

This section is concerned with the existence of solutions for the boundary multivalued problem with nonlinear boundary conditions and impulsive effects given by

$$\begin{gathered} y'(t) \in F(t,y(t)), \quad t \in J = [0,T],\ t \neq t_k,\ k = 1,\dots,m, \\ y(t_k^+) = I_k(y(t_k^-)), \quad k = 1,\dots,m, \\ L(y(0),y(T)) = 0, \end{gathered} \tag{6.93}$$

where $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a compact convex-valued, multivalued map, and $L : \mathbb{R}^2 \to \mathbb{R}$ is a single-valued map, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $I_k \in C(\mathbb{R},\mathbb{R})$ $(k = 1,2,\dots,m)$ are bounded, and $y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$.

Let us start by defining what we mean by a solution of problem (6.93).

Definition 6.13. A function $y \in \mathrm{PC}(J,\mathbb{R}) \cap \mathrm{AC}(J',\mathbb{R})$ is said to be a solution of (6.93) if y satisfies the inclusion $y'(t) \in F(t,y(t))$ a.e. on J' and the conditions $y(t_k^+) = I_k(y(t_k^-))$, $k = 1,\dots,m$, and $L(y(0),y(T)) = 0$.

The following concept of lower and upper solutions for (6.93) was introduced by Benchohra and Boucherif [34] for initial value problems for impulsive differential inclusions of first order. These will be the basic tools in the approach that follows.

Definition 6.14. A function $\alpha \in \mathrm{PC}(J,\mathbb{R}) \cap \mathrm{AC}(J',\mathbb{R})$ is said to be a lower solution of (6.93) if there exists $v_1 \in L^1(J,\mathbb{R})$ such that $v_1(t) \in F(t,\alpha(t))$ a.e. on J, $\alpha'(t) \le v_1(t)$ a.e. on J', $\alpha(t_k^+) \le I_k(\alpha(t_k^-))$ $k = 1,\dots,m$, and $L(\alpha(0),\alpha(T)) \le 0$.

Similarly a function $\beta \in \mathrm{PC}(J,\mathbb{R}) \cap \mathrm{AC}(J',\mathbb{R})$ is said to be an upper solution of (6.93) if there exists $v_2 \in L^1(J,\mathbb{R})$ such that $v_2(t) \in F(t,\beta(t))$ a.e. on J, $\beta'(t) \ge v_2(t)$ a.e. on J', $\beta(t_k^+) \ge I_k(\beta(t_k^-))$, $k = 1,\dots,m$, and $L(\beta(0),\beta(T)) \ge 0$.

We are now in a position to state and prove our existence result for the problem (6.93).

Theorem 6.15. *Assume the following hypotheses hold.*

(6.12.1) *$F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is an L^1-Carathéodory multivalued map.*

(6.12.2) *There exist α and $\beta \in \mathrm{PC}(J,\mathbb{R}) \cap \mathrm{AC}((t_k,t_{k+1}),\mathbb{R})$, $k = 0,\dots,m$, lower and upper solutions for the problem* (6.93) *such that $\alpha \le \beta$.*

(6.12.3) *L is a continuous single-valued map in $(x,y) \in [\alpha(0),\beta(0)] \times [\alpha(T),\beta(T)]$ and nonincreasing in $y \in [\alpha(T),\beta(T)]$.*

(6.12.4)

$$\alpha(t_k^+) \leq \min_{y \in [\alpha(t_k^-),\beta(t_k^-)]} I_k(y) \leq \max_{y \in [\alpha(t_k^-),\beta(t_k^-)]} I_k(y) \leq \beta(t_k^+), \quad k = 1,\dots,m. \tag{6.94}$$

Then the problem (6.93) *has at least one solution y such that*

$$\alpha(t) \leq y(t) \leq \beta(t), \quad \forall t \in J. \tag{6.95}$$

Proof. Transform the problem (6.93) into a fixed point problem. Consider the following modified problem:

$$\begin{gathered} y'(t) + y(t) \in F_1(t,y(t)), \quad \text{a.e. } t \in J,\ t \neq t_k,\ k = 1,\dots,m, \\ y(t_k^+) = I_k(\tau(t_k^-, y(t_k^-))), \quad k = 1,\dots,m, \\ y(0) = \tau(0, y(0) - L(\overline{y}(0),\overline{y}(T))), \end{gathered} \tag{6.96}$$

where $F_1(t,y) = F(t,\tau(t,y)) + \tau(t,y)$, $\tau(t,y) = \max(\alpha(t),\min(y,\beta(t)))$, and $\overline{y}(t) = \tau(t,y)$. A solution to (6.96) is a fixed point of the operator $N : \mathrm{PC}(J,\mathbb{R}) \to \mathcal{P}(\mathrm{PC}(J,\mathbb{R}))$ defined by

$$N(y) = \left\{ h \in \mathrm{PC}(J,\mathbb{R}) : h(t) = \left\{ \begin{aligned} & y(0) + \int_0^t [g(s) + \overline{y}(s) - y(s)]ds \\ & + \sum_{0<t_k<t} I_k(\tau(t_k^-, y(t_k^-))) \end{aligned} \right\} \right\}, \tag{6.97}$$

where $g \in \widetilde{S}^1_{F,\overline{y}}$, and

$$\begin{gathered} \widetilde{S}_{F,\overline{y}} = \{v \in S_{F,\overline{y}} : v(t) \geq v_1(t) \text{ a.e. on } A_1,\ v(t) \leq v_2(t) \text{ a.e. on } A_2\}, \\ S_{F,\overline{y}} = \{v \in L^1(J,\mathbb{R}) : v(t) \in F(t,\overline{y}(t)) \text{ for a.e. } t \in J\}, \\ A_1 = \{t \in J : y(t) < \alpha(t) \leq \beta(t)\}, \qquad A_2 = \{t \in J : \alpha(t) \leq \beta(t) < y(t)\}. \end{gathered} \tag{6.98}$$

Remark 6.16. (i) Notice that F_1 is an L^1-Carathéodory multivalued map with compact convex values, and there exists $\varphi \in L^1(J,\mathbb{R})$ such that

$$\|F_1(t,y)\| \leq \varphi(t) + \max\left(\sup_{t\in J} |\alpha(t)|, \sup_{t\in J} |\beta(t)|\right). \tag{6.99}$$

(ii) By the definition of τ, it is clear that

$$\begin{gathered}\alpha(0) \leq y(0) \leq \beta(0), \\ \alpha(t_k^+) \leq I_k(\tau(t_k, y(t_k))) \leq \beta(t_k^+), \quad k = 1,\dots,m.\end{gathered} \tag{6.100}$$

We will show that N satisfies the assumptions of Theorem 1.7. The proof will be given in several steps.

Step 1. $N(y)$ is convex for each $y \in \mathrm{PC}(J,\mathbb{R})$.

Indeed, if h_1, h_2 belong to $N(y)$, then there exist $g_1, g_2 \in \widetilde{S}^1_{F,\overline{y}}$ such that, for each $t \in J$, we have

$$h_i(t) = y(0) + \int_0^t [g_i(s) + \overline{y}(s) - y(s)]ds + \sum_{0<t_k<t} I_k(\tau(t_k^-, y(t_k^-))), \quad i = 1,2. \tag{6.101}$$

Let $0 \leq d \leq 1$. Then, for each $t \in J$, we have

$$\begin{aligned}(dh_1 + (1-d)h_2)(t) = {} & \int_0^t [dg_1(s) + (1-d)g_2(s) + \overline{y}(s) - y(s)]ds \\ & + \sum_{0<t_k<t} I_k(\tau(t_k^-, y(t_k^-))).\end{aligned} \tag{6.102}$$

Since $\widetilde{S}^1_{F_1,\overline{y}}$ is convex (because F_1 has convex values), then

$$dh_1 + (1-d)h_2 \in N(y). \tag{6.103}$$

Step 2. N maps bounded sets into bounded sets in $\mathrm{PC}(J,\mathbb{R})$.

Indeed, it is enough to show that for each $q > 0$, there exists a positive constant ℓ such that, for each $y \in B_q = \{y \in \mathrm{PC}(J,\mathbb{R}) : \|y\|_{\mathrm{PC}} \leq q\}$, one has $\|N(y)\|_{\mathrm{PC}} \leq \ell$.

Let $y \in B_q$ and $h \in N(y)$. Then there exists $g \in \widetilde{S}^1_{F,\overline{y}}$ such that, for each $t \in J$, we have

$$h(t) = y(0) + \int_0^t [g(s) + \overline{y}(s) - y(s)]ds + \sum_{0<t_k<t} I_k(\tau(t_k^-, y(t_k^-))). \tag{6.104}$$

By (6.12.1), we have that, for each $t \in J$,

$$\begin{aligned}|h(t)| \leq {} & |y(0)| + \int_0^T [|g(s)| + |\overline{y}(s)| + |y(s)|]ds + \sum_{0<t_k<t} |I_k(\tau(t_k, y(t_k)))| \\ & \leq \max(|\alpha(0)|, |\beta(0)|) + \|\varphi_q\|_{L^1} + T\max\left(q, \sup_{t\in J}|\alpha(t)|, \sup_{t\in J}|\beta(t)|\right) \\ & \quad + Tq\sum_{k=1}^m \max(q, |\alpha(t_k^-)|, |\beta(t_k^-)|).\end{aligned} \tag{6.105}$$

In particular, if

$$\ell = \max\left(|\alpha(0)|, |\beta(0)|\right) + \|\varphi_q\|_{L^1} + T\max\left(q, \sup_{t\in J}|\alpha(t)|, \sup_{t\in J}|\beta(t)|\right) + Tq + \sum_{k=1}^{m}\max\left(q, |\alpha(t_k^-)|, |\beta(t_k^-)|\right), \tag{6.106}$$

then $\|N(y)\|_{\mathrm{PC}} \le \ell$.

Step 3. N maps bounded set into equicontinuous sets of $\mathrm{PC}(J,\mathbb{R})$.

Let $u_1, u_2 \in J'$, $u_1 < u_2$, and let B_q be a bounded set of $\mathrm{PC}(J,\mathbb{R})$ as in Step 2. Let $y \in B_q$ and $h \in N(y)$. Then there exists $g \in \widetilde{S}^1_{F,\overline{y}}$ such that, for each $t \in J$, we have

$$h(t) = y(0) + \int_0^t [g(s) + \overline{y}(s) - y(s)]ds + \sum_{0<t_k<t} I_k(\tau(t_k^-, y(t_k^-))). \tag{6.107}$$

Then

$$|h(u_2) - h(u_1)| \le \int_{u_1}^{u_2}\phi_q(s)ds + (u_2 - u_1)\max\left(q, \sup_{t\in J}|\alpha(t)|, \sup_{t\in J}|\beta(t)|\right) + (u_2 - u_1)q + \sum_{u_1<t_k<u_2}\max\left(q, |\alpha(t_k^-)|, |\beta(t_k^-)|\right). \tag{6.108}$$

As $u_2 \to u_1$ the right-hand side of the above inequality tends to zero. This proves the equicontinuity for the case where $t \neq t_i$, $i = 1,\dots,m$. The proof of the equicontinuity at $t = t_i$ is similar to that given in Theorem 4.3.

As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $N : \mathrm{PC}(J,\mathbb{R}) \to \mathcal{P}(\mathrm{PC}(J,\mathbb{R}))$ is a completely continuous multivalued map, and therefore a condensing map.

Step 4. N has a closed graph.

Let $y_n \to y_*$, $h_n \in N(y_n)$, and $h_n \to h_*$. We will prove that $h_* \in N(y_*)$.

$h_n \in N(y_n)$ means that there exists $g_n \in \widetilde{S}^1_{F,\overline{y}_n}$ such that, for each $t \in J$,

$$h_n(t) = y_n(0) + \int_0^t [g_n(s) + \overline{y}_n(s) - y_n(s)]ds + \sum_{0<t_k<t} I_k(\tau(t_k^-, y_n(t_k))). \tag{6.109}$$

We must prove that there exists $g_* \in \widetilde{S}^1_{F,\overline{y}_*}$ such that, for each $t \in J$,

$$h_*(t) = y_*(0) + \int_0^t [g_*(s) + \overline{y}_*(s) - y_*(s)]ds + \sum_{0<t_k<t} I_k(\tau(t_k, y_*(t_k))). \tag{6.110}$$

Since τ and I_k, $k = 1, \dots, m$, are continuous, we have

$$\begin{aligned}
&\Bigg\| \Bigg(h_n - y_n(0) - \sum_{0<t_k<t} I_k\big(\tau(t_k^-, y_n(t_k))\big) - \int_0^t \overline{y}_n(s) - y_n(s)ds \Bigg) \\
&\quad - \Bigg(h_* - y_*(0) - \sum_{0<t_k<t} I_k\big(\tau(t_k^-, y_*(t_k))\big) - \int_0^t \overline{y}_*(s) - y_*(s)ds \Bigg) \Bigg\|_{\mathrm{PC}} \longrightarrow 0,
\end{aligned} \tag{6.111}$$

as $n \to \infty$.

Consider the linear continuous operator

$$\begin{aligned}
\Gamma : L^1(J, \mathbb{R}) &\longrightarrow C(J, \mathbb{R}), \\
g \longmapsto (\Gamma g)(t) &= \int_0^t g(s)ds.
\end{aligned} \tag{6.112}$$

From Lemma 1.28, it follows that $\Gamma \circ \widetilde{S}_F$ is a closed graph operator.

Moreover, we have that

$$\Bigg(h_n(t) - y_n(0) - \int_0^t [\overline{y}_n(s) - y_n(s)]ds - \sum_{0<t_k<t} I_k\big(\tau(t_k^-, y_n(t_k))\big) \Bigg) \in \Gamma(\widetilde{S}^1_{F,\overline{y}_n}). \tag{6.113}$$

Since $y_n \to y_*$, it follows from Lemma 1.28 that

$$\Bigg(h_*(t) - y_*(0) - \int_0^t [\overline{y}_*(s) - y_*(s)]ds - \sum_{0<t_k<t} I_k\big(\tau(t_k, y_*(t_k))\big) \Bigg) = \int_0^t g_*(s)ds \tag{6.114}$$

for some $g_* \in \widetilde{S}^1_{F,y_*}$.

Step 5. Now it remains to show that the set

$$\mathcal{M} := \{ y \in \mathrm{PC}(J, \mathbb{R}) : y \in \lambda N(y), \text{ for some } 0 < \lambda < 1 \} \tag{6.115}$$

is bounded.

Let $y \in \mathcal{M}$. Then $y \in \lambda N(y)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$,

$$y(t) = \lambda \Bigg[y(0) + \int_0^t [g(s) - \overline{y}(s) - y(s)]ds + \sum_{0<t_k<t} I_k\big(\tau(t_k^-, y(t_k))\big) \Bigg]. \tag{6.116}$$

This implies by (6.12.2)–(6.12.4) that, for each $t \in J$, we have

$$\begin{aligned}|y(t)| &\leq |y(0)| + \int_0^t [|g(s)| + |\overline{y}(s)| + |y(s)|]ds + \sum_{k=1}^{m} |I_k(\tau(t_k^-, y(t_k)))| \\ &\leq \max(|\alpha(0)|, |\beta(0)|) + \|\varphi\|_{L^1} + T\max\left(\sup_{t\in J}|\alpha(t)|, \sup_{t\in J}|\beta(t)|\right) \\ &\quad + \int_0^t |y(s)|ds + \sum_{k=1}^{m} \max(|\alpha(t_k^-)|, |\beta(t_k^-)|).\end{aligned} \tag{6.117}$$

Set

$$\begin{aligned}z_0 = {}& \max(|\alpha(0)|, |\beta(0)|) + \|\varphi\|_{L^1} + T\max\left(\sup_{t\in J}|\alpha(t)|, \sup_{t\in J}|\beta(t)|\right) \\ &+ \sum_{k=1}^{m} \max(|\alpha(t_k^-)|, |\beta(t_k^-)|).\end{aligned} \tag{6.118}$$

Using Gronwall's lemma (see [160, page 36]), we get that, for each $t \in J$,

$$|y(t)| \leq z_0 e^t. \tag{6.119}$$

Thus

$$\|y\|_{\mathrm{PC}} \leq z_0 e^T. \tag{6.120}$$

This shows that $\mathcal{M}$ is bounded.

Set $X := \mathrm{PC}(J, \mathbb{R})$. As a consequence of Theorem 1.7, we deduce that N has a fixed point which is a solution of (6.96).

Step 6. The solution y of (6.96) satisfies

$$\alpha(t) \leq y(t) \leq \beta(t), \quad \forall t \in J. \tag{6.121}$$

Let y be a solution to (6.96). We prove that

$$y(t) \leq \beta(t), \quad \forall t \in J. \tag{6.122}$$

Assume that $y - \beta$ attains a positive maximum on $[t_k^+, t_{k+1}^-]$ at $\overline{t}_k \in [t_k^+, t_{k+1}^-]$ for some $k = 0, \dots, m$, that is,

$$(y - \beta)(\overline{t}_k) = \max\{y(t) - \beta(t) : t \in [t_k^+, t_{k+1}^-],\ k = 0, \dots, m\} > 0. \tag{6.123}$$

We distinguish the following cases.

Case 1. If $\bar{t}_k \in (t_k^+, t_{k+1}^-]$, there exists $t_k^* \in [t_k^+, \bar{t}_k)$ such that

$$0 < y(t) - \beta(t) \le y(\bar{t}_k) - \beta(\bar{t}_k), \quad \forall t \in [t_k^*, \bar{t}_k]. \tag{6.124}$$

By the definition of τ, one has

$$y'(t) + y(t) \in F(t, \beta(t)) + \beta(t) \quad \text{a.e. on } [t_k^*, \bar{t}_k]. \tag{6.125}$$

Thus there exist $v(t) \in F(t, \beta(t))$ a.e. on $[t_k^*, \bar{t}_k]$, with $v(t) \le v_2(t)$ a.e. on $[t_k^*, \bar{t}_k]$ such that

$$y'(t) + y(t) = v(t) + \beta(t) \quad \text{a.e on } [t_k^*, \bar{t}_k]. \tag{6.126}$$

An integration on $[t_k^*, \bar{t}_k]$ yields

$$\begin{aligned} y(\bar{t}_k) - y(t_k^*) &= \int_{t_k^*}^{\bar{t}_k} (v(s) - y(s) + \beta(s))\,ds \\ &\le \int_{t_k^*}^{\bar{t}_k} \big(v_2(s) - \big(y(s) - \beta(s)\big)\big)\,ds. \end{aligned} \tag{6.127}$$

Using the fact that β is an upper solution to (6.93), the above inequality yields

$$\begin{aligned} y(\bar{t}_k) - y(t_k^*) &\le \beta(\bar{t}_k) - \beta(t_k^*) - \int_{t_k^*}^{\bar{t}_k} (y(s) - \beta(s))\,ds \\ &< \beta(\bar{t}_k) - \beta(t_k^*). \end{aligned} \tag{6.128}$$

Thus we obtain the contradiction

$$y(\bar{t}_k) - y(t_k^*) < \beta(\bar{t}_k) - \beta(t_k^*). \tag{6.129}$$

Case 2. $\bar{t}_k = t_k^+$, $k = 1, \dots, m$.
Then

$$\beta(t_k^+) < I_k\big(\tau(t_k^-, y(t_k^-))\big) \le \beta(t_k^+) \tag{6.130}$$

which is a contradiction. Thus

$$y(t) \le \beta(t), \quad \forall t \in [0, T]. \tag{6.131}$$

Analogously, we can prove that

$$y(t) \ge \alpha(t), \quad \forall t \in J. \tag{6.132}$$

This shows that the problem (6.96) has a solution in the interval $[\alpha, \beta]$.

Finally, we prove that every solution of (6.96) is also a solution to (6.93). We only need to show that

$$\alpha(0) \le y(0) - L(\overline{y}(0), \overline{y}(T)) \le \beta(0). \tag{6.133}$$

Notice first that we can prove

$$\alpha(T) \le y(T) \le \beta(T). \tag{6.134}$$

Suppose now that $y(0) - L(\overline{y}(0), \overline{y}(T)) < \alpha(0)$. Then $y(0) = \alpha(0)$ and

$$y(0) - L(y(T), \overline{y}(0)) < \alpha(0). \tag{6.135}$$

Since L is nonincreasing in y, we have

$$\alpha(0) \le \alpha(0) - L(\alpha(0), \alpha(T)) \le \alpha(0) - L(\alpha(0), \overline{y}(T)) < \alpha(0) \tag{6.136}$$

which is a contradction. Analogously, we can prove that

$$y(0) - L(\overline{y}(0), \overline{y}(T)) \le \beta(0). \tag{6.137}$$

Then y is a solution to (6.93). □

Remark 6.17. Observe that if $L(x, y) = ax - by - c$, then Theorem 6.15 gives an existence result for the problem

$$\begin{gathered} y'(t) \in F(t, y(t)), \quad t \in J = [0, T],\ t \neq t_k,\ k = 1, \dots, m, \\ y(t_k^+) = I_k(y(t_k^-)), \quad k = 1, \dots, m, \\ ay(0) - by(T) = c, \end{gathered} \tag{6.138}$$

with $a, b \ge 0$, $a + b > 0$, which includes the periodic case ($a = b = 1,\ c = 0$) and the initial and the terminal problems.

6.4. Second-order boundary value problems

In this section, we will be concerned with the existence of solutions of the second-order boundary value problem for the impulsive functional differential inclusion,

$$\begin{gathered} y''(t) \in F(t, y_t), \quad t \in J = [0, T],\ t \neq t_k,\ k = 1, \dots, m, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \\ \Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \\ y(t) = \phi(t), \quad t \in [-r, 0],\ y(T) = y_T, \end{gathered} \tag{6.139}$$

where $F : J \times \mathcal{D} \to \mathcal{P}(E)$ is a given multivalued map with compact and convex values, $\mathcal{D} = \{\psi : [-r, 0] \to E \mid \psi$ is continuous everywhere except for a finite number of points s at which $\psi(s)$ and the right limit $\psi(s^+)$ exist, and $\psi(s^-) = \psi(s)\}$, $\phi \in \mathcal{D}$, $(0 < r < \infty)$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $I_k, \bar{I}_k \in C(E, E)$ $(k = 1, 2, \ldots, m)$ are bounded, $y_T \in E$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $\Delta y'|_{t=t_k} = y'(t_k^+) - y'(t_k^-)$, and $y(t_k^-)$, $y(t_k^+)$, $y'(t_k^-)$, and $y'(t_k^+)$ represent the left and right limits of $y(t)$ and $y'(t)$, respectively, at $t = t_k$, and E is a real separable Banach space with norm $|\cdot|$.

The notations from Section 3.2 are used in the sequel.

Definition 6.18. A function $y \in \Omega \cap AC^1(J', E)$ is said to be a solution of (6.139) if y satisfies the differential inclusion $y''(t) \in F(t, y_t)$ a.e. on J' and the conditions $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $\Delta y'|_{t=t_k} = \bar{I}_k(y(t_k^-))$, $k = 1, \ldots, m$.

In what follows, we will use the notation $\sum_{0<t_k<t}[y(t_k^+) - y(t_k)]$ to mean 0, when $k = 0$ and $0 < t < t_1$, and to mean $\sum_{i=1}^{k}[y(t_k^+) - y(t_k)]$, when $k \geq 1$ and $t_k < t \leq t_{k+1}$.

Theorem 6.19. *Suppose that the following hold.*

(6.16.1) *$F : J \times \mathcal{D} \to \mathcal{P}_{b,cp,cv}(E)$ is an L^1-Carathéodory multivalued map.*

(6.16.2) *There exist constants c_k, d_k such that $|I_k(y)| \leq c_k$, $|\bar{I}_k(y)| \leq d_k$, $k = 1, \ldots, m$, for each $y \in E$.*

(6.16.3) *There exists a function $m \in L^1(J, \mathbb{R}^+)$ such that*

$$\|F(t, u)\| := \sup\{|v| : v \in F(t, u)\} \leq m(t) \tag{6.140}$$

for almost all $t \in J$ and for all $u \in \mathcal{D}$.

(6.16.4) *For each bounded $B \subseteq \Omega$, and for each $t \in J$, the set*

$$\begin{aligned}\Big\{\frac{T-t}{T}\phi(0) + \frac{t}{T}y_T + \int_0^T H(t,s)g(s)\,ds \\ + \sum_{0<t_k<t}\left[I_k(y(t_k)) + (t - t_k)\bar{I}_k(y(t_k))\right] \\ - \frac{t}{T}\sum_{k=1}^{m}\left[I_k(y(t_k)) + (T - t_k)\bar{I}_k(y(t_k))\right] : g \in S_{F,B}\Big\}\end{aligned} \tag{6.141}$$

is relatively compact in E, where $S_{F,B} = \{S_{F,y} : y \in B\}$ and

$$H(t,s) = \begin{cases} \dfrac{t}{T}(s - T), & 0 \leq s \leq t \leq T, \\[2mm] \dfrac{s}{T}(t - T), & 0 \leq t < s \leq T. \end{cases} \tag{6.142}$$

Then the impulsive boundary value problem (6.139) *has at least one solution on* $[-r, T]$.

Proof. Transform the problem (6.139) into a fixed point problem. Consider the multivalued map $G:\Omega\to\mathcal{P}(\Omega)$ defined by

$$G(y)=\left\{h\in\Omega: h(t)=\begin{cases}\phi(t), & t\in[-r,0],\\ \dfrac{T-t}{T}\phi(0)+\dfrac{t}{T}y_T+\displaystyle\int_0^T H(t,s)g(s)ds & \\ \quad+\displaystyle\sum_{0<t_k<t}\left[I_k(y(t_k))\right. & \\ \quad\left.+(t-t_k)\bar{I}_k(y(t_k))\right] & \\ \quad-\dfrac{t}{T}\displaystyle\sum_{k=1}^{m}\left[I_k(y(t_k))\right. & \\ \quad\left.+(T-t_k)\bar{I}_k(y(t_k))\right], & t\in J,\end{cases}\right\} \tag{6.143}$$

where $g\in S_{F,y}$.

Indeed, assume that $y\in\Omega$ is a fixed point of G. It is clear that

$$\begin{gathered}y(t)=\phi(t)\quad\text{for each } t\in[-r,0],\ y(T)=y_T,\\ \Delta y|_{t=t_k}=I_k(y(t_k^-)),\quad k=1,\dots,m.\end{gathered}\tag{6.144}$$

By performing direct differentiation twice, we find

$$\begin{gathered}y'(t)=\frac{-1}{T}\phi(0)+\frac{1}{T}y_T+\int_0^T H_t'(t,s)g(s)ds\\ +\sum_{0<t_k<t}\bar{I}_k(y(t_k))-\frac{1}{T}\sum_{k=1}^{m}\left[I_k(y(t_k))+(T-t_k)\bar{I}_k(y(t_k))\right],\quad t\neq t_k,\\ y''(t)=g(t),\quad t\neq t_k,\end{gathered}\tag{6.145}$$

which imply that y is a solution of BVP (6.139).

We will now show that G satisfies the assumptions of Theorem 1.7. The proof will be given in several steps.

Step 1. $G(y)$ is convex for each $y\in\Omega$.

Indeed, if h_1, h_2 belong to $G(y)$, then there exist $g_1,g_2\in S_{F,y}$ such that, for each $t\in J$, we have

$$\begin{aligned}h_i(t)=\frac{T-t}{T}\phi(0)&+\frac{t}{T}y_T+\int_0^T H(t,s)g_i(s)ds\\ &+\sum_{0<t_k<t}\left[I_k(y(t_k))+(t-t_k)\bar{I}_k(y(t_k))\right]\\ &-\frac{t}{T}\sum_{k=1}^{m}\left[I_k(y(t_k))+(T-t_k)\bar{I}_k(y(t_k))\right],\quad i=1,2.\end{aligned}\tag{6.146}$$

Let $0 \leq d \leq 1$. Then, for each $t \in J$, we have

$$\begin{aligned}
(dh_1 + (1-d)h_2)(t) &= \frac{T-t}{T}\phi(0) + \frac{t}{T}y_T + \int_0^T H(t,s)[dg_1(s) + (1-d)g_2(s)]ds \\
&\quad + \sum_{0<t_k<t} [I_k(y(t_k)) + (t-t_k)\overline{I}_k(y(t_k))] \\
&\quad - \frac{t}{T}\sum_{k=1}^{m} [I_k(y(t_k)) + (T-t_k)\overline{I}_k(y(t_k))].
\end{aligned} \tag{6.147}$$

Since $S_{F,y}$ is convex (because F has convex values), then

$$dh_1 + (1-d)h_2 \in G(y). \tag{6.148}$$

Step 2. G maps bounded sets into bounded sets in Ω.

Indeed, it is enough to show that there exists a positive constant ℓ such that, for each $h \in G(y)$ with $y \in B_q = \{y \in \Omega : \|y\| \leq q\}$, one has $\|h\| \leq \ell$. If $h \in G(y)$, then there exists $g \in S_{F,y}$ such that, for each $t \in J$, we have

$$\begin{aligned}
h(t) &= \frac{T-t}{T}\phi(0) + \frac{t}{T}y_T + \int_0^T H(t,s)g(s)ds \\
&\quad + \sum_{0<t_k<t} [I_k(y(t_k)) + (t-t_k)\overline{I}_k(y(t_k))] \\
&\quad - \frac{t}{T}\sum_{k=1}^{m} [I_k(y(t_k)) + (T-t_k)\overline{I}_k(y(t_k))].
\end{aligned} \tag{6.149}$$

By (6.16.2) and (6.16.3), we have that, for each $t \in J$,

$$\begin{aligned}
|h(t)| &\leq \|\phi\|_{\mathcal{D}} + |y_T| + \sup_{(t,s)\in J\times J} |H(t,s)| \int_0^T |g(s)|ds \\
&\quad + \sum_{0<t_k<t} [|I_k(y(t_k))| + |(t-t_k)||\overline{I}_k(y(t_k))|] \\
&\quad + \sum_{k=1}^{m} [I_k(y(t_k)) + (T-t_k)\overline{I}_k(y(t_k))] \\
&\leq \|\phi\|_{\mathcal{D}} + |y_T| + \sup_{(t,s)\in J\times J} |H(t,s)| \int_0^T g_q(s)ds \\
&\quad + \sum_{k=1}^{m} \Big[2\sup\{|I_k(|y|)| : \|y\| \leq q\} \\
&\qquad + 2(T-t_k)\sup\{|\overline{I}_k(|y|)| : \|y\| \leq q\} \Big] = \ell.
\end{aligned} \tag{6.150}$$

Step 3. G maps bounded sets into equicontinuous sets of Ω.

Let $r_1, r_2 \in J'$, $r_1 < r_2$, and let $B_q = \{y \in \Omega : \|y\| \leq q\}$ be a bounded set of Ω. For each $y \in B_q$ and $h \in G(y)$, there exists $g \in S_{F,y}$ such that

$$\begin{aligned} h(t) = {} & \frac{T-t}{T}\phi(0) + \frac{t}{T}y_T + \int_0^T H(t,s)g(s)ds \\ & + \sum_{0<t_k<t} \left[I_k(y(t_k)) + (t-t_k)\bar{I}_k(y(t_k))\right] \\ & - \frac{t}{T}\sum_{k=1}^{m} \left[I_k(y(t_k)) + (T-t_k)\bar{I}_k(y(t_k))\right]. \end{aligned} \tag{6.151}$$

Thus

$$\begin{aligned} |h(r_2) - h(r_1)| \leq {} & (r_2 - r_1)|\phi(0)| + (r_2 - r_1)\frac{|y_T|}{T} \\ & + \int_0^T |H(r_2,s) - H(r_1,s)| g_q(s)ds \\ & + \sum_{0<t_k<r_2-r_1} \left[I_k(y(t_k)) + (r_2 - r_1)\bar{I}_k(y(t_k))\right] \\ & - \frac{r_2 - r_1}{T}\sum_{k=1}^{m} \left[I_k(y(t_k)) + (T-t_k)\bar{I}_k(y(t_k))\right]. \end{aligned} \tag{6.152}$$

As $r_2 \to r_1$, the right-hand side of the above inequality tends to zero. This proves the equicontinuity for the case where $t \neq t_i$, $i = 1, \dots, m$. The proof of the equicontinuity at $t = t_i$ is similar to that given in Theorem 4.3.

The equicontinuity for the cases $r_1 < r_2 \leq 0$ and $r_1 \leq 0 \leq r_2$ are obvious.

Step 4. G has a closed graph.

Let $y_n \to y_*$, $h_n \in G(y_n)$, and $h_n \to h_*$. We will prove that $h_* \in G(y_*)$.

$h_n \in G(y_n)$ means that there exists $g_n \in S_{F,y_n}$ such that, for each $t \in J$,

$$\begin{aligned} h_n(t) = {} & \frac{T-t}{T}\phi(0) + \frac{t}{T}y_T + \int_0^T H(t,s)g_n(s)ds \\ & + \sum_{0<t_k<t} \left[I_k(y_n(t_k)) + (t-t_k)\bar{I}_k(y_n(t_k))\right] \\ & - \frac{t}{T}\sum_{k=1}^{m} \left[I_k(y_n(t_k)) + (T-t_k)\bar{I}_k(y_n(t_k))\right]. \end{aligned} \tag{6.153}$$

We must prove that there exists $g_* \in S_{F,y_*}$ such that, for each $t \in J$,

$$\begin{aligned} h_*(t) = {} & \frac{T-t}{T}\phi(0) + \frac{t}{T}y_T + \int_0^T H(t,s)g_*(s)ds \\ & + \sum_{0<t_k<t} \left[I_k(y_*(t_k)) + (t-t_k)\overline{I}_k(y_*(t_k))\right] \\ & - \frac{t}{T}\sum_{k=1}^{m} \left[I_k(y_*(t_k)) + (T-t_k)\overline{I}_k(y_*(t_k))\right]. \end{aligned} \tag{6.154}$$

Clearly since I_k and $\overline{I}_k$, $k = 1,\dots,m$, are continuous, we have that

$$\begin{aligned} \Bigg\| & \Bigg(h_n - \frac{T-t}{T}\phi(0) - \frac{t}{T}y_T - \sum_{0<t_k<t}\left[I_k(y_n(t_k)) + (t-t_k)\overline{I}_k(y_n(t_k))\right] \\ & + \frac{t}{T}\sum_{k=1}^{m}\left[I_k(y_n(t_k)) + (T-t_k)\overline{I}_k(y_n(t_k))\right]\Bigg) \\ & - \Bigg(h_* - \frac{T-t}{T}\phi(0) - \frac{t}{T}y_T - \sum_{0<t_k<t}\left[I_k(y_*(t_k)) + (t-t_k)\overline{I}_k(y_*(t_k))\right] \\ & + \frac{t}{T}\sum_{k=1}^{m}\left[I_k(y_*(t_k)) + (T-t_k)\overline{I}_k(y_*(t_k))\right]\Bigg)\Bigg\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{6.155}$$

Consider the linear continuous operator

$$\begin{aligned} \Gamma : L^1(J,E) & \longrightarrow C(J,E), \\ g \longmapsto \Gamma(g)(t) & = \int_0^T H(t,s)g(s)ds. \end{aligned} \tag{6.156}$$

From Lemma 1.28, it follows that $\Gamma \circ S_F$ is a closed graph operator.

Moreover, we have that

$$\begin{aligned} & \Bigg(h_n(t) - \frac{T-t}{T}\phi(0) - \frac{t}{T}y_T - \sum_{0<t_k<t}\left[I_k(y_n(t_k)) + (t-t_k)\overline{I}_k(y_n(t_k))\right] \\ & + \frac{t}{T}\sum_{k=1}^{m}\left[I_k(y_n(t_k)) + (T-t_k)\overline{I}_k(y_n(t_k))\right]\Bigg) \in \Gamma(S_{F,y_n}). \end{aligned} \tag{6.157}$$

Since $y_n \to y_*$, it follows from Lemma 1.28 that

$$\begin{aligned}&\left(h_*(t) - \frac{T-t}{T}\phi(0) - \frac{t}{T}y_T - \sum_{0<t_k<t}\left[I_k(y_*(t_k)) + (t-t_k)\overline{I}_k(y_*(t_k))\right]\right.\\&\left. + \frac{t}{T}\sum_{k=1}^{m}\left[I_k(y_*(t_k)) + (T-t_k)\overline{I}_k(y_*(t_k))\right]\right) = \int_0^T H(t,s)g_*(s)ds\end{aligned} \tag{6.158}$$

for some $g_* \in S_{F,y_*}$.
Step 5. Now it remains to show that the set

$$\mathcal{M} := \{y \in \Omega : \lambda y \in G(y), \text{ for some } \lambda > 1\} \tag{6.159}$$

is bounded.

Let $y \in \mathcal{M}$. Then $\lambda y \in G(y)$ for some $\lambda > 1$. Thus there exists $g \in S_{F,y}$ such that

$$\begin{aligned}y(t) = {}& \lambda^{-1}\frac{T-t}{T}\phi(0) + \lambda^{-1}\frac{t}{T}y_T + \lambda^{-1}\int_0^T H(t,s)g(s)ds\\&+\lambda^{-1}\sum_{0<t_k<t}\left[I_k(y(t_k)) + (t-t_k)\overline{I}_k(y(t_k))\right]\\&-\lambda^{-1}\frac{t}{T}\sum_{k=1}^{m}\left[I_k(y(t_k)) + (T-t_k)\overline{I}_k(y(t_k))\right].\end{aligned} \tag{6.160}$$

This implies by (6.16.2)-(6.16.3) that, for each $t \in J$, we have

$$\begin{aligned}|y(t)| \le {}& \|\phi\|_{\mathcal{D}} + |y_T| + \sup_{(t,s)\in J\times J}|H(t,s)|\int_0^T m(s)ds\\&+\sum_{k=1}^{m}\left[2c_k + 2(T-t_k)d_k\right] = b.\end{aligned} \tag{6.161}$$

This inequality implies that there exists a constant b depending only on T and on the function m such that

$$|y(t)| \le b \quad \text{for each } t \in J. \tag{6.162}$$

Hence

$$\|y\| \le \max(\|\phi\|_{\mathcal{D}}, b). \tag{6.163}$$

This shows that $\mathcal{M}$ is bounded.

Set $X := \Omega$. As a consequence of Theorem 1.7, we deduce that G has a fixed point y which is a solution of (6.139). $\square$

Remark 6.20. We can analogously (with obvious modifications) study the boundary value problem

$$\begin{aligned}
y'' &\in F(t, y_t), \quad t \in J = [0, T],\ t \neq t_k,\ k = 1, \dots, m, \\
\Delta y|_{t=t_k} &= I_k(y(t_k^-)), \quad k = 1, \dots, m, \\
\Delta y'|_{t=t_k} &= \overline{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \\
y(t) &= \phi(t), \quad t \in [-r, 0],\ y'(T) = y_T.
\end{aligned} \tag{6.164}$$

6.5. Notes and remarks

Sections 6.2 and 6.3 are based on upper- and lower-solutions methods for first-order impulsive differential inclusions. The results of Section 6.2, which address periodic multivalued problems, are adapted from Benchohra et al. [50, 59], and the results of Section 6.3, which deal with multivalued impulsive boundary value problems with nonlinear boundary conditions, are adapted from Benchohra et al. [52]. The material of Section 6.4 on second-order impulsive boundary value problems is taken from Benchohra et al. [58].

7 Nonresonance impulsive differential inclusions

7.1. Introduction

This chapter is devoted to impulsive differential inclusions satisfying periodic boundary conditions. These problems are termed as being *nonresonant*, because the linear operator involved will be invertible in the absence of impulses. The first problem addressed concerns first-order problems. A result from [51] that generalizes a paper by Nieto [199] is presented. The methods used involve the Martelli fixed point theorem (Theorem 1.7) and the Covitz-Nadler fixed point theorem (Theorem 1.11).

The second part of the chapter is focused on a second-order problem, and a result of [55] is obtained which is an extension of the first-order result. Again the method used involves an application of Theorem 1.7. Then, the final section of the chapter is a successful extension of these results to nth order nonresonance problems, which were first established in [63]. Also, an initial value function is introduced for the higher-order consideration.

7.2. Nonresonance first-order impulsive functional differential inclusions with periodic boundary conditions

This section is concerned with the existence of solutions for the nonresonance problem for functional differential inclusions with impulsive effects as

$$y'(t) - \lambda y(t) \in F(t, y_t), \quad t \in J = [0, T],\ t \neq t_k,\ k = 1, \dots, m, \tag{7.1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{7.2}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{7.3}$$

$$\phi(0) = y(0) = y(T), \tag{7.4}$$

where $\lambda \neq 0$ and λ is not an eigenvalue of y', $F : J \times \mathcal{D} \to \mathcal{P}(E)$ is a compact convex-valued multivalued map, $\mathcal{D} = \{\psi : [-r, 0] \to E \mid \psi$ is continuous everywhere except for a finite number of points s at which $\psi(s)$ and the right limit $\psi(s^+)$

exist and $\psi(s^-) = \psi(s)\}$, $\phi \in \mathcal{D}$, $(0 < r < \infty)$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $I_k \in C(E,E)$ $(k = 1,2,\dots,m)$ are bounded, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively, and E is a real separable Banach space with norm $|\cdot|$ and $J' = J \setminus \{t_1,\dots,t_k\}$.

Definition 7.1. A function $y \in \Omega \cap \mathrm{AC}(J',E)$ is said to be a solution of (7.1)–(7.4) if y satisfies the inclusion $y'(t) - \lambda y(t) \in F(t,y_t)$ a.e on $J \setminus \{t_1,\dots,t_m\}$ and the conditions $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $k = 1,\dots,m$, and $y(0) = y(T)$.

We now consider the following "linear problem" (7.2), (7.3), (7.4), (7.5), where (7.5) is the equation

$$y'(t) - \lambda y(t) = g(t), \quad t \neq t_k,\ k = 1,\dots,m, \tag{7.5}$$

where $g \in L^1(J_k,E)$, $k = 1,\dots,m$. For short, we will refer to (7.2), (7.3), (7.4), (7.5) as (LP). Note that (LP) is not really a linear problem since the impulsive functions are not necessarily linear. However, if I_k, $k = 1,\dots,m$, are linear, then (LP) is a linear impulsive problem.

We need the following auxiliary result.

Lemma 7.2. *$y \in \Omega \cap \mathrm{AC}(J',E)$ is a solution of* (LP) *if and only if $y \in \Omega \cap \mathrm{AC}(J',E)$ is a solution of the impulsive integral equation*

$$y(t) = \int_0^T H(t,s)g(s)\,ds + \sum_{k=1}^m H(t,t_k)I_k(y(t_k)), \tag{7.6}$$

where

$$H(t,s) = \left(e^{-\lambda T} - 1\right)^{-1}\begin{cases} e^{-\lambda(T+s-t)}, & 0 \le s \le t \le T, \\ e^{-\lambda(s-t)}, & 0 \le t < s \le T. \end{cases} \tag{7.7}$$

Proof. First, suppose that $y \in \Omega \cap \mathrm{AC}(J',E)$ is a solution of (LP). Then

$$y'(t) - \lambda y(t) = g(t), \quad t \neq t_k, \tag{7.8}$$

that is,

$$\left(e^{-\lambda t}y(t)\right)' = e^{-\lambda t}g(t), \quad t \neq t_k. \tag{7.9}$$

Assume that $t_k < t \le t_{k+1}$, $k = 0, \dots, m$. By integration of (7.9), we obtain

$$\begin{aligned} e^{-\lambda t_1} y(t_1) - y(0) &= \int_0^{t_1} e^{-\lambda s} g(s)\,ds, \\ e^{-\lambda t_2} y(t_2) - e^{-\lambda t_1} y(t_1^+) &= \int_{t_1}^{t_2} e^{-\lambda s} g(s)\,ds, \\ &\vdots \\ e^{-\lambda t_k} y(t_k) - e^{-\lambda t_{k-1}} y(t_{k-1}^+) &= \int_{t_{k-1}}^{t_k} e^{-\lambda s} g(s)\,ds, \\ e^{-\lambda t} y(t) - e^{-\lambda t_k} y(t_k^+) &= \int_{t_k}^{t} e^{-\lambda s} g(s)\,ds. \end{aligned} \tag{7.10}$$

Adding these together, we get

$$e^{-\lambda t} y(t) - y(0) = \sum_{0<t_k<t} e^{-\lambda t_k} y(t_k^+) \sum_{0<t_k<t} e^{-\lambda t_k} y(t_k) + \int_0^t e^{-\lambda s} g(s)\,ds, \tag{7.11}$$

that is,

$$e^{-\lambda t} y(t) = y(0) + \sum_{0<t_k<t} e^{-\lambda t_k} I_k(y(t_k)) + \int_0^t e^{-\lambda s} g(s)\,ds. \tag{7.12}$$

In view of (7.12) with $y(0) = y(T)$, we get

$$e^{-\lambda T} y(0) = y(0) + \sum_{k=1}^{m} e^{-\lambda t_k} I_k(y(t_k)) + \int_0^T e^{-\lambda s} g(s)\,ds. \tag{7.13}$$

Hence

$$y(0) = (e^{-\lambda T} - 1)^{-1} \left[\sum_{k=1}^{m} e^{-\lambda t_k} I_k((t_k)) + \int_0^T e^{-\lambda s} g(s)\,ds \right]. \tag{7.14}$$

Substituting (7.14) in (7.12), we obtain

$$\begin{aligned} e^{-\lambda t} y(t) = {} & (e^{-\lambda T} - 1)^{-1} \left[\sum_{k=1}^{m} e^{-\lambda t_k} I_k(y(t_k)) + \int_0^T e^{-\lambda s} g(s)\,ds \right] \\ & + \sum_{0<t_k<t} e^{-\lambda t_k} I_k(y(t_k)) + \int_0^t e^{-\lambda s} g(s)\,ds. \end{aligned} \tag{7.15}$$

Using (7.15) and the fact that

$$\sum_{k=1}^{m} I_k(y(t_k)) = \sum_{0<t_k<T} I_k(y(t_k)) = \sum_{0<t_k<t} I_k(y(t_k)) + \sum_{t\le t_k<T} I_k(y(t_k)), \tag{7.16}$$

we get

$$\begin{aligned}
e^{-\lambda t}y(t) &= (e^{-\lambda T}-1)^{-1}\Bigg[\sum_{0<t_k<t} e^{-\lambda t_k}I_k(y(t_k)) + \sum_{t\le t_k<T} e^{-\lambda t_k}I_k(y(t_k)) \\
&\qquad + \int_0^t e^{-\lambda s}g(s)ds + \int_t^T e^{-\lambda s}g(s)ds \\
&\qquad + (e^{-\lambda T}-1)\sum_{0<t_k<t} e^{-\lambda t_k}I_k(y(t_k)) \\
&\qquad + (e^{-\lambda T}-1)\int_0^t e^{-\lambda s}g(s)ds\Bigg] \\
&= (e^{-\lambda T}-1)^{-1}\Bigg[e^{-\lambda T}\sum_{0<t_k<t} e^{-\lambda t_k}I_k(y(t_k)) + \sum_{t\le t_k<T} e^{-\lambda t_k}I_k(y(t_k)) \\
&\qquad + e^{-\lambda T}\int_0^t e^{-\lambda s}g(s)ds + \int_t^T e^{-\lambda s}g(s)ds\Bigg].
\end{aligned} \tag{7.17}$$

Thus

$$\begin{aligned}
y(t) &= (e^{-\lambda T}-1)^{-1}\Bigg[\int_0^t e^{-\lambda(T+s-t)}g(s)ds + \int_t^T e^{-\lambda(s-t)}g(s)ds \\
&\qquad + \sum_{0<t_k<t} e^{-\lambda(T+t_k-t)}I_k(y(t_k)) + \sum_{t\le t_k<T} e^{-\lambda(t_k-t)}I_k(y(t_k))\Bigg] \\
&= \int_0^T H(t,s)g(s)ds + \sum_{k=1}^m H(t,t_k)I_k(y(t_k)),
\end{aligned} \tag{7.18}$$

that is, y is a solution of (7.6).

Conversely, assume that y is a solution of (7.6). Direct differentiation on (7.6) implies, for $t \neq t_k$,

$$\begin{aligned}
y'(t) &= \int_0^T \frac{\partial H(t,s)}{\partial t}g(s)ds + \sum_{k=1}^m \left[\frac{\partial H(t,t_k)}{\partial t}I_k(y(t_k))\right] \\
&= g(t) + \int_0^T \lambda H(t,s)g(s)ds + \sum_{k=1}^m \lambda H(t,t_k)I_k(y(t_k)) \\
&= g(t) + \lambda\left[\int_0^T H(t,s)g(s)ds + \sum_{k=1}^m H(t,t_k)I_k(y(t_k))\right] \\
&= g(t) + \lambda y(t).
\end{aligned} \tag{7.19}$$

It is easy to see that

$$\Delta\left[\sum_{k=1}^{m} H(t,t_k)I_k\right]\Bigg|_{t=t_k} = I_k. \tag{7.20}$$

Moreover, we have

$$\Delta y|_{t=t_k} = I_k(y(t_k)). \tag{7.21}$$

Making use of the fact $H(0,s) = H(T,s)$ for $s \in J$, we obtain that $y(0) = y(T)$. Hence y is a solution of the impulsive periodic problem (LP). □

We are now in a position to state and prove our existence result for problem (7.1)–(7.4).

Theorem 7.3. *Assume that*

(7.3.1) *$F : J \times \mathcal{D} \longrightarrow \mathcal{P}(E)$ is an L^1-Carathéodory multivalued map;*

(7.3.2) *there exist constants c_k such that $|I_k(y)| \le c_k$, $k = 1,\dots,m$, for each $y \in E$;*

(7.3.3) *there exists $m \in L^1(J,\mathbb{R})$ such that*

$$||F(t,y_t)|| := \sup\{|v| : v \in F(t,y_t)\} \le m(t) \tag{7.22}$$

for almost all $t \in J$ and all $y \in \Omega$;

(7.3.4) *for each bounded $B \subseteq \Omega$ and $t \in J$, the set*

$$\left\{\int_0^T H(t,s)g(s)ds + \sum_{k=1}^{m} H(t,t_k)I_k(y(t_k)) : g \in S_{F,B}\right\} \tag{7.23}$$

is relatively compact in E, where $S_{F,B} = \cup\{S_{F,y} : y \in B\}$.

Then problem (7.1)–(7.4) *has at least one solution on* $[-r,T]$.

Proof. Transform problem (7.1)–(7.4) into a fixed point problem. Consider the multivalued operator $N : \Omega \to \mathcal{P}(\Omega)$ defined by

$$N(y) = \left\{h \in \Omega : h(t) = \begin{cases} y(0) & \text{if } t \in [-r,0], \\ \displaystyle\int_0^T H(t,s)g(s)ds & \\ \displaystyle\quad + \sum_{k=1}^{m} H(t,t_k)I_k(y(t_k)) & \text{if } t \in J, \end{cases}\right\} \tag{7.24}$$

where $g \in S_{F,y}$.

We will show that N satisfies the assumptions of Theorem 1.7. The proof will be given in several steps.

Step 1. $N(y)$ is convex, for each $y \in \Omega$.

Indeed, if h_1, h_2 belong to $N(y)$, then there exist $g_1, g_2 \in S_{F,y}$ such that, for each $t \in J$, we have

$$h_i(t) = \int_0^T H(t,s)g_i(s)ds + \sum_{k=1}^m H(t,t_k)I_k(y(t_k)), \quad i = 1,2. \tag{7.25}$$

Let $0 \le d \le 1$. Then, for each $t \in J$, we have

$$(dh_1 + (1-d)h_2)(t) = \int_0^T H(t,s)[dg_1(s)+(1-d)g_2(s)]ds + \sum_{k=1}^m H(t,t_k)I_k(y(t_k)). \tag{7.26}$$

Since $S_{F,y}$ is convex (because F has convex values), then

$$dh_1 + (1-d)h_2 \in N(y). \tag{7.27}$$

Step 2. N maps bounded sets into bounded sets in Ω.

Indeed, it is enough to show that there exists a positive constant ℓ such that, for each $y \in B_q = \{y \in \Omega : \|y\| \le q\}$, one has $\|N(y)\| \le \ell$.

Let $y \in B_q$ and $h \in N(y)$. Then there exists $g \in S_{F,y}$ such that, for each $t \in J$, we have

$$h(t) = \int_0^T H(t,s)g(s)ds + \sum_{k=1}^m H(t,t_k)I_k(y(t_k)). \tag{7.28}$$

By (7.3.1), we have, for each $t \in J$,

$$\begin{aligned} |h(t)| &\le \int_0^T |H(t,s)|\,|g(s)|\,ds + \sum_{k=1}^m |H(t,t_k)|\,|I_k(y(t_k))| \\ &\le \int_0^T |H(t,s)|\,l_q(s)ds + \sum_{k=1}^m |H(t,t_k)| \sup\{|I_k(|y|)| : \|y\| \le q\}. \end{aligned} \tag{7.29}$$

Then, for each $h \in N(B_q)$, we have

$$\begin{aligned} \|h\|_\Omega \le{} & \sup_{(t,s)\in J\times J} |H(t,s)| \int_0^T l_q(s)ds \\ & + \sum_{k=1}^m \sup_{t\in J} |H(t,t_k)| \sup\{|I_k(|y|)| : \|y\| \le q\} = \ell. \end{aligned} \tag{7.30}$$

Step 3. N maps bounded set into equicontinuous sets of Ω.

Let $\tau_1, \tau_2 \in J'$, $\tau_1 < \tau_2$, and let B_q be a bounded set of Ω as in Step 1. Let $y \in B_q$ and $h \in N(y)$. Then there exists $g \in S_{F,y}$ such that, for each $t \in J$, we have

$$h(t) = \int_0^T H(t,s)g(s)ds + \sum_{k=1}^{m} H(t,t_k)I_k(y(t_k)). \tag{7.31}$$

Then

$$\begin{aligned} |h(\tau_2) - h(\tau_1)| \le &\int_0^T |H(\tau_2,s) - H(\tau_1,s)|\, l_q(s)ds \\ &+ \sum_{k=1}^{m} |H(\tau_2,t_k) - H(\tau_1,t_k)|\,|I_k(y(t_k))|. \end{aligned} \tag{7.32}$$

As $\tau_2 \to \tau_1$, the right-hand side of the above inequality tends to zero. This proves the equicontinuity for the case, where $t \neq t_i$, $i = 1, \ldots, m$. The proof of the equicontinuity at $t = t_i$ is similar to that given in Theorem 4.3.

As a consequence of Steps 1–3, and (7.3.4) together with the Arzelá-Ascoli theorem, we can conclude that $N : \Omega \to \mathcal{P}(\Omega)$ is completely continuous multivalued, and therefore a condensing map.

Step 4. N has a closed graph.

Let $y_n \to y_*$, $h_n \in N(y_n)$, and $h_n \to h_*$. We will prove that $h_* \in N(y_*)$.

$h_n \in N(y_n)$ means that there exists $g_n \in S_{F,y_n}$ such that, for each $t \in J$,

$$h_n(t) = \int_0^T H(t,s)g_n(s)ds + \sum_{k=1}^{m} H(t,t_k)I_k(y_n(t_k)). \tag{7.33}$$

We must prove that there exists $g_* \in S_{F,y_*}$ such that, for each $t \in J$,

$$h_*(t) = \int_0^T H(t,s)g_*(s)ds + \sum_{k=1}^{m} H(t,t_k)I_k(y_*(t_k)). \tag{7.34}$$

Clearly since I_k, $k = 1, \ldots, m$, are continuous, we have that

$$\left\| \left(h_n - \sum_{k=1}^{m} H(t,t_k)I_k(y_n(t_k)) \right) \left(h_* - \sum_{k=1}^{m} H(t,t_k)I_k(y_*(t_k)) \right) \right\| \longrightarrow 0, \tag{7.35}$$

as $n \to \infty$. Consider the linear continuous operator

$$\begin{gathered} \Gamma : L^1(J,E) \longrightarrow C(J,E), \\ g \longmapsto \Gamma(g)(t) = \int_0^T H(t,s)g(s)ds. \end{gathered} \tag{7.36}$$

From Lemma 1.28, it follows that $\Gamma \circ S_F$ is a closed graph operator.

Moreover, we have that

$$\left(h_n(t)-\sum_{k=1}^{m}H(t,t_k)I_k(y_n(t_k))\right)\in\Gamma(S_{F,y_n}). \tag{7.37}$$

Since $y_n\to y_*$, it follows from Lemma 1.28 that

$$\left(h_*(t)-\sum_{k=1}^{m}H(t,t_k)I_k(y_*(t_k))\right)=\int_0^T H(t,s)g_*(s)ds \tag{7.38}$$

for some $g_*\in S_{F,y_*}$.

Step 5. Now it remains to show that the set

$$\mathcal{M}:=\{y\in\Omega:\lambda y\in N(y),\ \text{for some }\lambda>1\} \tag{7.39}$$

is bounded.

Let $y\in\mathcal{M}$. Then $y\in\lambda N(y)$ for some $0<\lambda<1$. Thus, for each $t\in J$,

$$y(t)=\lambda\int_0^T H(t,s)g(s)ds+\lambda\sum_{k=1}^{m}H(t,t_k)I_k(y(t_k)). \tag{7.40}$$

This implies by (7.3.2)-(7.3.3) that, for each $t\in J$, we have

$$\begin{aligned}|y(t)|&\le\int_0^T|H(t,s)g(s)|ds+\sum_{k=1}^{m}|H(t,t_k)||I_k(y(t_k))|\\&\le\sup_{(t,s)\in J\times J}|H(t,s)|\int_0^T m(s)ds+\sum_{k=1}^{m}\sup_{t\in J}|H(t,t_k)|c_k:=b,\end{aligned} \tag{7.41}$$

where b depends only on T and on the function m. This shows that $\mathcal{M}$ is bounded.

Set $X:=\Omega$. As a consequence of Theorem 1.7, we deduce that N has a fixed point which is a solution of (7.1)–(7.4). □

Theorem 7.4. *Assume the following conditions are satisfied:*

(7.4.1) $F:[0,T]\times\mathcal{D}\to\mathcal{P}_{\mathrm{cp,cv}}(E)$ *has the property that* $F(\cdot,u):[0,T]\to\mathcal{P}_{\mathrm{cp}}(E)$ *is measurable, for each* $u\in\mathcal{D}$;

(7.4.2) *there exists* $l\in L^1([0,T],\mathbb{R}^+)$ *such that*

$$H_d(F(t,u),F(t,\overline{u}))\le l(t)\|u-\overline{u}\|_{\mathcal{D}}, \tag{7.42}$$

for each $t\in[0,T]$ *and* $u,\overline{u}\in\mathcal{D}$, *and*

$$d(0,F(t,0))\le l(t),\quad \textit{for almost each } t\in J; \tag{7.43}$$

(7.4.3) $|I_k(y)-I_k(\overline{y})|\le c_k\|y-\overline{y}\|_{\mathcal{D}}$, *for each* $y,\overline{y}\in E$, $k=1,\dots,m$, *where* c_k *are nonnegative constants.*

Let $h_0 = \sup_{(t,s)\in J\times J} |H(t,s)|$ *and* $l^* = \int_0^T l(t)\,\mathrm{dt}$. *If*

$$h_0 l^* + h_0 \sum_{k=1}^{m} c_k < 1, \tag{7.44}$$

then problem (7.1)–(7.4) *has at least one solution on* $[-r,T]$.

Proof. Transform problem (7.1)–(7.4) into a fixed point problem. It is clear from Lemma 7.2 that solutions of problem (7.1)–(7.4) are fixed points of the multivalued operator $N : \Omega \to \mathcal{P}(\Omega)$ defined by

$$N(y) := \left\{ h \in \Omega : h(t) = \begin{cases} y(0) & \text{if } t \in [-r,0], \\ \displaystyle\int_0^T H(t,s)v(s)ds & \\ \displaystyle\quad + \sum_{k=1}^{m} H(t,t_k)I_k(y(t_k^-)) & \text{if } t \in J, \end{cases} \right\} \tag{7.45}$$

where $v \in S_{F,y}$.

We will show that N satisfies the assumptions of Theorem 1.11. The proof will be given in two steps.

Step 1. $N(y) \in P_{\text{cl}}(\Omega)$, for each $y \in \Omega$.

Indeed, let $(y_n)_{n\geq 0} \in N(y)$ such that $y_n \to \tilde{y}$ in Ω. Then $\tilde{y} \in \Omega$ and, for each $t \in J$,

$$y_n(t) \in \int_0^T H(t,s)F(s,y_s)ds + \sum_{k=1}^{m} H(t,t_k)I_k(y(t_k^-)). \tag{7.46}$$

Using the fact that F has compact values and from (7.4.2), we may pass to a subsequence, if necessary, to get that g_n converges to g in $L^1(J,E)$, and hence $g \in S_{F,y}$. Then, for each $t \in [0,b]$,

$$y_n(t) \longrightarrow \tilde{y}(t) = \int_0^T H(t,s)F(s,y_s)ds + \sum_{k=1}^{m} H(t,t_k)I_k(y(t_k^-)). \tag{7.47}$$

So, $\tilde{y} \in N(y)$.

Step 2. $H(N(y),N(\overline{y})) \leq \gamma\|y-\overline{y}\|$, for each $y,\overline{y} \in \Omega$ (where $\gamma < 1$).

Let $y,\overline{y} \in \Omega$ and $h_1 \in N(y)$. Then there exists $v_1(t) \in F(t,y_t)$ such that, for each $t \in J$,

$$h_1(t) = \int_0^T H(t,s)v_1(s)ds + \sum_{k=1}^{m} H(t,t_k)I_k(y(t_k^-)). \tag{7.48}$$

From (7.4.2), it follows that

$$H(F(t,y_t),F(t,\overline{y}_t)) \leq l(t)\|y_t - \overline{y}_t\|_{\mathcal{D}}, \quad t \in J. \tag{7.49}$$

Hence there is $w \in F(t, \overline{y}_t)$ such that

$$|v_1(t) - w| \leq l(t)\|y_t - \overline{y}_t\|_{\mathcal{D}}, \quad t \in J. \tag{7.50}$$

Consider $U : J \to \mathcal{P}(E)$, given by

$$U(t) = \{w \in E : |v_1(t) - w| \leq l(t)\|y_t - \overline{y}_t\|_{\mathcal{D}}\}. \tag{7.51}$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \overline{y}_t)$ is measurable (see [119, Proposition III.4]), there exists $v_2(t)$, which is a measurable selection for V. So, $v_2(t) \in F(t, \overline{y}_t)$ and

$$|v_1(t) - v_2(t)| \leq l(t)\|y - \overline{y}\|_{\mathcal{D}}, \quad \text{for each } t \in J. \tag{7.52}$$

Let us define, for each $t \in J$,

$$h_2(t) = \int_0^T H(t,s)v_2(s)ds + \sum_{k=1}^{m} H(t,t_k)I_k(\overline{y}(t_k^-)). \tag{7.53}$$

Then we have

$$\begin{aligned} |h_1(t) - h_2(t)| &\leq \int_0^T |H(t,s)|\,|v_1(s) - v_2(s)|\,ds \\ &\quad + \sum_{k=1}^{m} |H(t,t_k)|\,|I_k(y(t_k^-)) - I_k(\overline{y}(t_k^-))| \\ &\leq h_0 \int_0^T l(s)\|y_s - \overline{y}_s\|_{\mathcal{D}}ds + h_0 \sum_{k=1}^{m} c_k |y(t_k^-) - \overline{y}(t_k^-)| \\ &\leq h_0 l^* \|y - \overline{y}\| + h_0 \sum_{k=1}^{m} c_k \|y - \overline{y}\|. \end{aligned} \tag{7.54}$$

Then

$$\|h_1 - h_2\|_{\Omega} \leq \left[h_0 l^* + h_0 \sum_{k=1}^{m} c_k\right]\|y - \overline{y}\|. \tag{7.55}$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H(N(y),N(\overline{y})) \leq \left[h_0 l^* + h_0 \sum_{k=1}^{m} c_k\right]\|y-\overline{y}\|. \tag{7.56}$$

So, N is a contraction and thus, by Theorem 1.11, N has a fixed point y, which is a solution to (7.1)–(7.4). □

7.3. Nonresonance second-order impulsive functional differential inclusions with periodic boundary conditions

This section is concerned with the existence of solutions for the nonresonance problem, for functional differential inclusions, with impulsive effects,

$$y''(t) - \lambda y(t) \in F(t, y_t), \quad t \in J = [0,T],\ t \neq t_k,\ k = 1,\dots,m, \tag{7.57}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m, \tag{7.58}$$

$$\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1,\dots,m, \tag{7.59}$$

$$y(t) = \phi(t), \quad t \in [-r,0], \tag{7.60}$$

$$y(0) - y(T) = \mu_0, \qquad y'(0) - y'(T) = \mu_1, \tag{7.61}$$

where $F : J \times \mathcal{D} \to \mathcal{P}(\mathbb{R})$ is a compact convex-valued multivalued map, $(0 < r < \infty)$, $\lambda \neq 0$ and λ is not an eigenvalue of y'', $\mu_0, \mu_1 \in \mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $I_k, \overline{I}_k \in C(\mathbb{R},\mathbb{R})$ $(k = 1,2,\dots,m)$ are bounded, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $\Delta y'|_{t=t_k} = y'(t_k^+) - y'(t_k^-)$, $y(t_k^-)$, $y(t_k^+)$, $y'(t_k^-)$, and $y'(t_k^+)$ represent the left and right limits of $y(t)$ and $y'(t)$, respectively, at $t = t_k$.

Note that when $\mu_0 = \mu_1 = 0$, we have periodic boundary conditions.

Definition 7.5. A function $y \in \Omega \cap AC^1(J',\mathbb{R})$ is said to be a solution of problem (7.57)–(7.61) if y satisfies conditions (7.57) to (7.61).

We now consider the "linear problem" (7.58), (7.59), (7.60), (7.61), (7.62), where (7.62) is the equation

$$y''(t) - \lambda y(t) = g(t), \quad t \neq t_k,\ k = 1,\dots,m, \tag{7.62}$$

where $g \in L^1(J_k,\mathbb{R})$, $k = 1,\dots,m$. For brevity, we will refer to (7.58), (7.59), (7.60), (7.61), (7.62) as (LP). Note that (LP) is not really a linear problem since the impulsive functions are not necessarily linear. However, if I_k, $\overline{I}_k$, $k = 1,\dots,m$ are linear, then (LP) is a linear impulsive problem.

We need the following auxiliary result.

Lemma 7.6. *$y \in \Omega \cap AC^1(J', \mathbb{R})$ is a solution of* (LP) *if and only if $y \in \Omega$ is a solution of the impulsive integral equation*

$$y(t) = \begin{cases} y(0), & t \in [-r, 0], \\ \displaystyle\int_0^T H(t,s)g(s)ds + H(t,0)\mu_1 + L(t,0)\mu_0 & \\ \quad \displaystyle + \sum_{k=1}^{m} \big[H(t,t_k)I_k(y(t_k)) & \\ \qquad + L(t,t_k)\bar{I}_k(y(t_k))\big], & t \in J, \end{cases} \tag{7.63}$$

where

$$\begin{gathered} H(t,s) = \frac{-1}{2\sqrt{\lambda}(e^{\sqrt{\lambda}T} - 1)} \begin{cases} e^{\sqrt{\lambda}(T+s-t)} + e^{\sqrt{\lambda}(t-s)}, & 0 \le s \le t \le T, \\ e^{\sqrt{\lambda}(T+t-s)} + e^{\sqrt{\lambda}(s-t)}, & 0 \le t < s \le T, \end{cases} \\ L(t,s) = \frac{\partial}{\partial t} H(t,s) = \frac{1}{2(e^{\sqrt{\lambda}T} - 1)} \begin{cases} e^{\sqrt{\lambda}(T+s-t)} - e^{\sqrt{\lambda}(t-s)}, & 0 \le s \le t \le T, \\ e^{\sqrt{\lambda}(s-t)} - e^{\sqrt{\lambda}(T+t-s)}, & 0 \le t < s \le T. \end{cases} \end{gathered} \tag{7.64}$$

Proof. We omit the proof since it is simple. □

We are now in a position to state and prove our existence result for problem (7.57)–(7.61).

Theorem 7.7. *Assume that (7.3.1)–(7.3.3) hold. Moreover, assume that*

(7.7.1) *there exist constants d_k such that $|\bar{I}_k(y)| \le d_k$, $k = 1, \dots, m$, for each $y \in \mathbb{R}$.*

Then problem (7.57)–(7.61) *has at least one solution on* $[-r, T]$.

Proof. Transform the problem (7.57)–(7.61) into a fixed point problem. Consider the multivalued operator $N : \Omega \to \mathcal{P}(\Omega)$ defined by

$$N(y) = \left\{ h \in \Omega : h(t) = \begin{cases} y(0), & t \in [-r, 0], \\ \displaystyle\int_0^T H(t,s)g(s)ds + H(t,0)\mu_1 + L(t,0)\mu_0 & \\ \quad \displaystyle + \sum_{k=1}^{m} \big[H(t,t_k)I_k(y(t_k)) & \\ \qquad + L(t,t_k)\bar{I}_k(y(t_k))\big], & t \in J, \end{cases} \right\} \tag{7.65}$$

where $g \in S_{F,y}$.

We will show that N satisfies the assumptions of Theorem 1.7. The proof will be given in several steps.

Step 1. $N(y)$ is convex, for each $y \in \Omega$.

Indeed, if h_1, h_2 belong to $N(y)$, then there exist $g_1, g_2 \in S_{F,y}$ such that, for each $t \in J$, we have

$$\begin{aligned} h_i(t) = &\int_0^T H(t,s)g_i(s)ds + H(t,0)\mu_1 + L(t,0)\mu_0 \\ &+ \sum_{k=1}^{m} \left[H(t,t_k)I_k(y(t_k)) + L(t,t_k)\overline{I}_k(y(t_k))\right], \quad i = 1,2. \end{aligned} \tag{7.66}$$

Let $0 \le d \le 1$. Then, for each $t \in J$, we have

$$\begin{aligned} (dh_1 + (1-d)h_2)(t) = &\int_0^T H(t,s)[dg_1(s) + (1-d)g_2(s)]ds + H(t,0)\mu_1 + L(t,0)\mu_0 \\ &+ \sum_{k=1}^{m} \left[H(t,t_k)I_k(y(t_k)) + L(t,t_k)\overline{I}_k(y(t_k))\right]. \end{aligned} \tag{7.67}$$

Since $S_{F,y}$ is convex (because F has convex values), then

$$dh_1 + (1-d)h_2 \in N(y). \tag{7.68}$$

Step 2. N maps bounded sets into bounded sets in Ω.

Indeed, it is enough to show that there exists a positive constant ℓ such that, for each $y \in B_q = \{y \in \Omega : \|y\| \le q\}$, one has $\|N(y)\| \le \ell$.

Let $y \in B_q$ and $h \in N(y)$. Then there exists $g \in S_{F,y}$ such that, for each $t \in J$, we have

$$\begin{aligned} h(t) = &\int_0^T H(t,s)g(s)ds + H(t,0)\mu_1 + L(t,0)\mu_0 \\ &+ \sum_{k=1}^{m} \left[H(t,t_k)I_k(y(t_k)) + L(t,t_k)\overline{I}_k(y(t_k))\right]. \end{aligned} \tag{7.69}$$

By (7.7.1), we have, for each $t \in J$,

$$\begin{aligned} |h(t)| \le &\int_0^T |H(t,s)||g(s)|ds + |H(t,0)||\mu_1| + |L(t,0)||\mu_0| \\ &+ \sum_{k=1}^{m} \left|H(t,t_k)I_k(y(t_k)) + L(t,t_k)\overline{I}_k(y(t_k))\right| \end{aligned}$$

$$\begin{aligned}
&\leq \int_0^T |H(t,s)|\,l_q(s)ds + |H(t,0)|\,|\mu_1| + |L(t,0)|\,|\mu_0| \\
&\quad + \sum_{k=1}^{m} \big[\,|H(t,t_k)| \sup\{|I_k(|y|)| : \|y\| \leq q\} \\
&\qquad\qquad + |L(t,t_k)| \sup\{|\bar{I}_k(|y|)| : \|y\| \leq q\}\big].
\end{aligned} \tag{7.70}$$

Then, for each $h \in N(B_q)$, we have

$$\begin{aligned}
\|h\|_\Omega \leq{}& \sup_{(t,s)\in J\times J} |H(t,s)| \int_0^T l_q(s)ds \\
&+ |\mu_1| \sup_{t\in J} |H(t,0)| + |\mu_0| \sup_{t\in J} |L(t,0)| \\
&+ \sum_{k=1}^{m} \Big[\sup_{t\in J} |H(t,t_k)| \sup\{|I_k(|y|)| : \|y\| \leq q\} \\
&\qquad + \sup_{t\in J} |L(t,t_k)| \sup\{|\bar{I}_k(|y|)| : \|y\| \leq q\}\Big] = \ell.
\end{aligned} \tag{7.71}$$

Step 3. N maps bounded sets into equicontinuous sets of Ω.

Let $\tau_1, \tau_2 \in J'$, $\tau_1 < \tau_2$, and let B_q be a bounded set of Ω as in Step 2. Let $y \in B_q$ and $h \in N(y)$. Then there exists $g \in S_{F,y}$ such that, for each $t \in J$, we have

$$\begin{aligned}
h(t) ={}& \int_0^T H(t,s)g(s)ds + H(t,0)\mu_1 + L(t,0)\mu_0 \\
&+ \sum_{k=1}^{m} [H(t,t_k)I_k(y(t_k)) + L(t,t_k)\bar{I}_k(y(t_k))].
\end{aligned} \tag{7.72}$$

Then

$$\begin{aligned}
|h(\tau_2) - h(\tau_1)| \leq{}& \int_0^T |H(\tau_2,s) - H(\tau_1,s)|\,l_q(s)ds \\
&+ |H(\tau_2,0) - H(\tau_1,0)|\,|\mu_1| + |L(\tau_2,0) - L(\tau_1,0)|\,|\mu_0| \\
&+ \sum_{k=1}^{m} [\,|H(\tau_2,t_k) - H(\tau_1,t_k)|\,c_k + |L(\tau_2,t_k) - L(\tau_1,t_k)|\,d_k].
\end{aligned} \tag{7.73}$$

As $\tau_2 \to \tau_1$, the right-hand side of the above inequality tends to zero. This proves the equicontinuity for the case where $t \neq t_i$, $i = 1,\dots,m$. The proof of the equicontinuity at $t = t_i$ is similar to that given in Theorem 4.3. The equicontinuity for the cases $\tau_1 < \tau_2 \leq 0$ or $\tau_1 \leq 0 \leq \tau_2$ are obvious.

As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $N : \Omega \to \mathcal{P}(\Omega)$ is completely continuous multivalued, and therefore a condensing multivalued map.

Step 4. N has a closed graph.

Let $y_n \to y_*$, $h_n \in N(y_n)$, and $h_n \to h_*$. We will prove that $h_* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $g_n \in S_{F,y_n}$ such that, for each $t \in J$,

$$\begin{aligned} h_n(t) = &\int_0^T H(t,s)g_n(s)ds + H(t,0)\mu_1 + L(t,0)\mu_0 \\ &+ \sum_{k=1}^{m} \big[H(t,t_k)I_k(y_n(t_k)) + L(t,t_k)\bar{I}_k(y_n(t_k))\big]. \end{aligned} \tag{7.74}$$

We must prove that there exists $g_* \in S_{F,y_*}$ such that, for each $t \in J$,

$$\begin{aligned} h_*(t) = &\int_0^T H(t,s)g_*(s)ds + H(t,0)\mu_1 + L(t,0)\mu_0 \\ &+ \sum_{k=1}^{m} \big[H(t,t_k)I_k(y_*(t_k)) + L(t,t_k)\bar{I}_k(y_*(t_k))\big]. \end{aligned} \tag{7.75}$$

Clearly since $I_k, \bar{I}_k$, $k = 1,\dots,m$, are continuous, we have that

$$\begin{aligned} &\Bigg\|\bigg(h_n - H(t,0)\mu_1 - L(t,0)\mu_0 - \sum_{k=1}^{m}\big[H(t,t_k)I_k(y_n(t_k)) - L(t,t_k)\bar{I}_k(y_n(t_k))\big]\bigg) \\ &- \bigg(h_* - H(t,0)\mu_1 - L(t,0)\mu_0 \\ &\quad - \sum_{k=1}^{m}\big[H(t,t_k)I_k(y_*(t_k)) - L(t,t_k)\bar{I}_k(y_*(t_k))\big]\bigg)\Bigg\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{7.76}$$

Consider the linear continuous operator

$$\begin{gathered} \Gamma : L^1(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R}), \\ g \longmapsto \Gamma(g)(t) = \int_0^T H(t,s)g(s)ds. \end{gathered} \tag{7.77}$$

From Lemma 1.28, it follows that $\Gamma \circ S_F$ is a closed graph operator.

Moreover, we have that

$$\begin{aligned} &h_n(t) - H(t,0)\mu_1 - L(t,0)\mu_0 \\ &\quad - \sum_{k=1}^{m}\big[H(t,t_k)I_k(y_n(t_k)) - L(t,t_k)\bar{I}_k(y_n(t_k))\big] \in \Gamma(S_{F,y_n}). \end{aligned} \tag{7.78}$$

Since $y_n \to y_*$, it follows from Lemma 1.28 that

$$\begin{aligned} &h_*(t) - H(t,0)\mu_1 - L(t,0)\mu_0 \\ &\quad - \sum_{k=1}^{m} \left[H(t,t_k)I_k(y_*(t_k)) - L(t,t_k)\bar{I}_k(y_*(t_k))\right] = \int_0^T H(t,s)g_*(s)ds \end{aligned} \tag{7.79}$$

for some $g_* \in S_{F,y_*}$.

Step 5. Now it remains to show that the set

$$\mathcal{M} := \{y \in \Omega : \beta y \in N(y), \text{ for some } \beta > 1\} \tag{7.80}$$

is bounded.

Let $y \in \mathcal{M}$. Then $\beta y \in N(y)$ for some $\beta > 1$. Thus, for each $t \in J$,

$$\begin{aligned} y(t) &= \beta^{-1}\int_0^T H(t,s)g(s)ds + \beta^{-1}H(t,0)\mu_1 + \beta^{-1}L(t,0)\mu_0 \\ &\quad + \beta^{-1}\sum_{k=1}^{m}\left[H(t,t_k)I_k(y(t_k)) + L(t,t_k)\bar{I}_k(y(t_k))\right]. \end{aligned} \tag{7.81}$$

This implies by (7.7.1) that, for each $t \in J$, we have

$$\begin{aligned} |y(t)| &\le \int_0^T |H(t,s)g(s)|ds + |H(t,0)||\mu_1| + |L(t,0)||\mu_0| \\ &\quad + \sum_{k=1}^{m} |H(t,t_k)I_k(y(t_k)) + L(t,t_k)\bar{I}_k(y(t_k))| \\ &\le \sup_{(t,s)\in J\times J}|H(t,s)|\int_0^T m(s)ds + |H(t,0)||\mu_1| + |L(t,0)||\mu_0| \\ &\quad + \sum_{k=1}^{m}|H(t,t_k)c_k + L(t,t_k)d_k|. \end{aligned} \tag{7.82}$$

Thus

$$\begin{aligned} \|y\|_\Omega &\le \sup_{(t,s)\in J\times J}|H(t,s)|\int_0^T m(s)ds + \sup_{t\in J}|H(t,0)||\mu_1| \\ &\quad + \sup_{t\in J}|L(t,0)||\mu_0| + \sum_{k=1}^{m}\left[\sup_{t\in J}|H(t,t_k)|c_k + \sup_{t\in J}|L(t,t_k)|d_k\right] := b, \end{aligned} \tag{7.83}$$

where b depends only on T and on the function m. This shows that $\mathcal{M}$ is bounded.

Set $X := \Omega$. As a consequence of Theorem 1.7, we deduce that N has a fixed point which is a solution of (7.57)–(7.61). □

Theorem 7.8. *Assume that (7.4.1)–(7.4.3) and the following are satisfied:*

(7.8.1) $|\overline{I}_k(y) - \overline{I}_k(\overline{y})| \leq d_k|y(t) - \overline{y}(t)|$, *for each* $y, \overline{y} \in \mathbb{R}$, $k = 1, \dots, m$, *where d_k are nonnegative constants.*

Let $m_0 = \sup_{(t,s)\in J\times J} |H(t,s)|$, $l_0 = \sup_{(t,s)\in J\times J} |L(t,s)|$. *If*

$$m_0 l^* + m_0 \sum_{k=1}^{m} c_k + l_0 \sum_{k=1}^{m} d_k < 1, \tag{7.84}$$

then problem (7.57)–(7.61) *has at least one solution on* $[-r, T]$.

Proof. Transform problem (7.57)–(7.61) into a fixed point problem. It is clear that the solutions of problem (7.57)–(7.61) are fixed points of the multivalued operator $\overline{N} : \Omega \to \mathcal{P}(\Omega)$ defined by

$$\overline{N}(y) := \left\{ h \in \Omega : h(t) = \begin{cases} y(0) & \text{if } t \in [-r, 0], \\ \displaystyle\int_0^T H(t,s)v(s)ds + H(t,0)\mu_1 + L(t,0)\mu_0 & \\ \displaystyle\quad + \sum_{k=1}^{m} [H(t,t_k)I_k(y(t_k)) & \\ \qquad + L(t,t_k)\overline{I}_k(y(t_k))] & \text{if } t \in J, \end{cases} \right\} \tag{7.85}$$

where $v \in S_{F,y}$.

We will show that $\overline{N}$ satisfies the assumptions of Theorem 1.11. The proof will be given in two steps.

Step 1. $\overline{N}(y) \in P_{\mathrm{cl}}(\Omega)$, for each $y \in \Omega$.

The proof is similar to that of Step 1 of Theorem 7.4.

Step 2. $H(\overline{N}(y), \overline{N}(\overline{y})) \leq \gamma\|y - \overline{y}\|$, for each $y, \overline{y} \in \Omega$ (where $\gamma < 1$).

Let $y, \overline{y} \in \Omega$ and $h_1 \in \overline{N}(y)$. Then there exists $v_1(t) \in F(t, y_t)$ such that, for each $t \in J$,

$$\begin{aligned} h_1(t) = &\int_0^T H(t,s)v_1(s)ds + H(t,0)\mu_1 + L(t,0)\mu_0 \\ &+ \sum_{k=1}^{m} [H(t,t_k)I_k(y(t_k)) + L(t,t_k)\overline{I}_k(y(t_k))]. \end{aligned} \tag{7.86}$$

From (7.4.2), it follows that

$$H(F(t, y_t), F(t, \overline{y}_t)) \leq l(t)\|y_t - \overline{y}_t\|_{\mathcal{D}}, \quad t \in J. \tag{7.87}$$

Hence there is $w \in F(t, \overline{y}_t)$ such that

$$|v_1(t) - w| \leq l(t)\|y_t - \overline{y}_t\|_{\mathcal{D}}, \quad t \in J. \tag{7.88}$$

Consider $U : J \to \mathcal{P}(\mathbb{R})$, given by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq l(t)\|y_t - \overline{y}_t\|_{\mathcal{D}}\}. \tag{7.89}$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \overline{y}_t)$ is measurable (see [119, Proposition III.4]), there exists $v_2(t)$, which is a measurable selection for V. So, $v_2(t) \in F(t, \overline{y}_t)$ and

$$|v_1(t) - v_2(t)| \leq l(t)\|y - \overline{y}\|_{\mathcal{D}}, \quad \text{for each } t \in J. \tag{7.90}$$

Let us define, for each $t \in J$,

$$\begin{aligned} h_2(t) = & \int_0^T H(t,s)v_2(s)ds + H(t,0)\mu_1 + L(t,0)\mu_0 \\ & + \sum_{k=1}^{m} [H(t,t_k)I_k(\overline{y}(t_k)) + L(t,t_k)\overline{I}_k(\overline{y}(t_k))]. \end{aligned} \tag{7.91}$$

Then we have

$$\begin{aligned} |h_1(t) - h_2(t)| \leq & \int_0^T |H(t,s)||v_1(s) - v_2(s)|ds \\ & + \sum_{k=1}^{m} |H(t,t_k)||I_k(y(t_k^-)) - I_k(\overline{y}(t_k^-))| \\ & + \sum_{k=1}^{m} |L(t,t_k)||\overline{I}_k(y(t_k^-)) - \overline{I}_k(\overline{y}(t_k^-))| \\ \leq & \, m_0 \int_0^T l(s)\|y_s - \overline{y}_s\|_{\mathcal{D}} ds + m_0 \sum_{k=1}^{m} c_k |y(t_k^-) - \overline{y}(t_k^-)| \\ & + l_0 \sum_{k=1}^{m} d_k |y(t_k^-) - \overline{y}(t_k^-)| \\ \leq & \, m_0 l^* \|y - \overline{y}\| + m_0 \sum_{k=1}^{m} c_k \|y - \overline{y}\| + l_0 \sum_{k=1}^{m} d_k \|y - \overline{y}\|. \end{aligned} \tag{7.92}$$

Then

$$||h_1 - h_2|| \le \left[m_0 l^* + m_0 \sum_{k=1}^{m} c_k + l_0 \sum_{k=1}^{m} d_k \right] \| y - \overline{y} \|. \tag{7.93}$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H(\overline{N}(y), \overline{N}(\overline{y})) \le \left[m_0 l^* + m_0 \sum_{k=1}^{m} c_k + l_0 \sum_{k=1}^{m} d_k \right] \| y - \overline{y} \|. \tag{7.94}$$

So, $\overline{N}$ is a contraction and thus, by Theorem 1.11, $\overline{N}$ has a fixed point y, which is a solution to (7.57)–(7.61). □

7.4. Nonresonance higher-order boundary value problems for impulsive functional differential inclusions

In the interval $J = [0, T]$, let $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$ be fixed. In this section, we are concerned with the existence of solutions for a nonresonance problem for the functional differential inclusion,

$$y^{(n)}(t) - \lambda y(t) \in F(t, y_t), \quad t \in J \setminus \{t_1, \ldots, t_m\}, \tag{7.95}$$

subject to the impulse effects

$$\Delta y^{(i)}(t_k) = I_k^i(y(t_k^-)), \quad 0 \le i \le n-1,\ 1 \le k \le m, \tag{7.96}$$

satisfying the initial condition

$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{7.97}$$

and satisfying the boundary conditions

$$y^{(i)}(0) - y^{(i)}(T) = \mu_i, \quad 0 \le i \le n-1, \tag{7.98}$$

where $F : J \times \mathcal{D} \to P(\mathbb{R})$ is a compact convex-valued multivalued map, $P(\mathbb{R})$ is the power set of $\mathbb{R}$, $\lambda \neq 0$ and λ is not an eigenvalue of y^n, $\mu_i \in \mathbb{R}$, $0 \le i \le n-1$, $I_k^i \in C(\mathbb{R}, \mathbb{R})$ are bounded, $0 \le i \le n-1$, $1 \le k \le m$, and $\Delta y^{(i)}(t_k) = \Delta y^{(i)}(t_k^+) - \Delta y^{(i)}(t_k^-)$, $0 \le i \le n-1$. As usual, for any continuous function y defined on $[-r, T] \setminus \{t_1, \ldots, t_m\}$ and any $t \in J$, we define $y_t \in \mathcal{D}$ by $y_t(\theta) = y(t + \theta), \theta \in [-r, 0]$.

We now define what we mean by a solution of problem (7.95)–(7.98).

Definition 7.9. A function $y \in \Omega \cap \mathrm{AC}^{n-1}(t_k, t_{k+1})$, $k = 0, \ldots, m$, is said to be a solution of problem (7.95)–(7.98) if y satisfies conditions (7.95) to (7.98).

Next, let $G(t,s)$ be Green's function for the periodic boundary value problem

$$y^{(n)}(t) - \lambda y(t) = 0, \quad y^{(i)}(0) - y^{(i)}(T) = 0, \quad 0 \le i \le n-1. \tag{7.99}$$

Among various properties of $G(t,s)$, we recall that

$$\frac{\partial^i}{\partial t^i} G(0,0) - \frac{\partial^i}{\partial t^i} G(T,0) = \begin{cases} 0, & 0 \le i \le n-2, \\ 1, & i = n-1. \end{cases} \tag{7.100}$$

We now consider the equation

$$y^{(n)}(t) - \lambda y(t) = g(t), \quad t \ne t_k,\ k = 1,\dots,m, \tag{7.101}$$

satisfying (7.96), (7.98), where $g \in L^1(J_k, \mathbb{R})$, $k = 1,\dots,m$. For brevity, we will refer to (7.96), (7.97), (7.98), (7.101), as (LP). Note that (LP) is not a linear problem, since the impulsive functions are not necessarily linear, however, if I_k^i, $0 \le i \le n-1$, $k = 1,\dots,m$, are linear, then (LP) is a linear impulsive problem.

The following is also fundamental in establishing solutions of (7.95)–(7.98). The proof is much along the lines of Dong's result [133], and we omit the proof.

Lemma 7.10. *A function $y \in \Omega \cap \mathrm{AC}^{n-1}(t_k, t_{k+1})$, $k = 1,\dots,m$, is a solution of* (LP) *if and only if $y \in \Omega$, and there exists $g \in S_{F,y}$ such that y is a solution of the impulsive integral equation*

$$y(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ \displaystyle\int_0^T G(t,s)g(s)\,ds + \sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,0)\mu_{n-i-1} & \\ \displaystyle\quad + \sum_{k=1}^{m} \sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,t_k) I_k^i(y(t_k)), & t \in J. \end{cases} \tag{7.102}$$

We provide constraints on F and the impulse operators I_k^i so that (7.95)–(7.98) has a solution. Our main tool will be Lemma 7.10.

Theorem 7.11. *Assume that conditions (7.3.1) and (7.3.3) are satisfied. Suppose also that*

(7.11.1) *for each $0 \le i \le n-1$, $1 \le k \le m$, there exist constants $d_k^i \ge 0$ such that $|I_k^i(y)| \le d_k^i$, for each $y \in \mathbb{R}$;*

(7.11.2) *for each $t \in J$, the multivalued map $F(t,\cdot) : \mathcal{D} \to \mathcal{P}(E)$ maps bounded sets into relatively compact sets.*

Then problem (7.95)–(7.98) *has at least one solution on* $[-r,T]$.

Proof. In order to apply the Martelli fixed point theorem, that is, Theorem 1.7, we define a multivalued operator $N : \Omega \to \mathcal{P}(\Omega)$ by

$$N(y) = \left\{ h \in \Omega : h(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \displaystyle\int_0^T G(t,s)g(s)ds + \sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,0)\mu_{n-i-1} & \\ \displaystyle\quad + \sum_{k=1}^{m}\sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,t_k) I_k^i(y(t_k)), & t \in J, \end{cases} \right\} \tag{7.103}$$

where $g \in S_{F,y}$. It is straightforward that fixed points of N are solutions of (7.95)–(7.98). In addition, Lasota and Opial [186] have proved that, for each $y \in \Omega$, the set $S_{F,y}$ is nonempty.

We now exhibit that N satisfies the conditions of Theorem 1.7. The proof will be done in several steps.

Our first step is to show that, for each $y \in \Omega$, the set $N(y)$ is convex. Indeed, if $h_1, h_2 \in N(y)$, then there exist $g_1, g_2 \in S_{F,y}$ such that, for each $t \in J$, we have

$$\begin{aligned} h_i(t) = & \int_0^T G(t,s)g_i(s)ds + \sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,0)\mu_{n-i-1} \\ & + \sum_{k=1}^{m}\sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,t_k) I_k^i(y(t_k)), \quad i = 1, 2. \end{aligned} \tag{7.104}$$

Then, for $0 \le d \le 1$ and $t \in J$, we have

$$\begin{aligned} (dh_1 + (1-d)h_2)(t) = & \int_0^T G(t,s)[dg_1(s) + (1-d)g_2(s)]ds \\ & + \sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,0)\mu_{n-i-1} + \sum_{k=1}^{m}\sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,t_k) I_k^i(y(t_k)). \end{aligned} \tag{7.105}$$

The convexity of F implies $S_{F,y}$ is convex, which in turn implies

$$dh_1 + (1-d)h_2 \in N(y); \tag{7.106}$$

that is, $N(y)$ is convex.

Our next step is to argue that N maps bounded sets into bounded sets in Ω. In particular, we show that, for each $y \in B_q = \{y \in \Omega : \|y\|_\Omega \le q\}$, there exists an $\ell > 0$ such that $\|N(y)\|_\Omega \le \ell$.

So, let $y \in B_q$ and $h \in N(y)$. Then there exists $g \in S_{F,y}$ such that, for each $t \in J$, we have

$$h(t) = \int_0^T G(t,s)g(s)ds + \sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,0)\mu_{n-i-1} + \sum_{k=1}^{m}\sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,t_k) I_k^i(y(t_k)). \tag{7.107}$$

By (7.11.1), we have, for each $t \in J$,

$$\begin{aligned} |h(t)| &\le \int_0^T |G(t,s)||g(s)|ds + \sum_{i=0}^{n-1} \left|\frac{\partial^i}{\partial t^i} G(t,0)\right| |\mu_{n-i-1}| \\ &\quad + \sum_{k=1}^{m}\sum_{i=0}^{n-1} \left|\frac{\partial^i}{\partial t^i} G(t,t_k) I_k^i(y(t_k))\right| \\ &\le \int_0^T |G(t,s)| l_q(s)ds + \sum_{i=0}^{n-1} \left|\frac{\partial^i}{\partial t^i} G(t,0)\right| |\mu_{n-i-1}| \\ &\quad + \sum_{k=1}^{m}\sum_{i=0}^{n-1} \left|\frac{\partial^i}{\partial t^i} G(t,t_k)\right| \sup\{|I_k^i(|y|)| : \|y\| \le q\}. \end{aligned} \tag{7.108}$$

Then, for each $h \in N(B_q)$, we have

$$\begin{aligned} \|h\|_\Omega \le \sup_{(t,s)\in J\times J} |G(t,s)| \int_0^T l_q(s)ds + \sum_{i=0}^{n-1} |\mu_{n-i-1}| \sup_{t\in J} \left|\frac{\partial^i}{\partial t^i} G(t,0)\right| \\ + \sum_{k=1}^{m}\sum_{i=0}^{n-1} \sup_{t\in J} \left|\frac{\partial^i}{\partial t^i} G(t,t_k)\right| \sup\{|I_k^i(|y|)| : \|y\| \le q\} := \ell. \end{aligned} \tag{7.109}$$

We next show that N maps bounded sets into equicontinuous sets of Ω. Let $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$, and B_q be a bounded set (as described above) in Ω. Choose $y \in B_q$ and $h \in N(y)$. Then there exists $g \in S_{F,y}$ such that, for each $t \in J$, we have

$$h(t) = \int_0^T G(t,s)g(s)ds + \sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,0)\mu_{n-i-1} + \sum_{k=1}^{m}\sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,t_k) I_k^i(y(t_k)), \tag{7.110}$$

which yields

$$\begin{aligned} |h(\tau_2) - h(\tau_1)| \le \int_0^T &|G(\tau_2,s) - G(\tau_1,s)| l_q(s)ds \\ &+ \sum_{i=0}^{n-1} \left|\frac{\partial^i}{\partial t^i} G(\tau_2,0) - \frac{\partial^i}{\partial t^i} G(\tau_1,0)\right| |\mu_{n-i-1}| \\ &+ \sum_{k=1}^{m}\sum_{i=0}^{n-1} \left|\frac{\partial^i}{\partial t^i} G(\tau_2,t_k) - \frac{\partial^i}{\partial t^i} G(\tau_1,t_k)\right| d_k^i. \end{aligned} \tag{7.111}$$

In the inequality, if we let $\tau_2 \to \tau_1$, the right side tends to zero. Also, the equicontinuity for the other cases, $\tau_1 < \tau_2 \le 0$ or $\tau_1 \le 0 \le \tau_2$, are straightforward.

As a consequence of the convexity of $N(y)$, for each $y \in \Omega$, and N mapping bounded sets into equicontinuous sets of Ω, when coupled with the Arzelá-Ascoli theorem, we conclude that $N : \Omega \to \mathcal{P}(\Omega)$ is completely continuous multivalued, and therefore, a condensing multivalued map.

The next step of our argument involves exhibiting that N has a closed graph. To that end, let $y_n \to y_*$, $h_n \in N(y_n)$, and $h_n \to h_*$. It remains to show that $h_* \in N(y_*)$.

Since $h_n \in N(y_n)$, there exists $g_n \in S_{F,y_n}$ such that, for each $t \in J$,

$$h_n(t) = \int_0^T G(t,s)g_n(s)ds + \sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i}G(t,0)\mu_{n-i-1} + \sum_{k=1}^{m}\sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i}G(t,t_k)I_k^i(y_n(t_k)). \tag{7.112}$$

Since each I_k^i is continuous, we have that

$$\begin{aligned}&\Bigg\|\left(h_n - \sum_{i=0}^{n-1} G(t,0)\mu_{n-i-1} - \sum_{k=1}^{m}\sum_{i=0}^{n-1}\frac{\partial^i}{\partial t^i}G(t,t_k)I_k^i(y_n(t_k))\right)\\ &\quad - \left(h_* - \sum_{i=0}^{n-1}\frac{\partial^i}{\partial t^i}G(t,0)\mu_{n-i-1} - \sum_{k=1}^{m}\sum_{i=0}^{n-1}\frac{\partial^i}{\partial t^i}G(t,t_k)I_k^i(y_*(t_k))\right)\Bigg\|_\infty \longrightarrow 0,\end{aligned} \tag{7.113}$$

as $n \to \infty$.

If we define a continuous linear operator $\Gamma : L^1(J,\mathbb{R}) \to C(J,\mathbb{R})$ by

$$\Gamma(g)(t) = \int_0^T G(t,s)g(s)ds, \tag{7.114}$$

then, by Lemma 1.28, it follows that $\Gamma \circ S_F$ is a closed graph operator. Moreover, we have that

$$h_n(t) - \sum_{i=0}^{n-1}\frac{\partial^i}{\partial t^i}G(t,0)\mu_{n-i-1} - \sum_{k=1}^{m}\sum_{i=0}^{n-1}\frac{\partial^i}{\partial t^i}G(t,t_k)I_k^i(y_n(t_k)) \in \Gamma(S_{F,y_n}). \tag{7.115}$$

Since $y_n \to y_*$, we also have from Lemma 1.28 that

$$h_*(t) - \sum_{i=0}^{n-1}\frac{\partial^i}{\partial t^i}G(t,0)\mu_{n-i-1} - \sum_{k=1}^{m}\sum_{i=0}^{n-1}\frac{\partial^i}{\partial t^i}G(t,t_k)I_k^i(y_*(t_k)) = \int_0^T G(t,s)g_*(s)ds \tag{7.116}$$

for some $g_* \in S_{F,y_*}$. In particular, $h_* \in N(y_*)$, and N has closed graph.

Our final step is to exhibit that the set

$$\mathcal{M} := \{y \in \Omega : \beta y \in N(y),\ \text{for some } \beta > 1\} \tag{7.117}$$

is bounded. So we choose $y \in \mathcal{M}$. Then $\beta y \in N(y)$, for some $\beta > 1$, and thus, for each $t \in J$,

$$y(t)=\beta^{-1}\left[\int_0^T G(t,s)g(s)ds+\sum_{i=0}^{n-1}\frac{\partial^i}{\partial t^i}G(t,0)\mu_{n-i-1}+\sum_{k=1}^{m}\sum_{i=0}^{n-1}\frac{\partial^i}{\partial t^i}G(t,t_k)I_k^i(y(t_k))\right], \tag{7.118}$$

and so, by (7.11.1), we have

$$\begin{aligned}|y(t)| \leq{}& \sup_{(t,s)\in J\times J}|G(t,s)|\int_0^T m(s)ds+\sum_{i=0}^{n-1}\sup_{t\in J}\left|\frac{\partial^i}{\partial t^i}G(t,0)\right||\mu_{n-i-1}|\\ &+\sum_{k=1}^{m}\sum_{i=0}^{n-1}\sup_{t\in J}\left|\frac{\partial^i}{\partial t^i}G(t,t_k)\right|d_k^i := b,\end{aligned} \tag{7.119}$$

where b depends only on T and on the function w. In particular, $\|y\| \leq b$, and $\mathcal{M}$ is bounded.

Set $X := \Omega$. As a consequence of Theorem 1.7, we deduce that N has a fixed point which is a solution of (7.95)–(7.98). □

In this section, we provide constraints on F and the impulse operators I_k^i so that (7.95)–(7.98) has a solution. This will be done by an application of Theorem 1.11.

Theorem 7.12. *Assume that (7.4.1)-(7.4.2) are satisfied. Suppose also that*

(7.12.1) *for each* $0 \leq i \leq n-1$, $1 \leq k \leq m$, *there exist constants* $d_k^i \geq 0$ *such that* $|I_k^i(y) - I_k^i(\overline{y})| \leq d_k^i|y-\overline{y}|$, *for each* $y, \overline{y} \in E$.

Then problem (7.95)–(7.98) *has at least one solution on* $[-r, T]$.

Proof. In order to apply the Covitz-Nadler fixed point theorem, that is, Theorem 1.11, we define a multivalued operator $N : \Omega \to \mathcal{P}(\Omega)$ by

$$N(y)=\left\{h\in\Omega : h(t)=\begin{cases}\phi(t), & t\in[-r,0],\\ \displaystyle\int_0^T G(t,s)v(s)ds+\sum_{i=0}^{n-1}\frac{\partial^i}{\partial t^i}G(t,0)\mu_{n-i-1} & \\ \displaystyle\quad+\sum_{k=1}^{m}\sum_{i=0}^{n-1}\frac{\partial^i}{\partial t^i}G(t,t_k)I_k^i(y(t_k)), & t\in J,\end{cases}\right\} \tag{7.120}$$

where $v \in S_{F,y}$. It is straightforward that fixed points of N are solutions of (7.95)–(7.98). In addition, by (7.12.1), F has a measurable selection from which Castaing and Valadier (see [119, Theorem III]) have proved that, for each $y \in \Omega$, the set $S_{F,y}$ is nonempty.

We now exhibit that N satisfies the conditions of Theorem 1.11, which will be done in a couple of steps.

Our first step is to show that, for each $y \in \Omega$, we have $N(y) \in P_{cl}(\Omega)$. Indeed, let $(y_n)_{n\geq 0} \in N(y)$ be such that $y_n \to \tilde{y}$ in Ω. Then $\tilde{y} \in \Omega$, and there exists $g_n \in S_{F,y}$ such that, for each $t \in J$,

$$y_n(t) \in \int_0^T G(t,s)g_n(s)ds + \sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,0)\mu_{n-i-1} + \sum_{k=1}^{m}\sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,t_k) I_k^i(y(t_k^-)). \tag{7.121}$$

Using the fact that F has compact values and from (7.12.1), we may pass to a subsequence if necessary to get that g_n converges to g in $L^1(J,E)$ and hence $g \in S_{F,y}$. Then, for each $t \in [0,b]$,

$$y_n(t) \to \tilde{y}(t) = \int_0^T G(t,s)g(s)ds + \sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,0)\mu_{n-i-1} + \sum_{k=1}^{m}\sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,t_k) I_k^i(y(t_k^-)). \tag{7.122}$$

So, $\tilde{y} \in N(y)$, and in particular, $N(y) \in \mathcal{P}_{cl}(\Omega)$.

Our second step is to show there exists a $0 \leq \gamma < 1$ such that $H_d(N(y), N(\overline{y})) \leq \gamma \| y - \overline{y}\|$, for each $y, \overline{y} \in \Omega$.

So, let $y, \overline{y} \in \Omega$ and $h_1 \in N(y)$. Then there exists $v_1(t) \in F(t, y_t)$ such that, for each $t \in J$,

$$h_1(t) = \int_0^T G(t,s)v_1(s)ds + \sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,0)\mu_{n-i-1} + \sum_{k=1}^{m}\sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,t_k) I_k^i(y(t_k^-)). \tag{7.123}$$

From (7.12.1), it follows that, for $t \in J$,

$$H_d(F(t,y_t), F(t,\overline{y}_t)) \leq l(t)\|y_t - \overline{y}_t\|_{\mathcal{D}}. \tag{7.124}$$

Hence there is $w \in F(t, \overline{y}_t)$ such that

$$|v_1(t) - w| \leq l(t)\|y_t - \overline{y}_t\|_{\mathcal{D}}, \quad t \in J. \tag{7.125}$$

Consider $U : J \to \mathcal{P}(E)$, defined by

$$U(t) = \{w \in E : |v_1(t) - w| \leq l(t)\|y_t - \overline{y}_t\|_{\mathcal{D}}\}. \tag{7.126}$$

By Castaing and Valadier (see [119, Proposition III.4]), the multivalued operator $V(t) = U(t) \cap F(t, \overline{y}_t)$ is measurable, and hence there exists a measurable selection for V; call it $v_2(t)$. So, $v_2(t) \in F(t, \overline{y}_t)$ and

$$|v_1(t) - v_2(t)| \leq l(t)\|y - \overline{y}\|_{\mathcal{D}}, \quad t \in J. \tag{7.127}$$

We define, for each $t \in J$,

$$h_2(t) = \int_0^T G(t,s)v_2(s)ds + \sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,0)\mu_{n-i-1} + \sum_{k=1}^{m}\sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,t_k) I_k^i(\overline{y}(t_k^-)). \tag{7.128}$$

Then, we have, for $t \in J$,

$$\begin{aligned} |h_1(t) - h_2(t)| &\leq \int_0^T |G(t,s)|\,|v_1(s) - v_2(s)|\,ds \\ &\quad + \sum_{k=1}^{m}\sum_{i=0}^{n-1} \left|\frac{\partial^i}{\partial t^i} G(t,t_k)\right| \left|I_k^i(y(t_k^-)) - I_k^i(\overline{y}(t_k^-))\right| \\ &\leq M_0 \int_0^T l(s)\|y_s - \overline{y}_s\|_{\mathcal{D}} ds + \sum_{k=1}^{m}\sum_{i=0}^{n-1} M_i d_k^i \left|y(t_k^-) - \overline{y}(t_k^-)\right| \\ &\leq \left[M_0 l^* + \sum_{k=1}^{m}\sum_{i=0}^{n-1} M_i d_k^i\right] \|y - \overline{y}\|. \end{aligned} \tag{7.129}$$

Then

$$\|h_1 - h_2\| \leq \left[M_0 l^* + \sum_{k=1}^{m}\sum_{i=0}^{n-1} M_i d_k^i\right] \|y - \overline{y}\|. \tag{7.130}$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H_d(N(y), N(\overline{y})) \leq \left[M_0 l^* + \sum_{k=1}^{m}\sum_{i=0}^{n-1} M_i d_k^i\right] \|y - \overline{y}\|. \tag{7.131}$$

So, N is a contraction and thus, by Theorem 1.11, N has a fixed point y, which is a solution to (7.95)–(7.98). □

By the help of Schaefer's fixed point theorem combined with the selection theorem of Bressan and Colombo for lower semicontinuous maps with decomposable values, we will present an existence result for problem (7.95)-(7.96), with a nonconvex valued right-hand side.

Theorem 7.13. *Suppose (7.3.3), (7.11.1), (7.11.2), and the following conditions are satisfied:*

(7.13.1) $F : [0,T] \times D \to \mathcal{P}(E)$ *is a nonempty, compact-valued, multivalued, map such that*
  (a) $(t,u) \mapsto F(t,u)$ *is* $\mathcal{L} \otimes \mathcal{B}$ *measurable;*
  (b) $u \mapsto F(t,u)$ *is lower semicontinuous for a.e.* $t \in [0,T]$;
(7.13.2) *for each* $q > 0$, *there exists a function* $h_q \in L^1([0,T],\mathbb{R}^+)$ *such that*

$$\|F(t,u)\| := \sup\{|v| : v \in F(t,u)\} \leq h_q(t) \quad \text{for a.e. } t \in [0,T], \tag{7.132}$$

and for $u \in D$ *with* $\|u\|_{\mathcal{D}} \leq q$.

Then problem (7.95)–(7.98) *has at least one solution on* $[-r,T]$.

Proof. Conditions (7.13.1) and (7.13.2) imply that F is of lower semicontinuous type. Then from Theorem 1.5, there exists a continuous function $f : \Omega \to L^1([0,T],E)$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in \Omega$.

Consider problem

$$\begin{gathered} y'(t) = f(y_t), \quad t \in [0,T],\ t \neq t_k,\ k = 1,\dots,m, \\ \Delta y^{(i)}(t_k) = I_k^i(y(t_k^-)), \quad 0 \leq i \leq n-1,\ 1 \leq k \leq m, \\ y(t) = \phi(t), \quad t \in [-r,0], \\ y^{(i)}(0) - y^{(i)}(T) = \mu_i, \quad 0 \leq i \leq n-1. \end{gathered} \tag{7.133}$$

Transform problem (7.133) into a fixed point problem. Consider the operator $N : \Omega \to \Omega$ defined by

$$N(y)(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ \displaystyle\int_0^T G(t,s) f(y_s)\,ds + \sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,0)\mu_{n-i-1} & \\ \displaystyle\quad + \sum_{k=1}^{m}\sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} G(t,t_k) I_k^i(y(t_k)), & t \in J. \end{cases} \tag{7.134}$$

We will show that N is completely continous; that is, continuous and sends bounded sets into relatively compact sets.

Step 1. N is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in Ω. Then

$$\begin{aligned} |N(y_n(t)) - N(y(t))| \leq{} & \int_0^T |G(t,s)|\,|f(y_{ns}) - f(y_s)|\,ds \\ & + \sum_{k=1}^{m}\sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} |G(t,t_k)|\,|I_k^i(y_n(t_k)) - I_k^i(y(t_k))|. \end{aligned} \tag{7.135}$$

Since the functions f and $I_k,\ k=1,\dots,m$, are continuous, then

$$\|N(y_n)-N(y)\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{7.136}$$

Step 2. N maps bounded sets into bounded sets in Ω.

Indeed, it is enough to show that there exists a positive constant ℓ such that, for each $y \in B_q = \{y \in \Omega : \|y\| \le q\}$, we have $\|N(y)\| \le \ell$.

By our assumptions, we have, for each $t \in J$,

$$\begin{aligned}
|h(t)| &\le \int_0^T |G(t,s)|\,|f(y_s)|\,ds + \sum_{i=0}^{n-1}\left|\frac{\partial^i}{\partial t^i}G(t,0)\right||\mu_{n-i-1}| \\
&\quad + \sum_{k=1}^{m}\sum_{i=0}^{n-1}\left|\frac{\partial^i}{\partial t^i}G(t,t_k)I_k^i(y(t_k))\right| \\
&\le \int_0^T |G(t,s)|\,l_q(s)ds + \sum_{i=0}^{n-1}\left|\frac{\partial^i}{\partial t^i}G(t,0)\right||\mu_{n-i-1}| \\
&\quad + \sum_{k=1}^{m}\sum_{i=0}^{n-1}\left|\frac{\partial^i}{\partial t^i}G(t,t_k)\right| \sup\{|I_k^i(|y|)| : \|y\| \le q\}.
\end{aligned} \tag{7.137}$$

Then, for each $h \in N(B_q)$, we have

$$\begin{aligned}
\|h\| &\le \sup_{(t,s)\in J\times J}|G(t,s)|\int_0^T l_q(s)ds + \sum_{i=0}^{n-1}|\mu_{n-i-1}|\sup_{t\in J}\left|\frac{\partial^i}{\partial t^i}G(t,0)\right| \\
&\quad + \sum_{k=1}^{m}\sum_{i=0}^{n-1}\sup_{t\in J}\left|\frac{\partial^i}{\partial t^i}G(t,t_k)\right| \sup\{|I_k^i(|y|)| : \|y\| \le q\} := \ell.
\end{aligned} \tag{7.138}$$

Step 3. N maps bounded sets into equicontinuous sets of Ω.

Let $\tau_1,\tau_2 \in J$, $\tau_1 < \tau_2$, and B_q be a bounded set (as described above) in Ω. Let $y \in B_q$. Then

$$\begin{aligned}
|h(\tau_2)-h(\tau_1)| &\le \int_0^T |G(\tau_2,s)-G(\tau_1,s)|\,l_q(s)ds \\
&\quad + \sum_{i=0}^{n-1}\left|\frac{\partial^i}{\partial t^i}G(\tau_2,0)-\frac{\partial^i}{\partial t^i}G(\tau_1,0)\right||\mu_{n-i-1}| \\
&\quad + \sum_{k=1}^{m}\sum_{i=0}^{n-1}\left|\frac{\partial^i}{\partial t^i}G(\tau_2,t_k)-\frac{\partial^i}{\partial t^i}G(\tau_1,t_k)\right|d_k^i.
\end{aligned} \tag{7.139}$$

In the inequality, if we let $\tau_2 \to \tau_1$, the right side tends to zero. Also, the equicontinuity for the other cases, $\tau_1 < \tau_2 \le 0$ or $\tau_1 \le 0 \le \tau_2$, are straightforward.

As a consequence of Steps 1 to 3, and (7.13.3) together with the Arzelá-Ascoli theorem, we conclude that $N : \Omega \to \Omega$ is completely continuous.

Step 4. Now it remains to show that the set

$$\mathcal{E}(N) := \{y \in \Omega : y = \beta N(y), \text{ for some } 0 < \beta < 1\} \tag{7.140}$$

is bounded.

So we choose $y \in \mathcal{E}(N)$. Then $y = \beta N(y)$, for some $0 < \beta < 1$, and thus, for each $t \in J$,

$$y(t) = \beta\left[\int_0^T G(t,s)f(y_s)ds + \sum_{i=0}^{n-1}\frac{\partial^i}{\partial t^i}G(t,0)\mu_{n-i-1} + \sum_{k=1}^{m}\sum_{i=0}^{n-1}\frac{\partial^i}{\partial t^i}G(t,t_k)I_k^i(y(t_k))\right], \tag{7.141}$$

and so, by (7.13.1) and (7.13.2), we have

$$\begin{aligned} |y(t)| \leq{} & \sup_{(t,s)\in J\times J}|G(t,s)|\int_0^T m(s)ds + \sum_{i=0}^{n-1}\sup_{t\in J}\left|\frac{\partial^i}{\partial t^i}G(t,0)\right||\mu_{n-i-1}| \\ & + \sum_{k=1}^{m}\sum_{i=0}^{n-1}\sup_{t\in J}\left|\frac{\partial^i}{\partial t^i}G(t,t_k)\right|d_k^i := b, \end{aligned} \tag{7.142}$$

where b depends only on T and on the function m. In particular, $\|y\| \leq b$, and $\mathcal{E}(N)$ is bounded.

With $X := \Omega$, we conclude by Schaefer's theorem that N has a fixed point which is a solution of (7.95)–(7.98). □

7.5. Notes and remarks

Chapter 7 deals with nonresonance problems for impulsive functional differential inclusions. The results of Section 7.1, on first-order inclusions, are adapted from Benchohra et al. [51, 60], while the results of Section 7.2, on second-order inclusions, are adapted from Benchohra et al. [56, 60]. Finally, the results of Section 7.4, on higher-order boundary value problems for impulsive functional differential inclusions, are taken from Benchohra et al. [44, 63].

8 Impulsive differential equations & inclusions with variable times

8.1. Introduction

The theory of impulsive differential equations with variable time is relatively less developed due to the diffculties created by the state-dependent impulses. Recently, some interesting extensions to impulsive differential equations with variable times have been done by Bajo and Liz [31], Frigon and O'Regan [150, 151], Kaul [173], Kaul et al. [174], and Benchohra et al. [43, 45, 70, 71, 91, 92].

8.2. First-order impulsive differential equations with variable times

This section is concerned with the existence of solutions, for initial value problems (IVP for short), for first-order functional differential equations with impulsive effects

$$\begin{gathered} y'(t) = f(t, y_t), \quad \text{a.e. } t \in J = [0, T],\ t \neq \tau_k(y(t)),\ k = 1, \ldots, m, \\ y(t^+) = I_k(y(t)), \quad t = \tau_k(y(t)),\ k = 1, \ldots, m, \\ y(t) = \phi(t), \quad t \in [-r, 0], \end{gathered} \tag{8.1}$$

where $f : J \times \mathcal{D} \to \mathbb{R}^n$ is a given function, $\mathcal{D} = \{\psi : [-r, 0] \to \mathbb{R}^n : \psi$ is continuous everywhere except for a finite number of points $\bar{t}$ at which $\psi(\bar{t})$ and $\psi(\bar{t}^+)$ exist, and $\psi(\bar{t}^-) = \psi(\bar{t})\}$, $\phi \in D$, $0 < r < \infty$, $\tau_k : \mathbb{R}^n \to \mathbb{R}$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, \ldots, m$, are given functions satisfying some assumptions that will be specified later.

The main theorem of this section extends the problem (8.1) considered by Benchohra et al. [46] when the impulse times are constant. Our approach is based on Schaefer's fixed point theorem.

Let us start by defining what we mean by a solution of problem (8.1).

Definition 8.1. A function $y \in \Omega \cap AC((t_k, t_{k+1}), \mathbb{R})$, $k = 0, \ldots, m$, is said to be a solution of (8.1) if y satisfies the equation $y'(t) = f(t, y_t)$ a.e. on J, $t \neq \tau_k(y(t))$, $k = 1, \ldots, m$, and the conditions $y(t^+) = I_k(y(t))$, $t = \tau_k(y(t))$, $k = 1, \ldots, m$, and $y(t) = \phi(t)$ on $[-r, 0]$.

We are now in a position to state and prove our existence result for the problem (8.1). Recall that throughout $\Omega = \mathrm{PC}([-r,T]\mathbb{R}^n)$.

Theorem 8.2. *Assume the following hypotheses are satisfied:*

(8.2.1) $f : J \times \mathcal{D} \to \mathbb{R}^n$ *is an* L^1*-Carathéodory function;*

(8.2.2) *the functions* $\tau_k \in C^1(\mathbb{R}^n,\mathbb{R})$ *for* $k = 1,\dots,m$. *Moreover,*

$$0 < \tau_1(x) < \cdots < \tau_m(x) < T, \quad \forall x \in \mathbb{R}^n; \tag{8.2}$$

(8.2.3) *there exist constants* c_k *such that* $|I_k(x)| \le c_k$, $k = 1,\dots,m$, *for each* $x \in \mathbb{R}^n$;

(8.2.4) *there exists a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *and* $p \in L^1(J,\mathbb{R}_+)$ *such that*

$$|f(t,u)| \le p(t)\psi(\|u\|_{\mathcal{D}}) \tag{8.3}$$

for a.e. $t \in J$ *and each* $u \in \mathcal{D}$ *with*

$$\int_1^\infty \frac{ds}{\psi(s)} = \infty; \tag{8.4}$$

(8.2.5) *for all* $(t,x) \in [0,T] \times \mathbb{R}^n$ *and for all* $y_t \in D$,

$$\langle \tau_k'(x), f(t,y_t)\rangle \ne 1, \quad \text{for } k = 1,\dots,m, \tag{8.5}$$

where $\langle\cdot,\cdot\rangle$ *denotes the scalar product in* $\mathbb{R}^n$;

(8.2.6) *for all* $x \in \mathbb{R}^n$,

$$\tau_k(I_k(x)) \le \tau_k(x) < \tau_{k+1}(I_k(x)), \quad \text{for } k = 1,\dots,m. \tag{8.6}$$

Then the IVP (8.1) *has at least one solution on* $[-r,T]$.

Proof. The proof will be given in several steps.
Step 1. Consider the problem

$$\begin{aligned} y'(t) &= f(t,y_t), \quad \text{a.e. } t \in [0,T], \\ y(t) &= \phi(t), \quad t \in [-r,0]. \end{aligned} \tag{8.7}$$

Transform the problem (8.7) into a fixed point problem. Consider the operator $N : \Omega \to \Omega$ defined by

$$N(y)(t) = \begin{cases} \phi(t) & \text{if } t \in [-r,0], \\ \phi(0) + \displaystyle\int_0^t f(s,y_s)\,ds & \text{if } t \in [0,T]. \end{cases} \tag{8.8}$$

We will show that the operator N is completely continuous.

Claim 1. N is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in Ω.

Then

$$|N(y_n)(t) - N(y)(t)| \leq \int_0^t |f(s, y_{ns}) - f(s, y_s)|\,ds \\ \leq \int_0^T |f(s, y_{ns}) - f(s, y_s)|\,ds. \tag{8.9}$$

Since f is an L^1-Carathéodory function, we have by the Lebesgue dominated convergence theorem

$$||N(y_n) - N(y)|| \leq ||f(\cdot, y_n) - f(\cdot, y)||_{L^1} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{8.10}$$

Claim 2. N maps bounded sets into bounded sets in Ω.

Indeed, it is enough to show that for any $q > 0$, there exists a positive constant ℓ such that, for each $y \in B_q = \{y \in \Omega : \|y\| \leq q\}$, we have $\|N(y)\| \leq \ell$. We have, for each $t \in [0, T]$,

$$|N(y)(t)| \leq |\phi(0)| + \int_0^t |f(s, y_s)|\,ds \leq \|\phi\|_{\mathcal{D}} + ||h_q||_{L^1}. \tag{8.11}$$

Thus

$$||N(y)||_{\Omega} \leq \|\phi\|_{\mathcal{D}} + ||h_q||_{L^1} := \ell. \tag{8.12}$$

Claim 3. N maps bounded sets into equicontinuous sets of Ω.

Let $l_1, l_2 \in [0, T]$, $l_1 < l_2$, and let B_q be a bounded set of Ω as in Claim 2, and let $y \in B_q$. Then

$$|N(y)(l_2) - N(y)(l_1)| \leq \int_{l_1}^{l_2} h_q(s)ds. \tag{8.13}$$

As $l_2 \to l_1$, the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $l_1 < l_2 \leq 0$ and $l_1 \leq 0 \leq l_2$ is obvious.

As a consequence of Claims 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $N : \Omega \to \Omega$ is completely continuous.

Claim 4. Now it remains to show that the set

$$\mathcal{E}(N) := \{y \in \Omega : y = \lambda N(y) \text{ for some } 0 < \lambda < 1\} \tag{8.14}$$

is bounded.

Let $y \in \mathcal{E}(N)$. Then $y = \lambda N(y)$ for some $0 < \lambda < 1$. Thus, for each $t \in [0, T]$,

$$y(t) = \lambda\left(\phi(0) + \int_0^t f(s, y_s)\,ds\right). \tag{8.15}$$

This implies by (8.2.2), (8.3.2) that, for each $t \in J$, we have

$$|y(t)| \leq \|\phi\|_{\mathcal{D}} + \int_0^t p(s)\psi(\|y_s\|_{\mathcal{D}})ds. \tag{8.16}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq T. \tag{8.17}$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0,T]$, by the previous inequality, we have, for $t \in [0,T]$,

$$\mu(t) \leq \|\phi\|_{\mathcal{D}} + \int_0^t p(s)\psi(\mu(s))ds. \tag{8.18}$$

If $t^* \in [-r,0]$, then $\mu(t) = \|\phi\|_{\mathcal{D}}$ and the previous inequality holds. Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$\begin{gathered} c = v(0) = \|\phi\|_{\mathcal{D}}, \qquad \mu(t) \leq v(t), \quad t \in [0,T], \\ v'(t) = p(t)\psi(\mu(t)), \quad \text{a.e. } t \in [0,T]. \end{gathered} \tag{8.19}$$

Using the nondecreasing character of ψ, we get

$$v'(t) \leq p(t)\psi(v(t)), \quad \text{a.e. } t \in [0,T]. \tag{8.20}$$

This implies that, for each $t \in [0,T]$,

$$\int_{v(0)}^{v(t)} \frac{ds}{\psi(s)} \leq \int_0^T p(s)ds < \int_{v(0)}^{\infty} \frac{ds}{\psi(s)}. \tag{8.21}$$

Thus there exists a constant K such that $v(t) \leq K$, $t \in [0,T]$, and hence $\mu(t) \leq K$, $t \in [0,T]$. Since for every $t \in [0,T]$, $\|y_t\|_{\mathcal{D}} \leq \mu(t)$, we have

$$\|y\| \leq K' = \max\{\|\phi\|_{\mathcal{D}}, K\}, \tag{8.22}$$

where K' depends on T and on the functions p and ψ. This shows that $\mathcal{E}(N)$ is bounded.

Set $X := \Omega$. As a consequence of Schaefer's fixed point theorem, Theorem 1.6, we deduce that N has a fixed point y which is a solution to problem (8.7). Denote this solution by y_1.

Define the function

$$r_{k,1}(t) = \tau_k(y_1(t)) - t, \quad \text{for } t \geq 0. \tag{8.23}$$

Hypothesis (8.2.1) implies that

$$r_{k,1}(0) \neq 0, \quad \text{for } k = 1, \dots, m. \tag{8.24}$$

If

$$r_{k,1}(t) \neq 0 \quad \text{on } [0,T], \text{ for } k = 1,\dots,m, \tag{8.25}$$

that is,

$$t \neq \tau_k\big(y_1(t)\big) \quad \text{on } [0,T] \text{ and for } k = 1,\dots,m, \tag{8.26}$$

then y_1 is a solution of the problem (8.1).

It remains to consider the case when

$$r_{k,1}(t) = 0, \quad \text{for some } t \in [0,T],\ k = 1,\dots,m. \tag{8.27}$$

Now since

$$r_{k,1}(0) \neq 0 \tag{8.28}$$

and $r_{k,1}$ is continuous, there exists $t_1 > 0$ such that

$$r_{k,1}(t_1) = 0, \quad r_{k,1}(t) \neq 0, \quad \forall t \in [0,t_1). \tag{8.29}$$

Step 2. Consider now the problem

$$y'(t) = f(t, y_t), \quad \text{a.e. } t \in [t_1, T], \tag{8.30}$$

$$y(t_1^+) = I_1\big(y_1(t_1)\big), \tag{8.31}$$

$$y(t) = y_1(t), \quad t \in [t_1 - r, t_1]. \tag{8.32}$$

Transform the problem (8.30)–(8.32) into a fixed point problem. Consider the operator $N_1 : \mathrm{PC}([t_1 - r, T], \mathbb{R}^n) \to \mathrm{PC}([t_1 - r, T], \mathbb{R}^n)$ defined by

$$N_1(y)(t) = \begin{cases} y(t_1), & t \in [t_1 - r, t_1], \\ I_1\big(y_1(t_1)\big) + \displaystyle\int_{t_1}^{t} f(s, y_s)\,ds, & t \in [t_1, T]. \end{cases} \tag{8.33}$$

As in Step 1, we can show that N_1 is completely continuous, and the set

$$\mathcal{E}(N_1) := \big\{y \in \mathrm{PC}\big([t_1 - r, T], \mathbb{R}^n\big) : y = \lambda N_1(y) \text{ for some } 0 < \lambda < 1\big\} \tag{8.34}$$

is bounded.

Set $X := \mathrm{PC}([t_1 - r, T], \mathbb{R}^n)$. As a consequence of Schaefer's theorem, we deduce that N_1 has a fixed point y which is a solution to problem (8.30)-(8.31). Denote this solution by y_2. Define

$$r_{k,2}(t) = \tau_k(y_2(t)) - t, \quad \text{for } t \geq t_1. \tag{8.35}$$

If

$$r_{k,2}(t) \neq 0 \quad \text{on } (t_1, T], \ \forall k = 1, \dots, m, \tag{8.36}$$

then

$$y(t) = \begin{cases} y_1(t) & \text{if } t \in [0, t_1], \\ y_2(t) & \text{if } t \in (t_1, T] \end{cases} \tag{8.37}$$

is a solution of problem (8.1).

It remains to consider the case when

$$r_{k,2}(t) = 0, \quad \text{for some } t \in (t_1, T], \ k = 2, \dots, m. \tag{8.38}$$

By (8.2.6), we have

$$\begin{aligned} r_{k,2}(t_1^+) &= \tau_k(y_2(t_1^+)) - t_1 = \tau_k(I_1(y_1(t_1))) - t_1 \\ &> \tau_{k-1}(y_1(t_1)) - t_1 \geq \tau_1(y_1(t_1)) - t_1 \\ &= r_{1,1}(t_1) = 0. \end{aligned} \tag{8.39}$$

Since $r_{k,2}$ is continuous, there exists $t_2 > t_1$ such that

$$\begin{gathered} r_{k,2}(t_2) = 0, \\ r_{k,2}(t) \neq 0, \quad \forall t \in (t_1, t_2). \end{gathered} \tag{8.40}$$

Suppose now that there is $\bar{s} \in (t_1, t_2]$ such that

$$r_{1,2}(\bar{s}) = 0. \tag{8.41}$$

From (8.2.6), it follows that

$$\begin{aligned} r_{1,2}(t_1^+) &= \tau_1(y_2(t_1^+)) - t_1 = \tau_1(I_1(y_1(t_1))) - t_1 \\ &\leq \tau_1(y_1(t_1)) - t_1 = r_{1,1}(t_1) = 0. \end{aligned} \tag{8.42}$$

Thus the function $r_{1,2}$ attains a nonnegative maximum at some point $s_1 \in (t_1, T]$. Since

$$y_2'(t) = f(t, (y_2)_t), \tag{8.43}$$

then

$$r_{1,2}'(s_1) = \tau_1'(y_2(s_1))y_2'(s_1) - 1 = 0. \tag{8.44}$$

Therefore

$$\langle \tau_1'(y_2(s_1)), f(s_1, (y_2)_{s_1})\rangle = 1, \tag{8.45}$$

which is a contradiction by (8.2.5).
Step 3. We continue this process taking into account that $y_{m+1} := y\,|_{[t_m,T]}$ is a solution to the problem

$$\begin{gathered} y'(t) = f(t, y_t), \quad \text{a.e. } t \in (t_m, T), \\ y(t_m^+) = I_m(y_{m-1}(t_m)), \\ y(t) = y_{m-1}(t), \quad t \in [t_m - r, t_m]. \end{gathered} \tag{8.46}$$

The solution y of the problem (8.1) is then defined by

$$y(t) = \begin{cases} y_1(t) & \text{if } t \in [-r, t_1], \\ y_2(t) & \text{if } t \in (t_1, t_2], \\ \vdots & \\ y_{m+1}(t) & \text{if } t \in (t_m, T]. \end{cases} \tag{8.47}$$

□

8.3. Higher-order impulsive differential equations with variable times

Consider now initial value problems (IVP for short), for higher-order functional differential equations with impulsive effects

$$\begin{gathered} y^{(n)}(t) = f(t, y_t), \quad \text{a.e. } t \in J = [0, T],\ t \neq \tau_k(y(t)),\ k = 1, \dots, m, \\ y^{(i)}(t^+) = I_{k,i}(y(t)), \quad t = \tau_k(y(t)),\ k = 1, \dots, m,\ i = 1, \dots, n-1, \end{gathered} \tag{8.48}$$

$$\begin{gathered} y^{(i)}(0) = y_i, \quad i = 1, 2, \dots, n-1, \\ y(t) = \phi(t), \quad t \in [-r, 0], \end{gathered} \tag{8.49}$$

where $n \in \mathbb{N}$, $f : J \times \mathcal{D} \to \mathbb{R}^n$ is a given function, $\mathcal{D} = \{\psi : [-r, 0] \to \mathbb{R}^n : \psi$ is continuous everywhere except for a finite number of points $\bar{t}$ at which $\psi(\bar{t})$ and $\psi(\bar{t}^+)$ exist, and $\psi(\bar{t}^-) = \psi(\bar{t})\}$, $\phi \in \mathcal{D}$, $0 < r < \infty$, $\tau_k : \mathbb{R}^n \to \mathbb{R}$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, \dots, m$, are given functions satisfying some assumptions that will be specified later. Here $y^{(i)}$ denotes the ith derivative of the function y.

The main theorem of this section extends the problem (8.48) for the particular case $n = 1$ considered by Benchohra et al. [46, 71] when the impulse times are constant and variable, respectively. Our approach is based on Schaefer's fixed point theorem.

Let us start by defining what we mean by a solution of problem (8.48).

Definition 8.3. A function $y \in \Omega \cap AC^{n-1}((t_k, t_{k+1}), \mathbb{R}^n)$, $k = 0, \dots, m$, is said to be a solution of (8.48) if y satisfies the equation $y^{(n)}(t) = f(t, y_t)$ a.e. on J, $t \neq \tau_k(y(t))$, $k = 1, \dots, m$, and the conditions $y^{(i)}(t^+) = I_{k,i}(y(t))$, $t = \tau_k(y(t))$, $k = 1, \dots, m$, $i = 1, 2, \dots, n-1$, $y^{(i)}(0) = y_i$, $i = 1, \dots, n-1$, and $y(t) = \phi(t)$ on $[-r, 0]$.

We are now in a position to state and prove our existence result for the problem (8.48).

Theorem 8.4. *Assume that conditions (8.2.1)–(8.2.3) and (8.2.5) hold. Suppose also the following is satisfied.*

(8.4.1) *For all $(t, \bar{s}, x) \in [0, T] \times [0, T] \times \mathbb{R}^n$ and for all $y_t \in D$,*

$$\left\langle \tau_k'(x), \sum_{i=2}^{n-1} I_{k,i}(\bar{s}) \frac{(t-\bar{s})^{i-2}}{(i-2)!} + \int_{\bar{s}}^{t} \frac{(t-s)^{n-2}}{(n-2)!} f(s, y_s)\, ds \right\rangle \neq 1 \tag{8.50}$$

for $k = 1, \dots, m$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$.
Then the IVP (8.48)–(8.49) *has at least one solution on* $[-r, T]$.

Proof. The proof will be given in several steps.
Step 1. Consider the following problem:

$$y^{(n)}(t) = f(t, y_t), \quad \text{a.e. } t \in [0, T], \tag{8.51}$$

$$y^{(i)}(0) = y_i, \quad i = 1, \dots, n-1, \tag{8.52}$$

$$y(t) = \phi(t), \quad t \in [-r, 0]. \tag{8.53}$$

Transform the problem (8.51)–(8.53) into a fixed point problem. Consider the operator $N : \Omega \to \Omega$ defined by

$$N(y)(t) = \begin{cases} \phi(t) & \text{if } t \in [-r, 0], \\ \phi(0) + \displaystyle\sum_{i=1}^{n-1} y_i \frac{t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s, y_s)\, ds & \text{if } t \in [0, T]. \end{cases} \tag{8.54}$$

We will show that the operator N is completely continuous.
Claim 1. N is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in Ω.
Then

$$\begin{aligned} |N(y_n)(t) - N(y)(t)| &\leq \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} |f(s, y_{ns}) - f(s, y_s)|\, ds \\ &\leq \frac{T^{n-1}}{(n-1)!} \int_0^T |f(s, y_{ns}) - f(s, y_s)|\, ds. \end{aligned} \tag{8.55}$$

Since f is an L^1-Carathéodory function, we have by the Lebesgue dominated convergence theorem

$$\|N(y_n) - N(y)\| \leq \frac{T^{n-1}}{(n-1)!} \|f(\cdot, y_{ns}) - f(\cdot, y_s)\|_{L^1} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{8.56}$$

Claim 2. N maps bounded sets into bounded sets in Ω.

Indeed, it is enough to show that for any $p^* > 0$, there exists a positive constant ℓ such that, for each $y \in B_{p^*} = \{y \in \Omega : \|y\| \leq p^*\}$, we have $\|N(y)\| \leq \ell$. We have, for each $t \in [0, T]$,

$$|N(y)(t)| \leq \|\phi\|_{\mathcal{D}} + \sum_{i=1}^{n-1} |y_i| \frac{t^i}{i!} + \frac{T^{n-1}}{(n-1)!} \int_0^t |f(s, y_s)| ds \tag{8.57}$$

$$\leq \|\phi\|_{\mathcal{D}} + \sum_{i=1}^{n-1} |y_i| \frac{T^i}{i!} + \frac{T^{n-1}}{(n-1)!} \|h_{p^*}\|_{L^1}. \tag{8.58}$$

Thus

$$\|N(y)\|_\infty \leq \|\phi\|_{\mathcal{D}} + \sum_{i=1}^{n-1} |y_i| \frac{T^i}{i!} + \frac{T^{n-1}}{(n-1)!} \|h_{p^*}\|_{L^1} := \ell. \tag{8.59}$$

Claim 3. N maps bounded sets into equicontinuous sets of Ω.

Let $l_1, l_2 \in [0, T]$, $l_1 < l_2$, and let B_{p^*} be a bounded set of Ω as in Claim 2, and let $y \in B_{p^*}$. Then

$$\begin{aligned} |N(y)(l_2) - N(y)(l_1)| \leq{}& \sum_{i=1}^{n-1} |y_i| \frac{{l_1}^i - {l_2}^i}{i!} + \int_{l_1}^{l_2} \frac{|l_2 - s|^{n-1}}{(n-1)!} h_{p^*}(s) ds \\ &+ \int_0^{l_1} \frac{\left|(l_2 - s)^{n-1} - (l_2 - s)^{n-1}\right|}{(n-1)!} h_{p^*}(s) ds. \end{aligned} \tag{8.60}$$

As $l_2 \to l_1$, the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $l_1 < l_2 \leq 0$ and $l_1 \leq 0 \leq l_2$ is obvious. As a consequence of Claims 1 to 3 together with the Arzela-Ascoli theorem, we can conclude that $N : \Omega \to \Omega$ is completely continuous.

Claim 4. Now it remains to show that the set

$$\mathcal{E}(N) := \{y \in \Omega : y = \lambda N(y) \text{ for some } 0 < \lambda < 1\} \tag{8.61}$$

is bounded.

Let $y \in \mathcal{E}(N)$. Then $y = \lambda N(y)$ for some $0 < \lambda < 1$. Thus, for each $t \in [0, T]$,

$$y(t) = \lambda\left(\phi(0) + \sum_{i=1}^{n-1} y_i \frac{t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s, y_s)\,ds\right). \tag{8.62}$$

This implies that, for each $t \in J$, we have

$$|y(t)| \leq \|\phi\|_{\mathcal{D}} + \sum_{i=1}^{n-1} |y_i| \frac{T^i}{i!} + \int_0^t \frac{T^{n-1}}{(n-1)!} p(s)\psi(\|y_s\|_{\mathcal{D}})\,ds. \tag{8.63}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq T. \tag{8.64}$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0, T]$, by the previous inequality we have, for $t \in [0, T]$,

$$\mu(t) \leq \|\phi\|_{\mathcal{D}} + \sum_{i=1}^{n-1} |y_i| \frac{T^i}{i!} + \int_0^t \frac{T^{n-1}}{(n-1)!} p(s)\psi(\mu(s))\,ds. \tag{8.65}$$

If $t^* \in [-r, 0]$, then $\mu(t) \leq \|\phi\|_{\mathcal{D}}$ and the previous inequality holds. Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$\begin{aligned} v(0) &= \|\phi\|_{\mathcal{D}} + \sum_{i=1}^{n-1} |y_i| \frac{T^i}{i!}, \qquad \mu(t) \leq v(t), \quad t \in [0, T], \\ v'(t) &= \frac{T^{n-1}}{(n-1)!} p(t)\psi(\mu(t)), \quad \text{a.e. } t \in [0, T]. \end{aligned} \tag{8.66}$$

Using the nondecreasing character of ψ, we get

$$v'(t) \leq \frac{T^{n-1}}{(n-1)!} p(t)\psi(v(t)), \quad \text{a.e. } t \in [0, T]. \tag{8.67}$$

This implies that, for each $t \in [0, T]$,

$$\int_{v(0)}^{v(t)} \frac{ds}{\psi(s)} \leq \frac{T^{n-1}}{(n-1)!} \int_0^T p(s)\,ds < +\infty. \tag{8.68}$$

Thus there exists a constant K such that $v(t) \leq K$, $t \in [0, T]$, and hence $\mu(t) \leq K$, $t \in [0, T]$. Since for every $t \in [0, T]$, $\|y_t\| \leq \mu(t)$, we have

$$\|y\| \leq K' = \max\{\|\phi\|_{\mathcal{D}}, K\}, \tag{8.69}$$

where K' depends on T and on the functions p and ψ. This shows that $\mathcal{E}(N)$ is bounded.

Set $X := \Omega$. As a consequence of Schaefer's theorem, we deduce that N has a fixed point y which is a solution to problem (8.51)-(8.52). Denote this solution by y_1.

Define the function

$$r_{k,1}(t) = \tau_k\big(y_1(t)\big) - t, \quad \text{for } t \geq 0. \tag{8.70}$$

Hypothesis (8.2.2) implies that

$$r_{k,1}(0) \neq 0, \quad \text{for } k = 1, \ldots, m. \tag{8.71}$$

If

$$r_{k,1}(t) \neq 0 \quad \text{on } [0, T], \text{ for } k = 1, \ldots, m, \tag{8.72}$$

that is,

$$t \neq \tau_k\big(y_1(t)\big) \quad \text{on } [0, T] \text{ and for } k = 1, \ldots, m, \tag{8.73}$$

then y_1 is a solution of the problem (8.48). It remains to consider the case when

$$r_{k,1}(t) = 0, \quad \text{for some } t \in [0, T],\ k = 1, \ldots, m. \tag{8.74}$$

Now since

$$r_{k,1}(0) \neq 0 \tag{8.75}$$

and $r_{k,1}$ is continuous, there exists $t_1 > 0$ such that

$$r_{k,1}(t_1) = 0, \quad r_{k,1}(t) \neq 0, \quad \forall t \in [0, t_1). \tag{8.76}$$

Step 2. Consider now the following problem:

$$y^{(n)}(t) = f(t, y_t), \quad \text{a.e. } t \in [t_1, T], \tag{8.77}$$

$$y^{(i)}(t_1^+) = I_{1,i}\big(y_1(t_1)\big), \quad i = 1, \ldots, n-1, \tag{8.78}$$

$$y(t) = y_1(t), \quad t \in [t_1 - r, t_1]. \tag{8.79}$$

Transform the problem (8.77)–(8.79) into a fixed point problem. Consider the operator $N_1 : \mathrm{PC}([t_1 - r, T], \mathbb{R}^n) \to \mathrm{PC}([t_1 - r, T], \mathbb{R}^n)$ defined by

$$N_1(y)(t) = \begin{cases} y_1(t), & t \in [t_1 - r, t_1] \\ \displaystyle\sum_{i=1}^{n-1} I_{1,i}\big(y_1(t_1)\big)\frac{(t-t_1)^i}{i!} + \int_{t_1}^{t} \frac{(t-s)^{n-1}}{(n-1)!} f(s, y_s)\,ds, & t \in [t_1, T]. \end{cases} \tag{8.80}$$

As in Step 1, we can show that N_1 is completely continuous, and the set

$$\mathcal{E}(N_1) := \{y \in \mathrm{PC}([t_1 - r, T], \mathbb{R}^n) : y = \lambda N_1(y) \text{ for some } 0 < \lambda < 1\} \tag{8.81}$$

is bounded.

Set $X := \mathrm{PC}([t_1 - r, T], \mathbb{R}^n)$. As a consequence of Schaefer's theorem, we deduce that N_1 has a fixed point y which is a solution to problem (8.53)-(8.77). Denote this solution by y_2. Define

$$r_{k,2}(t) = \tau_k(y_2(t)) - t, \quad \text{for } t \geq t_1. \tag{8.82}$$

If

$$r_{k,2}(t) \neq 0 \quad \text{on } (t_1, T], \ \forall k = 1, \dots, m, \tag{8.83}$$

then

$$y(t) = \begin{cases} y_1(t) & \text{if } t \in [0, t_1], \\ y_2(t) & \text{if } t \in (t_1, T], \end{cases} \tag{8.84}$$

is a solution of the problem (8.48)-(8.49). It remains to consider the case when

$$r_{k,2}(t) = 0, \quad \text{for some } t \in (t_1, T], \ k = 2, \dots, m. \tag{8.85}$$

By (8.2.6), we have

$$\begin{aligned} r_{k,2}(t_1^+) &= \tau_k(y_2(t_1^+)) - t_1 = \tau_k(I_{1,1}(y_1(t_1))) - t_1 \\ &> \tau_{k-1}(y_1(t_1)) - t_1 \geq \tau_1(y_1(t_1)) - t_1 = r_{1,1}(t_1) = 0. \end{aligned} \tag{8.86}$$

Since $r_{k,2}$ is continuous, there exists $t_2 > t_1$ such that

$$\begin{gathered} r_{k,2}(t_2) = 0, \\ r_{k,2}(t) \neq 0, \quad \forall t \in (t_1, t_2). \end{gathered} \tag{8.87}$$

Suppose now that there is $\bar{s} \in (t_1, t_2]$ such that

$$r_{1,2}(\bar{s}) = 0. \tag{8.88}$$

From (8.2.6), it follows that

$$\begin{aligned} r_{1,2}(t_1^+) &= \tau_1(y_2(t_1^+)) - t_1 = \tau_1(I_{1,1}(y_1(t_1))) - t_1 \\ &\leq \tau_1(y_1(t_1)) - t_1 = r_{1,1}(t_1) = 0. \end{aligned} \tag{8.89}$$

Thus the function $r_{1,2}$ attains a nonnegative maximum at some point $s_1 \in (t_1, T]$. Since

$$y_2'(t) = \sum_{i=2}^{n-1} I_{1,i}(y_1(t_1))\frac{(t-t_1)^{i-2}}{(i-2)!} + \int_{t_1}^{t} \frac{(t-s)^{n-2}}{(n-2)!} f(s, y_s)\,ds, \tag{8.90}$$

then

$$r_{1,2}'(s_1) = \tau_1'(y_2(s_1))y_2'(s) - 1 = 0. \tag{8.91}$$

Therefore

$$\left\langle \tau_1'(y_2(s_1)), \sum_{i=2}^{n-1} I_{1,i}(y_1(t_1))\frac{(s_1-t_1)^{i-2}}{(i-2)!} + \int_{t_1}^{s_1} \frac{(s_1-s)^{n-2}}{(n-2)!} f(s, y_s)\,ds \right\rangle = 1, \tag{8.92}$$

which is a contradiction by (8.4.1).

Step 3. We continue this process taking into account that $y_m := y\,|_{[t_m,T]}$ is a solution to the problem

$$\begin{aligned} y^{(n)}(t) &= f(t, y_t), \quad \text{a.e. } t \in (t_m, T),\\ y^{(i)}(t_m^+) &= I_{m,i}(y_{m-1}(t_m)), \quad i = 1, \dots, n-1,\\ y(t) &= y_{m-1}(t), \quad t \in [t_m - r, t_m],\ i = 1, \dots, n-1. \end{aligned} \tag{8.93}$$

The solution y of the problem (8.48) is then defined by

$$y(t) = \begin{cases} y_1(t) & \text{if } t \in [-r, t_1],\\ y_2(t) & \text{if } t \in (t_1, t_2],\\ \vdots & \\ y_m(t) & \text{if } t \in (t_m, T]. \end{cases} \tag{8.94}$$

$\square$

8.4. Boundary value problems for differential inclusions with variable times

This section is concerned with the existence of solutions for first-order boundary value problems with impulsive effects as

$$\begin{aligned} y'(t) &\in F(t, y(t)), \quad t \in J = [0, T],\ t \neq \tau_k(y(t)),\ k = 1, \dots, m,\\ y(t^+) &= I_k(y(t)), \quad t = \tau_k(y(t)),\ k = 1, \dots, m,\\ L(y(0), y(T)) &= 0, \end{aligned} \tag{8.95}$$

where $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a compact convex-valued multivalued map, and $L : \mathbb{R}^2 \to \mathbb{R}$ is a single-valued map, $\tau_k : \mathbb{R} \to \mathbb{R}$, $I_k \in C(\mathbb{R}, \mathbb{R})$ ($k = 1, 2, \dots, m$), are bounded maps, $y(t^-)$ and $y(t^+)$ represent the left and right limits of $y(s)$ at $s = t$, respectively.

So let us start by defining what we mean by a solution of problem (8.95).

Definition 8.5. A function $y \in \mathrm{PC}(J,\mathbb{R}) \cap (\mathrm{AC}(t_k, t_{k+1}),\mathbb{R})$, $k = 0,\ldots,m$, is said to be a solution of (8.95) if there exists $v \in L^1(J,\mathbb{R})$ with $v(t) \in F(t, y(t))$ for a.e. $t \in J$ such that y satisfies the differential equation $y'(t) = v(t)$ a.e. on J, $t \neq \tau_k(y(t))$, $k = 1,\ldots,m$, and the conditions $y(t^+) = I_k(y(t))$, $t = \tau_k(y(t))$, $k = 1,\ldots,m$, and $L(y(0), y(T)) = 0$.

The following concept of lower and upper solutions for (8.95) has been introduced by Benchohra et al. [53] for periodic boundary value problems for impulsive differential inclusions at fixed moments (see also [35]). It will be the basic tool in the approach that follows.

Definition 8.6. A function $\alpha \in \mathrm{PC}(J,\mathbb{R}) \cap (\mathrm{AC}(t_k, t_{k+1}),\mathbb{R})$, $k = 0,\ldots,m$, is said to be a lower solution of (8.95) if there exists $v_1 \in L^1(J,\mathbb{R})$ such that $v_1(t) \in F(t,\alpha(t))$ a.e. on J, $\alpha'(t) \le v_1(t)$ a.e. on J, $t \neq \tau_k(\alpha(t))$, $\alpha(t^+) \le I_k(\alpha(t^-))$, $t = \tau_k(\alpha(t))$, $k = 1,\ldots,m$, and $L(\alpha(0),\alpha(T)) \le 0$.

Similarly a function $\beta \in \mathrm{PC}(J,\mathbb{R}) \cap (\mathrm{AC}(t_k, t_{k+1}),\mathbb{R})$, $k = 0,\ldots,m$, is said to be an upper solution of (8.95) if there exists $v_2 \in L^1(J,\mathbb{R})$ such that $v_2(t) \in F(t,\beta(t))$ a.e. on J, $\beta'(t) \ge v_2(t)$ a.e. on J, $t_k \neq \tau_k(\beta(t))$, $\beta(t^+) \ge I_k(\beta(t^-))$, $t = \tau(\beta_k(t))$, $k = 1,\ldots,m$, and $L(\beta(0),\beta(T)) \ge 0$.

We are now in a position to state and prove our existence result for the problem (8.95).

Theorem 8.7. *Assume that the following hypotheses hold.*

(8.7.1) $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is an L^1-Carathéodory multivalued map.*

(8.7.2) *There exist α and $\beta \in \mathrm{PC}(J,\mathbb{R})$, lower and upper solutions for the problem* (8.95) *such that $\alpha \le \beta$.*

(8.7.3) *L is a continuous single-valued map in $(x, y) \in [\alpha(0),\beta(0)] \times [\alpha(T), \beta(T)]$, nonincreasing and linear in $y \in [\alpha(T),\beta(T)]$, and $L(x,0) = 0$ for each $x \in \mathbb{R}$.*

(8.7.4) *For each $k = 1,\ldots,m$, the function I_k is nondecreasing.*

(8.7.5) *The functions $\tau_k \in C^1(\mathbb{R},\mathbb{R})$ for $k = 1,\ldots,m$. Moreover,*

$$0 = \tau_0(x) < \tau_1(x) < \cdots < \tau_m(x) < \tau_{m+1}(x) = T, \quad \forall x \in \mathbb{R}. \tag{8.96}$$

(8.7.6) *For all $y \in C([0,T],\mathbb{R})$ and for all $v \in S_{F,y}$,*

$$\tau_k'(y(t))v(t) \neq 1, \quad \text{for } t \in [0,T],\ k = 1,\ldots,m. \tag{8.97}$$

(8.7.7) *For all $x \in \mathbb{R}$,*

$$\tau_k(I_k(x)) \le \tau_k(x) < \tau_{k+1}(I_k(x)), \quad \text{for } k = 1,\ldots,m. \tag{8.98}$$

Then the problem (8.95) *has at least one solution y such that*

$$\alpha(t) \le y(t) \le \beta(t), \quad \forall t \in J. \tag{8.99}$$

Proof. The proof will be given in several steps.
Step 1. Consider the following problem:

$$\begin{gathered} y'(t) \in F(t, y(t)), \quad \text{a.e. } t \in [0, T], \\ L(y(0), y(T)) = 0. \end{gathered} \tag{8.100}$$

Transform the problem (8.100) into a fixed point problem. Consider the modified problem

$$y'(t) + y(t) \in F_1(t, y(t)), \quad \text{a.e.} t \in J, \tag{8.101}$$

$$y(0) = \gamma(0, y(0) - L(\overline{y}(0), \overline{y}(T))), \tag{8.102}$$

where $F_1(t, y) = F(t, \gamma(t, y)) + \gamma(t, y)$, $\gamma(t, y) = \max(\alpha(t)), \min(y, \beta(t))$, and $\overline{y}(t) = \gamma(t, y)$. A solution to (8.101)-(8.102) is a fixed point of the operator $N : \mathrm{PC}(J, \mathbb{R}) \to \mathcal{P}(\mathrm{PC}(J, \mathbb{R}))$ defined by

$$N(y) = \left\{ h \in \mathrm{PC}(J, \mathbb{R}) : h(t) = y(0) + \int_0^t [g(s) + \overline{y}(s) - y(s)] ds \right\}, \tag{8.103}$$

where $g \in \widetilde{S}_{F,\overline{y}}$, and

$$\begin{gathered} \widetilde{S}_{F,\overline{y}} = \{ v \in S_{F,\overline{y}} : v(t) \ge v_1(t) \text{ a.e. on } A_1 \text{ and } v(t) \le v_2(t) \text{ a.e. on } A_2 \}, \\ S_{F,\overline{y}} = \{ v \in L^1(J, \mathbb{R}) : v(t) \in F(t, \overline{y}(t)) \text{ for a.e. } t \in J \}, \\ A_1 = \{ t \in J : y(t) < \alpha(t) \le \beta(t) \}, \qquad A_2 = \{ t \in J : \alpha(t) \le \beta(t) < y(t) \}. \end{gathered} \tag{8.104}$$

Remark 8.8. (i) Notice that F_1 is an L^1-Carathéodory multivalued map with compact convex values and there exists $\varphi \in L^1(J, \mathbb{R})$ such that

$$\|F_1(t, y)\| \le \varphi(t) + \max\left(\sup_{t \in J} |\alpha(t)|, \sup_{t \in J} |\beta(t)| \right). \tag{8.105}$$

(ii) By the definition of γ, it is clear that

$$\begin{gathered} \alpha(0) \le y(0) \le \beta(0), \\ I_k(\alpha(t)) \le I_k(\gamma(t, y(t))) \le I_k(\beta(t)), \quad k = 1, \ldots, m. \end{gathered} \tag{8.106}$$

In order to apply the nonlinear alternative of Leray-Schauder type, we will first show that N is completely continuous with convex values. The proof will be given in several claims.

Claim 1. $N(y)$ is convex for each $y \in \mathrm{PC}(J,\mathbb{R})$.

Indeed, if h_1, h_2 belong to $N(y)$, then there exist $g_1, g_2 \in \widetilde{S}_{F,\overline{y}}$ such that, for each $t \in J$, we have

$$h_i(t) = y(0) + \int_0^t [g_i(s) + \overline{y}(s) - y(s)]ds, \quad i = 1,2. \tag{8.107}$$

Let $0 \le d \le 1$. Then, for each $t \in J$, we have

$$(dh_1 + (1-d)h_2)(t) = \int_0^t [dg_1(s) + (1-d)g_2(s) + \overline{y}(s) - y(s)]ds. \tag{8.108}$$

Since $\widetilde{S}_{F_1,\overline{y}}$ is convex (because F_1 has convex values), then

$$dh_1 + (1-d)h_2 \in N(y). \tag{8.109}$$

Claim 2. N maps bounded sets into bounded sets in $\mathrm{PC}(J,\mathbb{R})$.

Indeed, it is enough to show that for each $q > 0$ there exists a positive constant ℓ such that for each $y \in B_q = \{y \in C(J,\mathbb{R}) : \|y\|_{\mathrm{PC}} \le q\}$, one has $\|N(y)\|_{\mathrm{PC}} := \sup\{\|h\|_{\mathrm{PC}} : h \in N(y)\} \le \ell$.

Let $y \in B_q$ and $h \in N(y)$, then there exists $g \in \widetilde{S}_{F,\overline{y}}$ such that, for each $t \in J$, we have

$$h(t) = y(0) + \int_0^t [g(s) + \overline{y}(s) - y(s)]ds. \tag{8.110}$$

By (8.7.1), we have, for each $t \in J$,

$$\begin{aligned} |h(t)| &\le |y(0)| + \int_0^T [|g(s)| + |\overline{y}(s)| + |y(s)|]ds \\ &\le \max(|\alpha(0)|, |\beta(0)|) + \|\phi_q\|_{L^1} \\ &\quad + T\max\left(q, \sup_{t\in J}|\alpha(t)|, \sup_{t\in J}|\beta(t)|\right) + Tq. \end{aligned} \tag{8.111}$$

Claim 3. N maps bounded set into equicontinuous sets of $\mathrm{PC}(J,\mathbb{R})$.

Let $u_1, u_2 \in J$, $u_1 < u_2$, and let B_q be a bounded set of $\mathrm{PC}(J,\mathbb{R})$ as in Claim 2. Let $y \in B_q$ and $h \in N(y)$. Then there exists $g \in \widetilde{S}_{F,\overline{y}}$ such that, for each $t \in J$, we have

$$h(t) = y(0) + \int_0^t [g(s) + \overline{y}(s) - y(s)]ds. \tag{8.112}$$

Then

$$
\begin{aligned}
|h(u_2)-h(u_1)| \le{}& \int_{u_1}^{u_2} \phi_q(s)ds + (u_2-u_1)\max\left(q, \sup_{t\in J}|\alpha(t)|, \sup_{t\in J}|\beta(t)|\right) \\
&+ (u_2-u_1)q.
\end{aligned}
\tag{8.113}
$$

As $u_2 \to u_1$, the right-hand side of the above inequality tends to zero.

As a consequence of Claims 1 to 3 together with the Arzela-Ascoli theorem, we can conclude that $N : \mathrm{PC}(J,\mathbb{R}) \to \mathcal{P}(\mathrm{PC}(J,\mathbb{R}))$ is a completely continuous multivalued map, and therefore, a condensing map.

Claim 4. N has a closed graph.

Let $y_n \to y_*$, $h_n \in N(y_n)$, and $h_n \to h_*$. We will prove that $h_* \in N(y_*)$.

$h_n \in N(y_n)$ means that there exists $g_n \in \widetilde{S}_{F,\overline{y}_n}$ such that for each $t \in J$,

$$
h_n(t) = y_n(0) + \int_0^t [g_n(s) + \overline{y}_n(s) - y_n(s)]ds. \tag{8.114}
$$

We must prove that there exists $g_* \in \widetilde{S}_{F,\overline{y}_*}$ such that, for each $t \in J$,

$$
h_*(t) = y_*(0) + \int_0^t [g_*(s) + \overline{y}_*(s) - y_*(s)]ds. \tag{8.115}
$$

Since y is continuous, then we have

$$
\begin{aligned}
&\left\|\left(h_n - y_n(0) - \int_0^t [\overline{y}_n(s) - y_n(s)]ds\right)\right. \\
&\quad \left.- \left(h_* - y_*(0) - \int_0^t [\overline{y}_*(s) - y_*(s)]ds\right)\right\|_\infty \longrightarrow 0,
\end{aligned}
\tag{8.116}
$$

as $n \to \infty$.

Consider the linear continuous operator

$$
\begin{gathered}
\Gamma : L^1(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R}), \\
g \longmapsto (\Gamma g)(t) = \int_0^t g(s)ds.
\end{gathered}
\tag{8.117}
$$

From Lemma 1.28, it follows that $\Gamma \circ \widetilde{S}_F$ is a closed graph operator.

Moreover, we have

$$
h_n(t) - y_n(0) - \int_0^t [\overline{y}_n(s) - y_n(s)]ds \in \Gamma(\widetilde{S}_{F,\overline{y}_n}). \tag{8.118}
$$

Since $y_n \to y_*$, it follows from Lemma 1.28 that

$$\left(h_*(t) - y_*(0) - \int_0^t [\overline{y}_*(s) - y_*(s)]ds\right) = \int_0^t g_*(s)ds \tag{8.119}$$

for some $g_* \in \widetilde{S}_{F,y_*}$.

Claim 5. A priori bounds on solutions.

Let y be such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. Then

$$y(t) = \lambda y(0) + \lambda \int_0^t [g(s) - \overline{y}(s) - y(s)]ds. \tag{8.120}$$

This implies by Remark 8.8 that, for each $t \in J$, we have

$$\begin{aligned} |y(t)| &\leq |y(0)| + \int_0^t [|g(s)| + |\overline{y}(s)| + |y(s)|]ds \\ &\leq \max(|\alpha(0)|, |\beta(0)|) + \|\varphi\|_{L^1} \\ &\quad + T\max\left(\sup_{t\in J}|\alpha(t)|, \sup_{t\in J}|\beta(t)|\right) + \int_0^t |y(s)|ds. \end{aligned} \tag{8.121}$$

Set

$$z_0 = \max(|\alpha(0)|, |\beta(0)|) + \|\varphi\|_{L^1} + T\max\left(\sup_{t\in J}|\alpha(t)|, \sup_{t\in J}|\beta(t)|\right). \tag{8.122}$$

Using Gronwall's lemma, we get, for each $t \in J$,

$$|y(t)| \leq z_0 e^t. \tag{8.123}$$

Thus

$$\|y\|_{\mathrm{PC}} \leq z_0 e^T. \tag{8.124}$$

Set

$$U = \{y \in \mathrm{PC}(J,\mathbb{R}) : \|y\|_{\mathrm{PC}} < z_0 e^T + 1\}, \tag{8.125}$$

and consider the operator N defined on $\overline{U}$. From the choice of U there is no $y \in \partial U$ such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray Schauder type [157], we deduce that N has a fixed point y_1 in U is a solution of the problem (8.101)-(8.102).

Claim 6. The solution y of (8.101)-(8.102) satisfies

$$\alpha(t) \leq y(t) \leq \beta(t), \quad \forall t \in J. \tag{8.126}$$

Let y be a solution to (8.101)-(8.102). We prove that

$$\alpha(t) \leq y(t), \quad \forall t \in J. \tag{8.127}$$

Suppose not. Then there exist $c_1, c_2 \in J$, $c_1 < c_2$, such that

$$y(c_1) = \alpha(c_1), \qquad y(t) < \alpha(t), \quad \forall t \in (c_1, c_2). \tag{8.128}$$

In view of the definition of γ, one has

$$y'(t) + y(t) \in F(t, \alpha(t)) + \alpha(t), \quad \text{a.e. on } (c_1, c_2). \tag{8.129}$$

Thus there exists $v(t) \in F(t, \alpha(t))$ a.e. on (c_1, c_2), $v(t) \geq v_1(t)$ a.e. on $(c_1, t]$ such that

$$y'(t) + y(t) = v(t) + \alpha(t) \quad \text{a.e on } (c_1, t]. \tag{8.130}$$

An integration on $(c_1, t]$ yields

$$y(t) - y(c_1) = \int_{c_1}^{t} (v(s) - y(s) + \alpha(s))ds > \int_{c_1}^{t} v(s)ds. \tag{8.131}$$

Using the fact that α is a lower solution to (8.95), we get

$$\alpha(t) - \alpha(c_1) \leq \int_{c_1}^{t} v_1(s)ds, \quad t \in (c_1, c_2). \tag{8.132}$$

It follows that from the facts $y(c_1) = \alpha(c_1)$, $v(t) \geq v_1(t)$, we get

$$y(t) > \alpha(t), \quad \text{for each } t \in (c_1, c_2), \tag{8.133}$$

which is a contradiction. Consequently,

$$\alpha(t) \leq y(t), \quad \forall t \in J. \tag{8.134}$$

Analogously, we can prove that

$$y(t) \leq \beta(t), \quad \forall t \in J. \tag{8.135}$$

This shows that the problem (8.101)-(8.102) has a solution in the interval $[\alpha, \beta]$.

Finally, we prove that every solution of (8.101)-(8.102) is also a solution to (8.95). We need to show only that

$$\alpha(0) \leq y(0) - L(\overline{y}(0), \overline{y}(T)) \leq \beta(0). \tag{8.136}$$

Notice first that we readily have

$$\alpha(T) \leq y(T) \leq \beta(T). \tag{8.137}$$

Suppose now that $y(0) - L(\overline{y}(0), \overline{y}(T)) \leq \alpha(0)$. Then $y(0) = \alpha(0)$ and

$$y(0) - L(\alpha(T), \overline{y}(0)) \leq \alpha(0). \tag{8.138}$$

Since L is nonincreasing in y, we have

$$\alpha(0) \leq \alpha(0) - L(\alpha(0), \alpha(T)) \leq \alpha(0) - L(\alpha(0), \overline{y}(T)) < \alpha(0), \tag{8.139}$$

which is a contradiction. Analogously, we can prove that

$$y(0) - L(\overline{y}(0), \overline{y}(T)) \leq \beta(0). \tag{8.140}$$

Then y is a solution to (8.100). Denote this solution by y_1. Define the function

$$r_{k,1}(t) = \tau_k(y_1(t)) - t, \quad \text{for } t \geq 0. \tag{8.141}$$

(8.7.5) implies that

$$r_{k,1}(0) \neq 0, \quad \text{for } k = 1, \ldots, m. \tag{8.142}$$

If

$$r_{k,1}(t) \neq 0 \quad \text{on } [0, T], \text{ for } k = 1, \ldots, m, \tag{8.143}$$

that is,

$$t \neq \tau_k(y_1(t)) \quad \text{on } [0, T] \text{ and for } k = 1, \ldots, m, \tag{8.144}$$

then y_1 is a solution of the problem (8.95).

It remains to consider the case when

$$r_{k,1}(t) = 0, \quad \text{for some } t \in [0, T]. \tag{8.145}$$

Now since

$$r_{k,1}(0) \neq 0 \tag{8.146}$$

and $r_{k,1}$ is continuous, there exists $t_1 > 0$ such that

$$r_{k,1}(t_1) = 0, \quad r_{k,1}(t) \neq 0, \quad \forall t \in [0, t_1). \tag{8.147}$$

Step 2. Consider now the problem

$$\begin{gathered} y'(t) \in F(t, y(t)), \quad \text{a.e. } t \in [t_1, T], \\ y(t_1^+) = I_1(y_1(t_1)). \end{gathered} \tag{8.148}$$

Transform the problem (8.148) into a fixed point problem. Consider the modified problem

$$y'(t) + y(t) \in F(t, \gamma(t, y)) + \gamma(t, y), \quad \text{a.e. } t \in [t_1, T], \tag{8.149}$$

$$y(t_1^+) = I_1(\gamma(t_1^-, y(t_1^-))). \tag{8.150}$$

A solution to (8.149)-(8.150) is a fixed point of the operator $N_1 : \mathrm{PC}([t_1, T], \mathbb{R}) \to \mathcal{P}(\mathrm{PC}([t_1, T], \mathbb{R}))$ defined by

$$\begin{aligned} &N_1(y) \\ &\quad = \left\{ h \in \mathrm{PC}([t_1, T], \mathbb{R}) : h(t) = I_1(\gamma(t_1^-, y(t_1^-))) + \int_{t_1}^{t} [g(s) + \overline{y}(s) - y(s)] ds \right\}, \end{aligned} \tag{8.151}$$

where $g \in \widetilde{S}_{F,\overline{y}}$.

As in Step 1, we can show that N_1 is completely continuous, and there exists a constant $M_1 > 0$ such that for any solution y of problem (8.149)-(8.150) one has

$$|y(t)| \leq M_1, \quad \text{for each } t \in [t_1, T]. \tag{8.152}$$

Let the set

$$U_2 = \{ y \in C([t_1, T], \mathbb{R}) : \|y\|_{\mathrm{PC}} < M_1 + 1 \}. \tag{8.153}$$

As in Step 1, we show that the operator $N_1 : \overline{U}_2 \to \mathcal{P}(\mathrm{PC}([t_1, T], \mathbb{R}^n))$ is completely continuous. From the choice of U_2 there is no $y \in \partial U_2$ such that $y \in \lambda N_2(y)$ for some $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Leray Schauder type [157] we deduce that N_2 has a fixed point y in U_2 which is a solution to problem (8.148). Note this solution by y_2. Define

$$r_{k,2}(t) = \tau_k(y_2(t)) - t, \quad \text{for } t \geq t_1. \tag{8.154}$$

If

$$r_{k,2}(t) \neq 0 \quad \text{on } (t_1, T], \ \forall k = 1, \dots, m, \tag{8.155}$$

then

$$y(t) = \begin{cases} y_1(t) & \text{if } t \in [0, t_1], \\ y_2(t) & \text{if } t \in (t_1, T], \end{cases} \tag{8.156}$$

is a solution of problem (8.95).

It remains to consider the case when

$$r_{k,2}(t) = 0, \quad \text{for some } t \in (t_1, T], \ k = 2, \dots, m. \tag{8.157}$$

By (8.7.7), we have

$$\begin{aligned} r_{k,2}(t_1^+) &= \tau_k(y_2(t_1^+)) - t_1 = \tau_k(I_1(y_1(t_1))) - t_1 \\ &\geq \tau_1(y_1(t_1)) - t_1 = r_{1,1}(t_1) = 0. \end{aligned} \tag{8.158}$$

Since $r_{k,2}$ is continuous, there exists $t_2 > t_1$ such that

$$\begin{gathered} r_{k,2}(t_2) = 0, \\ r_{k,2}(t) \neq 0, \quad \forall t \in (t_1, t_2). \end{gathered} \tag{8.159}$$

Suppose now that there is $\bar{s} \in (t_1, t_2]$ such that

$$r_{1,2}(\bar{s}) = 0. \tag{8.160}$$

From (8.7.5), it follows that

$$\begin{aligned} r_{1,2}(t_1^+) &= \tau_1(y_2(t_1^+)) - t_1 = \tau_1(I_1(y_1(t_1))) - t_1 \\ &\leq \tau_1(y_1(t_1)) - t_1 = r_{1,1}(t_1) = 0. \end{aligned} \tag{8.161}$$

Thus the function $r_{1,2}$ attains a nonnegative maximum at some point $s_1 \in (t_1, T]$. Since

$$y_2'(t) \in F(t, y_2(t)), \quad \text{a.e. } t \in (t_1, T), \tag{8.162}$$

then there exist $v(\cdot) \in L^1((t_1, T))$ with $v(t) \in F(t, y_2(t))$, a.e. $t \in (t_1, T)$ such that

$$y_2'(t) = v(t), \quad \text{a.e. } t \in (t_1, T]. \tag{8.163}$$

Thus

$$r_{1,2}'(s_1) = \tau_1'(y_2(s_1))v(s_1) - 1 = 0. \tag{8.164}$$

Therefore

$$\tau_1'(y_2(s_1))v(s_1) = 1, \tag{8.165}$$

which contradicts (8.7.6).

Step 3. We continue this process taking into account that $y_m := y\,|_{[t_m,T]}$ is a solution to the problem

$$\begin{gathered} y'(t) \in F(t, y(t)), \quad \text{a.e. } t \in (t_m, T), \\ y(t_m^+) = I_m(y_{m-1}(t_m^-)). \end{gathered} \tag{8.166}$$

Consider the modified problem

$$\begin{gathered} y'(t) + y(t) \in F(t, \gamma(t, y)) + \gamma(t, y), \quad \text{a.e. } t \in [t_m, T], \\ y(t_m^+) = I_m(\gamma(t_m^-, y(t_m^-))). \end{gathered} \tag{8.167}$$

Transform the problem into a fixed point problem. Consider the operator N_m : $\mathrm{PC}([t_m, T], \mathbb{R}) \to \mathcal{P}(\mathrm{PC}([t_m, T], \mathbb{R}))$ defined by

$$N_m(y) = \left\{h \in C([t_m, T], \mathbb{R}) : h(t) = I_m(\gamma(t_m^-, y(t_m^-))) + \int_{t_m}^t [g(s) + \overline{y}(s) - y(s)]ds\right\}, \tag{8.168}$$

where $g \in \widetilde{S}_{F,\overline{y}}$. By Remark 8.8 and using Gronwall's lemma there exists M_m such that for every possible solution y of problem (8.167), we have

$$\|y\|_{\mathrm{PC}} \le M_m. \tag{8.169}$$

Let the set

$$C_1 = \{y \in \mathrm{PC}([t_m, T], \mathbb{R}^n) : L(y_1(0), y(T)) = 0\}. \tag{8.170}$$

From (8.7.3), C_1 is convex. Set

$$U_m = \{y \in C_1 : \|y\|_{\mathrm{PC}} < M_m + 1\}. \tag{8.171}$$

As in Step 1, we show that the operator $N_m : \overline{U}_m \to \mathcal{P}(\mathrm{PC}([t_m, T], \mathbb{R}))$ is completely continuous. From the choice of U_m there is no $y \in \partial U_m$ such that $y \in \lambda N_m(y)$ for some $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Leray-Schauder type, we deduce that N_m has a fixed point y in U_m which is a solution of the problem (8.166), and

$$\alpha(t) \le y(t) \le \beta(t), \quad t \in [t_m, T]. \tag{8.172}$$

Since $y(t, y) = y$ for all $y \in [\alpha, \beta]$, then y is a solution to the problem (8.102)–(8.149). Denote this solution by y_m.

The solution y of the problem (8.95) is then defined by

$$y(t) = \begin{cases} y_1(t) & \text{if } t \in [0, t_1], \\ y_2(t) & \text{if } t \in (t_1, t_2], \\ \vdots & \\ y_m(t) & \text{if } t \in (t_m, T]. \end{cases} \tag{8.173}$$

□

8.5. Notes and remarks

The results of Section 8.2 were obtained by Benchohra et al. [71]. Section 8.3 appeared in [70]. The results of Section 8.4 were obtained by Benchohra et al. [45].

9 Nondensely defined impulsive differential equations & inclusions

9.1. Introduction

This chapter deals with semilinear functional differential equations and functional differential inclusions involving linear operators that are nondensely defined on a Banach space. This chapter extends several previous results of this book that were devoted to semilinear problems with densely defined operators. Some of the results of this chapter were first presented in the work by Benchohra et al. [76].

9.2. Nondensely defined impulsive semilinear differential equations with nonlocal conditions

In this section, we will prove existence results for an evolution equation with nonlocal conditions of the form

$$y'(t) = Ay(t) + F(t, y(t)), \quad t \in J := [0, T],\ t \neq t_k,\ k = 1, \dots, m, \tag{9.1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{9.2}$$

$$y(0) + g(y) = y_0, \tag{9.3}$$

where $A : D(A) \subset E \to E$ is a nondensely defined closed linear operator, $F : J \times E \to E$ is continuous, $g : C(J', E) \to E$, $(J' = J \setminus \{t_1, \dots, t_m\})$, $I_k : E \to \overline{D(A)}$, $k = 1, \dots, m$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^+} y(t_k - h)$, and E is a separable Banach space with norm $|\cdot|$.

As indicated in [112, 115, 126] and the references therein, the nonlocal condition $y(0) + g(y) = y_0$ can be applied to physics with better effect than the classical initial condition $y(0) = y_0$. For example, in [126], the author used

$$g(y) = \sum_{k=1}^{p} c_i y(t_i), \tag{9.4}$$

where c_i, $i = 1, \dots, p$, are given constants and $0 < t_1 < t_2 < \cdots < t_p \leq T$, to describe the diffusion phenomenon of a small amount of gas in a transparent tube. In this case, (9.4) allows the additional measurements at t_i, $i = 1, \dots, p$.

When operator A generates a C_0 semigroup, or equivalently, when a closed linear operator A satisfies

(i) $\overline{D(A)} = E$, (D means domain),

(ii) the Hille-Yosida condition; that is, there exists $M \geq 0$ and $\tau \in \mathbb{R}$ such that $(\tau, \infty) \subset \rho(A)$, $\sup\{(\lambda I - \tau)^n |(\lambda I - A)^{-n}| : \lambda > \tau,\ n \in \mathbb{N}\} \leq M$,

where $\rho(A)$ is the resolvent operator set of A and I is the identity operator, then (9.1) with nonlocal conditions has been studied extensively. Existence, uniqueness, and regularity, among other things, are derived; see [112–115, 126, 205].

However, as indicated in [124], we sometimes need to deal with nondensely defined operators. For example, when we look at a one-dimensional heat equation with Dirichlet conditions on $[0,1]$ and consider $A = \partial^2/\partial x^2$ in $C([0,1],\mathbb{R})$, in order to measure the solutions in the sup-norm, then the domain

$$D(A) = \{\phi \in C^2([0,1],\mathbb{R}) : \phi(0) = \phi(1) = 0\} \tag{9.5}$$

is not dense in $C([0,1],\mathbb{R})$ with the sup-norm. See [124] for more examples and remarks concerning nondensely defined operators.

Our purpose here is to extend the results of densely defined impulsive evolution equations with nonlocal conditions. We use Schaefers fixed point theorem and integrated semigroups to derive the existence of integral solutions (when the operator is nondensely defined).

In order to define the solution of (9.1)–(9.3) we will consider the following space:

$$\begin{aligned} \Omega = \{y : [0,T] \longrightarrow E : y_k \in C(J_k, E),\ k = 0,\dots,m, \text{ and there exist} \\ y(t_k^-),\ y(t_k^+),\ k = 1,\dots,m \text{ with } y(t_k^-) = y(t_k)\}, \end{aligned} \tag{9.6}$$

which is a Banach space with the norm

$$\|y\|_\Omega = \max\{\|y_k\|_{J_k},\ k = 0,\dots,m\}, \tag{9.7}$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}]$, $k = 0,\dots,m$.

Consider the initial value problem

$$y'(t) - Ay(t) = f(t), \quad t \in [0,T], \quad y(0) = y_0, \tag{9.8}$$

and let $(S(t))_{t\geq 0}$ be the integrated semigroup generated by A. Then since A satisfies the Hille-Yosida condition, $\|S'(t)\|_{B(E)} \leq Me^{\omega t}$, $t \geq 0$, where M and ω are from the Hille-Yosida condition (see [21, 175]).

Theorem 9.1. *Let $f : [0,T] \to E$ be a continuous function. Then, for $y_0 \in \overline{D(A)}$, there exists a unique continuous function $y : [0,T] \to E$ such that*

(i) $\int_0^t y(s)ds \in D(A)$, $t \in [0,T]$,

(ii) $y(t) = y_0 + A\int_0^t y(s)ds + \int_0^t f(s)ds$, $t \in [0,T]$,

(iii) $|y(t)| \leq Me^{\omega t}(|y_0| + \int_0^t e^{-\omega s}|f(s)|ds)$, $t \in [0,T]$.

Moreover, y satisfies the following variation of constants formula:

$$y(t) = S'(t)y_0 + \frac{d}{dt}\int_0^t S(t-s)f(s)ds, \quad t \geq 0. \tag{9.9}$$

Let $B_\lambda = \lambda R(\lambda, A) := \lambda(\lambda I - A)^{-1}$. Then (see [175]) for all $x \in \overline{D(A)}$, $B_\lambda x \to x$ as $\lambda \to \infty$. Also from the Hille-Yosida condition (with $n = 1$) it easy to see that $\lim_{\lambda\to\infty} |B_\lambda x| \leq M|x|$, since

$$|B_\lambda| = |\lambda(\lambda I - A)^{-1}| \leq \frac{M\lambda}{\lambda - \omega}. \tag{9.10}$$

Thus $\lim_{\lambda\to\infty} |B_\lambda| \leq M$. Also if y satisfies (9.9), then

$$y(t) = S'(t)y_0 + \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda f(s)ds, \quad t \geq 0. \tag{9.11}$$

Definition 9.2. Given $F \in L^1(J \times E, E)$ and $y_0 \in E$, say that $y : J \to E$ is an integral solution of (9.1)–(9.3) if

(i) $y \in \Omega$,

(ii) $\int_0^t y(s)ds \in D(A)$ for $t \in J$,

(iii) $y(t) = y_0 - g(y) + A\int_0^t y(s)ds + \int_0^t F(s, y(s))ds + \sum_{0<t_k<t} I_k(y(t_k^-))$, $t \in J$.

From (ii) it follows that $y(t) \in \overline{D(A)}$, for all $t \geq 0$. Also from (iii) it follows that $y_0 - g(y) \in \overline{D(A)}$. So, if we assume that $y_0 \in \overline{D(A)}$, we conclude that $g(y) \in \overline{D(A)}$.

Here and hereafter we assume that

(H1) A satisfies the Hille-Yosida condition.

Lemma 9.3. *If y is an integral solution of* (9.1)–(9.3), *then it is given by*

$$\begin{aligned} y(t) = {} & S'(t)[y_0 - g(y)] + \frac{d}{dt}\int_0^t S(t-s)F(s, y(s))ds \\ & + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)), \quad \textit{for } t \in J. \end{aligned} \tag{9.12}$$

Proof. Let y be a solution of problem (9.1)–(9.3). Define $w(s) = S(t-s)y(s)$. Then we have

$$\begin{aligned} w'(s) &= -S'(t-s)y(s) + S(t-s)y'(s) \\ &= -AS(t-s)y(s) - y(s) + S(t-s)y'(s) \\ &= S(t-s)[y'(s) - Ay(s)] - y(s) \\ &= S(t-s)F(s, y(s)) - y(s). \end{aligned} \tag{9.13}$$

Consider $t_k < t,\ k = 1,\dots,m$. Then integrating the previous equation we have

$$\int_0^t w'(s)ds = \int_0^t S(t-s)F(s,y(s))ds - \int_0^t y(s)ds. \tag{9.14}$$

For $k = 1$,

$$w(t) - w(0) = \int_0^t S(t-s)F(s,y(s))ds - \int_0^t y(s)ds \tag{9.15}$$

or

$$\begin{aligned}\int_0^t y(s) &= S(t)y(0) + \int_0^t S(t-s)F(s,y(s))ds \\ &= S(t)(y_0 - g(y)) + \int_0^t S(t-s)F(s,y(s))ds.\end{aligned} \tag{9.16}$$

Now, for $k = 2,\dots,m$, we have that

$$\begin{aligned}&\int_0^{t_1} w'(s)ds + \int_{t_1}^{t_2} w'(s)ds + \cdots + \int_{t_k}^t w'(s)ds \\ &\quad= \int_0^t S(t-s)F(s,y(s))ds - \int_0^t y(s)ds \\ &\quad\Longleftrightarrow w(t_1^-) - w(0) + w(t_2^-) - w(t_1^+) + \cdots + w(t_k^+) - w(t) \\ &\quad= \int_0^t S(t-s)F(s,y(s))ds - \int_0^t y(s)ds,\end{aligned}$$

$$\begin{aligned}\int_0^t y(s)ds &= w(0) + \sum_{0<t_k<t} [w(t_k^+) - w(t_k^-)] + \int_0^t S(t-s)F(s,y(s))ds \\ &= S(t)(y_0 - g(y)) + \sum_{0<t_k<t} S(t-t_k)I(y(t_k^-)) + \int_0^t S(t-s)F(s,y(s))ds.\end{aligned} \tag{9.17}$$

By differentiating the above equation we have that

$$\begin{aligned}y(t) &= S'(t)(y_0 - g(y)) \\ &\quad + \sum_{0<t_k<t} S'(t-t_k)I(y(t_k^-)) + \frac{d}{dt}\int_0^t S(t-s)F(s,y(s))ds,\end{aligned} \tag{9.18}$$

which proves the lemma. □

We set $\Omega' = \Omega \cap C(J, \overline{D(A)})$.

Now we are able to state and prove our main theorem in this section.

Theorem 9.4. *Assume that (H1) holds. Suppose also that*

(9.4.1) *for each $t \in J$, the function $F(t,\cdot)$ is continuous and for each y, the function $F(\cdot, y)$ is measurable;*

(9.4.2) *the operator $S'(t)$ is compact in $\overline{D(A)}$ whenever $t > 0$;*

(9.4.3) *there exist a continuous function $p : [0,T] \to \mathbb{R}^+$ and a continuous nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ such that*

$$|F(t,x)| \leq p(t)\psi(|x|), \quad t \in J,\ x \in E; \tag{9.19}$$

(9.4.4) *$g : \Omega' \to \overline{D(A)}$ is completely continuous (i.e., continuous and takes a bounded set into a compact set) and there exists $G > 0$ such that $|g(y)| \leq G$, for all $y \in \Omega$;*

(9.4.5) *$I_k : E \to \overline{D(A)}$ are completely continuous and there exist constants d_k, $k = 1, \ldots, m$, such that*

$$|I_k(y)| \leq d_k, \quad y \in \overline{D(A)}; \tag{9.20}$$

(9.4.6) *$y_0 \in \overline{D(A)}$ and*

$$\int_0^T \max(\omega, Mp(s))\,ds < \int_c^\infty \frac{ds}{s + \psi(s)}, \tag{9.21}$$

where $c = M(|y_0| + G + \sum_{k=1}^m e^{-\omega t_k} d_k)$ and M and ω are from the Hille-Yosida condition.

Then problem (9.1)–(9.3) *has at least one integral solution on J.*

Proof. Consider the operator $N : \Omega' \to \Omega'$ defined by

$$\begin{aligned} N(y)(t) = {} & S'(t)[y_0 - g(y)] + \frac{d}{dt}\int_0^t S(t-s)F(s,y(s))\,ds \\ & + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)), \quad t \in J. \end{aligned} \tag{9.22}$$

Step 1. N is continuous.

Let $\{y_n\}$ be a sequence in Ω' with $\lim_{n\to\infty} y_n = y$ in Ω'. By the continuity of F with respect to the second argument, we deduce that for each $s \in J$, $F(s, y_n(s))$ converges to $F(s, y(s))$ in E, and we have that

$$\begin{aligned} |N(y_n)(t) - Ny(t)| \leq e^{\omega T}\bigg[& |g(y_n) - g(y)| + \int_0^T e^{-\omega s}|F(s,y_n(s)) - F(s,y(s))|\,ds \\ & + \sum_{k=1}^m e^{-\omega t_k}|I_k(y_n(t_k^-)) - I_k(y(t_k^-))|\bigg]. \end{aligned} \tag{9.23}$$

The sequence $\{y_n\}$ is bounded in Ω'. Then by assumption (9.4.5), and using Lebesgue's dominated convergence theorem and the continuity of g, we obtain that

$$\lim_{n\to\infty} N(y_n) = N(y), \quad \text{in } \Omega', \tag{9.24}$$

which implies that the mapping N is continuous on Ω'.

Step 2. N maps bounded sets into compact sets.

First, we will prove that $\{Ny(t) : y \in B\}$ is relatively compact in E, where B is a bounded set in Ω'. Let $t \in J$ be fixed.

If $t = 0$, then $\{Ny(0) : y \in B\} = \{y_0 - g(y) : y \in B\}$ is relatively compact since we assumed that g is completely continuous.

If $t \in (0, T]$, choose ϵ such that $0 < \epsilon < t$. Then

$$\begin{aligned}
N(y)(t) &= S'(t)[y_0 - g(y)] + \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda F(s, y(s))\,ds \\
&\quad + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)) \\
&= S'(t)[y_0 - g(y)] + S'(\epsilon)\lim_{\lambda\to\infty}\int_0^{t-\epsilon} S'(t-\epsilon-s)\times B_\lambda F(s, y(s))\,ds \\
&\quad + \lim_{\lambda\to\infty}\int_{t-\epsilon}^t S'(t-s)B_\lambda F(s, y(s))\,ds + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)).
\end{aligned} \tag{9.25}$$

Since $S'(t)$ is compact, we deduce that there exists a compact set W_1 such that

$$S'(\epsilon)\lim_{\lambda\to\infty}\int_0^{t-\epsilon} S'(t-\epsilon-s)B_\lambda F(s, y(s))\,ds \in W_1, \tag{9.26}$$

for $y \in B$. Furthermore, by (9.4.4), there exists a positive constant b_1 such that

$$\left|\lim_{\lambda\to\infty}\int_{t-\epsilon}^t S'(t-s)B_\lambda F(s, y(s))\,ds\right| \le b_1\epsilon, \quad \text{for } y \in B. \tag{9.27}$$

Moreover, by (9.4.4) and since $S'(t)$ is compact, the set

$$S'(t)[y_0 - g(y)] + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)) : y \in B \tag{9.28}$$

is relatively compact. We conclude that $\{Ny(t) : y \in B\}$ is totally bounded and therefore, it is relatively compact in E.

Finally, let us show that NB is equicontinuous. For every $0 < \tau_0 < \tau \leq T$ and $y \in B$,

$$
\begin{aligned}
&|Ny(\tau) - Ny(\tau_0)| \\
&\quad = |(S'(\tau) - S'(\tau_0))[y_0 - g(y)]| \\
&\qquad + \left| \lim_{\lambda\to\infty} \int_0^{\tau_0} [S'(\tau - s) - S'(\tau_0 - s)]B_\lambda F(s, y(s))\,ds \right| \\
&\qquad + \left| \lim_{\lambda\to\infty} \int_{\tau_0}^{\tau} S'(\tau - s)B_\lambda F(s, y(s))\,ds \right| \\
&\qquad + \left| \sum_{0<t_k<\tau_0} [S'(\tau - t_k) - S'(\tau_0 - t_k)]I_k(y(t_k^-)) \right| \\
&\qquad + \left| \sum_{\tau_0<t_k<\tau} S'(\tau - t_k)I_k(y(t_k^-)) \right| \\
&\quad \leq |[S'(\tau) - S'(\tau_0)][y_0 - g(y)]| \\
&\qquad + \left| [S'(\tau - \tau_0) - I] \lim_{\lambda\to\infty} \int_0^{\tau_0} S'(\tau_0 - s)B_\lambda F(s, y(s))\,ds \right| \\
&\qquad + e^{\omega T} \lim_{\lambda\to\infty} \int_{\tau_0}^{\tau} e^{-\omega s} p(s)\psi(|y(s)|)\,ds \\
&\qquad + \sum_{0<t_k<\tau_0} \|S'(\tau - t_k) - S'(\tau_0 - t_k)\|_{B(E)} d_k \\
&\qquad + e^{\omega T} \sum_{\tau_0<t_k<\tau} e^{-\omega t_k} d_k.
\end{aligned}
\tag{9.29}
$$

The right-hand side tends to zero as $\tau \to \tau_0$, since $S'(t)$ is strongly continuous, and the compactness of $S'(t)$, $t > 0$, implies the continuity in the uniform operator topology. This proves the equicontinuity for the case where $t \neq t_i$, $i = 1, \dots, m$. It remains to examine the equicontinuity at $t = t_i$. The proof is similar to that given in Theorem 4.3. Thus, NB is equicontinuous.

The equicontinuity for $\tau_0 = 0$ is obvious. As a consequence of the above steps and the Arzelá-Ascoli theorem, we deduce that N maps B into precompact sets in $\overline{D(A)}$.

Step 3. The set

$$\Phi = \{x \in \Omega' : x = \sigma N x \text{ for some } 0 < \sigma < 1\} \tag{9.30}$$

is bounded.

For $y \in \Phi$, there exists $\sigma \in (0, 1)$ such that $y = \sigma N y$; that is,

$$
\begin{aligned}
y(t) = \sigma S'(t)[y_0 - g(y)] &+ \sigma \frac{d}{dt} \int_0^t S(t - s)F(s, y(s))\,ds \\
&+ \sigma \sum_{0<t_k<t} S'(t - t_k)I_k(y(t_k^-)), \quad t \in J.
\end{aligned}
\tag{9.31}
$$

Using assumptions (9.4.3)–(9.4.6), we get

$$e^{-\omega t}|y(t)| \le M\left[|y_0| + G + \int_0^t e^{-\omega s}p(s)\psi(|y(s)|)ds + \sum_{k=1}^{m} e^{-\omega t_k}d_k\right]. \tag{9.32}$$

Let $v(t)$ denote the right-hand side of the above inequality, then

$$\begin{aligned} v'(t) &= Me^{-\omega t}p(t)\psi(|y(t)|), \quad \text{for } t \in J, \\ v(0) &= M\left(y_0 + G + \sum_{k=1}^{m} e^{-\omega t_k}d_k\right). \end{aligned} \tag{9.33}$$

From (9.32), we have that $|y(t)| \le e^{-\omega t}v(t)$. Then

$$v'(t) \le Me^{-\omega t}p(t)\psi(e^{\omega t}v(t)), \quad t \in J. \tag{9.34}$$

Accordingly, we have that

$$(e^{\omega t}v(t))' \le \max\{\omega, Mp(t)\}(e^{\omega t}v(t) + \psi(e^{\omega t}v(t))), \quad t \in J, \tag{9.35}$$

which implies that

$$\int_c^{e^{\omega t}v(t)} \frac{ds}{s+\psi(s)} \le \int_0^T \max(\omega, Mp(s))ds < \int_c^\infty \frac{ds}{s+\psi(s)}, \quad t \in J. \tag{9.36}$$

Using (9.4.7) we deduce that there exists a positive constant α which depends on T and the functions p, ψ such that $|y(t)| \le \alpha$ for all $y \in \Phi$, which implies that Φ is bounded.

Consequently, the mapping N is completely continuous and Theorem 1.6 implies that N has at least one fixed point, which gives rise to an integral solution of problem (9.1)–(9.3). □

9.2.1. A special case

In this section, we suppose that the nonlocal condition is given by

$$g(y) = \sum_{k=1}^{m+1} c_k y(\eta_k), \tag{9.37}$$

where $c_k,\ k = 1, \dots, m+1$, are nonnegative constants and $0 \le \eta_1 < t_1 < \eta_2 < t_2 < \cdots < t_m < \eta_{m+1} \le T$.

Lemma 9.5. *Assume that*

(9.5.1) *there exists a bounded operator $B : E \to E$ such that*

$$B = \left(I + \sum_{k=1}^{m+1} c_k S'(\eta_k)\right)^{-1}. \tag{9.38}$$

If y is an integral solution of (9.1), (9.2), (9.4), *then it is given by*

$$\begin{aligned} y(t) = S'(t)B\Bigg[& y_0 - \sum_{k=2}^{m+1} c_k \sum_{\lambda=1}^{k-1} S'(\eta_k - t_j) I_j(y(t_j^-)) \\ & - \sum_{k=1}^{m+1} c_k \int_0^{\eta_k} S'(\eta_k - s) F(s, y(s))\Bigg] \\ & + \frac{d}{dt}\int_0^t S(t-s)F(s,y(s))\,ds + \sum_{0<t_k<t} S'(t-t_k) I_k(y(t_k^-)), \quad t \in J. \end{aligned} \tag{9.39}$$

Proof. Let y be a solution of problem (9.1), (9.2), (9.4). As in Lemma 9.3 we conclude that

$$\int_0^t y(s)ds = w(0) + \sum_{0<t_k<t} S(t-t_k) I_k(y(t_k^-)) + \int_0^t S(t-s)F(s,y(s))ds, \tag{9.40}$$

where $w(0) = S(t)y(0) = S(t)[y_0 - \sum_{k=1}^{m+1} c_k y(\eta_k)]$.

It remains to find $y(\eta_k)$. The proof follows the steps of Lemma 4.2, with the necessary modifications of integrated semigroups, and for this reason is omited. □

Now we are able to state and prove our main theorem in this section.

Theorem 9.6. *Assume that assumptions (H1), (9.4.1), (9.4.2), (9.4.5), (9.5.1) hold. Also assume that*

(9.6.1) *there exist a continuous nondecreasing function $\psi : \mathbb{R}_+ \to (0,\infty)$, a function $p \in L^1(J,\mathbb{R}_+)$, and a constant $M > 0$ such that*

$$||F(t,y)|| \le p(t)\psi(|y|) \tag{9.41}$$

for almost all $t \in J$ and all $y \in E$, and

$$\frac{M}{\alpha + P + Q} > 1, \tag{9.42}$$

where

$$\alpha = Me^{\omega T}\left(\|B\|_{B(E)}\,|y_0| + M\|B\|_{B(E)}\sum_{k=2}^{m+1}|c_k|\sum_{\mu=1}^{k-1}e^{-\omega t_k}d_\mu + \sum_{k=1}^{m}e^{-\omega t_k}d_k\right),$$

$$P = M^2e^{\omega T}\|B\|_{B(E)}\sum_{k=1}^{m+1}|c_k|\,\psi(M)\int_0^{\eta_k}p(t)dt, \tag{9.43}$$

$$Q = Me^{\omega T}\int_0^b p(s)\psi(M)ds';$$

(9.6.2) *the set* $\{y_0 - \sum_{k=1}^{m+1}c_k y(\eta_k)\}$ *is relatively compact.*
Then problem (9.1), (9.2), (9.4) *has at least one integral solution on* J.

Proof. Consider the operator $\overline{N}:\Omega' \to \Omega'$ defined by

$$\begin{aligned}\overline{N}(y) = S'(t)B\Bigg[&y_0 - \sum_{k=2}^{m+1}c_k\sum_{\mu=1}^{k-1}S'(\eta_k - t_\mu)I_\mu(y(t_\mu^-))\\ &- \sum_{k=1}^{m+1}c_k\lim_{\lambda\to\infty}\int_0^{\eta_k}S'(\eta_k - s)B_\lambda F(s,y(s))\Bigg]\\ &+ \frac{d}{dt}\int_0^t S(t-s)F(s,y(s))ds + \sum_{0<t_k<t}S'(t-t_k)I_k(y(t_k^-)),\quad t\in J.\end{aligned} \tag{9.44}$$

We will prove that $\overline{N}$ is compact. Let $\{y_n\}$ be a sequence in Ω' with $\lim_{n\to\infty} y_n = y$ in Ω'. By the continuity of F with respect to the second argument, we deduce that, for each $s\in J$, $F(s,y_n(s))$ converges to $F(s,y(s))$ in E, and we have that

$$\begin{aligned}&|\overline{N}(y_n)(t) - \overline{N}(y)(t)|\\ &\quad\le M\|B\|_{B(E)}\Bigg[\sum_{k=2}^{m+1}|c_k|\sum_{\mu=1}^{k-1}e^{\omega(\eta_k - t_\mu)}|I_\mu(y_n(t_\mu^-)) - I_\mu(y(t_\mu^-))|\\ &\qquad\qquad + \sum_{k=1}^{m+1}|c_k|e^{\omega\eta_k}\int_0^{\eta_k}e^{-\omega s}|F(s,y_n(s)) - F(s,y(s))|ds\Bigg]\\ &\qquad + e^{\omega T}\Bigg[\int_0^T e^{-\omega s}|F(s,y_n(s)) - F(s,y(s))|ds\\ &\qquad\qquad + \sum_{k=1}^{m}e^{-\omega t_k}|I_k(y_n(t_k^-)) - I_k(y(t_k^-))|\Bigg].\end{aligned} \tag{9.45}$$

The sequence $\{y_n\}$ is bounded in Ω'. Then by using the Lebesgue dominated convergence theorem we obtain that

$$\lim_{n\to\infty} \overline{N}(y_n) = \overline{N}(y) \quad \text{in } \Omega', \tag{9.46}$$

which implies that the mapping $\overline{N}$ is continuous on Ω'.

Next, we use Arzelá-Ascoli's theorem to prove that $\overline{N}$ maps every bounded set into a compact set. Let B be a bounded set of Ω' and let $t \in J$ be fixed. Then we need to prove that $\{\overline{N}(y)(t) : y \in B\}$ is relatively compact in $\overline{D(A)}$. If $t = 0$, then from hypothesis (9.6.2) we have that $\{\overline{N}(y)(0) : y \in B\} = \{y_0 - \sum_{k=1}^{m+1} c_k y(\eta_k) : y \in B\}$ is relatively compact. If $t \in (0, T]$, the proof of relative compactness and equicontinuity is similar to that given in Theorem 9.4.

It remains to prove that the set $\Phi = \{x \in \Omega' : x = \sigma\overline{N}x \text{ for some } 0 < \sigma < 1\}$ is bounded. For $y \in \Phi$, there exists $\sigma \in (0,1)$ such that $y = \sigma\overline{N}y$; that is,

$$\begin{aligned} y(t) = \sigma S'(t)B\Bigg[y_0 &- \sum_{k=2}^{m+1} c_k \sum_{\mu=1}^{k-1} S'(\eta_k - t_\mu) I_\mu(y(t_\mu^-)) \\ &- \sum_{k=1}^{m+1} c_k \lim_{\lambda\to\infty} \int_0^{\eta_k} S'(\eta_k - s) B_\lambda F(s, y(s)) \Bigg] \\ &+ \sigma \frac{d}{dt}\int_0^t S(t-s)F(s,y(s))\,ds + \sigma \sum_{0<t_k<t} S'(t-t_k) I_k(y(t_k^-)), \quad t \in J. \end{aligned} \tag{9.47}$$

Using assumptions (9.5.1) and (9.6.1), we get

$$\begin{aligned} |y(t)| &\le Me^{\omega t}\|B\|_{B(E)}\Bigg[|y_0| + M\sum_{k=2}^{m+1} |c_k| \sum_{\mu=1}^{k-1} e^{-\omega t_k} d_\mu \\ &\qquad + M\sum_{k=1}^{m+1} |c_k| \int_0^{\eta_k} m(s)\psi(|y(s)|)\,ds \Bigg] \\ &\quad + Me^{\omega t}\int_0^t e^{-\omega s} m(s)\psi(|y(s)|)\,ds + Me^{\omega t}\sum_{k=1}^{m} e^{-\omega t_k} d_k \\ &\le Me^{\omega T}\|B\|_{B(E)}\Bigg[|y_0| + M\sum_{k=2}^{m+1} |c_k| \sum_{\mu=1}^{k-1} e^{-\omega t_k} d_\mu \\ &\qquad + M\sum_{k=1}^{m+1} |c_k| \int_0^{\eta_k} m(s)\psi(|y(s)|)\,ds \Bigg] \\ &\quad + Me^{\omega T}\int_0^t e^{-\omega s} m(s)\psi(|y(s)|)\,ds + Me^{\omega T}\sum_{k=1}^{m} e^{-\omega t_k} d_k. \end{aligned} \tag{9.48}$$

Consequently,

$$\frac{\|y\|_{\mathrm{PC}}}{\alpha+P+Q} \leq 1. \tag{9.49}$$

Then, by (9.6.1), there exists M such that $\|y\|_{\mathrm{PC}} \neq M$. Set

$$U = \{y \in \mathrm{PC}(J,E) : \|y\|_{\mathrm{PC}} < M+1\}. \tag{9.50}$$

The operator $\overline{N}$ is continuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y = \sigma\overline{N}(y)$ for some $\sigma \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type (Theorem 1.8), we deduce that $\overline{N}$ has at least one fixed point, which gives rise to an integral solution of problem (9.1), (9.2), (9.4). □

9.3. Nondensely defined impulsive semilinear differential inclusions with nonlocal conditions

In this section, we will prove existence results for evolution impulsive differential inclusions, with nonlocal conditions, of the form

$$\begin{gathered} y'(t) \in Ay(t) + F(t,y(t)), \quad t \in J := [0,b],\ t \neq t_k,\ k = 1,\dots,m, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m, \\ y(0) + g(y) = y_0, \end{gathered} \tag{9.51}$$

where $A : D(A) \subset E \to E$ is a nondensely defined closed linear operator, $F : J\times E \to \mathcal{P}(E)$ is a multivalued map ($\mathcal{P}(E)$ is the family of all subsets of E), $g : C(J',E) \to E$, ($J' = J\backslash\{t_1,\dots,t_m\}$), $I_k : E \to \overline{D(A)}$, $k = 1,\dots,m$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k+h)$ and $y(t_k^-) = \lim_{h\to 0^+} y(t_k-h)$, $y_0 \in E$, and E is a separable Banach space with norm $|\cdot|$.

Lemma 9.7. *If y is an integral solution of*

$$\begin{gathered} y'(t) = Ay(t) + f(t), \quad t \in J = [0,b],\ t \neq t_k,\ k = 1,\dots,m, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m, \\ y(0) + g(y) = y_0, \end{gathered} \tag{9.52}$$

where $F : J \times E \to E$ and A, g, I_k, $k = 1,\dots,m$, are as in problem (9.51), *then y is given by*

$$\begin{aligned} y(t) = S'(t)[y_0 - g(y)] &+ \frac{d}{dt}\int_0^t S(t-s)f(s)ds \\ &+ \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)), \quad \text{for } t \in J. \end{aligned} \tag{9.53}$$

Definition 9.8. Say that $y : J \to E$ is an integral solution of (9.51) if

(i) $y \in \Omega$,

(ii) $\int_0^t y(s)ds \in D(A)$ for $t \in J$,

(iii) there exists a function $f \in L^1(J,E)$ such that $f(t) \in F(t, y(t))$ a.e. in J and $y(t) = y_0 - g(y) + A\int_0^t y(s)ds + \int_0^t f(s)ds + \sum_{0<t_k<t} I_k(y(t_k^-))$, $t \in J$.

From (ii) it follows that $y(t) \in \overline{D(A)}$, for all $t \geq 0$. Also from (iii) it follows that $y_0 - g(y) \in \overline{D(A)}$. So, if we assume that $y_0 \in \overline{D(A)}$, we conclude that $g(y) \in \overline{D(A)}$.

Definition 9.9. If y is an integral solution of (9.51), then y is given by

$$\begin{aligned} y(t) = S'(t)(y_0 - g(y)) &+ \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)) \\ &+ \frac{d}{dt}\int_0^t S(t-s)f(s)ds, \quad t \in J. \end{aligned} \tag{9.54}$$

9.3.1. Existence result: the convex case

In this section, we are concerned with the existence of solutions for problem (9.51). Recall that

$$\Omega' = \Omega \cap C(J, \overline{D(A)}). \tag{9.55}$$

Now we are able to state and prove our main theorem in this section.

Theorem 9.10. *Assume that (H1), (9.4.2), (9.4.4), (9.4.5), and the following assumptions hold:*

(9.10.1) *let $F : J \times E \to P_{cp}(E)$; $(t, y) \mapsto F(t, y)$ be measurable with respect to t, for each $y \in E$, u.s.c., with respect to y, for each $t \in J$;*

(9.10.2) *there exist a continuous function $p : [0, b] \to \mathbb{R}^+$ and a continuous nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ such that*

$$\|F(t, y)\| := \sup\{|v| : v \in F(t, y)\} \leq p(t)\psi(|y|), \quad t \in J,\ y \in E, \tag{9.56}$$

with

$$\int_0^T m(s)ds < \int_c^\infty \frac{ds}{\psi(s)}, \tag{9.57}$$

where

$$m(t) = M^* e^{-\omega s} p(t), \qquad c = M^*\left(|y_0| + L + \sum_{k=1}^m e^{-\omega t_k} d_k\right), \tag{9.58}$$

and $M^* = M\max\{e^{\omega b}, 1\}$.

Then problem (9.51) *has at least one integral solution on J.*

Proof. Consider the operator $N : \Omega' \to \mathcal{P}(\Omega')$ defined by

$$N(y) = \left\{ h \in \Omega' : h(t) = S'(t)[y_0 - g(y)] + \frac{d}{dt}\int_0^t S(t-s)f(s)ds \right.$$
$$\left. + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)),\ f \in S_{F,y} \right\}, \quad t \in J. \tag{9.59}$$

Let

$$K = \{y \in \Omega' : \|y\|_{\Omega'} \leq \alpha(t),\ t \in J\}, \tag{9.60}$$

where

$$\alpha(t) = I^{-1}\left(\int_0^t m(s)ds\right),$$
$$I(z) = \int_c^z \frac{du}{\psi(u)}. \tag{9.61}$$

It is clear that K is a closed convex and bounded set. □

Step 1. $N(K) \subset K$.

For $y \in K$ and $h \in N(y)$, there exists a function $f \in S_{F,y}$ such that, for every $t \in J$, we have

$$h(t) = S'(t)(y_0 - g(y)) + \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda f(s)ds + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)). \tag{9.62}$$

Thus

$$\begin{aligned}
|h(t)| &\leq Me^{\omega t}(|y_0| + L) + Me^{\omega t}\int_0^t e^{-\omega s}p(s)\psi(|y(s)|)ds + Me^{\omega t}\sum_{k=1}^m e^{-\omega t_k}d_k \\
&\leq M^*(|y_0| + L) + M^*\int_0^t e^{-\omega s}p(s)\psi(\alpha(s))ds + M^*\sum_{k=1}^m e^{-\omega t_k}d_k \\
&\leq M^*(|y_0| + L) + \int_0^t m(s)\psi(\alpha(s))ds + M^*\sum_{k=1}^m e^{-\omega t_k}d_k \\
&= M^*\left(|y_0| + L + \sum_{k=1}^m e^{-\omega t_k}d_k\right) + \int_0^t \alpha'(s)ds = \alpha(t),
\end{aligned} \tag{9.63}$$

since

$$\int_c^{\alpha(s)} \frac{du}{\psi(u)} = \int_0^t m(s)ds. \tag{9.64}$$

Thus $N(y) \in K$.

Step 2. $N(K)$ is relatively compact.

Since K is bounded and $N(K) \subset K$, it is clear that $N(K)$ is bounded.

Let $t \in (0, b]$ be fixed and let ε be a real number satisfying $0 < \varepsilon < t$. For $y \in K$ and $h \in N(y)$, there exists a function $f \in S_{F,y}$ such that

$$\begin{aligned} h(t) = {} & S'(t)(y_0 - g(y)) + \lim_{\lambda\to\infty} \int_0^{t-\varepsilon} S'(t-s)B_\lambda f(s)ds \\ & + \lim_{\lambda\to\infty} \int_{t-\varepsilon}^{t} S'(t-s)B_\lambda f(s)ds + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)). \end{aligned} \tag{9.65}$$

Define

$$\begin{aligned} h_\varepsilon(t) &= S'(t)(y_0 - g(y)) + \lim_{\lambda\to\infty} \int_0^{t-\varepsilon} S'(t-s)B_\lambda f(s)ds + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)) \\ &= S'(t)(y_0 - g(y)) + S'(\varepsilon) \lim_{\lambda\to\infty} \int_0^{t-\varepsilon} S'(t-\epsilon-s)B_\lambda f(s)ds \\ &\quad + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)). \end{aligned} \tag{9.66}$$

Since $S'(t)$, $t > 0$, is compact, the set $H_\varepsilon(t) = \{h_\varepsilon(t) : h_\varepsilon \in N(y)\}$ is precompact in $\overline{D(A)}$ for every ε, $0 < \varepsilon < t$. Moreover, for every $h \in N(y)$,

$$|h(t) - h_\varepsilon(t)| \le M^* \int_{t-\varepsilon}^{t} e^{-\omega s} p(s)\psi(|y(s)|)ds \le M^* \int_{t-\varepsilon}^{t} e^{-\omega s} p(s)\psi(\alpha(s))ds. \tag{9.67}$$

Therefore there are precompact sets arbitrarily close to the set $\{h(t) : h \in N(y)\}$. Hence the set $\{h(t) : h \in N(y)\}$ is precompact in $\overline{D(A)}$.

Step 3. N(K) is equicontinuous.

Let $\tau_1, \tau_2 \in J'$, $\tau_1 < \tau_2$. Let $y \in K$ and $h \in N(y)$. Then there exists $f \in S_{F,y}$ such that, for each $t \in J$, we have

$$h(t) = S'(t)(y_0 - g(y)) + \lim_{\lambda\to\infty} \int_0^{t} S'(t-s)B_\lambda f(s)ds + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)). \tag{9.68}$$

Then

$$\begin{aligned}
&|h(\tau_2)-h(\tau_1)|\\
&\quad\le |[S'(\tau_2)-S'(\tau_1)](y_0-g(y))|\\
&\qquad+\left|\lim_{\lambda\to\infty}\int_0^{\tau_1}[S'(\tau_2-s)-S'(\tau_1-s)]B_\lambda f(s)ds\right|\\
&\qquad+\left|\lim_{\lambda\to\infty}\int_{\tau_1}^{\tau_2}S'(\tau_2-s)B_\lambda f(s)ds\right|\\
&\qquad+\left|\sum_{0<t<\tau_1}[S'(\tau_2-t_k)-S'(\tau_1-t_k)]I_k(y(t_k^-))\right|\\
&\qquad+\left|\sum_{\tau_1<t<\tau_2}S'(\tau_2-t_k)I_k(y(t_k^-))\right|\\
&\quad\le |[S'(\tau_2)-S'(\tau_1)](y_0-g(y))|\\
&\qquad+\left|[S'(\tau_2-\tau_1)-I]\lim_{\lambda\to\infty}\int_0^{\tau_1}S'(\tau_1-s)B_\lambda f(s)ds\right|\\
&\qquad+M^*\int_{\tau_1}^{\tau_2}e^{-\omega s}p(s)\psi(\alpha(s))ds\\
&\qquad+\sum_{0<t_k<\tau_1}\|S'(\tau_2-t_k)-S'(\tau_1-t_k)\|_{B(E)}d_k+M^*\sum_{\tau_1<t_k<\tau_2}e^{-\omega t_k}d_k.
\end{aligned}\tag{9.69}$$

The right-hand side tends to zero as $\tau_2-\tau_1\to 0$, since $S'(t)$ is strongly continuous, and the compactness of $S'(t)$, $t>0$, implies the continuity in the uniform operator topology.

This proves the equicontinuity for the case where $t\neq t_i$, $i=1,\dots,m$. It remains to examine the equicontinuity at $t=t_i$. The proof is similar to that given in Theorem 4.3.

As a consequence of Steps 2-3 and the Arzelá-Ascoli theorem, we deduce that N maps K into precompact sets in $\overline{D(A)}$.

Step 4. N has closed graph.

Let $y_n\to y_*$, $h_n\in N(y_n)$, $y_n\in K$ and $h_n\to h_*$. We will prove that $h_*\in N(y_*)$.

$h_n\in N(y_n)$ means that there exists $v_n\in S_{F,y_n}$ such that, for each $t\in J$,

$$\begin{aligned}
h_n(t)=S'(t)[y_0-g(y_n)]&+\lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda v_n(s)ds\\
&+\sum_{0<t_k<t}S'(t-t_k)I_k(y_n(t_k^-)).
\end{aligned}\tag{9.70}$$

We must prove that there exists $v_* \in S_{F,y_*}$ such that, for each $t \in J$,

$$\begin{aligned} h_*(t) &= S'(t)[y_0 - g(y_*)] + \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda v_*(s)ds \\ &\quad + \sum_{0<t_k<t} S'(t-t_k)I_k(y_*(t_k^-)). \end{aligned} \tag{9.71}$$

Clearly since I_k, $k = 1,\dots,m$, and g are continuous, we have that

$$\begin{aligned} &\bigg\| \bigg(h_n - S'(t)[y_0 - g(y_n)] - \sum_{0<t_k<t} S'(t-t_k)I_k(y_n(t_k^-)) \bigg) \\ &\quad - \bigg(h_* - S'(t)[y_0 - g(y_*)] - \sum_{0<t_k<t} S'(t-t_k)I_k(y_*(t_k^-)) \bigg) \bigg\|_{\Omega'} \longrightarrow 0, \end{aligned} \tag{9.72}$$

as $n \to \infty$.

Consider the linear continuous operator

$$\begin{aligned} \Gamma : L^1(J,E) &\longrightarrow C(J,E), \\ v \longmapsto \Gamma(v)(t) &= \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda v(s)ds. \end{aligned} \tag{9.73}$$

From Lemma 1.28, it follows that $\Gamma \circ S_F$ is a closed graph operator. Moreover, we have that

$$h_n(t) - S'(t)[y_0 - g(y_n)] - \sum_{0<t_k<t} S'(t-t_k)I_k(y_n(t_k^-)) \in \Gamma(S_{F,y_n}). \tag{9.74}$$

Since $y_n \to y_*$, it follows from Lemma 1.28 that

$$\begin{aligned} h_*(t) &- S'(t)[y_0 - g(y_*)] - \sum_{0<t_k<t} S'(t-t_k)I_k(y_*(t_k^-)) \\ &= \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda v_*(s)ds \end{aligned} \tag{9.75}$$

for some $v_* \in S_{F,y_*}$.

As a consequence of Theorem 1.9, we deduce that N has a fixed point which gives rise to an integral solution of problem (9.51).

Our next result in this section is based on Covitz and Nadler's fixed point theorem for contraction multivalued operators.

Theorem 9.11. *Assume that (H1) and the following hypotheses hold:*

(9.11.1) $F : [0,b] \times E \to \mathcal{P}_{\text{cp,cv}}(E)$ *has the property that* $F(\cdot,y) : [0,b] \to \mathcal{P}_{\text{cp}}(E)$ *is measurable for each* $y \in E$;

(9.11.2) *there exists* $l \in L^1([0,b],\mathbb{R}^+)$ *such that*

$$H_d(F(t,y),F(t,\overline{y})) \le l(t)|y-\overline{y}|, \quad \text{for almost each } t \in [0,b] \tag{9.76}$$

and $y, \overline{y} \in E$, *and*

$$d(0,F(t,0)) \le \ell(t), \quad \text{for almost each } t \in [0,b]; \tag{9.77}$$

(9.11.3) *there exist constants* d_k *such that*

$$\|I_k(y_1) - I_k(y_2)\|_{\overline{D(A)}} \le d_k' \,|\, y_1 - y_2 \,|, \quad \forall y_1, y_2 \in E; \tag{9.78}$$

(9.11.4) g *is continuous and there exists constant a* $c' > 0$ *such that*

$$|g(y_1) - g(y_2)| \le c'\|y_1 - y_2\|_{\Omega'}, \quad \forall y_1, y_2 \in \Omega'; \tag{9.79}$$

(9.11.5) *for* $M^* = M\max\{e^{\omega b}, 1\}$, *and* M *is from the Hille-Yosida condtion,*

$$M^*\left(c' + \int_0^b e^{-\omega s} l(s)ds + \sum_{k=1}^{m} e^{-\omega t_k} d_k'\right) < 1. \tag{9.80}$$

Then the IVP (9.51) *has at least one integral solution on* $[0,b]$.

Proof. Transform problem (9.51) into a fixed point problem. Consider the multivalued operator N defined in Theorem 9.10.

We will show that N satisfies the assumptions of Theorem 1.11. The proof will be given in two steps.

Step 1. $N(y) \in P_{cl}(\Omega')$ for each $y \in \Omega'$.

Indeed, let $(y_n)_{n\ge 0} \in N(y)$ such that $y_n \to \tilde{y}$ in Ω'. Then $\tilde{y} \in \Omega'$ and there exists $f_n \in S_{F,y}$ such that, for every $t \in [0,b]$,

$$y_n(t) = S'(t)[y_0 - g(y)] + \frac{d}{dt}\int_0^t S(t-s)f_n(s)ds + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)). \tag{9.81}$$

Using the fact that F has compact values and from (9.11.2), we may pass to a subsequence if necessary to get that f_n converges to f in $L^1([0,b],E)$ and hence

$f \in S_{F,y}$. Then, for each $t \in [0,b]$,

$$y_n(t) \longrightarrow \widetilde{y}(t) = S'(t)[y_0 - g(y)] + \frac{d}{dt}\int_0^t S(t-s)f(s)ds + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)). \tag{9.82}$$

So, $\widetilde{y} \in N(y)$.

Step 2. $H_d(N(y_1),N(y_2)) \le \gamma\|y_1 - y_2\|_{\Omega'}$ for each $y_1, y_2 \in \Omega'$ (where $\gamma < 1$).

Let $y_1, y_2 \in \Omega'$ and $h_1 \in N(y_1)$. Then there exists $f_1(t) \in F(t, y_1(t))$ such that

$$h_1(t) = S'(t)[y_0 - g(y_1)] + \frac{d}{dt}\int_0^t S(t-s)f_1(s)ds + \sum_{0<t_k<t} S'(t-t_k)I_k(y_1(t_k^-)), \quad t \in [0,b]. \tag{9.83}$$

From (9.11.2) it follows that

$$H_d(F(t,y_1(t)),F(t,y_2(t))) \le l(t)|y_1(t) - y_2(t)|, \quad t \in [0,b]. \tag{9.84}$$

Hence there is $w \in F(t, y_2(t))$ such that

$$|f_1(t) - w| \le l(t)|y_1(t) - y_2(t)|, \quad t \in [0,b]. \tag{9.85}$$

Consider $U : [0,b] \to \mathcal{P}(E)$, given by

$$U(t) = \{w \in E : |f_1(t) - w| \le l(t)|y_1(t) - y_2(t)|\}. \tag{9.86}$$

Since the multivalued operator $V(t) = U(t) \cap F(t, y_2(t))$ is measurable (see [119, Proposition III.4]), there exists $f_2(t)$ a measurable selection for V. So, $f_2(t) \in F(t, y_2(t))$ and

$$|f_1(t) - f_2(t)| \le l(t)|y_1(t) - y_2(t)|, \quad \text{for each } t \in [0,b]. \tag{9.87}$$

Let us define, for each $t \in [0,b]$,

$$h_2(t) = S'(t)[y_0 - g(y_2)] + \frac{d}{dt}\int_0^t S(t-s)f_2(s)ds + \sum_{0<t_k<t} S'(t-t_k)I_k(y_2(t_k^-)). \tag{9.88}$$

Then we have

$$
\begin{aligned}
&|h_1(t)-h_2(t)|\\
&\quad\le \left| S'(t)[g(y_1)-g(y_2)] + \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda[f_1(s)-f_2(s)]ds \right.\\
&\qquad \left. + \sum_{0<t_k<t} S'(t-t_k)\big(I_k(y_2(t_k^-)) - I_k(y_1(t_k^-))\big)\right|\\
&\quad\le M^*c'\|y_1-y_2\|_{\Omega'} + M^*\int_0^t e^{-\omega s}\ell(s)|y_1(s)-y_2(s)|ds\\
&\qquad + M^*\sum_{k=1}^m e^{-\omega t_k}d_k'|y_1(t_k^-)-y_2(t_k^-)|\\
&\quad\le M^*c'\|y_1-y_2\|_{\Omega'} + M^*\|y_1-y_2\|_{\Omega'}\int_0^t e^{-\omega s}\ell(s)ds\\
&\qquad + M^*\|y_1-y_2\|_{\Omega'}\sum_{k=1}^m e^{-\omega t_k}d_k'\\
&\quad\le \left[M^*c' + M^*\int_0^b e^{-\omega s}\ell(s)ds + M^*\sum_{k=1}^m e^{-\omega t_k}d_k'\right]\times \|y_1-y_2\|_{\Omega'}.
\end{aligned}
\tag{9.89}
$$

Then

$$\|h_1-h_2\|_{\Omega'} \le M^*\left(c' + \int_0^b e^{-\omega s}\ell(s)ds + \sum_{k=1}^m e^{-\omega t_k}d_k'\right)\|y_1-y_2\|_{\Omega'}. \tag{9.90}$$

By the analogous relation, obtained by interchanging the roles of y_1 and y_2, it follows that

$$H_d(N(y_1),N(y_2)) \le M^*\left(c' + \int_0^b e^{-\omega s}\ell(s)ds + \sum_{k=1}^m e^{-\omega t_k}d_k'\right)\|y_1-y_2\|_{\Omega'}. \tag{9.91}$$

From (9.11.5) we have that

$$\gamma := M^*\left(c' + \int_0^b e^{-\omega s}\ell(s)ds + \sum_{k=1}^m e^{-\omega t_k}d_k'\right) < 1. \tag{9.92}$$

Then N is a contraction and thus, by Theorem 1.11, N has a fixed point y, which is a mild solution to (9.51). □

9.3.2. Existence results: the nonconvex case

In this section, we consider the problems (9.51), with a nonconvex-valued right-hand side.

By the help of the Schaefer's fixed point theorem, combined with the selection theorem of Bressan and Colombo for lower semicontinuous maps with decomposable values, we will present a second existence result for problem (9.51).

Theorem 9.12. *Suppose, in addition to hypotheses (H1), (9.4.2), (9.4.4), (9.4.5), (9.10.2), the following also hold:*

(9.12.1) $F : [0,b] \times E \to \mathcal{P}(E)$ *is a nonempty compact-valued multivalued map such that*

(a) $(t,y) \mapsto F(t,y)$ *is* $\mathcal{L} \otimes \mathcal{B}$ *measurable,*

(b) $y \mapsto F(t,y)$ *is lower semicontinuous for a.e.* $t \in [0,b]$*;*

(9.12.2) *for each* $r > 0$*, there exists a function* $h_r \in L^1([0,b],\mathbb{R}^+)$ *such that*

$$\|F(t,y)\| := \sup\{|v| : v \in F(t,y)\} \le h_r(t) \quad \text{for a.e. } t \in [0,b],\ y \in E \text{ with } |y| \le r; \tag{9.93}$$

(9.12.3)

$$\int_0^b m(s)ds < \int_{c_1}^{\infty} \frac{ds}{s+\psi(s)}, \tag{9.94}$$

where M *and* ω *are from the Hille-Yosida condition and*

$$m(t) = \max\{\omega, Mp(t)\}, \quad t \in [0,b], \quad c_1 = M\left(|y_0| + L + \sum_{k=1}^{m} e^{-\omega t_k} d_k\right). \tag{9.95}$$

Then the initial value problem (9.51) *has at least one integral solution on* $[0,b]$.

Proof. Hypotheses (9.12.1) and (9.12.2) imply that F is of lower semicontinuous type. Then, from Theorem 1.5, there exists a continuous function $h : \Omega' \to L^1([0,b],E)$ such that $h(y) \in \mathcal{F}(y)$ for all $y \in \Omega'$.

We consider the problem

$$\begin{gathered} y'(t) = Ay(t) + h(y)(t), \quad t \in J = [0,b],\ t \neq t_k,\ k = 1,\dots,m, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m, \\ y(0) + g(y) = y_0. \end{gathered} \tag{9.96}$$

We remark that if $y \in \Omega'$ is a solution of the problem (9.96), then y is a solution to problem (9.51).

Transform problem (9.96) into a fixed point problem by considering the operator $N_1 : \Omega' \to \Omega'$ defined by

$$\begin{aligned} N_1(y) = {} & S'(t)[y_0 - g(y)] + \frac{d}{dt}\int_0^t S(t-s)h(y)(s)ds \\ & + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)), \quad t \in J. \end{aligned} \tag{9.97}$$

Step 1. N_1 is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in Ω'. Then

$$\begin{aligned}|N_1(y_n)(t) - N_1(y)(t)| \leq{}& M^*|g(y_n) - g(y)| + M^*\int_0^t e^{-\omega s}B_\lambda|f_n(s) - f(s)|ds\\ &+ M^*\sum_{0<t_k<t} e^{-\omega t_k}||I_k(y_n(t_k^-)) - I_k(y(t_k^-))||_{\overline{D(A)}}.\end{aligned} \tag{9.98}$$

Since the functions f, g are continuous, then

$$||N_1(y_n) - N_1(y)||_{\Omega'} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{9.99}$$

Step 2. N_1 maps bounded sets into bounded sets in Ω'.

Indeed, it is enough to show that for any $q > 0$ there exists a positive constant ℓ such that, for each $y \in B_q = \{y \in \Omega' : \|y\|_{\Omega'} \leq q\}$, we have $\|N_1(y)\|_{\Omega'} \leq \ell$. For each $t \in [0, b]$, we have that

$$\begin{aligned}|N_1(y)(t)| &= \left|S'(t)(y_0 - g(y)) + \frac{d}{dt}\int_0^t S(t-s)f(s,y(s))ds\right.\\ &\qquad \left.+ \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-))\right|\\ &\leq M^*\left[|y_0| + L + \int_0^t e^{-\omega s}h_q(s)ds + \sum_{k=1}^m e^{-\omega t_k}d_k\right]\\ &\leq M^*\left[|y_0| + L + N||h_q||_{L^1} + \sum_{k=1}^m e^{-\omega t_k}d_k\right],\end{aligned} \tag{9.100}$$

where $N = \max\{1, e^{-\omega b}\}$.

Thus

$$||N_1(y)||_{\Omega'} \leq M^*\left[|y_0| + L + N||h_q||_{L^1} + \sum_{k=1}^m e^{-\omega t_k}d_k\right] := \ell. \tag{9.101}$$

Step 3. N_1 maps bounded sets into equicontinuous sets of Ω'.

Let $0 < \tau_1 < \tau_2 \in J'$, $\tau_1 < \tau_2$, and let $\mathcal{B}_q$ be a bounded set of Ω as in Step 2. Let $y \in \mathcal{B}_q$. Then, for each $t \in J$, we have

$$\begin{aligned}N_1(y)(t) ={}& S'(t)(y_0 - g(y)) + \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda h(y)(s)ds\\ &+ \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)).\end{aligned} \tag{9.102}$$

Then

$$\begin{aligned}
&|N_1(y)(\tau_2) - N_1(y)(\tau_1)| \\
&\quad \leq |[S'(\tau_2) - S'(\tau_1)](y_0 - g(y))| \\
&\qquad + \left| \lim_{\lambda\to\infty} \int_{\tau_1}^{\tau_2} S'(\tau_2 - s)B_\lambda h(y)(s)ds \right| \\
&\qquad + \left| \lim_{\lambda\to\infty} \int_0^{\tau_1} (S'(\tau_2 - s) - S'(\tau_1 - s))B_\lambda h(y)(s)ds \right| \\
&\qquad + \sum_{0<t_k<\tau_1} d_k |S'(\tau_2 - t_k) - S'(\tau_1 - t_k)| \\
&\qquad + e^{\omega\tau_2} \sum_{\tau_1<t_k<\tau_2} d_k e^{-\omega t_k}.
\end{aligned} \tag{9.103}$$

The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$, since $S'(t)$ is strongly continuous, and the compactness of $S'(t)$, $t > 0$, implies the continuity in the uniform operator topology.

This proves the equicontinuity for the case where $t \neq t_i$, $i = 1, \dots, m$. It remains to examine the equicontinuity at $t = t_i$. The proof is similar to that given in Theorem 4.3.

As a consequence of Steps 1 to 3 and (9.4.4), together with the Arzelá-Ascoli theorem, we can conclude that $N_1 : \Omega' \to \Omega'$ is a completely continuous operator.

Step 4. Now it remains to show that the set

$$\mathcal{E}(N_1) := \{y \in \Omega' : y = \sigma N_1(y) \text{ for some } 0 < \sigma < 1\} \tag{9.104}$$

is bounded.

Let $y \in \mathcal{E}(N_1)$. Then $y = \sigma N_1(y)$ for some $0 < \sigma < 1$. Thus for each $t \in J$,

$$\begin{aligned}
y(t) = \sigma\bigg(S'(t)(y_0 - g(y)) &+ \frac{d}{dt}\int_0^t S(t-s)h(y)(s)ds \\
&+ \sum_{0<t_k<t} S'(t - t_k)I_k(y(t_k^-))\bigg).
\end{aligned} \tag{9.105}$$

This implies that, for each $t \in J$, we have

$$|y(t)| \leq Me^{\omega t}(|y_0| + L) + Me^{\omega t}\int_0^t e^{-\omega s}p(s)\psi(|y(s)|)ds + Me^{\omega t}\sum_{k=1}^m e^{-\omega t_k}d_k \tag{9.106}$$

or

$$e^{-\omega t}|y(t)| \leq M(|y_0| + L) + M\int_0^t e^{-\omega s}p(s)\psi(|y(s)|)ds + M\sum_{k=1}^m e^{-\omega t_k}d_k. \tag{9.107}$$

Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$\begin{gathered} |y(t)| \leq e^{\omega t}v(t), \quad \forall t \in J = [0,b], \\ v(0) = M\left(|y_0| + L + \sum_{k=1}^{m} e^{-\omega t_k} d_k\right), \\ v'(t) = Me^{-\omega t}p(t)\psi(|y(t)|) \leq Me^{-\omega t}p(t)\psi(e^{\omega t}v(t)), \quad t \in J = [0,b]. \end{gathered} \tag{9.108}$$

Then, for each $t \in [0,b]$, we have

$$\begin{aligned} \left(e^{\omega t}v(t)\right)' &= \omega e^{\omega t}v(t) + v'(t)e^{\omega t} \leq \omega e^{\omega t}v(t) + Mp(t)\psi(e^{\omega t}v(t)) \\ &\leq m(t)\left[e^{\omega t}v(t) + \psi(e^{\omega t}v(t))\right], \quad t \in [0,b]. \end{aligned} \tag{9.109}$$

Thus

$$\int_{v(0)}^{e^{\omega t}v(t)} \frac{du}{u+\psi(u)} \leq \int_0^b m(s)ds < \int_{v(0)}^{\infty} \frac{du}{u+\psi(u)}. \tag{9.110}$$

Consequently, there exists a constant d such that $v(t) \leq d$, $t \in [0,b]$, and hence $\|y\|_{\Omega'} \leq d$ where d depends only on the constants M, ω, d_k and the functions p and ψ. This shows that $\mathcal{E}(N_1)$ is bounded.

As a consequence of Schaefer's theorem (Theorem 1.6), we deduce that N_1 has a fixed point y which is a solution to problem (9.96). Then y is a solution to problem (9.51). □

9.4. Nondensely defined impulsive semilinear functional differential equations

In this section, we will be concerned with the existence of integral solutions for first-order impulsive semilinear functional and neutral functional differential equations in Banach spaces. First, we will consider first-order impulsive semilinear functional differential equations of the form

$$\begin{gathered} y'(t) - Ay(t) = f(t,y_t), \quad \text{a.e. } t \in J = [0,T]\backslash\{t_1,\dots,t_m\}, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m, \\ y(t) = \phi(t), \quad t \in [-r,0], \end{gathered} \tag{9.111}$$

where $f : [0,T] \times \mathcal{D} \to E$ is a function, $\mathcal{D} = \{\psi : [-r,0] \to E : \psi$ is continuous everywhere except for a finite number of points $\bar{t}$ at which $\psi(\bar{t})$ and $\psi(\bar{t}^+)$ exist and $\psi(\bar{t}^-) = \psi(\bar{t})\}$ $(0 < r < \infty)$, $A : D(A) \subset E \to E$ is a nondensely defined closed linear operator on E, $\phi \in \mathcal{D}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $I_k \in C(E, E)(k = 1, \ldots, m)$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k - h)$, and E is a real separable Banach space with norm $|\cdot|$.

Next, we study the first-order impulsive semilinear neutral functional differential equations of the form

$$\begin{gathered}
\frac{d}{dt}\big[y(t) - g(t, y_t)\big] = Ay(t) + f(t, y_t), \quad \text{a.e. } t \in J = [0, T]\backslash\{t_1, \ldots, t_m\}, \\
\Delta y|_{t=t_k} = I_k\big(y(t_k^-)\big), \quad k = 1, \ldots, m, \\
y(t) = \phi(t), \quad t \in [-r, 0],
\end{gathered} \tag{9.112}$$

where f, I_k, A, and ϕ are as in problem (9.111), $g : [0, T] \times \mathcal{D} \to \overline{D(A)}$ is a given function.

Definition 9.13. The map $f : J \times \mathcal{D} \to E$ is said to be an L^1-Carathéodory if
- (i) $t \mapsto f(t, u)$ is measurable for each $u \in \mathcal{D}$;
- (ii) $u \mapsto f(t, u)$ is continuous for all $t \in J$;
- (iii) for each $\rho > 0$, there exists $\varphi_\rho \in L^1(J, \mathbb{R}_+)$ such that

$$\big|f(t, u)\big| \le \varphi_\rho(t), \quad \forall \|u\|_{\mathcal{D}} \le \rho \text{ for a.e. } t \in J. \tag{9.113}$$

9.4.1. Existence results for functional differential equations

In this section we are concerned with the existence of integral solutions for problem (9.111). Here we use again the symbol Ω for the space,

$$\begin{aligned}
\Omega = \{&y : [-r, T] \longrightarrow E : y_k \in C(J_k, E),\ k = 0, \ldots, m\ \exists\ y(t_k^-), \\
&y(t_k^+),\ k = 1, \ldots, m \text{ with } y(t_k^-) = y(t_k)\},
\end{aligned} \tag{9.114}$$

which is a Banach space with the norm

$$\|y\|_\Omega = \max\big\{\|y_k\|_{J_k},\ k = 0, \ldots, m\big\}, \tag{9.115}$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}]$, $k = 0, \ldots, m$.

Let us start by defining what we mean by an integral solution of problem (9.111).

Definition 9.14. A function $y \in \Omega$ is said to be an integral solution of (9.111) if y is the solution of the impulsive integral equation

$$y(t) = S'(t)\phi(0) + A\int_0^t y(s)ds + \int_0^t f(s, y_s)ds + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k)),$$
$$\int_0^t y(s)ds \in D(A), \quad t \in [0,T], \quad y(t) = \phi(t), \quad t \in [-r,0]. \tag{9.116}$$

Theorem 9.15. *Assume that (H1), (9.4.2), (9.4.5) hold and that f is an L^1-Carathéodory function. Also we suppose that*

(9.15.1) $\phi(0) \in \overline{D(A)}$;

(9.15.2) *there exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and $p \in L^1(J,\mathbb{R}_+)$ such that*

$$|f(t,u)| \le p(t)\psi(\|u\|_{\mathcal{D}}) \quad \textit{for a.e. } t \in J, \textit{ each } u \in \mathcal{D}, \tag{9.117}$$

with

$$\int_0^b m(s)ds < \int_c^\infty \frac{du}{u+\psi(u)}, \tag{9.118}$$

where

$$m(s) = \max(\omega, Mp(s)), \qquad c = M\left(\|\phi\| + \sum_{k=1}^m e^{-\omega t_k}c_k\right). \tag{9.119}$$

Then the IVP (9.111) *has at least one integral solution on* $[-r,T]$.

Proof. Transform problem (9.111) into a fixed point problem. Consider the operator $N : \Omega \to \Omega$ defined by

$$N(y)(t) = \begin{cases} \phi(t) & \text{if } t \in [-r,0], \\ S'(t)\phi(0) + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)f(s,y_s)ds & \\ \quad + \displaystyle\sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)) & \text{if } t \in [0,T]. \end{cases} \tag{9.120}$$

We will show that N is completely continuous. The proof will be given in several steps.

Step 1. N is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in Ω. Then

$$\begin{aligned} |N(y_n)(t) - N(y)(t)| &\leq \left| \frac{d}{dt} \int_0^t S(t-s)[f(s, y_{ns}) - f(s, y_s)]ds \right| \\ &\quad + \sum_{k=1}^{m} |S'(t-t_k)| \, |I_k(y_n(t_k)) - I_k(y(t_k^-))| \\ &\leq Me^{\omega T} \int_0^T e^{-\omega s} |f(s, y_{ns}) - f(s, y_s)| ds \\ &\quad + Me^{\omega T} \sum_{k=1}^{m} |I_k(y_n(t_k)) - I_k(y(t_k^-))|. \end{aligned} \tag{9.121}$$

Since f is an L^1-Carathéodory function, we have by the Lebesgue dominated convergence theorem

$$\begin{aligned} &||N(y_n) - N(y)||_\Omega \\ &\quad \leq Me^{\omega T} \left[||f(\cdot, y_n) - f(\cdot, y)||_{L^1} + \sum_{k=1}^{m} |I_k(y_n(t_k^-)) - I_k(y(t_k^-))| \right] \longrightarrow 0, \end{aligned} \tag{9.122}$$

as $n \to \infty$. Thus N is continuous.

Step 2. N maps bounded sets into bounded sets in Ω.

Indeed, it is enough to show that for any $q > 0$ there exists a positive constant ℓ such that, for each $y \in B_q = \{y \in \Omega : \|y\|_\Omega \leq q\}$, we have $\|N(y)\|_\Omega \leq \ell$. Then we have, for each $t \in [0, T]$,

$$\begin{aligned} |N(y)(t)| &= \left| S'(t)\phi(0) + \frac{d}{dt} \int_0^t S(t-s) f(s, y_s) ds + \sum_{0<t_k<t} S'(t-t_k) I_k(y(t_k^-)) \right| \\ &\leq Me^{\omega t_1} \left[\|\phi\|_{\mathcal{D}} + \int_0^{t_1} e^{-\omega s} \varphi_q(s) ds + \sum_{k=1}^{m} e^{-\omega t_k} |I_k(y(t_k^-))| \right] \\ &\leq Me^{\omega T} \left[\|\phi\|_{\mathcal{D}} + ||\varphi_q|| + \sum_{k=1}^{m} e^{-\omega t_k} c_k \right]. \end{aligned} \tag{9.123}$$

Thus

$$||N(y)||_\Omega \leq Me^{\omega T} \left[\|\phi\|_{\mathcal{D}} + ||\varphi_q||_{L^1} + \sum_{k=1}^{m} e^{-\omega t_k} c_k \right] := \ell. \tag{9.124}$$

Step 3. N maps bounded sets into equicontinuous sets of Ω.

Let $0 < \tau_1 < \tau_2 \in J$, $\tau_1 < \tau_2$, and let $\mathcal{B}_q$ be a bounded set of Ω as in Step 2. Let $y \in \mathcal{B}_q$. Then for each $t \in J$ we have

$$N(y)(t) = S'(t)\phi(0) + \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda f(s,y_s)\,ds + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)). \tag{9.125}$$

Then

$$\begin{aligned}
&|N(y)(\tau_2) - N(y)(\tau_1)| \\
&\quad\le |[S'(\tau_2) - S'(\tau_1)]\phi(0)| \\
&\qquad+ \left|\lim_{\lambda\to\infty}\int_{\tau_1}^{\tau_2} S'(\tau_2-s)B_\lambda f(s,y_s)\,ds\right| \\
&\qquad+ \left|\lim_{\lambda\to\infty}\int_0^{\tau_1} (S'(\tau_2-s) - S'(\tau_1-s))B_\lambda f(s,y_s)\,ds\right| \\
&\qquad+ \sum_{0<t_k<\tau_1} c_k|S'(\tau_2-t_k) - S'(\tau_1-t_k)| \\
&\qquad+ \sum_{\tau_1<t_k<\tau_2} c_k|S'(\tau_2-t_k)|.
\end{aligned} \tag{9.126}$$

The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$, since $S'(t)$ is strongly continuous, and the compactness of $S'(t)$, $t > 0$, implies the continuity in the uniform operator topology.

This proves the equicontinuity for the case where $t \ne t_i$, $i = 1,\dots,m$. It remains to examine the equicontinuity at $t = t_i$. The proof is similar to that given in Theorem 4.3.

As a consequence of Steps 1 to 3 and (9.15.2) together with the Arzelá-Ascoli theorem, it suffices to show that the operator N maps B_q into a precompact set in $\overline{D(A)}$. Let $0 < t \le T$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B_q$ we define

$$\begin{aligned}
N_\epsilon(y)(t) &= S'(t)\phi(0) + S'(\epsilon)\lim_{\lambda\to\infty}\int_0^{t-\epsilon} S'(t-s-\epsilon)B_\lambda f(s,y_s)\,ds \\
&\qquad + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)).
\end{aligned} \tag{9.127}$$

Since $S'(t)$ is a compact operator, the set $H_\epsilon(t) = \{N_\epsilon(y)(t) : y \in B_q\}$ is precompact in E for every ϵ, $0 < \epsilon < t$. Moreover, for every $y \in B_q$, we have

$$|N_\epsilon(y)(t) - N(y)(t)| \le M\left|\lim_{\lambda\to\infty}\int_{t-\epsilon}^{t} (S'(t-s-\epsilon) - S(t-s))B_\lambda f(s,y_s)\,ds\right|. \tag{9.128}$$

Therefore there are precompact sets arbitrarily close to the set $\{N(y)(t) : y \in B_q\}$. Hence the set $\{N(y)(t) : y \in B_q\}$ is precompact in $\overline{D(A)}$. Thus we can conclude that $N : \Omega \to \Omega$ is a completely continuous operator.

Step 4. Now it remains to show that the set

$$\mathcal{E}(N) := \{y \in \Omega : y = \sigma N(y) \text{ for some } 0 < \sigma < 1\} \tag{9.129}$$

is bounded.

Let $y \in \mathcal{E}(N)$. Then $y = \sigma N(y)$ for some $0 < \sigma < 1$. Thus, for each $t \in J$,

$$y(t) = \sigma\left(S'(t)\phi(0) + \frac{d}{dt}\int_0^t S(t-s)f(s, y_s)\,ds + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-))\right). \tag{9.130}$$

This implies by (9.15.3) that for each $t \in J$ we have

$$|y(t)| \le Me^{\omega t}\left[|\phi(0)| + \int_0^t e^{-\omega s}p(s)\psi(\|y_s\|_{\mathcal{D}})\,ds + \sum_{k=1}^m e^{-\omega t_k}c_k\right]. \tag{9.131}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \le s \le t\}, \quad 0 \le t \le T. \tag{9.132}$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0, T]$, by the previous inequality we have, for $t \in [0, T]$,

$$e^{-\omega t}\mu(t) \le M\left[\|\phi\|_{\mathcal{D}} + \sum_{k=1}^m e^{-\omega t_k}c_k + \int_0^t e^{-\omega s}p(s)\psi(\mu(s))\,ds\right]. \tag{9.133}$$

If $t^* \in [-r, 0]$, then $\mu(t) = \|\phi\|_{\mathcal{D}}$ and the previous inequality holds. Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$\begin{gathered}\mu(t) \le e^{\omega t}v(t), \quad \forall t \in [0, T],\\ v(0) = M\left(\|\phi\|_{\mathcal{D}} + \sum_{k=1}^m e^{-\omega t_k}c_k\right), \qquad v'(t) = Me^{-\omega t}p(t)\psi(\mu(t)), \quad t \in [0, T].\end{gathered} \tag{9.134}$$

Using the increasing character of ψ, we get

$$v'(t) \le Me^{-\omega t}p(t)\psi(e^{\omega t}v(t)), \quad \text{a.e. } t \in [0, T]. \tag{9.135}$$

Then for each $t \in [0,T]$ we have

$$\begin{aligned}(e^{\omega t}v(t))' &= \omega e^{\omega t}v(t) + v'(t)e^{\omega t} \\ &\le \omega e^{\omega t}v(t) + Mp(t)\psi(e^{\omega t}v(t)) \\ &\le m(t)[e^{\omega t}v(t) + \psi(e^{\omega t}v(t))], \quad t \in [0,T].\end{aligned} \tag{9.136}$$

Thus

$$\int_{v(0)}^{e^{\omega t}v(t)} \frac{du}{u+\psi(u)} \le \int_0^T m(s)ds < \int_{v(0)}^{\infty} \frac{du}{u+\psi(u)}. \tag{9.137}$$

Consequently, there exists a constant d such that $e^{\omega t}v(t) \le d$, $t \in [0,T]$, and hence $\|y\|_\Omega \le \max(\|\phi\|_{\mathcal{D}}, d)$ where d depends only on the constant M, ω, c_k and the functions p and ψ. This shows that $\mathcal{E}(N)$ is bounded. As a consequence of Schaefer's theorem we deduce that N has a fixed point which is an integral solution of (9.111). □

9.4.2. Impulsive neutral functional differential equations

In this section, we study problem (9.112). We give first the definition of integral solution of problem (9.112).

Definition 9.16. A function $y \in \Omega$ is said to be an integral solution of (9.112) if $y(t) = \phi(t)$, $t \in [-r,0]$, $\int_0^t y(s)ds \in D(A)$, $t \in [0,T]$, and y is the solution of impulsive integral equation

$$\begin{aligned}y(t) = S'(t)[\phi(0) - g(0,\phi(0))] + g(t,y_t) + A\int_0^t y(s)ds \\ + \int_0^t f(s,y_s)ds + \sum_{0<t_k<t} I_k(y(t_k^-)).\end{aligned} \tag{9.138}$$

Theorem 9.17. *Assume (H1), (9.4.2), (9.4.5), f is an L^1-Carathéodory function, and the following conditions hold:*

(9.17.1) *there exist constants* $0 \le c_1 < 1$, $c_2 \ge 0$ *such that*

$$|g(t,u)| \le c_1\|u\|_{\mathcal{D}} + c_2, \quad a.e.\ t \in [0,T],\ u \in D; \tag{9.139}$$

(9.17.2) (i) *the function g is completely continuous,*
(ii) *for any bounded set B in $C([-r,T],E)$, the set $\{t \to g(t,y_t) : y \in B\}$ is equicontinuous in Ω;*

(9.17.3) *there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and $p \in L^1([0,T],\mathbb{R}_+)$ such that*

$$|f(t,u)| \le p(t)\psi(\|u\|_{\mathcal{D}}) \quad \text{for a.e. } t \in [0,T] \text{ and each } u \in \mathcal{D}, \tag{9.140}$$

with

$$\int_0^T \overline{m}(s)ds < \int_c^\infty \frac{du}{u+\psi(u)}, \tag{9.141}$$

where

$$\begin{gathered}\overline{m}(t) = \max\left\{\omega, \frac{M}{1-c_1}p(t)\right\},\\ c = \frac{M}{1-c_1}\left[(1+c_1)\|\phi\|_{\mathcal{D}} + \frac{c_2}{M} + \sum_{k=1}^{m} e^{-\omega t_k}c_k\right].\end{gathered} \tag{9.142}$$

Then the IVP (9.112) *has at least one integral solution on* $[-r,T]$.

Proof. Transform problem (9.112) into a fixed point problem. Consider the operator $\overline{N}:\Omega\to\Omega$ defined by

$$\overline{N}(y)(t) := \begin{cases}\phi(t) & \text{if } t\in[-r,0],\\ S'(t)[\phi(0)-g(0,\phi(0))]+g(t,y_t) & \\ \quad +\dfrac{d}{dt}\displaystyle\int_0^t S(t-s)f(s,y_s)ds & \\ \quad +\displaystyle\sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)) & \text{if } t\in[0,T].\end{cases} \tag{9.143}$$

Let $\widetilde{N}:\Omega\to\Omega$ be defined by

$$\widetilde{N}(y)(t) := \begin{cases}\phi(t) & \text{if } t\in[-r,0],\\ S'(t)\phi(0)+\dfrac{d}{dt}\displaystyle\int_0^t S(t-s)f(s,y_s)ds & \\ \quad +\displaystyle\sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)) & \text{if } t\in[0,T].\end{cases} \tag{9.144}$$

As in the proof of Theorem 9.15, we can prove that $\widetilde{N}$ is completely continuous and by (9.17.2) $\overline{N}$ is completely continuous.

Now we prove only that the set

$$\mathcal{E}(\overline{N}) := \{y\in\Omega : y=\sigma\overline{N}(y) \text{ for some } 0<\sigma<1\} \tag{9.145}$$

is bounded. Let $y\in\mathcal{E}(\overline{N})$. Then $\sigma\overline{N}(y)=y$, for some $0<\sigma<1$ and

$$\begin{aligned}y(t) = \sigma\bigg[& S'(t)[\phi(0)-g(0,\phi(0))]+g(t,y_t)+\frac{d}{dt}\int_0^t S(t-s)f(s,y_s)ds\\ &+\sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-))\bigg].\end{aligned} \tag{9.146}$$

This implies that, for each $t \in [0,T]$, we have

$$
\begin{aligned}
|y(t)| \leq{}& Me^{\omega t}\big[(1+c_1)\|\phi\|_{\mathcal{D}}+c_2\big]+c_1\|y_t\|_{\mathcal{D}}+c_2 \\
&+Me^{\omega t}\int_0^t e^{-\omega s}p(s)\psi(\|y_s\|_{\mathcal{D}})\,ds+Me^{\omega t}\sum_{k=1}^{m}e^{-\omega t_k}c_k.
\end{aligned}
\tag{9.147}
$$

We consider the function μ defined by

$$
\mu(t) := \sup\{|y(s)| : -r \leq s \leq t\}, \quad t \in [0,T]. \tag{9.148}
$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0,T]$, by the previous inequality, we have, for $t \in [0,T]$,

$$
\begin{aligned}
(1-c_1)\mu(t) \leq{}& Me^{\omega t}\big[(1+c_1)\|\phi\|_{\mathcal{D}}+c_2\big]+c_2 \\
&+Me^{\omega t}\int_0^t e^{-\omega s}p(s)\psi(\mu(s))\,ds+Me^{\omega t}\sum_{k=1}^{m}e^{-\omega t_k}c_k
\end{aligned}
\tag{9.149}
$$

or

$$
e^{-\omega t}\mu(t) \leq \frac{M}{1-c_1}\left[(1+c_1)\|\phi\|_{\mathcal{D}}+c_2+\frac{c_2}{M}+\int_0^t e^{-\omega s}p(s)\psi(\mu(s))\,ds+\sum_{k=1}^{m}e^{-\omega t_k}c_k\right]. \tag{9.150}
$$

If $t^* \in [-r,0]$, then $\mu(t) = \|\phi\|$ and the inequality holds. Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$
\begin{aligned}
v(0) &= \frac{M}{1-c_1}\left[(1+c_1)\|\phi\|_{\mathcal{D}}+c_2+\frac{c_2}{M}+\sum_{k=1}^{m}e^{-\omega t_k}c_k\right], \\
v'(t) &= \frac{M}{1-c_1}e^{-\omega t}p(t)\psi(\mu(t)) \leq \frac{M}{1-c_1}e^{-\omega t}p(t)\psi(e^{\omega t}v(t)), \quad t \in [0,T].
\end{aligned}
\tag{9.151}
$$

Then for each $t \in [0,T]$ we have

$$
\begin{aligned}
\big(e^{\omega t}v(t)\big)' &= \omega e^{\omega t}v(t)+v'(t)e^{\omega t} \leq \omega e^{\omega t}v(t)+\frac{M}{1-c_1}p(t)\psi(e^{\omega t}v(t)) \\
&\leq \overline{m}(t)\big[e^{\omega t}v(t)+\psi(e^{\omega t}v(t))\big], \quad t \in [0,T].
\end{aligned}
\tag{9.152}
$$

By using (9.17.3) we then have

$$
\int_{v(0)}^{e^{\omega t}v(t)}\frac{du}{u+\psi(u)} \leq \int_0^t \overline{m}(s)\,ds \leq \int_0^T \overline{m}(s)\,ds < \int_{v(0)}^{\infty}\frac{du}{u+\psi(u)}. \tag{9.153}
$$

This inequality implies that there exists a constant b depending only on T, M, c_k and on the functions p and ψ such that

$$|y(t)| \leq b, \quad \text{for each } t \in [0,T]. \tag{9.154}$$

Hence

$$\|y\|_{\Omega} \leq \max\left(\|\phi\|_{\mathcal{D}}, b\right). \tag{9.155}$$

This shows that $\mathcal{E}(\overline{N})$ is bounded. Set $X := \Omega$. As a consequence of Schaefer's theorem we deduce that $\overline{N}$ has a fixed point which is an integral solution of problem (9.112). □

9.5. Nondensely defined impulsive semilinear functional differential inclusions

In this section, we will be concerned with the existence of integral solutions for first-order impulsive semilinear functional and neutral functional differential inclusions in Banach spaces. First, we will consider first-order impulsive semilinear functional differential inclusions of the form

$$\begin{gathered} y'(t) - Ay(t) \in F(t, y_t), \quad \text{a.e. } t \in J = [0,T]\setminus\{t_1,\dots,t_m\}, \\ \Delta y|_{t=t_k} = I_k\left(y(t_k^-)\right), \quad k = 1,\dots,m, \\ y(t) = \phi(t), \quad t \in [-r, 0], \end{gathered} \tag{9.156}$$

where $F : [0,T] \times \mathcal{D} \to P(E)$ is a function, $\mathcal{D} = \{\psi : [-r,0] \to E : \psi$ is continuous everywhere except for a finite number of points $\bar{t}$ at which $\psi(\bar{t})$ and $\psi(\bar{t}^+)$ exist and $\psi(\bar{t}^-) = \psi(\bar{t})\}$, $(0 < r < \infty)$, $A : D(A) \subset E \to E$ is a nondensely defined closed linear operator on E, $\phi \in \mathcal{D}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $I_k \in C(E,E)$ $(k = 1,\dots,m)$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k - h)$, and E is a real separable Banach space with norm $|\cdot|$.

Later, we study first-order impulsive semilinear neutral functional differential equations of the form

$$\begin{gathered} \frac{d}{dt}\left[y(t) - g(t, y_t)\right] - Ay(t) \in F(t, y_t), \quad \text{a.e. } t \in J = [0,T]\setminus\{t_1,\dots,t_m\}, \\ \Delta y|_{t=t_k} = I_k\left(y(t_k^-)\right), \quad k = 1,\dots,m, \\ y(t) = \phi(t), \quad t \in [-r, 0], \end{gathered} \tag{9.157}$$

where F, I_k, A, and ϕ are as in problem (9.156), $g : [0,T] \times \mathcal{D} \to E$ is a given function.

9.5.1. Impulsive functional differential inclusions

Let us start by defining what we mean by an integral solution of problem (9.156).

Definition 9.18. A function $y \in \Omega$ is said to be an integral solution of (9.156) if there exists $f(t) \in F(t, y_t)$ a.e. on J such that y is the solution of the impulsive integral equation

$$y(t) = \begin{cases} S'(t)\phi(0) + A\displaystyle\int_0^t y(s)ds + \int_0^t f(s)ds \\ \quad + \displaystyle\sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)), \int_0^t y(s)ds \in D(A), & t \in [0,T], \\ \phi(t), & t \in [-r, 0]. \end{cases} \tag{9.158}$$

By the definition, it follows that $y(t) \in \overline{D(A)}$, $t \geq 0$. Moreover, y satisfies the following variation of constants formula:

$$y(t) = S'(t)y_0 + \frac{d}{dt}\int_0^t S(t-s)f(s, y_s)ds + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)), \quad t \geq 0. \tag{9.159}$$

Let $B_\lambda = \lambda R(\lambda, A)$. Then for all $x \in \overline{D(A)}$, $B_\lambda x \to x$ as $\lambda \to \infty$. As a consequence, if y satisfies (9.159), then

$$y(t) = S'(t)y_0 + \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda f(s, y_s)ds + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)), \quad t \geq 0. \tag{9.160}$$

Theorem 9.19. *Assume that (H1), (9.4.2), (9.4.5), (9.15.2) and the following conditions hold:*

(9.19.1) $F : [0,T] \times D \to \mathcal{P}(\overline{D(A)})$ *is a nonempty compact valued multivalued map such that*
(a) $(t,u) \mapsto F(t,u)$ *is* $\mathcal{L} \otimes \mathcal{B}$ *measurable,*
(b) $u \mapsto F(t,u)$ *is lower semicontinuous for a.e.* $t \in [0,T]$*;*

(9.19.2) *for each* $q > 0$*, there exists a function* $h_q \in L^1([0,T], \mathbb{R}^+)$ *such that*

$$||F(t,u)|| := \sup\{|v| : v \in F(t,u)\} \leq h_q(t) \quad \text{for a.e. } t \in [0,T], \tag{9.161}$$

and for $u \in D$ *with* $\|u\|_{\mathcal{D}} \leq q$.

Then the IVP (9.156) *has at least one integral solution.*

Proof. Hypotheses (9.19.1) and (9.19.2) imply by Frigon [148, Lemma 2.2] that F is of lower semicontinuous type. Then from Theorem 1.5, there exists a continuous function $f : \Omega \to L^1([0,T], \overline{D(A)})$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in \Omega$.

Consider the following problem:

$$\begin{gathered} y'(t) - Ay(t) = f(y_t), \quad t \in [0,T],\ t \neq t_k,\ k = 1,\dots,m, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m, \\ y(t) = \phi(t), \quad t \in [-r,0]. \end{gathered} \tag{9.162}$$

Clearly, if $y \in \Omega$ is an integral solution of problem (9.162), then y is a solution to problem (9.156).

Transform problem (9.162) into a fixed point problem. Consider the operator $N : \Omega \to \Omega$ defined by

$$N(y)(t) = \begin{cases} \phi(t) & \text{if } t \in [-r,0], \\ S'(t)\phi(0) + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)f(y_s)\,ds & \\ \quad + \displaystyle\sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)) & \text{if } t \in [0,T]. \end{cases} \tag{9.163}$$

The proof that N has a fixed point is similar to that of Theorem 9.15 and we omit the details. □

9.5.2. Impulsive neutral functional differential inclusions

In this section, we study problem (9.157). We give first the definition of an integral solution of problem (9.157).

Definition 9.20. A function $y \in \Omega$ is said to be an integral solution of (9.157) if there exists $f(t) \in F(t, y_t)$ a.e. on $[0,T]$ such that y is the solution of impulsive integral equation

$$\begin{aligned} y(t) = {} & S'(t)[\phi(0) - g(0,\phi(0))] + g(t,y_t) + A\int_0^t y(s)\,ds \\ & + \int_0^t f(s)\,ds + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)), \quad t \in [0,T], \end{aligned} \tag{9.164}$$

and $y(t) = \phi(t)$, $t \in [-r,0]$, $\int_0^t y(s)ds \in D(A)$, $t \in [0,T]$.

Theorem 9.21. *Assume (H1), (9.4.2), (9.4.5), (9.17.2), (9.19.1), (9.19.2) and the following conditions hold:*

(9.21.1) *there exist constants* $0 \le c_1 < 1$, $c_2 \ge 0$ *such that*

$$|g(t,u)| \le c_1\|u\|_{\mathcal{D}} + c_2, \quad \text{a.e. } t \in [0,T],\ u \in D; \tag{9.165}$$

(9.21.2) *there exist a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *and* $p \in L^1([0,T],\mathbb{R}_+)$ *such that*

$$\|F(t,u)\| := \sup\{|v| : v \in F(t,u)\} \le p(t)\psi(\|u\|_{\mathcal{D}}) \tag{9.166}$$

for a.e. $t \in [0,T]$ *and each* $u \in \mathcal{D}$ *with*

$$\int_0^T m(s)ds < \int_c^\infty \frac{du}{u+\psi(u)}, \tag{9.167}$$

where

$$\begin{gathered}\overline{m}(t) = \max\left\{\omega, \frac{M}{1-c_1}p(t)\right\},\\ c = \frac{M}{1-c_1}\left[(1+c_1)\|\phi\|_{\mathcal{D}} + \frac{c_2}{M} + \sum_{k=1}^m e^{-\omega t_k}c_k\right].\end{gathered} \tag{9.168}$$

Then the IVP (9.157) *has at least one integral solution.*

Proof. Let $f : \Omega \to L^1([0,T],\overline{D(A)})$ be a selection of F, and consider the problem

$$\begin{gathered}\frac{d}{dt}[y(t) - g(t,y_t)] - Ay(t) = f(y_t), \quad t \in J,\ t \ne t_k,\ k = 1,\dots,m,\\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m,\\ y(t) = \phi(t), \quad t \in [-r,0].\end{gathered} \tag{9.169}$$

Consider the operator $\overline{N} : \Omega \to \Omega$ defined by

$$\overline{N}(y)(t) := \begin{cases}\phi(t) & \text{if } t \in [-r,0],\\ S'(t)[\phi(0) - g(0,\phi(0))] + g(t,y_t) & \\ \quad +\dfrac{d}{dt}\displaystyle\int_0^t S(t-s)f(y_s)ds & \\ \quad +\displaystyle\sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)) & \text{if } t \in [0,T].\end{cases} \tag{9.170}$$

Let $\tilde{N} : \Omega \to \Omega$ be defined by

$$\tilde{N}(y)(t) := \begin{cases}\phi(t) & \text{if } t \in [-r,0],\\ S'(t)\phi(0) + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)f(y_s)ds & \\ \quad +\displaystyle\sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)) & \text{if } t \in [0,T].\end{cases} \tag{9.171}$$

As in the proof of Theorem 9.19, we can prove that $\tilde{N}$ is completely continuous and by using (9.19.2) $\overline{N}$ is completely continuous.

Now we prove only that the set

$$\mathcal{E}(\overline{N}) := \{y \in \Omega : y = \sigma\overline{N}(y) \text{ for some } 0 < \sigma < 1\} \tag{9.172}$$

is bounded. Let $y \in \mathcal{E}(\overline{N})$. Then $\sigma\overline{N}(y) = y$ for some $0 < \sigma < 1$ and

$$\begin{aligned} y(t) = \sigma\bigg[& S'(t)[\phi(0) - g(0,\phi(0))] + g(t,y_t) + \frac{d}{dt}\int_0^t S(t-s)f(y_s)\,ds \\ & + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-))\bigg]. \end{aligned} \tag{9.173}$$

This implies that for each $t \in [0,T]$ we have

$$\begin{aligned} |y(t)| \le {} & Me^{\omega t}[(1+c_1)\|\phi\|_{\mathcal{D}} + c_2] + c_1\|y_t\|_{\mathcal{D}} + c_2 \\ & + Me^{\omega t}\int_0^t e^{-\omega s}p(s)\psi(\|y_s\|_{\mathcal{D}})\,ds + Me^{\omega t}\sum_{k=1}^m e^{-\omega t_k}c_k. \end{aligned} \tag{9.174}$$

We consider the function μ defined by

$$\mu(t) := \sup\{|y(s)| : -r \le s \le t\}, \quad t \in [0,T]. \tag{9.175}$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0,T]$, by the previous inequality we have, for $t \in [0,T]$,

$$\begin{aligned} (1-c_1)\mu(t) \le {} & Me^{\omega t}[(1+c_1)\|\phi\|_{\mathcal{D}} + c_2] + c_2 \\ & + Me^{\omega t}\int_0^t e^{-\omega s}p(s)\psi(\mu(s))\,ds + Me^{\omega t}\sum_{k=1}^m e^{-\omega t_k}c_k \end{aligned} \tag{9.176}$$

or

$$e^{-\omega t}\mu(t) \le \frac{M}{1-c_1}\bigg[(1+c_1)\|\phi\|_{\mathcal{D}} + c_2 + \frac{c_2}{M} + \int_0^t e^{-\omega s}p(s)\psi(\mu(s))\,ds + \sum_{k=1}^m e^{-\omega t_k}c_k\bigg]. \tag{9.177}$$

If $t^* \in [-r,0]$, then $\mu(t) = \|\phi\|_{\mathcal{D}}$ and the inequality holds. Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$\begin{aligned} v(0) &= \frac{M}{1-c_1}\bigg[(1+c_1)\|\phi\|_{\mathcal{D}} + c_2 + \frac{c_2}{M} + \sum_{k=1}^m e^{-\omega t_k}c_k\bigg], \\ v'(t) &= \frac{M}{1-c_1}e^{-\omega t}p(t)\psi(\mu(t)) \le \frac{M}{1-c_1}e^{-\omega t}p(t)\psi(e^{\omega t}v(t)), \quad t \in [0,T]. \end{aligned} \tag{9.178}$$

Then for each $t \in [0, T]$ we have

$$\begin{aligned}(e^{\omega t}v(t))' &= \omega e^{\omega t}v(t) + v'(t)e^{\omega t} \le \omega e^{\omega t}v(t) + \frac{M}{1-c_1}p(t)\psi(e^{\omega t}v(t)) \\ &\le \overline{m}(t)\left[e^{\omega t}v(t) + \psi(e^{\omega t}v(t))\right], \quad t \in [0, T].\end{aligned} \tag{9.179}$$

By using (A3) we then have

$$\int_{v(0)}^{e^{\omega t}v(t)} \frac{du}{u+\psi(u)} \le \int_0^t \overline{m}(s)ds \le \int_0^T \overline{m}(s)ds < \int_{v(0)}^{\infty} \frac{du}{u+\psi(u)}. \tag{9.180}$$

This inequality implies that there exists a constant b depending only on T, M, c_k and on the functions p and ψ such that

$$|y(t)| \le b, \quad \text{for each } t \in [0, T]. \tag{9.181}$$

Hence

$$\|y\|_\Omega \le \max\left(\|\phi\|_{\mathcal{D}}, b\right). \tag{9.182}$$

This shows that $\mathcal{E}(\overline{N})$ is bounded. Set $X := \Omega$. As a consequence of Schaefer's theorem we deduce that $\overline{N}$ has a fixed point which is an integral solution of problem (9.157). □

9.6. Notes and remarks

The results of Section 9.2 are taken from [38] and concern nondensely defined evolution equations with nonlocal conditions. These results are extended in Section 9.3, for nondensely defined impulsive differential inclusions, where the results from Benchohra et al. [39] are presented. Sections 9.4 and 9.5 deal with nondensely defined semilinear functional and neutral functional differential equations and inclusions, respectively. The material of Section 9.4 is taken from Benchohra et al. [42], and Section 9.5 contains results from Benchohra et al. [76].

10 Hyperbolic impulsive differential inclusions

10.1. Introduction

In this chapter, we will be concerned with the existence of solutions for second-order impulsive hyperbolic differential inclusions in a separable Banach space. More precisely, we will consider impulsive hyperbolic differential inclusions of the form

$$\begin{gathered}
\frac{\partial^2 u(t,x)}{\partial t \partial x} \in F(t,x,u(t,x)), \quad \text{a.e. } (t,x) \in J_a \times J_b,\ t \neq t_k,\ k = 1,\dots,m, \\
\Delta u(t_k,x) = I_k(u(t_k,x)), \quad k = 1,\dots,m, \\
u(t,0) = \psi(t), \quad t \in J_a, \qquad u(0,x) = \phi(x), \quad x \in J_b,
\end{gathered} \tag{10.1}$$

where $J_a = [0,a]$, $J_b = [0,b]$, $F : J_a \times J_b \times E \to \mathcal{P}(E)$ is a multivalued map ($\mathcal{P}(E)$ is the family of all nonempty subsets of E), $\phi \in C(J_a,E)$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = a$, $I_k \in C(E,E)$ $(k = 1,\dots,m)$, $\Delta u|_{t=t_k} = u(t_k^+,y) - u(t_k^-,y)$, $u(t_k^+,y) = \lim_{(h,x)\to(0^+,y)} u(t_k+h,x)$ is the right limit and $u(t_k^-,y) = \lim_{(h,x)\to(0^+,y)} u(t_k-h,x)$ is left limit of $u(t,x)$ at (t_k,x), and E is a real separable Banach space with norm $|\cdot|$.

In the last few years impulsive differential and partial differential equations have become the object of increasing investigation in some mathematical models of real world phenomena, especially in biological or medical domain; see the monographs by Baĭnov and Simeonov [29], Lakshmikantham et al. [180], Samoĭlenko and Perestyuk [217].

In the last three decades several papers have been devoted to the study of hyperbolic partial differential equations with local and nonlocal initial conditions; see for instance [113, 115, 182] and the references cited therein. For similar results with set-valued right-hand side, we refer to the papers by Byszewski and Papageorgiou [116], Papageorgiou [208], and Benchohra and Ntouyas [33, 81, 83, 84].

Here we will present three existence results for problem (10.1) in the cases when F has convex and nonconvex values. In the convex case, an existence result will be given by means of the nonlinear alternative of Leray-Schauder type for multivalued maps. In the nonconvex, case two results will be presented. The first one relies on a fixed point theorem due to Covitz and Nadler for contraction multivalued maps and the second one on the nonlinear alternative of Leray-Schauder type for single-valued maps combined with a selection theorem due to Bressan and Colombo [105] for lower semicontinuous multivalued operators with closed and decomposable values. Our results extend to the multivalued case some ones considered in the previous literature.

10.2. Preliminaries

We will briefly recall some basic definitions and facts from multivalued analysis that we will use in the sequel.

$C(J_a \times J_b, E)$ is the Banach space of all continuous functions from $J_a \times J_b$ into E with the norm

$$\|u\|_\infty = \sup\{|u(t,s)| : (t,s) \in J_a \times J_b\}. \tag{10.2}$$

A measurable function $z : J_a \times J_b \to E$ is Bochner integrable if and only if $|z|$ is Lebesgue integrable. (For properties of the Bochner integral, see, e.g., Yosida [230].)

$L^1(J_a \times J_b, E)$ denotes the Banach space of functions $z : J_a \times J_b \to E$ which are Bochner integrable normed by

$$\|z\|_{L^1} = \int_0^a \int_0^b |z(t,s)|\,dt\,ds. \tag{10.3}$$

A multivalued map $N : J_a \times J_b \times E \to \mathcal{P}_{\mathrm{cl}}(E)$ is said to be measurable, if for every $w \in E$, the function $t \mapsto d(w, N(t,x,u)) = \inf\{\|w - v\| : v \in N(t,x,u)\}$ is measurable, where d is the metric induced from the Banach space E.

Definition 10.1. The multivalued map $F : J_a \times J_b \times E \to \mathcal{P}(E)$ is said to be L^1-Carathéodory, if

(i) $(t,x) \mapsto F(t,x,u)$ is measurable for each $u \in E$;
(ii) $u \mapsto F(t,x,u)$ is upper semicontinuous for almost all $(t,x) \in J_a \times J_b$;
(iii) for each $\rho > 0$, there exists $\varphi_\rho \in L^1(J_a \times J_b, \mathbb{R}_+)$ such that

$$||F(t,x,u)|| = \sup\{|v| : v \in F(t,x,u)\} \le \varphi_\rho(t,x) \tag{10.4}$$

for all $|u| \le \rho$ and for a.e. $(t,x) \in J_a \times J_b$.

For each $u \in C(J_a \times J_b, E)$, define the set of *selections* of F by

$$S_{F,u} = \{v \in L^1(J_a \times J_b, E) : v(t,s) \in F(t,x,u(t,x)) \text{ a.e. } (t,x) \in J_a \times J_b\}. \tag{10.5}$$

The following lemma can be reduced easily from the corresponding one in [186].

Lemma 10.2 (see [186]). *Let X be a Banach space. Let $F : J_a \times J_b \times X \to \mathcal{P}_{\text{cp,cv}}(X)$ be an L^1-Carathéodory multivalued map with $S_F \neq \emptyset$, and let Ψ be a linear continuous mapping from $L^1(J_a \times J_b, X)$ to $C(J \times J_b, X)$. Then the operator*

$$\Psi \circ S_F : C(J_a \times J_b, X) \longrightarrow \mathcal{P}_{\text{cp},c}(C(J_a \times J_b, X)), \qquad u \longmapsto (\Psi \circ S_F)(u) := \Psi(S_{F,u}) \tag{10.6}$$

is a closed graph operator in $C(J_a \times J_b, X) \times C(J_a \times J_b, X)$.

10.3. Main results

In this section, we are concerned with the existence of solutions for problem (10.1) when the right-hand side has convex as well as nonconvex values. First, we assume that $F : J_a \times J_b \times E \to \mathcal{P}(E)$ is a compact and convex valued multivalued map. In order to define the solution of (10.1) we will consider the space

$$\begin{aligned} \Omega = \{u : J_a \times J_b \longrightarrow E : u_k \in C(\Gamma_k, E),\ k = 0, \dots, m, \\ \exists u(t_k^-, \cdot), u(t_k^+, \cdot),\ k = 1, \dots, m, \text{ with } u(t_k^-, \cdot) = u(t_k, \cdot)\} \end{aligned} \tag{10.7}$$

which is a Banach space with the norm

$$\|u\|_\Omega = \max\{\|u_k\|,\ k = 0, \dots, m\}, \tag{10.8}$$

where u_k is the restriction of u to $\Gamma_k := (t_k, t_{k+1}) \times [0, b]$, $k = 0, \dots, m$.

Definition 10.3. A function $u \in \Omega \cap AC^1(\Gamma_k, E)$, $k = 1, \dots, m$, is said to be a solution of (10.1) if there exists $v \in L^1(J_a \times J_b, E)$ such that $v(t,x) \in F(t, x, u(t,x))$ a.e. on $J_a \times J_b$, and

$$u(t,x) = z(t,x) + \int_0^t \int_0^x v(s,\tau)\,ds\,d\tau + \sum_{0<t_k<t} I_k(u(t_k, x)), \tag{10.9}$$

where $z(t,x) = \psi(t) + \phi(x) - \psi(0)$.

Theorem 10.4. *Assume that the following conditions are satisfied:*

(10.4.1) *$F : J_a \times J_b \times E \to P_{b,\text{cp,cv}}(E)$ is an L^1-Carathéodory multimap;*

(10.4.2) *there exist constants c_k, d_k such that*

$$|I_k(u)| \le c_k, \quad \text{for each } u \in E,\ k = 1, \dots, m; \tag{10.10}$$

(10.4.3) *there exist functions $p, q \in L^1(J_a \times J_b, \mathbb{R}_+)$ such that*

$$\|F(t,x,u)\| \le p(t,x) + q(t,x)|u| \tag{10.11}$$

for a.e. $(t,x) \in J_a \times J_b$ and each $u \in E$;

(10.4.4) *for each bounded $\mathcal{B} \subseteq \Omega$ and $t \in J$, the set*

$$\left\{z(t,x) + \int_0^t \int_0^x v(s,\tau)ds\,d\tau + \sum_{0<t_k<t} I_k(u(t_k,x)),\ v \in S_{F,\mathcal{B}}\right\} \tag{10.12}$$

is relatively compact in E, where $S_{F,\mathcal{B}} = \cup\{S_{F,y} : y \in \mathcal{B}\}$. Then problem (10.1) *has at least one solution.*

Proof. Transform the problem (10.1) into a fixed point problem. Consider the multivalued operator $N : \Omega \to \mathcal{P}(\Omega)$ defined by

$$\begin{aligned} N(u) = \Bigg\{h \in \Omega : h(t,x) = z(t,x) + \int_0^t \int_0^x v(s,\tau)ds\,d\tau \\ + \sum_{0<t_k<t} I_k(u(t_k,x)),\ v \in S_{F,u}\Bigg\}. \end{aligned} \tag{10.13}$$

We will show that N satisfies the assumptions of Theorem 1.8. The proof will be given in several steps.

Step 1. $N(u)$ is convex for each $u \in \Omega$.

Indeed, if h_1, h_2 belong to $N(u)$, then there exist $v_1, v_2 \in S_{F,u}$ such that for each $(t,x) \in J_a \times J_b$ we have

$$h_i(t,x) = z(t,x) + \int_0^t \int_0^x v_i(s,\tau)ds\,d\tau + \sum_{0<t_k<t} I_k(u(t_k,x)), \quad i = 1,2. \tag{10.14}$$

Let $0 \leq d \leq 1$. Then for each $(t,x) \in J_a \times J_b$ we have

$$\begin{aligned} &(dh_1 + (1-d)h_2)(t) \\ &\quad = z(t,x) + \int_0^t \int_0^x [dv_1(s,\tau) + (1-d)v_2(s,\tau)]ds\,d\tau + \sum_{0<t_k<t} I_k(u(t_k,x)). \end{aligned} \tag{10.15}$$

Since $S_{F,u}$ is convex (because F has convex values), then

$$dh_1 + (1-d)h_2 \in N(u). \tag{10.16}$$

Step 2. N maps bounded sets into bounded sets in Ω.

Indeed, it is enough to show that there exists a positive constant ℓ such that, for each $u \in \mathcal{B}_r = \{u \in \Omega : \|u\|_\Omega \leq r\}$, one has $\|N(u)\|_\Omega \leq \ell$.

Let $u \in \mathcal{B}_r$ and $h \in N(u)$. Then by (10.4.2)-(10.4.3) we have, for each $(t,x) \in J_a \times J_b$,

$$\begin{aligned} |h(t,x)| &\leq |z(t,x)| + \int_0^a \int_0^b |p(t,x)| + |q(t,x)|\,|u(t,x)|\,ds + \sum_{k=1}^m c_k \\ &\leq \|z\|_\infty + \|p\|_{L^1} + r\|q\|_{L^1} + \sum_{k=1}^m c_k := \ell. \end{aligned} \tag{10.17}$$

Step 3. N maps bounded sets into equicontinuous sets of Ω.

Let $(\tau_1, x_1), (\tau_2, x_2) \in J_a \times J_b$, $\tau_1 < \tau_2$, $x_1 < x_2$, and $\mathcal{B}_q$ be a bounded set of Ω as in Step 2. Then

$$\begin{aligned} |h(\tau_2,x_2) - h(\tau_1,x_1)| \leq{}& |z_0(\tau_2,x_2) - z_0(\tau_1,x_1)| + \int_0^{\tau_1}\int_{x_1}^{x_2} \phi_q(t,s)dt\,ds \\ &+ \int_0^{\tau_2}\int_{x_1}^{x_2} \phi_q(t,s)dt\,ds + \int_{\tau_2}^{\tau_1}\int_{x_1}^{x_2} \phi_q(t,s)dt + \sum_{0<t<\tau_2-\tau_1} c_k. \end{aligned} \tag{10.18}$$

The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$, $x_2 - x_1 \to 0$.

As a consequence of Steps 1 to 3 and (10.4.4) together with the Arzelá-Ascoli theorem we can conclude that $N : \Omega \to P(\Omega)$ is a completely continuous multivalued map.

Step 4. N has a closed graph.

Let $u_n \to u_*$, $h_n \in N(u_n)$, and $h_n \to h_*$. We will prove that $h_* \in N(u_*)$.

$h_n \in N(u_n)$ means that there exists $v_n \in S_{F,u_n}$ such that for each $(t,x) \in J_a \times J_b$,

$$h_n(t,x) = z(t,x) + \int_0^t \int_0^x v_n(s,x)ds + \sum_{0<t_k<t} I_k(u_n(t_k,x)). \tag{10.19}$$

We must prove that there exists $v_* \in S_{F,u_*}$ such that for each $(t,x) \in J_a \times J_b$,

$$h_*(t,x) = z(t,x) + \int_0^t \int_0^x v_*(s,x)ds + \sum_{0<t_k<t} I_k(u_*(t_k,x)). \tag{10.20}$$

Clearly since I_k, $k = 1, \dots, m$, and ϕ are continuous, we have that

$$\begin{aligned} &\Bigg\|\Bigg(h_n - z(t,x) - \sum_{0<t_k<t} I_k(u_n(t_k,x))\Bigg) \\ &\quad - \Bigg(h_* - z(t,x) - \sum_{0<t_k<t} I_k(u_*(t_k,x))\Bigg)\Bigg\|_\infty \longrightarrow 0, \end{aligned} \tag{10.21}$$

as $n \to \infty$.

Consider the linear continuous operator

$$\begin{aligned}\Psi : L^1(J_a \times J_b, E) &\longrightarrow C(J_a \times J_b, E),\\ v \longmapsto \Psi(v)(t,x) &= \int_0^t \int_0^x v(s,\tau)\,ds\,d\tau.\end{aligned} \tag{10.22}$$

From Lemma 10.2, it follows that $\Psi \circ S_F$ is a closed graph operator. Moreover, we have that

$$h_n(t,x) - z(t,x) - \sum_{0<t_k<t} I_k(u_n(t_k,x)) \in \Psi(S_{F,u_n}). \tag{10.23}$$

Since $u_n \to u_*$, it follows from Lemma 10.2 that

$$h_*(t,x) = z(t,x) + \int_0^t \int_0^x v_*(s,x)\,ds + \sum_{0<t_k<t} I_k(u_*(t_k,x)), \tag{10.24}$$

for some $v_* \in S_{F,u_*}$.
Step 5. A priori bounds on solutions.

Let $u \in \Omega$ be such that $u \in \lambda N(u)$ for some $\lambda \in (0,1)$. Then by (10.4.2)-(10.4.3) for each $(t,x) \in J_a \times J_b$ we have

$$\begin{aligned}|u(t,x)| &\le \|z\|_\infty + \int_0^t \int_0^x [|p(s,\tau)| + |q(s,\tau)||u(s,\tau)|]\,ds\,d\tau + \sum_{k=1}^m c_k\\ &\le \|z\|_\infty + \int_0^t \int_0^x |q(s,\tau)||u(s,\tau)|\,ds\,d\tau + \|p\|_{L^1} + \sum_{k=1}^m c_k.\end{aligned} \tag{10.25}$$

Let

$$z_0 = \|z\|_\infty + \|p\|_{L^1} + \sum_{k=1}^m c_k. \tag{10.26}$$

Then, for $(t,x) \in J_a \times J_b$,

$$u(t,x) \le z_0 + \int_0^t \int_0^x |q(s,\tau)||u(s,\tau)|\,ds\,d\tau. \tag{10.27}$$

Invoking Gronwall's inequality (see, e.g., [160]), we get that

$$u(t,x) \le z_0 e^{\|q\|_{L^1}} := M. \tag{10.28}$$

Then

$$\|u\|_\Omega < M. \tag{10.29}$$

Set

$$U = \{u \in \Omega : \|u\|_\Omega < M + 1\}. \tag{10.30}$$

From the choice of U there is no $u \in \partial U$ such that $u \in \lambda N(u)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [157] we deduce that N has a fixed point u in U which is a solution of problem (10.1). □

Theorem 10.5. *Suppose that the following hypotheses are satisfied:*

(10.5.1) $F : J_a \times J_b \times E \to \mathcal{P}_{\text{cp,cv}}(E)$; $(t,x,u) \mapsto F(t,x,u)$ *is measurable for each* $u \in E$;

(10.5.2) *there exist constants* c_k^* *such that*

$$\left| I_k(u) - I_k(\overline{u}) \right| \leq c_k^* |u - \overline{u}|, \tag{10.31}$$

for each $k = 1,\dots,m$, *and for all* $u, \overline{u} \in E$;

(10.5.3) *there exists a function* $l \in L^1(J_a \times J_b, \mathbb{R}^+)$ *such that*

$$H_d\big(F(t,x,u), F(t,x,\overline{u})\big) \leq l(t,s)|u - \overline{u}|, \tag{10.32}$$

for a.e. $(t,x) \in J_a \times J_b$ *and all* $u, \overline{u} \in E$, *and*

$$d\big(0, F(t,x,0)\big) \leq l(t,s) \quad \text{for a.e. } (t,x) \in J_a \times J_b. \tag{10.33}$$

If

$$\|l\|_{L^1} + \sum_{k=1}^m c_k^* < 1, \tag{10.34}$$

then problem (10.1) *has at least one solution.*

Proof. Transform the problem (10.1) into a fixed point problem. Let the multivalued operator $N : \Omega \to P(\Omega)$ defined as in Theorem 10.4. We will show that N satisfies the assumptions of Theorem 1.11. The proof will be given in two steps.

Step 1. $N(u) \in P_{\text{cl}}(\Omega)$ for each $u \in \Omega$.

Indeed, let $(u_n)_{n\geq 0} \in N(u)$ such that $u_n \to \tilde{u}$ in Ω. Then there exists $v_n \in S_{F,u}$ such that for each $(t,x) \in J_a \times J_b$,

$$u_n(t,x) = z(t,x) + \int_0^t \int_0^x v_n(s,\tau)\,ds\,d\tau + \sum_{0<t_k<t} I_k\big(u(t_k,x)\big). \tag{10.35}$$

Using the fact that F has compact values, and from (10.5.3), we may pass to a subsequence if necessary to get that v_n converges to v in $L^1(J_a \times J_b, E)$ and hence

$v \in S_{F,u}$. Then, for each $(t,x) \in J_a \times J_b$,

$$u_n(t,x) \longrightarrow \widetilde{u}(t,x) = z(t,x) + \int_0^t \int_0^x v(s,\tau)ds\,d\tau + \sum_{0<t_k<t} I_k\big(u(t_k,x)\big). \qquad (10.36)$$

So $\widetilde{u} \in N(u)$.

Step 2. There exists $\gamma < 1$ such that

$$H_d\big(N(u), N(\overline{u})\big) \le \gamma \|u - \overline{u}\|_\Omega \quad \text{for each } u, \overline{u} \in \Omega. \qquad (10.37)$$

Let $u, \overline{u} \in \Omega$ and $h \in N(u)$. Then there exists $v(\cdot,\cdot) \in F(\cdot,\cdot,u(\cdot,\cdot))$ such that, for each $(t,x) \in J_a \times J_b$,

$$h(t,x) = z(t,x) + \int_0^t \int_0^x v(s,\tau)ds\,d\tau + \sum_{0<t_k<t} I_k\big(u(t_k,x)\big). \qquad (10.38)$$

From (10.5.3) it follows that

$$H_d\big(F(t,x,u(t,x)), F(t,x,\overline{u}(t,x))\big) \le l(t,x)\,|u(t,x) - \overline{u}(t,x)|. \qquad (10.39)$$

Hence there is $w \in F(t,x,\overline{u}(t,x))$ such that

$$|v(t,x) - w| \le l(t,x)\,|u(t,x) - \overline{u}(t,x)|, \quad (t,x) \in J_a \times J_b. \qquad (10.40)$$

Consider $U : J_a \times J_b \to \mathcal{P}(E)$ given by

$$U(t,x) = \{w \in E : |v(t,x) - w| \le l(t,x)\,|u(t,x) - \overline{u}(t,x)|\}. \qquad (10.41)$$

Since the multivalued operator $V(t,x) = U(t,x) \cap F(t,x,\overline{u}(t,x))$ is measurable (see [119, Proposition III.4]), there exists a function $\overline{v}(t,x)$, which is a measurable selection for V. So, $\overline{v}(t,x) \in F(t,x,\overline{u}(t,x))$ and

$$|v(t,x) - \overline{v}(t,x)| \le l(t,x)\,|u(t,x) - \overline{u}(t,x)|, \quad \text{for each } (t,x) \in J_a \times J_b. \qquad (10.42)$$

Let us define, for each $(t,x) \in J_a \times J_b$,

$$\overline{h}(t,x) = z(t,x) + \int_0^t \int_0^x \overline{v}(s,\tau)ds\,d\tau + \sum_{0<t_k<t} I_k\big(\overline{u}(t_k,x)\big). \qquad (10.43)$$

Then we have

$$\begin{aligned}
|h(t,x)-\overline{h}(t,x)| &\leq \int_0^t\int_0^x l(s,\tau)\,|u(s,\tau)-\overline{u}(s,\tau)|\,ds\,d\tau \\
&\quad + \sum_{k=1}^m |I_k(u(t_k,x))-I_k(\overline{u}(t_k,x))| \\
&\leq \int_0^a\int_0^b l(s,\tau)\,|u(s,\tau)-\overline{u}(s,\tau)|\,ds\,d\tau \\
&\quad + \sum_{k=1}^m c_k^*\,|u(t_k,x)-\overline{u}(t_k,x)| \\
&\leq \left(\|l\|_{L^1}+\sum_{k=1}^m c_k^*\right)\|u-\overline{u}\|_{\Omega}.
\end{aligned} \tag{10.44}$$

By an analogous relation, obtained by interchanging the roles of u and $\overline{u}$, it follows that

$$H_d(N(u),N(\overline{u})) \leq \left(\|l\|_{L^1}+\sum_{k=1}^m c_k^*\right)\|u-\overline{u}\|_{\Omega}. \tag{10.45}$$

So, N is a contraction and thus, by Theorem 1.11, N has a fixed point u, which is a solution to (10.1). □

We present now a result for the problem (10.1) in the spirit of the nonlinear alternative of Leray-Schauder type for single-valued maps combined with a selection theorem due to Bressan and Colombo.

Let $\mathcal{A}$ be a subset of $J_a\times J_b\times E$. $\mathcal{A}$ is $\mathcal{L}\otimes\mathcal{B}$ measurable if $\mathcal{A}$ belongs to the σ-algebra generated by all sets of the form $\mathcal{N}\times D$, where $\mathcal{N}$ is Lebesgue measurable in $J_a\times J_b$ and D is Borel measurable in E. A subset $\mathcal{I}$ of $L^1(J_a\times J_b,E)$ is decomposable if, for all $u,v\in\mathcal{I}$ and $\mathcal{N}\subset J_a\times J_b$ measurable, the function $u\chi_{\mathcal{N}}+v\chi_{J_a\times J_b-\mathcal{N}}\in\mathcal{I}$, where $\chi_{J_a\times J_b}$ stands for the characteristic function of $J_a\times J_b$.

Let E be a Banach space, X a nonempty closed subset of E, and $G: X\to\mathcal{P}(E)$ a multivalued operator with nonempty closed values. G is lower semicontinuous (l.s.c.), if the set $\{x\in X : G(x)\cap B\neq\varnothing\}$ is open for any open set B in E.

Definition 10.6. Let Y be a separable metric space and let $N: Y\to\mathcal{P}(L^1(J_a\times J_b,E))$ be a multivalued operator. Say N has property (BC) if

(1) N is lower semicontinuous (l.s.c.);
(2) N has nonempty closed and decomposable values.

Let $F: J_a\times J_b\times E\to\mathcal{P}(E)$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator

$$\mathcal{F}:\Omega\longrightarrow\mathcal{P}(L^1(J_a\times J_b,E)) \tag{10.46}$$

by letting

$$\mathcal{F}(u) = \{w \in L^1(J_a \times J_b, E) : w(t,x) \in F(t,x,u(t,x)) \text{ for a.e. } (t,x) \in J_a \times J_b\}. \tag{10.47}$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated to F.

Definition 10.7. Let $F : J_a \times J_b \times E \to \mathcal{P}(E)$ be a multivalued function with nonempty compact values. Say F is of lower semicontinuous type (l.s.c. type) if its associated Niemytzki operator $\mathcal{F}$ is lower semicontinuous and has nonempty closed and decomposable values.

Next we state a selection theorem due to Bressan and Colombo.

Theorem 10.8 ([105]). *Let Y be separable metric space and let $N : Y \to \mathcal{P}(L^1(J_a \times J_b, E))$ be a multivalued operator which has property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1(J_a \times J_b, E)$ such that $g(u) \in N(u)$ for every $u \in Y$.*

Theorem 10.9. *Suppose that hypotheses (10.4.2)–(10.4.4) and the following hold:*

(10.9.1) *$F : J_a \times J_b \times E \to \mathcal{P}(E)$ is a nonempty compact valued multivalued map such that*
(a) *$(t,x,u) \mapsto F(t,x,u)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable,*
(b) *$u \mapsto F(t,x,u)$ is lower semicontinuous for a.e. $(t,x) \in J_a \times J_b$.*
Then problem (10.1) *has at least one solution.*

Proof. Hypotheses (10.4.3) and (10.9.1) imply by Lemma 2.2 in Frigon [148] that F is of lower semicontinuous type. Then from Theorem 10.8 there exists a continuous function $g : \Omega \to L^1(J_a \times J_b, E)$ such that $g(u) \in \mathcal{F}(u)$ for all $u \in \Omega$. Consider the problem

$$\begin{gathered}
\frac{\partial^2 u(t,x)}{\partial t \partial x} = g(t,x,u(t,x)), \quad \text{a.e. } (t,x) \in J_a \times J_b,\ t \neq t_k,\ k = 1,\dots,m, \\
\Delta u(t_k,x) = I_k(u(t_k,x)), \quad k = 1,\dots,m, \\
u(t,0) = \psi(t), \quad t \in J_a, \qquad u(0,x) = \phi(x), \quad x \in J_b.
\end{gathered} \tag{10.48}$$

Clearly, if $u \in \Omega$ is a solution of the problem (10.48), then u is a solution to the problem (10.1). Transform the problem (10.48) into a fixed point problem. Consider the operator $N_1 : \Omega \to \Omega$ defined by

$$N_1(u)(t,x) = z(t,x) + \int_0^t \int_0^x g(u(s,\tau))\,ds\,d\tau + \sum_{0<t_k<t} I_k(u(t_k,x)). \tag{10.49}$$

We can easily show as in Theorem 10.4 that N_1 is completely continuous and there is no $u \in \partial U$ such that $u = \lambda N_1(u)$ for some $\lambda \in (0,1)$. We omit the details and give only the proof that N_1 *is continuous*.

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in Ω. Then

$$\begin{aligned} |N_1(u_n(t,x)) - N_1(u(t,x))| &\leq \int_0^t\int_0^x |g(u_n(s,\tau)) - g(u(s,\tau))|\,ds\,d\tau \\ &\quad + \sum_{0<t_k<t} |I_k(u_n(t_k,x)) - I_k(u(t_k,x))| \\ &\leq \int_0^a\int_0^b |g(u_n(s,\tau)) - g(u(s,\tau))|\,ds\,d\tau \\ &\quad + \sum_{0<t_k<t} |I_k(u_n(t_k,x)) - I_k(u(t_k,x))|. \end{aligned} \tag{10.50}$$

Since the functions g and I_k, $k = 1,\dots,m$, are continuous, then

$$\begin{aligned} \|N_1(u_n) - N_1(u)\|_\Omega &\leq \|g(u_n(\cdot,x)) - g(u(\cdot,x))\|_{L^1} \\ &\quad + \sum_{k=1}^m |I_k(u_n(t_k,x)) - I_k(u(t_k,x))| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{10.51}$$

As a consequence of the nonlinear alternative of Leray-Schauder type, we deduce that N_1 has a fixed point u in U, which is a solution of the problem (10.48). Hence u is a solution to the problem (10.1). $\square$

In the rest of this section, we will be concerned with the existence of solutions for the second-order impulsive hyperbolic differential inclusion with variable times,

$$\frac{\partial^2 u(t,x)}{\partial t\partial x} \in F(t,x,u(t,x)), \quad \text{a.e. } (t,x) \in J_a \times J_b,\ t \neq \tau_k(u(t,x)),\ k = 1,\dots,m, \tag{10.52}$$

$$u(t_k^+,x) = I_k(u(t,x)), \quad t = \tau_k(u(t,x)),\ k = 1,\dots,m, \tag{10.53}$$

$$u(t,0) = \psi(t), \quad t \in J_a, \qquad u(0,x) = \phi(x), \quad x \in J_b, \tag{10.54}$$

where $F : J_a \times J_b \times \mathbb{R}^n \to P(\mathbb{R}^n)$ is a multivalued map with compact values, $J := J_a \times J_b := [0,a] \times [0,b]$, $I_k \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $\phi \in C(J_a, \mathbb{R}^n)$, $u(t^+,y) = \lim_{(h,x)\to(0^+,y)} u(t+h,x)$ and $u(t^-,y) = \lim_{(h,x)\to(0^+,y)} u(t-h,x)$, and $\mathbb{R}^n$ is Euclidean space with norm $|\cdot|$.

So let us start by defining what we mean by a solution of problem (10.52)–(10.54).

Definition 10.10. A function $u \in \Omega \cap AC^1(\Gamma_k, \mathbb{R}^n)$, $k = 1,\dots,m$, is said to be a solution of (10.52)–(10.54) if there exist $v \in L^1(J_a \times J_b)$ such that $v(t,x) \in F(t,x,u(t,x))$ is satisfied a.e. on $J_a \times J_b$, $\partial^2 u(t,x)/\partial t\partial x = v(t,x)$ a.e. on $J_a \times J_b$, and the conditions (10.53)-(10.54).

Theorem 10.11. *Assume that the following hypotheses are satisfied:*

(10.11.1) *there exist constants c_k such that $|I_k(u)| \le c_k$, $k = 1,\dots,m$, for each $u \in \mathbb{R}^n$;*

(10.11.2) *there exist functions $p, q \in L^1(J_a \times J_b, \mathbb{R}_+)$ such that*

$$\|F(t,x,u)\| \le p(t,x) + q(t,x)|u| \tag{10.55}$$

for a.e. $(t,x) \in J_a \times J_b$ and each $u \in \mathbb{R}^n$;

(10.11.3) *the functions $\tau_k \in C^1(\mathbb{R}^n, \mathbb{R})$ for $k = 1,\dots,m$. Moreover,*

$$0 < \tau_1(x) < \cdots < \tau_m(x) < a, \quad \forall x \in \mathbb{R}^n; \tag{10.56}$$

(10.11.4) *for all $u \in C(J_a \times J_b, \mathbb{R}^n)$ and all $v \in S_{F,u}$,*

$$\left\langle \tau_k'(x), \int_{\bar{t}}^{t} v(s,x)\,ds \right\rangle \ne 1, \quad \forall (t,\bar{t},x) \in J_a \times J_a \times \mathbb{R}^n \tag{10.57}$$

and $k = 0,\dots,m$, where $\langle \cdot,\cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$;

(10.11.5) *for all $x \in \mathbb{R}^n$,*

$$\tau_k(I_k(x)) \le \tau_k(x) < \tau_{k+1}(I_k(x)) \quad \text{for } k = 1,\dots,m. \tag{10.58}$$

Then the IVP (10.52)–(10.54) *has at least one solution.*

Proof. The proof will be given in several steps.
Step 1. Consider the problem

$$\begin{gathered} \frac{\partial^2 u(t,x)}{\partial t \partial x} \in F(t,x,u(t,x)), \quad \text{a.e. } (t,x) \in J_a \times J_b, \\ u(t,0) = \psi(t), \quad t \in J_a, \qquad u(0,x) = \phi(x), \quad x \in J_b. \end{gathered} \tag{10.59}$$

A solution to problem (10.59) is a fixed point of the operator $N : C(J_a \times J_b, \mathbb{R}^n) \to \mathcal{P}(C(J_a \times J_b, \mathbb{R}^n))$ defined by

$$N(u) = \left\{ h \in C(J_a \times J_b, \mathbb{R}^n) : h(t,x) = z_0(t,x) + \int_0^t \int_0^x v(s,y)\,ds\,dy,\ v \in S_{F,u} \right\}, \tag{10.60}$$

where $z_0(t,x) := \psi(t) + \phi(x) - \psi(0)$. The proof will be given in several claims.

Claim 1. $N(u)$ is convex for each $u \in \Omega$.

Indeed, if h_1, h_2 belong to $N(u)$, then there exist $v_1, v_2 \in S_{F,u}$ such that for each $(t,x) \in J_a \times J_b$ we have

$$h_i(t,x) = z_0(t,x) + \int_0^t \int_0^x v_i(s,y)\,ds\,dy, \quad i = 1,2. \tag{10.61}$$

Let $0 \le d \le 1$. Then for each $(t,x) \in J_a \times J_b$ we have

$$(dh_1 + (1-d)h_2)(t) = z_0(t,x) + \int_0^t \int_0^x [dv_1(s,y) + (1-d)v_2(s,y)]\,ds\,dy. \tag{10.62}$$

Since $S_{F,u}$ is convex (because F has convex values), then

$$dh_1 + (1-d)h_2 \in N(u). \tag{10.63}$$

Claim 2. N maps bounded sets into bounded sets in $C(J_a \times J_b, \mathbb{R}^n)$.

Indeed, it is enough to show that there exists a positive constant ℓ such that, for each $u \in \mathcal{B}_q = \{u \in C(J_a \times J_b, \mathbb{R}^n) : \|u\|_\infty \le q\}$, one has $\|N(u)\|_\infty \le \ell$.

Let $h \in N(u)$, then there exist $v \in S_{F,u}$ such that

$$h(t,x) = z_0(t,x) + \int_0^t \int_0^x v(s,y)\,ds\,dy. \tag{10.64}$$

Since F is L^1-Carathéodory we have for each $(t,x) \in J_a \times J_b$,

$$\begin{aligned} |h(t,x)| &\le |z_0(t,x)| + \int_0^a \int_0^b |\varphi_q(t,x)|\,ds \\ &\le \|z_0\|_\infty + \|\varphi_q\|_{L^1} := \ell. \end{aligned} \tag{10.65}$$

Claim 3. N maps bounded sets into equicontinuous sets of $C(J_a \times J_b, \mathbb{R}^n)$.

Let $(\bar{t}_1, x_1), (\bar{t}_2, x_2) \in J_a \times J_b$, $\bar{t}_1 < \bar{t}_2$, $x_1 < x_2$, and $\mathcal{B}_q$ be a bounded set of $C(J_a \times J_b, \mathbb{R}^n)$, with each as in Claim 2. Then

$$\begin{aligned} |h(\bar{t}_2, x_2) - h(\bar{t}_1, x_1)| &\le |z_0(\bar{t}_2, x_2) - z_0(\bar{t}_1, x_1)| \\ &\quad + \int_0^{\bar{t}_2} \int_{x_1}^{x_2} \varphi_q(t,s)\,dt\,ds + \int_{\bar{t}_1}^{\bar{t}_2} \int_0^{x_1} \varphi_q(t,s)\,dt. \end{aligned} \tag{10.66}$$

The right-hand side tends to zero as $\bar{t}_2 - \bar{t}_1 \to 0$, $x_2 - x_1 \to 0$.

As a consequence of Claims 2 to 3 with the Arzela-Ascoli theorem, we can conclude that $N : C(J_a \times J_b, \mathbb{R}^n) \to C(J_a \times J_b, \mathbb{R}^n)$ is completely continuous.

Claim 4. N has a closed graph.

Let $u_n \to u_*$, $h_n \in N(u_n)$ and $h_n \to h_*$. We will prove that $h_* \in N(u_*)$.

$h_n \in N(u_n)$ means that there exists $v_n \in S_{F,u_n}$ such that, for each $t \in J$,

$$h_n(t,x) = z_0(t,x) + \int_0^t \int_0^x v_n(s,x)ds. \tag{10.67}$$

We must prove that there exists $v_* \in S_{F,u_*}$ such that, for each $(t,x) \in J_a \times J_b$,

$$h_*(t,x) = z_0(t,x) + \int_0^t \int_0^x v_*(s,x)ds. \tag{10.68}$$

Clearly since ϕ is continuous, we have that

$$\left\|\left(h_n - z_0(t,x)\right) - \left(h_* - z_0(t,x)\right)\right\|_\infty \longrightarrow 0, \quad \text{as } n \to \infty. \tag{10.69}$$

Consider the linear continuous operator

$$\begin{aligned} \Psi : L^1(J_a \times J_b, \mathbb{R}^n) &\longrightarrow C(J_a \times J_b, \mathbb{R}^n), \\ v \longmapsto \Psi(v)(t,x) &= \int_0^t \int_0^x v(s,\tau)ds\,d\tau. \end{aligned} \tag{10.70}$$

From Lemma 1.28, it follows that $\Psi \circ S_F$ is a closed graph operator. Moreover, we have that

$$(h_n(t,x) - z_0(t,x)) \in \Psi(S_{F,u_n}). \tag{10.71}$$

Since $u_n \to u_*$, it follows from Lemma 1.28 that

$$h_*(t,x) = z_0(t,x) + \int_0^t \int_0^x v_*(s,y)ds\,dy, \tag{10.72}$$

for some $v_* \in S_{F,u_*}$.

Claim 5. A priori bounds on solutions.

Let $u \in \Omega$ by a possible solution to (10.59). Then there exists $v \in S_{F,u}$ such that, for each $(t,x) \in J_a \times J_b$,

$$u(t,x) = z_0(t,x) + \int_0^t \int_0^x v(s,y)ds\,dy. \tag{10.73}$$

This implies by (10.11.2)–(10.11.4) that for each $(t,x) \in J_a \times J_b$ we have

$$\begin{aligned} |u(t,x)| &\le \|z_0\|_\infty + \int_0^t \int_0^x [|p(s,\tau)| + |q(s,\tau)||u(s,\tau)|]ds\,d\tau \\ &\le \|z_0\|_\infty + \int_0^t \int_0^x |q(s,\tau)||u(s,\tau)|ds\,d\tau + \|p\|_{L^1}. \end{aligned} \tag{10.74}$$

Invoking Gronwall's inequality (see, e.g., [160]), we get that

$$|u(t,x)| \leq [||z_0||_\infty + \|p\|_{L^1}] \exp(\|q\|_{L^1}) := M. \tag{10.75}$$

Then

$$\|u\|_\Omega < M. \tag{10.76}$$

Set

$$U_1 = \{u \in C(J_a \times J_b, \mathbb{R}^n) : \|u\|_\infty < M + 1\}. \tag{10.77}$$

$N : \overline{U}_1 \to P(C(J_a \times J_b, \mathbb{R}^n))$ is completely continuous. From the choice of U_1 there is no $u \in \partial U_1$ such that $u \in \lambda N(u)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray Schauder type, we deduce that N has a fixed point u in U_1, which is a solution of (10.59). Denote this solution by u_1.

Define the function

$$r_{k,1}(t,x) = \tau_k(u_1(t,x)) - t \quad \text{for } t \geq 0. \tag{10.78}$$

Hypothesis (10.11.3) implies that

$$r_{k,1}(0,0) \neq 0 \quad \text{for } k = 1,\dots,m. \tag{10.79}$$

If

$$r_{k,1}(t,x) \neq 0 \quad \text{on } J_a \times J_b, \text{ for } k = 1,\dots,m, \tag{10.80}$$

that is,

$$t \neq \tau_k(u_1(t,x)) \quad \text{on } J_a \times J_b, \text{ for } k = 1,\dots,m, \tag{10.81}$$

then u_1 is a solution of the problem (10.52)–(10.54).

It remains to consider the case when

$$r_{1,1}(t,x) = 0 \quad \text{for some } (t,x) \in J_a \times J_b. \tag{10.82}$$

Now since

$$r_{1,1}(0,0) \neq 0 \tag{10.83}$$

and $r_{1,1}$ is continuous, there exist $t_1 > 0$ and $x_1 > 0$ such that

$$r_{1,1}(t_1,x_1) = 0, \quad r_{1,1}(t,x) \neq 0, \quad \forall (t,x) \in [0,t_1) \times [0,x_1]. \tag{10.84}$$

Thus by (10.11.4) we have

$$r_{1,1}(t_1,x_1)=0,\quad r_{1,1}(t,x)\neq 0,\quad \forall (t,x)\in[0,t_1)\times[0,x_1]\cup(x_1,b]. \tag{10.85}$$

Suppose that there exists $(\bar{t},\bar{x})\in[0,t_1)\times[0,x_1)\cup(x_1,b]$ such that $r_{1,1}(\bar{t},\bar{x})=0$. The function $r_{1,1}$ attains a maximum at some point $(s,\bar{s})\in[0,t_1]\times J_b$. Since

$$\frac{\partial^2 u(t,x)}{\partial t\partial x}\in F(t,x,u_1(t,x)),\quad \text{a.e. } (t,x)\in J_a\times J_b, \tag{10.86}$$

then there exists $v(\cdot,\cdot)\in L^1(J_a\times J_b)$ with $v(t,x)\in F(t,x,u_1(t,x))$, a.e. $(t,x)\in J_a\times J_b$ such that

$$\begin{gathered}\frac{\partial^2 u(t,x)}{\partial t\partial x}=v(t,x)\quad \text{a.e. } t\in J_a\times J_b;\\ \frac{\partial u_1(t,x)}{\partial t},\quad \frac{\partial u_1(t,x)}{\partial x}\quad \text{exist.}\end{gathered} \tag{10.87}$$

Then

$$\frac{\partial r_{1,1}(s,\bar{s})}{\partial t}=\tau_1'(u_1(s,\bar{s}))\frac{\partial u_1(s,\bar{s})}{\partial t}-1=0. \tag{10.88}$$

Since

$$\frac{\partial u_1(t,x)}{\partial t}=\int_0^t v(s,x,u_1(s,x))\,ds, \tag{10.89}$$

then

$$\tau_1'(u_1(s,\bar{s}))\int_0^s v(\tau,\bar{s})\,d\tau-1=0. \tag{10.90}$$

Therefore

$$\left\langle \tau_1'(u_1(s,\bar{s})),\int_0^s v(\tau,\bar{s})\,d\tau\right\rangle=1, \tag{10.91}$$

which contradicts (10.11.4). From (10.11.3) we have

$$r_{k,1}(t,x)\neq 0,\quad \forall t\in[0,t_1)\times J_b,\ k=1,\dots,m. \tag{10.92}$$

Step 2. Consider now the following problem:

$$\begin{gathered}\frac{\partial^2 u(t,x)}{\partial t\partial x}\in F(t,x,u(t,x)),\quad \text{a.e. } t\in[t_1,a]\times J_b,\\ u(t_1^+,x)=I_1(u_1(t_1,x)),\quad u(t,0)=\psi(t).\end{gathered} \tag{10.93}$$

Transform the problem (10.93) into a fixed point problem. Consider the operator $N_1 : C([t_1, a] \times J_b, \mathbb{R}^n) \to C([t_1, a] \times J_b, \mathbb{R}^n)$ defined by

$$N_1(u) = \Big\{ h \in C([t_1, a] \times J_b, \mathbb{R}^n) : h(t,x) = I_1(u_1(t_1, x)) \\ + \psi(t) - \psi(t_1) + \int_{t_1}^{t} \int_0^x v(s,y)\,ds\,dy,\ v \in S_{F,u} \Big\}. \tag{10.94}$$

As in Step 1 we can show that N_1 is completely continuous, and each possible solution of (10.93) is a priori bounded by a constant M_2. Set

$$U_2 := \{u \in C([t_1, a] \times J_b, \mathbb{R}^n) : \|u\|_\infty < M_2 + 1\}. \tag{10.95}$$

From the choice of U_2 there is no $u \in \partial U_2$ such that $u = \lambda N_1(u)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray Schauder type [157] we deduce that N_1 has a fixed point u in U_2 which is a solution of (10.93). Denote this solution by u_2. Define

$$r_{k,2}(t,x) = \tau_k(u_2(t,x)) - t \quad \text{for } (t,x) \in [t_1, a] \times J_b. \tag{10.96}$$

If

$$r_{k,2}(t,x) \neq 0 \quad \text{on } (t_1, a] \times J_b,\ \forall k = 1, \dots, m, \tag{10.97}$$

then

$$u(t,x) = \begin{cases} u_1(t,x) & \text{if } (t,x) \in [0, t_1) \times J_b, \\ u_2(t,x) & \text{if } (t,x) \in [t_1, a] \times J_b, \end{cases} \tag{10.98}$$

is a solution of the problem (10.52)–(10.54).

It remains to consider the case when

$$r_{2,2}(t,x) = 0, \quad \text{for some } (t,x) \in (t_1, a] \times J_b. \tag{10.99}$$

By (10.11.5) we have

$$\begin{aligned} r_{2,2}(t_1^+, x_1) &= \tau_2(u_2(t_1^+, x_1)) - t_1 \\ &= \tau_2(I_1(u_1(t_1, x_1))) - t_1 \\ &> \tau_1(u_1(t_1, x_1)) - t_1 \\ &= r_{1,1}(t_1, x_1) = 0. \end{aligned} \tag{10.100}$$

Since $r_{2,2}$ is continuous and by (10.11.3), there exist $t_2 > t_1$ and $x_2 > x_1$ such that

$$\begin{gathered} r_{2,2}(u_2(t_2, x_2)) = 0, \\ r_{2,2}(t,x) \neq 0, \quad \forall\ (t,x) \in (t_1, t_2) \times J_b. \end{gathered} \tag{10.101}$$

It is clear by (10.11.3) that

$$r_{k,2}(t,x) \neq 0, \quad \forall\ (t,x) \in (t_1,t_2) \times J_b,\ k = 2,\dots,m. \tag{10.102}$$

Suppose now that there is $(s,\bar{s}) \in (t_1,t_2] \times [0,x_2) \cup (x_2,b]$ such that

$$r_{1,2}(s,\bar{s}) = 0. \tag{10.103}$$

From (10.11.5) it follows that

$$\begin{aligned} r_{1,2}(t_1^+,x_1) &= \tau_1(u_2(t_1^+,x_1)) - t_1 \\ &= \tau_1(I_1(u_1(t_1,x_1))) - t_1 \\ &\leq \tau_1(u_1(t_1,x_1)) - t_1 \\ &= r_{1,1}(t_1,x_1) = 0. \end{aligned} \tag{10.104}$$

Thus the function $r_{1,2}$ attains a nonnegative maximum at some point $(s_1,\bar{s}_1) \in (t_1,a] \times [0,x_2) \cup (x_2,b]$. Since

$$\frac{\partial^2 u_2(t,x)}{\partial t\partial x} \in F(t,x,u_2(t,x)), \tag{10.105}$$

then there exist $v(t,x) \in F(t,x,u_2(t,x))$ a.e. $(t,x) \in [t_1,a] \times J_b$ such that

$$\frac{\partial^2 u_2(t,x)}{\partial t\partial x} = v(t,x), \quad (t,x) \in [t_1,a] \times J_b. \tag{10.106}$$

Then we have

$$r'_{1,2}(t,x) = \tau'_1(u_2(t,x))\frac{\partial u_2(t,x)}{\partial t} - 1 = 0. \tag{10.107}$$

Therefore

$$\left\langle \tau'_1(u_2(s_1,\bar{s}_1)), \int_{t_1}^{s_1} v(s,\bar{s}_1)\,ds \right\rangle = 1, \tag{10.108}$$

which contradicts (10.11.4).
Step 3. We continue this process, and taking into account that $u_m := y|_{[t_m,a]\times J_b}$ is a solution to the problem

$$\begin{aligned} &\frac{\partial^2 u(t,x)}{\partial t\partial x} \in F(t,x,u(t,x)), \quad \text{a.e. } t \in (t_m,a] \times (0,b], \\ &u(t_m^+,x) = I_m(u_{m-1}(t_m^-,x)), \quad u(t,0) = \psi(t). \end{aligned} \tag{10.109}$$

The solution u of the problem (10.52)–(10.54) is then defined by

$$u(t,x) = \begin{cases} u_1(t,x) & \text{if } t \in [0,t_1) \times J_b, \\ u_2(t,x) & \text{if } t \in [t_1,t_2) \times J_b, \\ \vdots & \\ u_m(t,x) & \text{if } t \in [t_m,a] \times J_b. \end{cases} \tag{10.110}$$

□

10.4. Notes and remarks

Impulsive differential and partial differential equations with fixed moments have become more important in recent years in theoretical developments as well as in some mathematical models of real phenomena. The results of Section 10.2 are taken from Benchohra et al. [41], and the results of Section 10.3 are from [43].

11 Impulsive dynamic equations on time scales

11.1. Introduction

In recent years dynamic equations on time scales have received much attention. We refer to the books by Agarwal and O'Regan [7], Bohner and Peterson [101, 102], and Lakshmikantham et al. [184], and the papers by Anderson [15, 18], Agarwal et al. [2, 3, 5], Bohner and Guseinov [100], Bohner and Eloe [99], and Erbe and Peterson [141, 142].

The time scales calculus has a tremendous potential for applications in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics, neural networks, social sciences, as is pointed out in the monographs of Aulbach and Hilger [24], Bohner and Peterson [101, 102], and Lakshmikantham et al. [184].

The existence of solutions of boundary value problem on a time scale was recently studied by Agarwal and O'Regan [7], Anderson [16, 17], Henderson [166], and Sun and Li [223]. In this chapter, dynamic equations on time scales are considered for both impulsive initial value problems and impulsive boundary value problems. The results here are based on work from [72, 165].

11.2. Preliminaries

We will introduce some basic definitions and facts from the time scale calculus that we will use in the sequel.

A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T} : s < t\} \tag{11.1}$$

(supplemented by $\inf \varnothing := \sup \mathbb{T}$ and $\sup \varnothing := \inf \mathbb{T}$) are well defined. The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_k := \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^k := \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^k = \mathbb{T}$. The notations $[c, d]$, $[c, d)$, and so on, will denote time scale intervals such as

$$[c,d] = \{t \in \mathbb{T} : c \leq t \leq d\}, \tag{11.2}$$

where $c, d \in \mathbb{T}$ with $c < \rho(d)$.

Definition 11.1. Let X be a Banach space. The function $f : \mathbb{T} \to X$ is called rd-continuous provided it is continuous at each right-dense point and has a left-sided limit at each point; write $f \in C_{\text{rd}}(\mathbb{T}) = C_{\text{rd}}(\mathbb{T}, X)$.

For $t \in \mathbb{T}^k$, let the Δ derivative of f at t, denoted by $f^{\Delta}(t)$, be the number (provided it exists) such that for all $\varepsilon > 0$ there exists a neighboord U of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s] \right| \leq \varepsilon \left| \sigma(t) - s \right| \tag{11.3}$$

for all $s \in U$.

A function F is called an antiderivative of $f : \mathbb{T} \to X$ provided

$$F^{\Delta}(t) = f(t), \quad \text{for each } t \in \mathbb{T}^k. \tag{11.4}$$

$C([a,b], \mathbb{R})$ is the Banach space of all continuous functions from $[a,b]$ into $\mathbb{R}$ where $[a,b] \subset \mathbb{T}$ with the norm

$$\|y\|_\infty = \sup\left\{ \left| y(t) \right| : t \in [a,b] \right\}. \tag{11.5}$$

Remark 11.2. (i) If f is continuous, then f is rd-continuous.

(ii) If f is delta differentiable at t, then f is continuous at t.

A function $p : \mathbb{T} \to \mathbb{R}$ is called *regressive* if

$$1 + \mu(t)p(t) \neq 0, \quad \forall t \in \mathbb{T}_k, \tag{11.6}$$

where $\mu(t) = \sigma(t) - t$, which is called the *graininess function*. The generalized exponential function e_p is defined as the unique solution $y(t) = e_p(t,a)$ of the initial value problem $y^{\Delta} = p(t)y$, $y(a) = 1$, where p is a regressive function. An explicit formula for $e_p(t,a)$ is given by

$$e_p(t,s) = \exp\left\{ \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau \right\} \quad \text{with } \xi_h(z) = \begin{cases} \dfrac{\operatorname{Log}(1+hz)}{h} & \text{if } h \neq 0, \\ z & \text{if } h = 0. \end{cases} \tag{11.7}$$

For more details, see [101]. Clearly, $e_p(t,s)$ never vanishes. We now give some fundamental properties of the exponential function. Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions. We define

$$p \oplus q = p + q + \mu pq, \quad \ominus p := -\frac{p}{1+\mu p}, \quad p \ominus q := p \oplus (\ominus q). \tag{11.8}$$

Theorem 11.3 (see [101]). *Assume that* $p, q : \mathbb{T} \to \mathbb{R}$ *are regressive functions. Then the following hold:*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;
(ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$;
(iii) $1/e_p(t,s) = e_{\ominus p}(t,s)$;
(iv) $e_p(t,s)(1/e_p(s,t)) = e_{\ominus p}(s,t)$;
(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;
(vi) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;
(vii) $e_p(t,s)/e_q(t,s) = e_{p\ominus q}(t,s)$.

11.3. First-order impulsive dynamic equations on time scales

This section is concerned with the existence of solutions of impulsive dynamic equations on time scales. We consider the problem

$$\begin{gathered} y^{\Delta}(t) - p(t)y^{\sigma}(t) = f(t, y(t)), \quad t \in J := [a,b], \\ t \neq t_k, \quad k = 1,\dots,m, \end{gathered} \tag{11.9}$$

$$y(t_k^+) = I_k(y(t_k^-)), \quad k = 1,\dots,m, \tag{11.10}$$

$$y(a) = \eta, \tag{11.11}$$

where $\mathbb{T}$ is a time scale, $[a,b] \subset \mathbb{T}$, $f : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$ is a given function, $I_k \in C(\mathbb{R},\mathbb{R})$ $t_k \in \mathbb{T}$, $a = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^+} y(t_k - h)$ represent the right and left limits of $y(t)$ at $t = t_k$ (with $y(t_k^+) = y(t_k)$ if t_k is right-scattered, and $y(t_k^-) = y(t_k)$ if t_k is left-scattered), σ is a function that will be defined later, and $y^{\sigma}(t) = y(\sigma(t))$.

We will prove our existence result for problem (11.9)–(11.11) by using the nonlinear alternative of Leray-Schauder type [157].

We will assume for the remainder of the paper that, for each $k = 1,\dots,m$, the points of impulse t_k are right-dense. In order to define the solution of (11.9)–(11.11), we will consider the space

$$\begin{aligned} \Omega = \Big\{ & y : [a,b] \longrightarrow \mathbb{R} : y_k \in C(J_k,\mathbb{R}),\ k = 0,\dots,m, \text{ and there exist} \\ & y(t_k^-),\ y(t_k^+),\ k = 1,\dots,m, \text{ with } y(t_k^-) = y(t_k) \Big\} \end{aligned} \tag{11.12}$$

which is a Banach space with the norm

$$\|y\|_{\Omega} = \max\left\{ \|y_k\|_{J_k},\ k = 0,\dots,m \right\}, \tag{11.13}$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}] \subset [a,b]$, $k = 1,\dots,m$, and $J_0 = [t_0, t_1]$. So let us start by defining what we mean by a solution of problem (11.9)–(11.11).

Definition 11.4. A function $y \in \Omega \cap C^1((t_k, t_{k+1}), \mathbb{R})$, $k = 0, \dots, m$, is said to be a solution of (11.9)–(11.11) if y satisfies the differential equation

$$y^{\Delta}(t) - p(t)y^{\sigma}(t) = f(t, y(t)) \quad \text{everywhere on } J \backslash \{t_k\},\ k = 1, \dots, m, \tag{11.14}$$

and for each $k = 1, \dots, m$, the function y satisfies the equations $y(t_k^+) = I_k(y(t_k^-)) = I_k(y(t_k))$, and $y(a) = \eta$.

We need the following auxiliary result. Its proof is given in [101].

Theorem 11.5. *Let $p : \mathbb{T} \to \mathbb{R}$ be* rd-*continuous and regressive. Suppose $f : \mathbb{T} \to \mathbb{R}$ is* rd-*continuous, $t_0 \in \mathbb{T}$, and $y_0 \in \mathbb{R}$. Then y is the unique solution of the initial value problem*

$$y^{\Delta}(t) - p(t)y^{\sigma}(t) = f(t), \qquad y(t_0) = y_0 \tag{11.15}$$

if and only if

$$y(t) = e_{\ominus p}(t, t_0)y_0 + \int_{t_0}^{t} e_{\ominus p}(t, s)f(s)\Delta s. \tag{11.16}$$

Theorem 11.6. *Suppose that the following hypotheses are satisfied.*

(11.6.1) *The function $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous.*

(11.6.2) *There exist constants c_k such that*

$$|I_k(y)| \le c_k, \quad \text{for each } k = 1, \dots, m,\ \forall y \in \mathbb{R}. \tag{11.17}$$

(11.6.3) *There exist a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$, a function $h \in C([a, b], \mathbb{R}_+)$, and for each $k = 0, \dots, m$, nonnegative numbers $r_k > 0$ such that*

$$\begin{gathered} \|f(t, y)\| \le h(t)\psi(|y|), \quad \text{for each } (t, y) \in [a, b] \times \mathbb{R}, \\ \frac{r_k}{\sup_{t \in J_k} e_{\ominus p}(t, t_k)\widetilde{c}_k + \psi(r_k)\sup_{t \in J_k} \int_{t_k}^{t_{k+1}} |e_{\ominus p}(t, s)h(s)|\Delta s} > 1, \end{gathered} \tag{11.18}$$

where $\widetilde{c}_0 = |\eta|$, $\widetilde{c}_k = c_k$, $k = 1, \dots, m$.

Then the impulsive IVP (11.9)–(11.11) *has at least one solution.*

Proof. The proof will be given in several steps.

Step 4. Consider problem

$$\begin{gathered} y^{\Delta}(t) - p(t)y^{\sigma}(t) = f(t, y(t)), \quad t \in (a, t_1), \\ y(a) = \eta. \end{gathered} \tag{11.19}$$

Transform problem into a fixed point problem. Consider the operator $N : C([a,t_1],\mathbb{R}) \to C([a,t_1],\mathbb{R})$ defined by

$$N(y)(t) = e_{\ominus p}(t,a)\eta + \int_a^t e_{\ominus p}(t,s) f(s,y(s))\Delta s. \tag{11.20}$$

Remark 11.7. From Theorem 11.5, the fixed points of N are solutions to (11.19).

In order to apply the nonlinear alternative of Leray-Schauder type, we first show that N is completely continuous.

Claim 1. N is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C([a,t_1],\mathbb{R})$. Then

$$|N(y_n)(t) - N(y)(t)| \le \int_a^{t_1} e_{\ominus p}(t,s)|f(s,y_n(s)) - f(s,y(s))|\Delta s. \tag{11.21}$$

Then

$$||N(y_n) - N(y)||_\infty \le ||f(\cdot,y_n(\cdot)) - f(\cdot,y(\cdot))||_\infty \sup_{t\in[a,t_1]} \int_a^{t_1} e_{\ominus p}(t,s)\Delta s. \tag{11.22}$$

Then

$$||N(y_n) - N(y)||_\infty \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{11.23}$$

Claim 2. N maps bounded sets into bounded sets in $C([a,t_1],\mathbb{R})$.

Indeed, it is enough to show that there exists a positive constant ℓ such that, for each $y \in \mathcal{B}_q = \{y \in C([a,t_1],\mathbb{R}) : \|y\|_\infty \le q\}$, one has $\|N(y)\|_\infty \le \ell$. Let $y \in B_q$. Then, for each $t \in [a,t_1]$, we have

$$(Ny)(t) = e_{\ominus p}(t,a)\eta + \int_a^t e_{\ominus p}(t,s) f(s,y(s))\Delta s. \tag{11.24}$$

By (H3), we have, for each $t \in [a,t_1]$,

$$|(Ny)(t)| \le \sup_{t\in[a,t_1]} e_{\ominus p}(t,s)|\eta| + \psi(q) \sup_{t\in[a,t_1]} h(t) \sup_{t\in[a,t_1]} \int_a^{t_1} e_{\ominus p}(t,s)\Delta s := \ell. \tag{11.25}$$

Claim 3. N maps bounded sets into equicontinuous sets of $C([a,t_1],\mathbb{R})$.

Let $u_1, u_2 \in [a,t_1]$, $u_1 < u_2$, and let B_q be a bounded set of $C([a,t_1],\mathbb{R})$ as in Claim 2. Let $y \in B_q$. Then

$$\begin{aligned} |(Ny)(u_2) - (Ny)(u_1)| \le{}& |e_{\ominus p}(u_2,a) - e_{\ominus p}(u_1,a)||\eta| + \psi(q) \\ & \times \sup_{t\in[a,t_1]} h(s) \int_{u_1}^{u_2} |e_{\ominus p}(u_1,s) - e_{\ominus p}(u_2,s)|\Delta s. \end{aligned} \tag{11.26}$$

The right-hand side tends to zero as $u_2 - u_1 \to 0$. As a consequence of Claims 1 to 3, together with the Arzelá-Ascoli theorem, we can conclude that $N : C([a, t_1], \mathbb{R}) \to C([a, t_1], \mathbb{R})$ is completely continuous.

Let y be such that $y = \lambda N y$, for some $\lambda \in (0, 1)$. Thus

$$y(t) = \lambda e_{\ominus p}(t, a)\eta + \lambda \int_a^t e_{\ominus p}(t, s) f(s, y(s))\Delta s. \tag{11.27}$$

This implies by (11.6.3) that, for each $t \in [a, t_1]$, we have

$$\begin{aligned} |y(t)| &\le \sup_{t\in[a,t_1]} e_{\ominus p}(t, a)|\eta| + \int_a^t e_{\ominus p}(t, s)h(s)\psi(|y(s)|)ds \\ &\le \sup_{t\in[a,t_1]} e_{\ominus p}(t, a)|\eta| + \psi(\|y\|_\infty) \sup_{t\in[a,t_1]} \int_a^{t_1} |e_{\ominus p}(t, s)h(s)|\Delta s. \end{aligned} \tag{11.28}$$

Consequently,

$$\frac{\|y\|_\infty}{\sup_{t\in[a,t_1]} e_{\ominus p}(t, a)|\eta| + \psi(\|y\|_\infty) \sup_{t\in[a,t_1]} \int_a^{t_1} |e_{\ominus p}(t, s)h(s)|\Delta s} \le 1. \tag{11.29}$$

Then, by (11.6.3), there exists r_0 such that $\|y\|_\infty \neq r_0$.

Set

$$U_1 = \{y \in C([a, t_1], \mathbb{R}) : \|y\|_\infty < r_0\}. \tag{11.30}$$

The operator $N : \overline{U}_1 \to C([a, t_1], \mathbb{R})$ is completely continuous. From the choice of U_1, there is no $y \in \partial U_1$ such that $y \in \lambda N(y)$, for some $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [157], we deduce that N has a fixed point y_1 in U_1 which is a solution of problem (11.19).

Step 2. Consider now problem

$$\begin{gathered} y^\Delta(t) - p(t)y^\sigma(t) = f(t, y(t)), \quad t \in (t_1, t_2), \\ y(t_1^+) = I_1(y_1(t_1)). \end{gathered} \tag{11.31}$$

Transform problem (11.31) into a fixed point problem. Let the operator $N_1 : C([t_1, t_2], \mathbb{R}) \to C([t_1, t_2], \mathbb{R})$ be defined by

$$N_1(y)(t) = e_{\ominus p}(t, t_1)I_1(y_1(t_1)) + \int_{t_1}^t e_{\ominus p}(t, s) f(s, y(s))\Delta s. \tag{11.32}$$

Let y be such that $y = \lambda N_1 y$, for some $\lambda \in (0, 1)$. Then

$$y(t) = \lambda e_{\ominus p}(t, t_1)I_1(y_1(t_1)) + \lambda \int_{t_1}^t e_{\ominus p(t,s)} f(s, y(s))\Delta s. \tag{11.33}$$

This implies by (H3) that, for each $t \in [t_1, t_2]$, we have

$$\begin{aligned} |y(t)| &\le \sup_{t\in[t_1,t_2]} e_{\ominus p}(t,t_1)\,|I_1(y_1(t_1))| + \int_{t_1}^{t} e_{\ominus p}(t,s)h(s)\psi(|y(s)|)\,ds \\ &\le \sup_{t\in[t_1,t_2]} e_{\ominus p}(t,t_1)c_1 + \psi(\|y\|_\infty) \sup_{t\in[t_1,t_2]} \int_{t_1}^{t_2} |e_{\ominus p}(t,s)h(s)|\,\Delta s. \end{aligned} \tag{11.34}$$

Hence

$$\frac{\|y\|_\infty}{\sup_{t\in[t_1,t_2]} e_{\ominus p}(t,t_1)c_1 + \psi(\|y\|_\infty)\sup_{t\in[t_1,t_2]} \int_{t_1}^{t_2} |e_{\ominus p}(t,s)h(s)|\,\Delta s} \le 1. \tag{11.35}$$

By (11.6.3), there exists r_1 such that $\|y\|_\infty \ne r_1$.

Set

$$U_2 = \{y \in C([t_1,t_2],\mathbb{R}) : \|y\|_\infty < r_1\}. \tag{11.36}$$

As in Step 4, we can show that the operator $N_1 : \overline{U}_2 \to C([t_1,t_2],\mathbb{R})$ is completely continuous. From the choice of U_2 there is no $y \in \partial U_2$ such that $y \in \lambda N_1(y)$, for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type, we deduce that N_1 has a fixed point y_2 in U_2 which is a solution of problem (11.19).

Step 3. Continue this process and construct solutions $y_k \in C(J_k,\mathbb{R})$, $k = 2,\dots,m$, to

$$\begin{gathered} y^\Delta(t) - p(t)y^\sigma(t) = f(t,y(t)), \quad t \in (t_k,t_{k+1}), \\ y(t_k^+) = I_k(y(t_k^-)). \end{gathered} \tag{11.37}$$

Then

$$y(t) = \begin{cases} y_1(t), & t \in [a,t_1], \\ y_2(t), & t \in (t_1,t_2], \\ \vdots & \\ y_{m-1}(t), & t \in (t_{m-1},t_m], \\ y_m(t), & t \in (t_m,b], \end{cases} \tag{11.38}$$

is a solution of (11.9)–(11.11). □

11.4. Impulsive functional dynamic equations on time scales with infinite delay

This section is concerned with the existence of solutions of impulsive functional dynamic equations on time scales with infinite delay. First, we consider the impulsive problem

$$\begin{gathered} y^{\Delta}(t) = f(t, y_t), \quad t \in J := [0, b], \ t \neq t_k, \ k = 1, \dots, m, \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad k = 1, \dots, m, \\ y_0 = \phi \in \mathcal{B}, \end{gathered} \tag{11.39}$$

where $\mathbb{T}$ is a time scale which has at least finitely many right-dense points, $[0, b] \subset (-\infty, b] \subset \mathbb{T}$, $f : \mathbb{T} \times \mathcal{B} \to \mathbb{R}$ is a given function, $I_k \in C(\mathbb{R}, \mathbb{R})$, $t_k \in \mathbb{T}$, $0 < t_1 < \cdots < t_m < t_{m+1} = b$, $\phi \in \mathcal{B}$, and $\mathcal{B}$ is called the phase space that will be defined later. $y(t_k^+)$ and $y(t_k^-)$ represent right and left limits with respect to the time scale, and in addition, if t_k is right-scattered, then $y(t_k^+) = y(t_k)$, whereas, if t_k is left-scattered, then $y(t_k^-) = y(t_k)$,

Next we consider first-order impulsive neutral functional dynamic equations on time scales of the form

$$[y(t) - g(t, y_t)]^{\Delta} = f(t, y_t), \quad t \in [0, b], \ t \neq t_k, \ k = 1, \dots, m, \tag{11.40}$$

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{11.41}$$

$$y_0 = \phi \in \mathcal{B}, \tag{11.42}$$

where f, ϕ, I_k are as in problem (11.39) and $g : J \times \mathcal{B} \to \mathbb{R}$.

The notion of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theories. A usual choice is a seminormed space satisfying suitable axioms, which was introduced by Hale and Kato [161] (see also Kappel and Schappacher [172]). For a detailed discussion on this topic we refer the reader to the book by Hino et al. [169]. In the case where the impulses are absent (i.e., $I_k = 0$, $k = 1, \dots, m$) an extensive theory is developed for problem (11.39). We refer to the monographs of Hale and Lunel [162], Hino et al. [169], and Lakshmikantham et al. [185], and the paper of Corduneanu and Lakshmikantham [122].

In order to define the phase space and the solution of (11.39) we will consider the space

$$\begin{aligned} \mathcal{B}_b = \Big\{ y : (-\infty, b] \longrightarrow \mathbb{R}^n \mid \exists\, t_0 < t_1 < \cdots < t_m < b \text{ such that} \\ y(t_k^-),\ y(t_k^+) \text{ exist, with } y(t_k) = y(t_k^-),\ 0 \le k \le m, \\ y(t) = \phi(t), t \le 0,\ y_k \in C(J_k, \mathbb{R}^n) \Big\}, \end{aligned} \tag{11.43}$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}]$, $k = 0, \dots, m$. Let $\|\cdot\|_b$ be the seminorm in $\mathcal{B}_b$ defined by

$$\|y\|_b = \|y_0\|_{\mathcal{B}} + \sup\{|y(s)| : 0 \le s \le b\}, \quad y \in \mathcal{B}_b. \tag{11.44}$$

We will assume that $\mathcal{B}$ satisfies the following axioms.

(A) If $y : (-\infty, b] \to \mathbb{R}$, $b > 0$ is such that $y|_{[0,b]} \in \mathcal{B}_b$ and $y_0 \in \mathcal{B}$, then, for every t in $[0, b)$ the following conditions hold:
 (i) y_t is in $\mathcal{B}$,
 (ii) $\|y_t\|_{\mathcal{B}} \le K(t)\sup\{|y(s)| : 0 \le s \le t\} + M(t)\|y_0\|_{\mathcal{B}}$, where $H \ge 0$ is a constant, $K : [0,\infty) \to [0,\infty)$ is continuous, $M : [0,\infty) \to [0,\infty)$ is locally bounded, and H, K, M are independent of $y(\cdot)$.

(A-1) For the function $y(\cdot)$ in (A), y_t is a $\mathcal{B}$-valued continuous function on $[0, b)$.

(A-2) The space $\mathcal{B}$ is complete.

Definition 11.8. A function $y \in \mathcal{B}_b$, is said to be a solution of (11.39) if y satisfies the dynamic equation

$$y^{\Delta}(t) = f(t, y_t) \quad \text{everywhere on } J\backslash\{t_k\},\ k = 1, \dots, m, \tag{11.45}$$

and for each $k = 1, \dots, m$, the function y satisfies the equations $y(t_k^+) - y(t_k^-) = I_k(y(t_k))$, and $y_0 = \phi \in \mathcal{B}$.

Theorem 11.9. *Suppose that the following hypotheses are satisfied.*

(11.9.1) *The function $f : [0, b] \times \mathcal{B} \to \mathbb{R}$ is continuous.*

(11.9.2) *There exist constants c_k such that*

$$|I_k(y)| \le c_k, \quad \text{for each } k = 1, \dots, m,\ \forall y \in \mathbb{R}. \tag{11.46}$$

(11.9.3) *There exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$, a function $p \in L^1([0, b], \mathbb{R}_+)$, and a constant $M > 0$ such that*

$$\begin{gathered} |f(t,u)| \le p(t)\psi(|u|_{\mathcal{B}}), \quad \text{for each } (t,u) \in [0,b] \times \mathcal{B}, \\ \frac{M}{K_b\left[\int_0^b p(s)\psi(M)\Delta s + \sum_{k=1}^m c_k\right] + K_b|\phi(0)| + M_b\|\phi\|_{\mathcal{B}}} > 1, \end{gathered} \tag{11.47}$$

where $K_b = \sup\{K(t) : t \in [0, b]\}$ and $M_b = \sup\{M(t) : t \in [0, b]\}$. Then the impulsive IVP (11.39) *has at least one solution.*

Proof. Transform problem (11.39) into a fixed point problem. We consider the operator $N : \mathcal{B}_b \to \mathcal{B}_b$ defined by

$$(Ny)(t) = \begin{cases} \phi(t) & \text{if } t \in (-\infty, 0], \\ \phi(0) + \displaystyle\int_0^t f(s, y_s)\Delta s + \sum_{0<t_k<t} I_k(y(t_k)) & \text{if } t \in [0, b]. \end{cases} \tag{11.48}$$

Clearly the fixed points of N are solutions to (11.39). So we will prove that N has a fixed point.

Let $x(\cdot) : (-\infty, b) \to \mathbb{R}$ be the function defined by

$$x(t) = \begin{cases} \phi(0) & \text{if } t \in [0, b], \\ \phi(t) & \text{if } t \in (-\infty, 0]. \end{cases} \tag{11.49}$$

Then $x_0 = \phi$. For each $z \in C([0, b], E)$ with $z_0 = 0$, we denote by $\bar{z}$ the function defined by

$$\bar{z}(t) = \begin{cases} z(t) & \text{if } t \in [0, b], \\ 0 & \text{if } t \in (-\infty, 0]. \end{cases} \tag{11.50}$$

If $y(\cdot)$ satisfies

$$y(t) = \phi(0) + \int_0^t f(s, y_s)\Delta s + \sum_{0<t_k<t} I_k(y(t_k^-)), \tag{11.51}$$

we can decompose it as $y(t) = \bar{z}(t) + x(t)$, $0 \le t \le b$, which implies $y_t = \bar{z}_t + x_t$, for every $0 \le t \le b$, and the function $z(\cdot)$ satisfies

$$z(t) = \int_0^t f(s, \bar{z}_s + x_s)\Delta s + \sum_{0<t_k<t} I_k(z(t_k^-) + x(t_k^-)). \tag{11.52}$$

Set

$$\mathcal{B}_b^0 = \{z \in \mathcal{B}_b : z_0 = 0\}. \tag{11.53}$$

For any $z \in \mathcal{B}_b^0$, we have

$$\|z\|_{\mathcal{B}_b^0} = \|z_0\|_{\mathcal{B}} + \sup\{|z(s)| : 0 \le s \le b\} = \sup\{|z(s)| : 0 \le s \le b\}. \tag{11.54}$$

Thus $(\mathcal{B}_b^0 \| \cdot \|_{\mathcal{B}_b^0})$ is a Banach space. Let the operator $P : \mathcal{B}_b^0 \to \mathcal{B}_b^0$ be defined by

$$(Pz)(t) = \begin{cases} 0, & t \le 0, \\ \displaystyle\int_0^t f(s, \bar{z}_s + x_s)\Delta s + \sum_{0<t_k<t} I_k(x(t_k^-) + z(t_k^-)), & t \in [0, b]. \end{cases} \tag{11.55}$$

Obviously that the operator N has a fixed point is equivalent to that P has one, so we turn to prove that P has a fixed point. We will use the Leray-Schauder alternative to prove that P has fixed point.

Step 1. P is continuous.

Let $\{z_n\}$ be a sequence such that $z_n \to z$ in $\mathcal{B}_b^0$. Then

$$\begin{aligned} |P(z_n)(t) - P(z)(t)| \leq & \int_0^b |f(s, \bar{z}_{n_s} + x_s) - f(s, \bar{z}_s + x_s)| \Delta s \\ & + \sum_{k=1}^{m} |I_k(z_n(t_k) + x(t_k)) - I_k(z(t_k) + x(t_k))|. \end{aligned} \tag{11.56}$$

Hence

$$\begin{aligned} \|P(z_n) - P(z)\|_{\mathcal{B}_b^0} \leq & \|f(\cdot, \bar{z}_n(\cdot) + x(\cdot)) - f(\cdot, \bar{z}(\cdot) + x(\cdot))\|_{L^1} \\ & + \sum_{k=1}^{m} |I_k(z_n(t_k) + x(t_k)) - I_k(z(t_k) + x(t_k))|. \end{aligned} \tag{11.57}$$

Thus

$$\|P(z_n) - P(z)\|_{\mathcal{B}_b^0} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{11.58}$$

Step 2. P sends bounded sets into bounded sets.

We will show that for any $q > 0$ there exists a positive constant ℓ such that, for each $z \in B_q = \{z \in \mathcal{B}_b^0 : \|z\|_{\mathcal{B}_b} \leq q\}$, one has $\|P\|_{\mathcal{B}_b} \leq \ell$. For every $x \in B_q$, we have

$$\begin{aligned} \|x_t + \bar{z}_t\|_{\mathcal{B}} & \leq \|x_t\|_{\mathcal{B}} + \|\bar{z}_t\|_{\mathcal{B}} \\ & \leq K(t) \sup\{|x(s)| : 0 \leq s \leq t\} + M(t)\|x_0\|_{\mathcal{B}} \\ & \quad + K(t) \sup\{|\bar{z}(s)| : 0 \leq s \leq t\} + M(t)\|z_0\|_{\mathcal{B}} \\ & \leq K_b q + K_b |\phi(0)| + M_b \|\phi\|_{\mathcal{B}} := q^*. \end{aligned} \tag{11.59}$$

By (11.9.1)–(11.9.3), for each $t \in J$, we have that

$$\begin{aligned} |(Pz)(t)| & \leq \int_0^t p(s)\psi(\|x_s + \bar{z}_s\|_{\mathcal{B}}) \Delta s + \sum_{k=1}^{m} c_k \\ & \leq \psi(q^*) \int_0^b p(s) \Delta s + \sum_{k=1}^{m} c_k. \end{aligned} \tag{11.60}$$

Then we have

$$\|P\|_{\mathcal{B}_b} \leq \psi(q^*) \int_0^b p(s) \Delta s + \sum_{k=1}^{m} c_k := \ell. \tag{11.61}$$

Step 3. P sends bounded sets into equicontinuous sets.

Let $\tau_1, \tau_2 \in J$, $0 < \tau_1 < \tau_2$. Then we have

$$\left|(Pz)(\tau_2) - (Pz)(\tau_1)\right| \leq \psi(r^*) \int_{\tau_1}^{\tau_2} p(s)\Delta s + \sum_{\tau_1 \leq t_k < \tau_2} c_k. \tag{11.62}$$

The right-hand side tends to zero as $\tau_2 \to \tau_1$.

As a consequence of Steps 2-3 together with the Arzelá-Ascoli theorem, it suffices to show that P maps B_q into precompact sets.

Step 4. A priori bounds on solutions.

Let z be a solution of the integral equation

$$z(t) = \int_0^t f(s, \bar{z}_s + x_s)\Delta s + \sum_{0<t_k<t} I_k\left(x(t_k^-) + z(t_k^-)\right). \tag{11.63}$$

By (11.9.2), we have that

$$\left|z(t)\right| \leq \int_0^t p(s)\psi\left(\left\|x_s + \bar{z}_s\right\|_{\mathcal{B}}\right)\Delta s + \sum_{0<t_k<t} c_k. \tag{11.64}$$

But

$$\begin{aligned}\left\|x_t + \bar{z}_t\right\|_{\mathcal{B}} &\leq \left\|x_t\right\|_{\mathcal{B}} + \left\|\bar{z}_t\right\|_{\mathcal{B}} \\ &\leq K(t)\sup\left\{\left|x(s)\right| : 0 \leq s \leq t\right\} + M(t)\left\|x_0\right\|_{\mathcal{B}} \\ &\quad + K(t)\sup\left\{\left|z(s)\right| : 0 \leq s \leq t\right\} + M(t)\left\|\bar{z}_0\right\|_{\mathcal{B}} \\ &\leq K_b \sup\left\{\left|z(s)\right| : 0 \leq s \leq t\right\} + K_b\left|\phi(0)\right| + M_b\|\phi\|_{\mathcal{B}}.\end{aligned} \tag{11.65}$$

If we name $w(t)$ the right-hand side of the above inequality, we have that

$$\left\|x_t + \bar{z}_t\right\|_{\mathcal{B}} \leq w(t), \tag{11.66}$$

and therefore (11.64) becomes

$$\left|z(t)\right| \leq \int_0^t p(s)\psi\left(w(s)\right)\Delta s + \sum_{0<t_k<t} c_k. \tag{11.67}$$

Using (11.67) in the definition of w, we have that

$$w(t) \leq K_b\left[\int_0^t p(s)\psi\left(w(s)\right)\Delta s + \sum_{0<t_k<t} c_k\right] + K_b\left|\phi(0)\right| + M_b\|\phi\|_{\mathcal{B}}. \tag{11.68}$$

Consequently,

$$\frac{\|w\|_\infty}{K_b\left[\int_0^b p(s)\psi(\|w\|_\infty)\Delta s+\sum_{0<t_k<t}c_k\right]+K_b\,|\phi(0)|+M_b\|\phi\|_{\mathcal{B}}}\leq 1. \tag{11.69}$$

Then by (11.9.3), there exists M such that $\|w\|_\infty \neq M$.

Set

$$U=\left\{z\in \mathcal{B}_b^0 : \|z\|_{\mathcal{B}_b^0}<M+1\right\}. \tag{11.70}$$

The operator $P:\overline{U}\to \mathcal{B}_b^0$ is completely continuous. From the choice of U, there is no $z\in\partial U$ such that $z=\lambda P(z)$, for some $\lambda\in(0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [157], we deduce that P has a fixed point z in U. Then problem (11.39) has at least one solution. □

We consider now neutral functional differential equations.

Definition 11.10. A function $y\in\mathcal{B}_b$ is said to be a solution of (11.40)–(11.42) if y satisfies the dynamic equation

$$\left[y(t)-g(t,y_t)\right]^\Delta=f(t,y_t)\quad \text{everywhere on } J\backslash\{t_k\},\ k=1,\dots,m, \tag{11.71}$$

and for each $k=1,\dots,m$, the function y satisfies the equations $y(t_k^+)-y(t_k^-)=I_k(y(t_k))$, and $y_0=\phi\in\mathcal{B}$.

Theorem 11.11. *Let $f:J\times\mathcal{B}\to\mathbb{R}$ be a continuous function. Assume (11.9.2) and the following conditions are satisfied.*

(11.11.1) *The function g is continuous and completely continuous, and for any bounded set $Q\subseteq C((-\infty,b],\mathbb{R})$, the set $\{t\to g(t,x_t):x\in Q\}$ is equicontinuous in $C([0,b],\mathbb{R}^n)$, and there exist constants $0\leq c_1<1$, $c_2\geq 0$ such that*

$$|g(t,u)|\leq c_1\|u\|_B+c_2,\quad t\in[0,b],\ u\in\mathcal{B}. \tag{11.72}$$

(11.11.2) *There exist a continuous nondecreasing function $\psi:[0,\infty)\to(0,\infty)$ and $p\in L^1(J,\mathbb{R}_+)$ such that*

$$|f(t,x)|\leq p(t)\psi(\|u\|_B),\quad \text{for a.e. } t\in[0,b] \text{ and each } u\in\mathcal{B}, \tag{11.73}$$

and there exists $M_*>0$ such that

$$\frac{M_*}{(1/(1-c_1K_b))\left[K_b\,|g(0,\phi(0))|+c_2K_b+\alpha+K_b\psi(M_*)\int_0^b p(s)\Delta s\right]}>1, \tag{11.74}$$

where $\alpha=K_b|\phi(0)|+M_b\|\phi\|_B$.

Then the IVP (11.40)–(11.42) has at least one solution.

Proof. In analogy to Theorem 11.9, we consider the operator $P^* : \mathcal{B}_b^0 \to \mathcal{B}_b^0$ defined by

$$(P^* z)(t) = \begin{cases} 0, & t \leq 0, \\ g(0,\phi(0)) - g(t, \overline{z}_t + x_t) + \int_0^t f(s, \overline{z}_s + x_s)\Delta s, & t \in [0,b]. \end{cases} \tag{11.75}$$

As in Theorem 11.9 we can prove that the operator P^* is completely continuous. In order to use the Leray-Schauder alternative, we will obtain a priori estimates for the solutions of the integral equation

$$z(t) = \lambda\left[g(0,\phi(0)) - g(t, \overline{z}_t + x_t) + \int_0^t f(s, \overline{z}_s + x_s)\Delta s\right], \tag{11.76}$$

where $z_0 = \lambda\phi$, for some $\lambda \in (0,1)$. Then

$$\begin{aligned} |z(t)| &\leq |g(0,\phi(0))| + |g(t, \bar{z}_t + x_t)| + \int_0^t p(s)\psi(\|\bar{z}_s + x_s\|_B)ds \\ &\leq |g(0,\phi(0))| + c_1\|\bar{z}_t + x_t\|_B + c_2 + \int_0^t p(s)\psi(\|\overline{z}_s + x_s\|_B)ds. \end{aligned} \tag{11.77}$$

If we put $\alpha = K_b|\phi(0)| + M_b\|\phi\|_B$, then

$$\begin{aligned} \|\bar{z}_t + x_t\|_B &\leq K_b \sup_{s\in[0,t]} |z(s)| + \alpha := w(t), \\ |z(t)| &\leq |g(0,\phi(0))| + c_1 w(t) + c_2 + \int_0^t p(s)\psi(w(s))\Delta s. \end{aligned} \tag{11.78}$$

But

$$w(t) \leq K_b|g(0,\phi(0))| + c_1K_b w(t) + c_2K_b + K_b\int_0^t p(s)\psi(w(s))\Delta s + \alpha, \tag{11.79}$$

or

$$w(t) \leq \frac{1}{1 - c_1K_b}\left[K_b|g(0,\phi(0))| + c_2K_b + \alpha + K_b\int_0^b p(s)\psi(w(s))\Delta s\right], \tag{11.80}$$

for $t \in [0,b]$. Hence

$$\frac{\|w\|_\infty}{(1/(1 - c_1K_b))[K_b|g(0,\phi)| + c_2K_b + \alpha + K_b\int_0^b p(s)\psi(\|w\|_\infty)\Delta s]} \leq 1. \tag{11.81}$$

Then, by (11.11.2), there exists M_* such that $\|w\|_\infty \neq M_*$.

Set

$$U_* = \{z \in \mathcal{B}_b^0 : \|z\|_{\mathcal{B}_b^0} < M_* + 1\}. \tag{11.82}$$

The operator $P^* : \overline{U}_* \to \mathcal{B}_b^0$ is completely continuous. From the choice of U_*, there is no $z \in \partial U_*$ such that $z = \lambda P^*(z)$, for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type, we deduce that P^* has a fixed point z in U_*. Then problem (11.40)–(11.42) has at least one solution. □

11.5. Second-order impulsive dynamic equations on time scales

This section is concerned with the existence of solutions for initial value problems for second-order impulsive dynamic equations on time scales. We consider the problem

$$y^{\Delta\Delta}(t) = f(t, y(t)), \quad t \in J := [0, b],\ t \neq t_k,\ k = 1, \dots, m, \tag{11.83}$$

$$y(t_k^+) - y(t_k) = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{11.84}$$

$$y^\Delta(t_k^+) - y^\Delta(t_k) = \bar{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{11.85}$$

$$y(0) = y_0, \qquad y^\Delta(0) = y_1, \tag{11.86}$$

where $\mathbb{T}$ is time scale, $[0, b] \subset \mathbb{T}$, $f : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$ is a given function, $I_k, \bar{I}_k \in C(\mathbb{R}, \mathbb{R})$ $y_0, y_1 \in \mathbb{R}, t_k \in \mathbb{T}, 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^+} y(t_k - h)$ represent the right and left limits of $y(t)$ at $t = t_k$.

Definition 11.12. A function $y \in \Omega \cap \bigcup_{k=0}^m C^2((t_k, t_{k+1}), \mathbb{R})$ is said to be a solution of (11.83)–(11.86) if it satisfies the dynamic equation

$$y^{\Delta\Delta}(t) = f(t, y(t)) \quad \text{everywhere on } J\backslash\{t_k\},\ k = 1, \dots, m, \tag{11.87}$$

and for each $k = 1, \dots, m$ the function y satisfies the conditions $y(t_k^+) - y(t_k) = I_k(y(t_k^-))$, $y^\Delta(t_k^+) - y^\Delta(t_k) = \bar{I}_k(y(t_k^-))$ and the initial conditions $y(0) = y_0$, and $y^\Delta(0) = y_1$.

We need the following auxiliary result.

Lemma 11.13. *Let y_0, $y_1 \in \mathbb{R}$ and let $f : \mathbb{T} \to \mathbb{R}$ be* rd-*continuous and regressive. Then y is the unique solution of the initial value problem*

$$\begin{gathered} y^{\Delta\Delta}(t) = f(t), \\ y(t_k^+) - y(t_k) = I_k(y(t_k^-)), \quad k = 1, \dots, m, \\ y^\Delta(t_k^+) - y^\Delta(t_k) = \bar{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \\ y(0) = y_0, \qquad y^\Delta(0) = y_1, \end{gathered} \tag{11.88}$$

if and only if

$$\begin{aligned} y(t) = y_0 + ty_1 + \int_0^t (t-s)f(s)\Delta s - \int_0^t \mu(s)f(s)\Delta s \\ + \sum_{0<t_k<t} \left[I_k(y(t_k)) + (t-t_k)\bar{I}_k(y(t_k))\right]. \end{aligned} \tag{11.89}$$

Proof. Let y be a solution of problem (11.88). Then

$$y^{\Delta\Delta}(t) = f(t), \quad \text{for } t \in [0, t_1] \subset \mathbb{T}. \tag{11.90}$$

An integration from 0 to t (here $t \in (0, t_1]$) of both sides of the above equality yields

$$\int_0^t y^{\Delta\Delta}(s)\Delta s = \int_0^t f(s)\Delta s, \qquad y^{\Delta}(t) - y^{\Delta}(0) = \int_0^t f(s)\Delta s. \tag{11.91}$$

Thus, for $t \in [0, t_1]$, we have

$$y^{\Delta}(t) = y^{\Delta}(0) + \int_0^t f(s)\Delta s. \tag{11.92}$$

We integrate both sides of the above equality to get

$$\begin{aligned} y(t) - y(0) &= ty_1 + \int_0^t \int_0^s f(u)\Delta u \Delta s \\ &= ty_1 + \int_0^t (t-s)f(s)\Delta s - \int_0^t \mu(s)f(s)\Delta s. \end{aligned} \tag{11.93}$$

Then, for $t \in [0, t_1]$, we have

$$y(t) = y_0 + ty_1 + \int_0^t (t-s)f(s)\Delta s - \int_0^t \mu(s)f(s)\Delta s. \tag{11.94}$$

If $t \in (t_1, t_2]$, then we have

$$\begin{gathered} \int_0^t y^{\Delta\Delta}(s)\Delta s = \int_0^t f(s)\Delta s, \\ \int_0^{t_1} y^{\Delta\Delta}(s)\Delta s + \int_{t_1}^t y^{\Delta\Delta}(s)\Delta s = \int_0^t f(s)\Delta s, \\ y^{\Delta}(t_1) - y^{\Delta}(0) + y^{\Delta}(t) - y^{\Delta}(t_1^+) = \int_0^t f(s)\Delta s, \\ y^{\Delta}(t) - \bar{I}_1(y(t_1)) - y_1 = \int_0^t f(s)\Delta s. \end{gathered} \tag{11.95}$$

An integration from t_1 to t of both sides of the above equality yields

$$
\begin{aligned}
\int_{t_1}^{t} [y^{\Delta}(s) - \bar{I}_1(y(t_1)) - y_1]\Delta s &= \int_{t_1}^{t}\int_0^s f(u)\Delta u\Delta s, \\
y(t) - y(t_1^+) - (t-t_1)\bar{I}_1(y(t_1)) - (t-t_1)y_1 &= \int_{t_1}^{t}\int_0^s f(u)\Delta u\Delta s, \\
y(t) - y(t_1^+) - (t-t_1)\bar{I}_1(y(t_1)) &- (t-t_1)y_1 \\
= \int_0^t tf(s)\Delta s - \int_0^{t_1} t_1 f(s)\Delta s &- \int_{t_1}^{t}\sigma(s)f(s)\Delta s.
\end{aligned}
\tag{11.96}
$$

Thus, for $t \in (t_1, t_2]$, we have

$$
\begin{aligned}
y(t) &= y(t_1^+) + (t-t_1)\bar{I}_1(y(t_1)) + (t-t_1)y_1 \\
&\quad + \int_0^t tf(s)\Delta s - \int_0^{t_1} t_1 f(s)\Delta s - \int_{t_1}^{t}\mu(s)f(s)\Delta s - \int_{t_1}^{t} sf(s)\Delta s \\
&= y(t_1) + I_1(y(t_1)) + (t-t_1)\bar{I}_1(y(t_1)) + (t-t_1)y_1 \\
&\quad + \int_0^t tf(s)\Delta s - \int_0^{t_1} t_1 f(s)\Delta s - \int_{t_1}^{t} sf(s)\Delta s - \int_{t_1}^{t}\mu(s)f(s)\Delta s \\
&= y_0 + t_1 y_1 + \int_0^{t_1}(t_1 - s)f(s)\Delta s - \int_0^{t_1}\mu(s)f(s)\Delta s \\
&\quad + \int_0^t tf(s)\Delta s - \int_0^{t_1} t_1 f(s)\Delta s - \int_{t_1}^{t} sf(s)\Delta s - \int_{t_1}^{t}\mu(s)f(s)\Delta s \\
&\quad + I_1(y(t_1)) + (t-t_1)\bar{I}_1(y(t_1)) + (t-t_1)y_1.
\end{aligned}
\tag{11.97}
$$

Hence, for $t \in [t_1, t_2]$, we have

$$
y(t) = y_0 + ty_1 + \int_0^t (t-s)f(s)\Delta s - \int_0^t \mu(s)f(s)\Delta s + I_1(y(t_1)) + (t-t_1)\bar{I}_1(y(t_1)).
\tag{11.98}
$$

Continue to obtain, for $t \in [0, b]$, that

$$
\begin{aligned}
y(t) = y_0 + ty_1 &+ \int_0^t (t-s)f(s)\Delta s - \int_0^t \mu(s)f(s)\Delta s \\
&+ \sum_{0<t_k<t}[I_k(y(t_k)) + (t-t_k)\bar{I}_k(y(t_k))].
\end{aligned}
\tag{11.99}
$$

Conversely, we prove that if y satisfies the integral equation (11.89), then y is solution of problem (11.86). Firstly $y(0) = y_0$. Let $t \in [0,b]\backslash\{t_1,\dots,t_m\}$ and

$$\begin{aligned} y(t) = y_0 + ty_1 + \int_0^t (t-s)f(s)\Delta s - \int_0^t \mu(s)f(s)\Delta s \\ + \sum_{0<t_k<t} \left[I_k(y(t_k)) + (t-t_k)\bar{I}_k(y(t_k))\right]. \end{aligned} \tag{11.100}$$

Then

$$\begin{aligned} y^{\Delta}(t) &= \Big[y_0 + ty_1 + \int_0^t (t-s)f(s)\Delta s - \int_0^t \mu(s)f(s)\Delta s \\ &\quad + \sum_{0<t_k<t} \left[I_k(y(t_k)) + (t-t_k)\bar{I}_k(y(t_k))\right]\Big]^{\Delta} \\ &= \left[y_0 + ty_1\right]^{\Delta} + \left[\int_0^t (t-s)f(s)\Delta s\right]^{\Delta} - \left[\int_0^t \mu(s)f(s)\Delta s\right]^{\Delta} \\ &\quad + \left[\sum_{0<t_k<t} \left[I_k(y(t_k)) + (t-t_k)\bar{I}_k(y(t_k))\right]\right]^{\Delta} \\ &= y_1 + \int_0^t f(s)\Delta s + \sigma(t)f(t) - tf(t) - \mu(t)f(t) + \sum_{0<t_k<t} \bar{I}_k(y(t_k)) \\ &= y_1 + \int_0^t f(s)\Delta s + \sum_{0<t_k<t} \bar{I}_k(y(t_k)). \end{aligned} \tag{11.101}$$

Thus

$$y^{\Delta\Delta}(t) = \left[y_1 + \int_0^t f(s)\Delta s + \sum_{0<t_k<t} \bar{I}_k(y(t_k))\right]^{\Delta} = f(t). \tag{11.102}$$

Clearly, we have $y^{\Delta}(0) = y_1$ and

$$y^{\Delta}(t_k^+) - y^{\Delta}(t_k^-) = \bar{I}_k(y(t_k)), \quad \text{for } k = 1,\dots,m. \tag{11.103}$$

From the definition of y we can prove that

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k)), \quad \text{for } k = 1,\dots,m. \tag{11.104}$$

In the proof of our main theorem, we use the following time scale version of the well-known Gronwall inequality. □

Lemma 11.14 (see [4]). *Let* $y, f : \mathbb{T} \to \mathbb{R}$ *be* rd-*continuous and* $p \in \mathcal{R}^+$ *regressive. Then*

$$y(t) \le f(t) + \int_a^t y(s)p(s)\Delta s, \quad \forall\, t \in \mathbb{T} \tag{11.105}$$

implies

$$y(t) \le f(t) + \int_a^t e_p(t, \sigma(s)) f(s)p(s)\Delta s, \quad \forall\, t \in \mathbb{T}, \tag{11.106}$$

where $\mathcal{R}^+$ *is the set of all* rd-*continuous functions and* p *satisfies* $1 + \mu(t)p(t) > 0$.

Theorem 11.15. *Suppose that the following hypotheses are satisfied.*

(11.15.1) *The function* $f : [0,b] \times \mathbb{R} \to \mathbb{R}$ *is continuous.*

(11.15.2) *There exist constants* c_k, $\bar{c}_k$ *such that*

$$|I_k(y)| \le c_k, \qquad |\bar{I}_k(y)| \le \bar{c}_k, \tag{11.107}$$

for each $k = 1, \ldots, m$, *and for all* $y \in \mathbb{R}$.

(11.15.3) *There exist continuous* $p, \bar{q} \in C([0,b], \mathbb{R}_+)$ *such that*

$$|f(t,y)| \le p(t)|y| + \bar{q}(t), \quad \text{for each } (t,y) \in [0,b] \times \mathbb{R}. \tag{11.108}$$

Then, if $|\sigma(b)| < \infty$, *the impulsive IVP* (11.83)–(11.85) *has at least one solution.*

Proof. Transform problem (11.83)–(11.85) into a fixed point problem. Consider the operator $G : \Omega \to \Omega$ defined by

$$\begin{aligned}(Gy)(t) = y_0 + t y_1 &+ \int_0^t (t-s) f(s, y(s))\Delta s - \int_0^t \mu(s) f(s, y(s))\Delta s \\ &+ \sum_{0<t_k<t} \left[I_k(y(t_k)) + (t - t_k)\bar{I}_k(y(t_k)) \right].\end{aligned} \tag{11.109}$$

We will show that G satisfies the assumptions of Schaefer's fixed point theorem. The proof will be given in several steps. We show first that G is continuous and completely continuous.

Step 1. G is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in Ω. Then

$$\begin{aligned}
&|G(y_n)(t) - G(y)(t)| \\
&\quad \le (b + |\sigma(b)|)\int_0^b |f(s, y_n(s)) - f(s, y(s))|\Delta s \\
&\qquad + \sum_{0<t_k<t} [|I_k(y_n(t_k)) - I_k(y(t_k))| + (b - t_k)|\bar{I}_k(y_n(t_k)) - \bar{I}_k(y(t_k))|].
\end{aligned} \tag{11.110}$$

Since $f, I_k, \bar{I}_k$ are continuous functions, then we have

$$\begin{aligned}
&||G(y_n) - G(y)||_\Omega \\
&\quad \le (b + |\sigma(b)|)||f(\cdot, y_n(\cdot)) - f(\cdot, y(\cdot))||_\infty \\
&\qquad + \sum_{0<t_k<t} [|I_k(y_n(t_k)) - I_k(y(t_k))| + b|\bar{I}_k(y_n(t_k)) - \bar{I}_k(y(t_k))|].
\end{aligned} \tag{11.111}$$

Thus

$$||G(y_n) - G(y)||_\Omega \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{11.112}$$

Step 2. G maps bounded sets into bounded sets in Ω.

Indeed, it is enough to show that there exists a positive constant ℓ such that, for each $y \in B_q = \{y \in \Omega : \|y\|_\Omega \le q\}$, one has $\|Gy\|_\Omega \le \ell$.

By (H2), (H3), we have

$$\begin{aligned}
|(Gy)(t)| &= \Big| y_0 + ty_1 + \int_0^t (t-s) f(s, y(s))\Delta s - \int_0^t \mu(s) f(s, y(s))\Delta s \\
&\qquad + \sum_{0<t_k<t} [I_k(y(t_k)) + (t - t_k)\bar{I}_k(y(t_k))]\Big| \\
&\le |y_0| + b|y_1| + \int_0^b b|f(s, y(s))|\Delta s + \int_0^t \mu(s)|f(s, y(s))|\Delta s \\
&\qquad + \sum_{k=0}^m [|I_k(y(t_k))| + (b - t_k)|\bar{I}_k(y(t_k))|] \\
&\le (b + |\sigma(b)|)q\int_0^b p(s)\Delta s + (b + |\sigma(b)|)\int_0^b \bar{q}(s)\Delta s \\
&\qquad + |y_0| + b|y_1| + \sum_{k=0}^m [c_k + (b - t_k)\bar{c}_k].
\end{aligned} \tag{11.113}$$

Thus

$$\begin{aligned}\|Gy\|_\Omega \leq q\big(b+|\sigma(b)|\big)\sup_{t\in[0,b]}p(t)+b\big(b+|\sigma(b)|\big)\sup_{t\in[0,b]}\bar{q}(t)\\ +|y_0|+b|y_1|+\sum_{k=0}^{m}\big[c_k+(b-t_k)\bar{c}_k\big]:=\ell.\end{aligned}\tag{11.114}$$

Step 3. G maps bounded sets into equicontinuous sets of Ω.

Let $r_1, r_2 \in J$, $r_1 < r_2$, and B_q be a bounded set of Ω as in Step 2. Let $y \in B_q$. Then

$$\begin{aligned}|(Gy)(r_2)-(Gy)(r_1)| \leq{} & |r_2-r_1||y_1|+\int_0^{r_1}(r_2-r_1)|f(s,y(s))|\Delta s\\ &+\int_{r_1}^{r_2}r_2|f(s,y(s))|\Delta s+\int_{r_1}^{r_2}|\mu(s)||f(s,y(s))|\Delta s\\ &+\sum_{0<t_k<r_2-r_1}\big[c_k+(r_2-r_1)\bar{c}_k\big].\\ \leq{} & \Big[|y_1|+(r_1q+r_2q)\sup_{t\in[0,b]}p(t)\\ &+(r_1+r_2)\sup_{t\in[0,b]}\bar{q}(t)\Big]|r_2-r_1|\\ &+\Big[|\sigma(b)|q\sup_{t\in[0,b]}p(t)+|\sigma(b)|\sup_{t\in[0,b]}\bar{q}(t)\Big]|r_2-r_1|\\ &+\sum_{0<t_k<r_2-r_1}\big[c_k+(r_2-r_1)\bar{c}_k\big].\end{aligned}\tag{11.115}$$

The right-hand side tends to zero as $r_2 - r_1 \to 0$. As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $G : \Omega \to \Omega$ is continuous and completely continuous.

Step 4. Now it remains to show that the set

$$\mathcal{E}(G) := \{y \in \Omega : y = \lambda G(y),\ \text{for some } 0 < \lambda < 1\}\tag{11.116}$$

is bounded. Let $y \in \mathcal{E}(G)$. Then there exists $0 < \lambda < 1$ such that $y = \lambda G(y)$, and so

$$\begin{aligned}(Gy)(t) = y_0+ty_1+\int_0^t(t-s)f(s,y(s))\Delta s-\int_0^t\mu(s)f(s,y(s))\Delta s\\ +\sum_{0<t_k<t}\big[I_k(y(t_k))+(t-t_k)\bar{I}_k(y(t_k))\big].\end{aligned}\tag{11.117}$$

By (H2), (H3), we have

$$
\begin{aligned}
|y(t)| &= \left| y_0 + ty_1 + \int_0^t (t-s)f(s,y(s))\Delta s - \int_0^t \mu(s)f(s,y(s))\Delta s \right. \\
&\qquad \left. + \sum_{0<t_k<t} \left[I_k(y(t_k)) + (t-t_k)\bar{I}_k(y(t_k)) \right] \right| \\
&\leq b\int_0^t \left[p(s)|y(s)| + \bar{q}(s) \right]\Delta s \\
&\qquad + \int_0^t |\sigma(b)|\left[p(s)|y(s)| + \bar{q}(s) \right]\Delta s \\
&\qquad + |y_0| + b|y_1| + \sum_{k=0}^{m} \left[c_k + b\bar{c}_k \right] \\
&\leq (b+|\sigma(b)|)\sup_{t\in[0,b]} p(t)\int_0^t |y(s)|\Delta s \\
&\qquad + b(b+|\sigma(b)|)\sup_{t\in[0,b]} \bar{q}(t) \\
&\qquad + |y_0| + b|y_1| + \sum_{k=0}^{m} \left[c_k + b\bar{c}_k \right].
\end{aligned}
\tag{11.118}
$$

Put $p_0 = (b + |\sigma(b)|)\sup_{t\in[0,b]} p(t)$. Then $p_0 \in \mathcal{R}^+$. Let $e_{p_0}(t,0)$ be the unique solution of problem

$$
y^{\Delta}(t) = p_0(t)y(t), \qquad y(0) = 1. \tag{11.119}
$$

Then, from the Gronwall's inequality, we have

$$
\begin{aligned}
|y(t)| \leq &\Bigg(|y_0| + b|y_1| + b(b+|\sigma(b)|)\sup_{t\in[0,b]} \bar{q}(t) \\
&\quad + \sum_{k=0}^{m} \left[c_k + b\bar{c}_k \right] \Bigg)(b+|\sigma(b)|)\sup_{t\in[0,b]} p(t)\int_0^t e_{p_0}(t,\sigma(s))\Delta s \\
&\quad + |y_0| + b|y_1| + b(b+|\sigma(b)|)\sup_{t\in[0,b]} \bar{q}(t) + \sum_{k=0}^{m} \left[c_k + b\bar{c}_k \right].
\end{aligned}
\tag{11.120}
$$

Thus

$$\begin{aligned}\|y\|_\Omega \leq \Bigg(& |y_0| + b|y_1| + b(b+|\sigma(b)|)\sup_{t\in[0,b]}\bar{q}(t) \\ & + \sum_{k=0}^{m}\left[c_k + b\bar{c}_k\right]\Bigg)(b+|\sigma(b)|)\sup_{t\in[0,b]}p(t)\sup_{t\in[0,b]}\int_0^b e_{p_0}(t,\sigma(s))\Delta s \\ & + |y_0| + b|y_1| + b(b+|\sigma(b)|)\sup_{t\in[0,b]}\bar{q}(t) + \sum_{k=0}^{m}\left[c_k + b\bar{c}_k\right].\end{aligned} \tag{11.121}$$

This shows that $\mathcal{E}(G)$ is bounded.

Set $X := \Omega$. As a consequence of Schaefer's theorem, we deduce that G has a fixed point y which is a solution to problem (11.83)–(11.85). □

Remark 11.16. A slight modification of the proof (i.e., in Step 4 use the usual Leray-Schauder alternative) guarantees that (11.15.3) could be replaced by

(11.15.3)* there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and $\bar{p} \in C([0,b],\mathbb{R}_+)$ such that

$$|f(t,y)| \leq \bar{p}(t)\psi(|y|), \quad \text{for each } (t,y) \in [0,b]\times\mathbb{R}, \tag{11.122}$$

and there exists a constant $M > 0$ with

$$\frac{M}{|y_0| + b|y_1| + \psi(M)\int_0^b (b+\sigma(b))\bar{p}(s)\Delta s + \sum_{k=0}^{m}\left[c_k + (b-t_k)\bar{c}_k\right]} > 1. \tag{11.123}$$

11.6. Existence results for second-order boundary value problems of impulsive dynamic equations on time scales

This section is concerned with the existence of solutions of boundary value problems for impulsive dynamic equations on time scales. We consider the boundary value problem

$$\begin{gathered}-y^{\Delta\Delta}(t) = f(t,y(t)), \quad t \in J := [0,1],\ t \neq t_k,\ k = 1,\dots,m, \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad k = 1,\dots,m, \\ y^{\Delta}(t_k^+) - y^{\Delta}(t_k^-) = \bar{I}_k(y(t_k^-)), \quad k = 1,\dots,m, \\ y(0) = y(1) = 0,\end{gathered} \tag{11.124}$$

where $\mathbb{T}$ is a time scale, $0,1 \in \mathbb{T}$, $[0,1] \subset \mathbb{T}$, $f : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$, is a given function, $I_k, \bar{I}_k \in C(\mathbb{R},\mathbb{R})$, $t_k \in \mathbb{T}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^+} y(t_k - h)$ represent the right and left limits of $y(t)$ at $t = t_k$.

We will assume for the remainder of the section that, for each $k = 1, \dots, m$, the points of impulse t_k are right-dense. In order to define the solution of (11.124) we will consider the notations of Section 11.3, with 0, 1 replacing a, b, respectively.

Definition 11.17. A function $y \in \Omega \cap C^2((t_k, t_{k+1}), \mathbb{R})$, $k = 0, \dots, m$, is said to be a solution of (11.124) if it satisfies the dynamic equation

$$-y^{\Delta\Delta}(t) = f(t, y(t)) \quad \text{everywhere on } J \backslash \{t_k\},\ k = 1, \dots, m, \tag{11.125}$$

and for each $k = 1, \dots, m$, the function y satisfies the conditions $y(t_k^+) - y(t_k) = I_k(y(t_k^-))$, $y^{\Delta}(t_k^+) - y^{\Delta}(t_k) = \bar{I}_k(y(t_k^-))$, and the boundary conditions $y(0) = y(1) = 0$.

Lemma 11.18. *Let $f : \mathbb{T} \to \mathbb{R}$ be* rd-*continuous. If y is a solution of the equation*

$$y(t) = \int_0^1 G(t,s) f(s) \Delta s + \sum_{k=1}^{m} W_k(t, y(t_k)), \tag{11.126}$$

where

$$\begin{aligned} G(t,s) &= \begin{cases} (1-t)\sigma(s) & \textit{if } 0 \le s \le t, \\ (1-\sigma(s))t & \textit{if } t \le s \le 1, \end{cases} \\ W_k(t, y(t_k)) &= \begin{cases} t\big[-I_k(y(t_k)) - (1-t_k)\bar{I}_k(y(t_k))\big] & \textit{if } 0 \le t \le t_k, \\ (1-t)\big[I_k(y(t_k)) - t_k \bar{I}_k(y(t_k))\big] & \textit{if } t_k < t \le 1, \end{cases} \end{aligned} \tag{11.127}$$

then y is a solution of the boundary value problem

$$\begin{gathered} -y^{\Delta\Delta}(t) = f(t), \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad k = 1, \dots, m, \\ y^{\Delta}(t_k^+) - y^{\Delta}(t_k^-) = \bar{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \\ y(0) = y(1) = 0. \end{gathered} \tag{11.128}$$

Proof. Let y satisfy the integral equation (11.126). Then y is solution of problem (11.128). Firstly $y(0) = y(1) = 0$. Let $t \in [0,1] \backslash \{t_1, \dots, t_m\}$. Then, we have

$$y(t) = \int_0^1 G(t,s) f(s) \Delta s + \sum_{k=1}^{m} W_k(t, y(t_k)). \tag{11.129}$$

Hence

$$\begin{aligned}
y^{\Delta}(t) &= \left[\int_0^1 G(t,s)f(s)\Delta s + \sum_{k=1}^{m} W_k(t_k, y(t_k))\right]^{\Delta} \\
&= \left[\int_0^1 G(t,s)f(s)\Delta s\right]^{\Delta} + \left[\sum_{k=1}^{m} W_k(t, y(t_k))\right]^{\Delta} \\
&= \left[\int_0^t (1-t)\sigma(s)f(s)\Delta s\right]^{\Delta} + \left[\int_t^1 (1-\sigma(s))tf(s)\Delta s\right]^{\Delta} + \sum_{k=1}^{m} W_k^{\Delta}(t,y) \\
&= -\int_0^t \sigma(s)f(s)\Delta s + \int_t^1 (1-\sigma(s))f(s)\Delta s + \sum_{k=1}^{m} W_k^{\Delta}(t,y),
\end{aligned} \tag{11.130}$$

where

$$W_k^{\Delta}(t,y) = \begin{cases} [-I_k(y(t_k)) - (1-t_k)\bar{I}_k(y(t_k))] & \text{if } 0 \le t \le t_k, \\ -[I_k(y(t_k)) - t_k\bar{I}_k(y(t_k))] & \text{if } t_k < t \le 1, \end{cases} \tag{11.131}$$

$$W_k^{\Delta\Delta}(t,y) = 0, \quad \text{for } k = 1,\dots,m.$$

Thus

$$\begin{aligned}
y^{\Delta\Delta}(t) &= \left[-\int_0^t \sigma(s)f(s)\Delta s + \int_t^1 (1-\sigma(s))f(s)\Delta s + \sum_{k=1}^{m} W_k^{\Delta}(t,y)\right]^{\Delta} \\
&= \left[-\int_0^t \sigma(s)f(s)\Delta s\right]^{\Delta} + \left[\int_t^1 (1-\sigma(s))f(s)\Delta s\right]^{\Delta} \\
&= f(t).
\end{aligned} \tag{11.132}$$

Clearly, we have

$$y^{\Delta}(t_k^+) - y^{\Delta}(t_k^-) = \bar{I}_k(y(t_k)), \quad \text{for } k = 1,\dots,m. \tag{11.133}$$

From the definition of y we can prove that

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k)), \quad \text{for } k = 1,\dots,m. \tag{11.134}$$

□

Theorem 11.19. *Assume the following hold.*

(11.19.1) *The function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.*

(11.19.2) *There exist constants c_k, $\bar{c}_k$ such that*

$$|I_k(y)| \le c_k, \qquad |\bar{I}_k(y)| \le \bar{c}_k, \tag{11.135}$$

for each $k = 1,\dots,m$, and for all $y \in \mathbb{R}$.

(11.19.3) *There exists a function* $p \in C([0,1],\mathbb{R}_+)$ *such that*

$$|f(t,y)| \le p(t), \quad \textit{for each } (t,y) \in [0,1]\times\mathbb{R}. \tag{11.136}$$

Then the impulsive BVP (11.124) *has at least one solution.*

Proof. Transform the BVP (11.124) into a fixed point problem. Consider the operator $N : \Omega \to \Omega$ defined by

$$(Ny)(t) = \int_0^1 G(t,s)f(s,y(s))\Delta s + \sum_{k=1}^m W_k(t,y(t_k)). \tag{11.137}$$

We will show that N satisfies the assumptions of Schaefer's fixed point theorem. The proof will be given in several steps. We show first that N is continuous and completely continuous.

Step 1. N is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in Ω. Then

$$\begin{aligned} |N(y_n)(t) - N(y)(t)| \le{}& \sup_{(t,s)\in J\times J} |G(t,s)| \int_0^1 |f(s,y_n(s)) - f(s,y(s))|\Delta s \\ &+ \sum_{k=1}^m |W_k(t,y_n(t_k)) - W_k(t,y(t_k))|. \end{aligned} \tag{11.138}$$

Since $f, I_k, \bar{I}_k$ are continuous, we have

$$\begin{aligned} ||N(y_n) - N(y)||_\Omega \le{}& \sup_{(t,s)\in J\times J} |G(t,s)|\, ||f(\cdot,y_n(\cdot)) - f(\cdot,y(\cdot))||_\infty \\ &+ 2\sum_{k=1}^m \big[|I_k(y_n(t_k)) - I_k(y(t_k))| \\ &\quad + |\bar{I}_k(y_n(t_k)) - \bar{I}_k(y(t_k))|\big]. \end{aligned} \tag{11.139}$$

Thus

$$||N(y_n) - N(y)||_\infty \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{11.140}$$

Step 2. N maps bounded sets into bounded sets in Ω.

Indeed, it is enough to show that there exists a positive constant ℓ such that, for each $y \in B_q = \{y \in \Omega : \|y\|_\Omega \le q\}$, one has $\|Ny\|_\Omega \le \ell$. For each $t \in [0,1]$, we have

$$(Ny)(t) = \int_0^1 G(t,s)f(s,y(s))\Delta s + \sum_{k=1}^m W_k(t,y(t_k)). \tag{11.141}$$

From (11.19.2), (11.19.3), we have

$$\begin{aligned}
|(Ny)(t)| &= \left| \int_0^1 G(t,s)f(s,y(s))\Delta s + \sum_{k=1}^{m} W_k(t,y(t_k)) \right| \\
&\leq \sup_{(t,s)\in J\times J} |G(t,s)| \int_0^1 |f(s,y(s))|\Delta s + \sum_{k=0}^{m} |W_k(t,y(t_k))| \\
&\leq \sup_{(t,s)\in J\times J} |G(t,s)| \int_0^1 p(s)\Delta s + 2\sum_{k=0}^{m} [c_k + \bar{c}_k].
\end{aligned} \tag{11.142}$$

Thus

$$\|Ny\|_\Omega \leq p_* + 2\sum_{k=0}^{m} [c_k + \bar{c}_k] := \ell, \tag{11.143}$$

where

$$p_* = \sup_{(t,s)\in J\times J} |G(t,s)| \sup_{t\in[0,1]} p(t). \tag{11.144}$$

Step 3. N maps bounded sets into equicontinuous sets of Ω.

Let $r_1, r_2 \in J$, $r_1 < r_2$, and let B_q be a bounded set of Ω as in Step 2. Let $y \in B_q$. Then

$$\begin{aligned}
|(Ny)(r_2) - (Ny)(r_1)| &\leq \int_0^1 |G(r_2,s) - G(r_1,s)| |f(s,y(s))|\Delta s \\
&\quad + \sum_{k=1}^{m} |W_k(r_2,y(t_k)) - W_k(r_1,y(t_k))| \\
&\leq \int_0^1 |G(r_2,s) - G(r_1,s)| p(s)\Delta s \\
&\quad + \sum_{k=1}^{m} |W_k(r_2,y(t_k)) - W_k(r_1,y(t_k))|.
\end{aligned} \tag{11.145}$$

The right-hand side tends to zero as $r_2 - r_1 \to 0$. As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $N : \Omega \to \Omega$ is continuous and completely continuous.

Step 4. Now it remains to show that the set

$$\mathcal{E}(N) := \{y \in \Omega : y = \lambda N(y), \text{ for some } 0 < \lambda < 1\} \tag{11.146}$$

is bounded. As in Step 2 we can prove that $\mathcal{E}(N)$ is bounded.

Set $X := \Omega$. As a consequence of Schaefer's fixed point theorem, we deduce that N has a fixed point y which is a solution to BVP problem (11.124). □

We present now a result for the BVP problem (11.124) in the spirit of the nonlinear alternative of Leray-Schauder type [157].

Theorem 11.20. *Suppose that hypotheses (11.19.1)–(11.19.2) and the following condition are satisfied.*

(11.20.1) *There exist a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$, $\bar{p} \in C([0,1],\mathbb{R}_+)$ *and a nonnegative number* $r > 0$ *such that*

$$|F(t,y)| \le \bar{p}(t)\psi(|y|), \quad \text{for each } y \in \mathbb{R},$$
$$\frac{r}{\sup_{(t,s)\in[0,1]\times[0,1]} |G(t,s)| \psi(r) \int_0^1 \bar{p}(s)\Delta s + 2\sum_{k=0}^m [c_k + \bar{c}_k]} > 1. \tag{11.147}$$

Then the impulsive BVP (11.124) *has at least one solution.*

Proof. Transform the BVP (11.124) into a fixed point problem. Consider the operator N defined in the proof of Theorem 11.19. We will show that N satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. Let y be such that $y = \lambda N y$, for some $\lambda \in (0,1)$. Thus

$$(Ny)(t) = \int_0^1 G(t,s) f(s,y(s))\Delta s + \sum_{k=1}^m W_k(t,y(t_k)). \tag{11.148}$$

From (11.6.2), (11.20.1), we have

$$\begin{aligned}
|y(t)| &= \lambda \left| \int_0^1 G(t,s) f(s,y(s))\Delta s + \sum_{k=1}^m W_k(t,y(t_k)) \right| \\
&\le \sup_{(t,s)\in[0,1]\times[0,1]} |G(t,s)| \int_0^1 p(s)\psi(|y(s)|)\Delta s + 2\sum_{k=0}^m [c_k + \bar{c}_k] \\
&\le \sup_{(t,s)\in[0,1]\times[0,1]} |G(t,s)| \int_0^1 p(s)\psi(\|y\|_\Omega)\Delta s + 2\sum_{k=0}^m [c_k + \bar{c}_k].
\end{aligned} \tag{11.149}$$

Consequently,

$$\frac{\|y\|_\Omega}{\sup_{(t,s)\in[0,1]\times[0,1]} |G(t,s)| \int_0^1 p(s)\psi(\|y\|_\Omega)\Delta s + 2\sum_{k=0}^m [c_k + \bar{c}_k]} \le 1. \tag{11.150}$$

Then, by (A1), there exists r such that $\|y\|_\Omega \ne r$.

Set

$$U = \{y \in C([0,1],\mathbb{R}) : \|y\|_\Omega < r\}. \tag{11.151}$$

As in Theorem 11.19 the operator $N : \overline{U} \to C([0,1)],\mathbb{R})$ is continuous and completely continuous. By the choice of U there is no $y \in \partial U$ such that $y = \lambda N(y)$, for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type, we deduce that N has a fixed point y in U, which is a solution of the BVP (11.124). □

11.7. Double positive solutions of impulsive dynamic boundary value problems

Let $\mathbb{T}$ be a time scale such that $0, 1 \in \mathbb{T}$. Throughout the section, all t-intervals $[a, b]$ should be interpreted as $[a, b] \cap \mathbb{T}$. Also throughout, let $\tau \in (0, 1)$ be fixed, and assume that τ is right-dense. In this section, we apply a double fixed point theorem, Theorem 1.16, to obtain at least two positive solutions of the nonlinear impulsive dynamic equation

$$y^{\Delta\Delta}(t) + f\big(y(\sigma(t))\big) = 0, \quad t \in [0,1] \setminus \{\tau\}, \tag{11.152}$$

subject to the underdetermined impulse condition

$$y(\tau^+) - y(\tau^-) = I\big(y(\tau)\big), \tag{11.153}$$

and satisfying the right focal boundary conditions

$$y(0) = y^{\Delta}\big(\sigma(1)\big) = 0, \tag{11.154}$$

where $f : \mathbb{R} \to [0, \infty)$ is continuous and $I : [0, \infty) \to [0, \infty)$ is continuous. By a positive solution, we will mean *positive with respect to a suitable cone*.

We note that, from the nonnegativity of f and I, a solution y of (11.152)–(11.154) is nonnegative and concave on each of $[0, \tau]$ and $(\tau, 1]$. We will apply Theorem 1.16 to a completely continuous integral operator whose kernel, $G(t, s)$, is Green's function for

$$-y^{\Delta\Delta} = 0, \tag{11.155}$$

satisfying (11.154). In this instance,

$$G(t,s) = \begin{cases} t, & 0 \le t \le s \le \sigma(1), \\ \sigma(s), & 0 \le \sigma(s) \le t \le \sigma^2(1). \end{cases} \tag{11.156}$$

Properties of $G(t, s)$ of which we will make use include

$$G(t,s) \le G\big(\sigma(s), s\big) = \sigma(s), \quad t \in \big[0, \sigma^2(1)\big],\ s \in \big[0, \sigma(1)\big], \tag{11.157}$$

and for each $0 < p < 1$,

$$G(t,s) \ge \frac{p}{\sigma^2(1)}\sigma(s), \quad t \in \big[p, \sigma^2(1)\big],\ s \in \big[0, \sigma(1)\big]. \tag{11.158}$$

To apply Theorem 1.16, we must define a suitable Banach space, $\mathcal{B}$, a cone, $\mathcal{P}$, and an operator A. In that direction, let

$$\mathcal{B} = \{y : [0,\sigma^2(1)] \longrightarrow \mathbb{R} \mid y \in C[0,\tau],\ y \in C(\tau,\sigma^2(1)],\ y(\tau^+) \in \mathbb{R}\}, \tag{11.159}$$

equipped with norm

$$\|y\| = \max\left\{\sup_{t\in[0,\tau]}|y(t)|,\ \sup_{t\in(\tau,\sigma^2(1)]}|y(t)|\right\}. \tag{11.160}$$

Of course, for $y \in \mathcal{B}$, we will consider in a piecewise manner that $y \in C[0,\tau]$ and $y \in C[\tau,\sigma^2(1)]$. Moreover, we note that if $y \in \mathcal{B}$, then $y(\tau^-) = \lim_{t\to\tau^-} y(t) = y(\tau)$. Next, let the cone $\mathcal{P} \subset \mathcal{B}$ be defined by

$$\begin{aligned}\mathcal{P} = \{&y \in \mathcal{B} \mid y \text{ is concave, nondecreasing, and nonnegative on each of}\\ &[0,\tau],\ [\tau,\sigma^2(1)],\ y(\tau^+) - y(\tau^-) \geq 0\}.\end{aligned} \tag{11.161}$$

We note that, for each $y \in \mathcal{P}$, $I(y(\tau)) \geq 0$. It follows that, for $y \in \mathcal{P}$,

$$\|y\| = \max\{y(\tau), y(\sigma^2(1))\} = y(\sigma^2(1)). \tag{11.162}$$

For the remainder, assume there exists

$$\eta = \inf\left[\frac{\tau+\sigma^2(1)}{2}, 1\right) \in \mathbb{T}, \tag{11.163}$$

and assume there exists $r \in \mathbb{T}$ with

$$\eta < r < 1, \tag{11.164}$$

which we fix. If $y \in \mathcal{P}$, then

$$\begin{aligned} y(t) &\geq \frac{1}{2}\sup_{s\in[\tau/2,\tau]} y(s) = \frac{1}{2}y(\tau), \quad t \in \left[\frac{\tau}{2},\tau\right],\\ y(t) &\geq \frac{1}{2}\sup_{s\in[\eta,\sigma^2(1)]} y(s) = \frac{1}{2}y(\sigma^2(1)), \quad t \in [\eta,\sigma^2(1)]. \end{aligned} \tag{11.165}$$

Now define nonnegative, increasing, continuous functionals γ, θ, and α on $\mathcal{P}$ by

$$\begin{aligned} \gamma(y) &= \min_{t\in[\eta,r]} y(t) = y(\eta),\\ \theta(y) &= \max_{t\in[\tau,\eta]} y(t) = y(\eta),\\ \alpha(y) &= \max_{t\in[\tau,r]} y(t) = y(r). \end{aligned} \tag{11.166}$$

Then, for each $y \in \mathcal{P}$,

$$\gamma(y) = \theta(y) \leq \alpha(y), \tag{11.167}$$

and $\gamma(y) = y(\eta) \geq (1/2)y(\sigma^2(1)) = (1/2)\|y\|$. So,

$$\|y\| \leq 2\gamma(y), \quad \forall y \in \mathcal{P}. \tag{11.168}$$

Moreover, we note that

$$\theta(\lambda y) = \lambda\theta(y), \quad 0 \leq \lambda \leq 1,\ y \in \partial\mathcal{P}(\theta, b). \tag{11.169}$$

For convenience, let

$$N = \int_0^{\sigma(1)} \sigma(s)\Delta s, \qquad M = \int_0^{\eta} \sigma(s)\Delta s. \tag{11.170}$$

We now state growth conditions on f and I so that (11.152)–(11.154) has at least two positive solutions.

Theorem 11.21. *Let* $0 < a < Mb/2N < \min\{Mc/4N, Mc/\eta(\sigma(1)-\eta)\} = Mc/4N$, *and suppose that f and I satisfy the following conditions:*

(A) $f(w) > c/\eta(\sigma(1)-\eta)$, *if* $c \leq w \leq 2c$,
(B) $f(w) < b/2N$, *if* $0 \leq w \leq 2b$,
(C) $f(w) > a/M$, *if* $0 \leq w \leq a$,
(D) $I(w) \leq b/2$, *if* $0 \leq w \leq b$.

Then the impulsive dynamic boundary value problem (11.152)–(11.154) *has at least two positive solutions, x_1 and x_2 such that*

$$\begin{aligned} &a < \max_{t\in[\tau,r]} x_1(t),\ \textit{with}\ \max_{t\in[\tau,\eta]} x_1(t) < b, \\ &b < \max_{t\in[\tau,\eta]} x_2(t),\ \textit{with}\ \min_{t\in[\eta,r]} x_2(t) < c. \end{aligned} \tag{11.171}$$

Proof. We begin by defining the completely continuous integral operator $A : \mathcal{B} \to \mathcal{B}$ by

$$Ax(t) = I\big(x(\tau)\big)\chi_{(\tau,\sigma^2(1)]}(t) + \int_0^{\sigma(1)} G(t,s)f\big(x(\sigma(s))\big)\Delta s, \quad x \in \mathcal{B},\ t \in [0, \sigma^2(1)], \tag{11.172}$$

where $\chi_{(\tau,\sigma^2(1)]}(t)$ is the characteristic function. Solutions of (11.152)–(11.154) are fixed points of A and conversely. We now show that the conditions of Theorem 1.16 are satisfied.

Let $x \in \overline{\mathcal{P}(\gamma, c)}$. By the nonnegativity of I, f, and G, for $t \in [0, \sigma^2(1)]$, $Ax(t) \geq 0$. Moreover, $(Ax)^{\Delta\Delta}(t) = -f(x(\sigma(t))) \leq 0$ on $[0,1] \setminus \{\tau\}$, which implies that $(Ax)(t)$ is concave on each of $[0,\tau]$ and $[\tau, \sigma^2(1)]$. In addition,

$$(Ax)^{\Delta}(t) = \int_0^{\sigma(1)} G^{\Delta}(t,s) f\big(x(\sigma(s))\big)\Delta s \geq 0 \quad \text{on } [0,\sigma(1)] \setminus \{\tau\}, \tag{11.173}$$

so that $(Ax)(t)$ is nondecreasing on each of $[0,\tau]$ and $[\tau, \sigma^2(1)]$. Since $(Ax)(0) = 0$, we have $(Ax)(t) \geq 0$ on $[0,\tau]$. Also, since $x \in \overline{\mathcal{P}(\gamma, c)}$,

$$(Ax)(\tau^+) - (Ax)(\tau) = I\big(x(\tau)\big) \geq 0. \tag{11.174}$$

This yields $(Ax)(\tau^+) \geq (Ax)(\tau) \geq 0$, and consequently $(Ax)(t) \geq 0$, $t \in [\tau, \sigma^2(1)]$, as well. Ultimately, we have $Ax \in \mathcal{P}$, and in particular, $A : \overline{\mathcal{P}(\gamma, c)} \to \mathcal{P}$.

We now verify that property (i) of Theorem 1.16 is satisfied. We choose $x \in \partial\mathcal{P}(\gamma, c)$. Then $\gamma(x) = \min_{t\in[\eta,r]} x(t) = x(\eta) = c$. Since $x \in \mathcal{P}$, $x(t) \geq c, t \in [\eta, \sigma^2(1)]$. Recalling that $\|x\| \leq 2\gamma(x) = 2c$, we have

$$c \leq x(t) \leq 2c, \quad t \in [\eta, \sigma^2(1)]. \tag{11.175}$$

Then, by hypothesis (A),

$$f\big(x(\sigma(s))\big) > \frac{c}{\eta(\sigma(1) - \eta)}, \quad s \in [\eta, \sigma(1)]. \tag{11.176}$$

By the above, $Ax \in \mathcal{P}$, and so

$$\begin{aligned}
\gamma(Ax) &= (Ax)(\eta) = I\big(x(\tau)\big)\chi_{(\tau,\sigma^2(1)]_T}(\eta) + \int_0^{\sigma(1)} G(\eta,s) f\big(x(\sigma(s))\big)\Delta s \\
&= \int_0^{\sigma(1)} G(\eta,s) f\big(x(\sigma(s))\big)\Delta s \geq \int_\eta^{\sigma(1)} G(\eta,s) f\big(x(\sigma(s))\big)\Delta s \\
&= \eta \int_\eta^{\sigma(1)} f\big(x(\sigma(s))\big)\Delta s > \eta\left(\frac{c}{\eta(\sigma(1) - \eta)}\right)\int_\eta^{\sigma(1)} \Delta s \\
&= c.
\end{aligned} \tag{11.177}$$

We conclude that Theorem 1.16(i) is satisfied.

We now turn to Theorem 1.16(ii). We choose $x \in \partial\mathcal{P}(\theta, b)$. Then $\theta(x) = \max_{t\in[\tau,\eta]} x(t) = x(\eta) = b$. Thus, $0 \leq x(t) \leq b$, $t \in (\tau, \eta]$. Since $x \in \mathcal{P}$ implies that $x(\tau) \leq x(\tau^+)$, and also $x(t)$ is nondecreasing on $[0,\tau]$, we have

$$x(t) \leq b, \quad t \in [0,\tau], \tag{11.178}$$

and so by hypothesis (D),

$$I\big(x(\tau)\big) \leq \frac{b}{2}. \tag{11.179}$$

If we recall that $\|x\| \le 2\gamma(x) \le 2\theta(x) = 2b$, then we have

$$0 \le x(t) \le 2b, \quad t \in [0, \sigma^2(1)], \tag{11.180}$$

and by (B),

$$f(x(\sigma(s))) < \frac{b}{2N}, \quad s \in [0, \sigma(1)]. \tag{11.181}$$

Then

$$\begin{aligned}\theta(Ax) = (Ax)(\eta) &= I(x(\tau))\chi_{(\tau,\sigma^2(1)]_T}(\eta) + \int_0^{\sigma(1)} G(\eta,s)f(x(\sigma(s)))\Delta s \\ &\le \frac{b}{2} + \int_0^{\sigma(1)} \sigma(s)f(x(\sigma(s)))\Delta s < \frac{b}{2} + \frac{b}{2N}\int_0^{\sigma(1)} \sigma(s)\Delta s \\ &= b.\end{aligned} \tag{11.182}$$

In particular, Theorem 1.16(ii) holds.

We finally consider Theorem 1.16(iii). The function $y(t) = a/2 \in \mathcal{P}(\alpha, a)$, and so $\mathcal{P}(\alpha, a) \neq \varnothing$.

Now choose $x \in \partial\mathcal{P}(\alpha, a)$. Then $\alpha(x) = \max_{t\in[\tau,r]} x(t) = x(r) = a$. This implies $0 \le x(t) \le a$, $t \in [\tau, r]$. Since x is nondecreasing and $x(\tau^+) \ge x(\tau)$,

$$0 \le x(t) \le a, \quad t \in [0, r]. \tag{11.183}$$

By assumption (C),

$$f(x(\sigma(s))) > \frac{a}{M}, \quad s \in [0, \eta]. \tag{11.184}$$

Then

$$\begin{aligned}\alpha(Ax) = (Ax)(r) &= I(x(\tau))\chi_{(\tau,\sigma^2(1)]_T}(r) + \int_0^{\sigma(1)} G(r,s)f(x(\sigma(s)))\Delta s \\ &\ge \int_0^{\sigma(1)} G(r,s)f(x(\sigma(s)))\Delta s \ge \int_0^{\eta} G(r,s)f(x(\sigma(s)))\Delta s \\ &= \int_0^{\eta} \sigma(s)f(x(\sigma(s)))\Delta s > \left(\frac{a}{M}\right)\int_0^{\eta} \sigma(s)\Delta s \\ &= a.\end{aligned} \tag{11.185}$$

Thus Theorem 1.16(iii) is satisfied. Hence there exist at least two fixed points of A which are solutions x_1 and x_2, belonging to $\overline{\mathcal{P}(\gamma, c)}$, of the impulsive dynamic boundary value problem (11.152)–(11.154) such that

$$\begin{aligned} a < \alpha(x_1) &\quad \text{with } \theta(x_1) < b, \\ b < \theta(x_2) &\quad \text{with } \gamma(x_2) < c. \end{aligned} \tag{11.186}$$

The proof is complete. □

Example 11.22. Let $\mathbb{T}$ be a measure chain with $0, \tau, \eta, r, 1 \in \mathbb{T}$, where $0 < \tau < \eta < r < 1$ are fixed and $\eta = \inf[(\tau + \sigma^2(1))/2, 1)$. For $0 < a < Mb/2N < Mc/4N$, where $N = \int_0^{\sigma(1)} \sigma(s)\Delta s$ and $M = \int_0^{\eta} \sigma(s)\Delta s$, define $f : \mathbb{R} \to [0, \infty)$ and $I : [0, \infty) \to [0, \infty)$ by

$$f(w) = \begin{cases} \dfrac{Mb + 2Na}{4NM}, & w \leq 2b, \\ \ell(w), & 2b \leq w \leq c, \\ \dfrac{c+1}{\eta(\sigma(1) - \eta)}, & c \leq w, \end{cases} \tag{11.187}$$

$$I(w) = \begin{cases} \dfrac{b}{2}, & 0 \leq w \leq b, \\ w - \dfrac{b}{2}, & b \leq w, \end{cases}$$

where $\ell(w)$ satisfies $\ell'' = 0$, $\ell(2b) = (Mb + 2Na)/4NM$, and $\ell(c) = (c+1)/\eta(\sigma(1) - \eta)$. Then, by Theorem 11.21, the impulsive dynamic boundary value problem (11.152)–(11.154) has at least two solutions belonging to $\overline{\mathcal{P}(\gamma, c)}$.

11.8. Notes and remarks

The study of dynamic equations on time scales is a fairly new area in mathematics, having only been in practice for about 15 years. Still largely theoretical, time scales serve as a binding force between continuous and discrete analysis. The results of Section 11.3 are adapted from Benchohra et al. [72], the results of Section 11.4 from Benchohra et al. [1], the results of Section 11.5 from Benchohra et al. [74], while the results of Section 11.6 from Benchohra et al. [88], and finally the source of Section 11.7 from Henderson [165]. The techniques in this chapter have been adapted from [3, 7, 101], where the nonimpulsive case was discussed.

12 On periodic boundary value problems of first-order perturbed impulsive differential inclusions

12.1. Introduction

In this chapter, we study the existence of solutions to periodic nonlinear boundary value problems for first-order Carathéodory impulsive ordinary differential inclusions with convex multifunctions. Given a closed and bounded interval $J := [0, T]$ in $\mathbb{R}$, and given the impulsive moments $t_1, t_2, \dots, t_p$ with $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T$, $J' = J \setminus \{t_1, t_2, \dots, t_p\}$, $J_j = (t_j, t_{j+1})$, consider the following periodic boundary value problem for impulsive differential inclusions (IDI):

$$x'(t) \in F(t, x(t)) + G(t, x(t)) \quad \text{a.e. } t \in J', \tag{12.1}$$

$$x(t_j^+) = x(t_j^-) + I_j(x(t_j^-)), \tag{12.2}$$

$$x(0) = x(T), \tag{12.3}$$

where $F, G : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ are impulsive multifunctions, $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1, 2, \dots, p$, are the impulse functions, and $x(t_j^+)$ and $x(t_j^-)$ are, respectively, the right and the left limits of x at $t = t_j$.

Let $C(J, \mathbb{R})$ and $L^1(J, \mathbb{R})$ denote the space of continuous and Lebesgue integrable real-valued functions on J. Consider the Banach space

$$X := \{x : J \longrightarrow \mathbb{R} : x \in C(J', \mathbb{R}),\ x(t_j^+),\ x(t_j^-) \text{ exist},\ x(t_j^-) = x(t_j),\ j = 1, 2, \dots, p\}, \tag{12.4}$$

equipped with the norm $\|x\| = \max\{|x(t)| : t \in J\}$, and the space

$$Y := \{x \in X : x \text{ is differentiable a.e. on } (0, T),\ x' \in L^1(J, \mathbb{R})\}. \tag{12.5}$$

By a solution of (12.1)–(12.3), we mean a function x in $Y_T := \{v \in Y : v(0) = v(T)\}$ that satisfies the differential inclusion (12.1) and the impulsive conditions (12.2).

Our aim is to provide sufficient conditions to the multifunctions F, G and the impulsive functions I_j that insure the existence of solutions of problem IDI (12.1)–(12.3).

The following form of a fixed point theorem of Dhage [127] will be used while proving our main existence result.

Theorem 12.1. *Let $B(0,r)$ and $B[0,r]$ denote, respectively, the open and closed balls in a Banach space E centered at the origin and of radius r, and let $A : E \to \mathcal{P}_{\mathrm{cl,cv},bd}(E)$ and $B : B[0,r] \to \mathcal{P}_{\mathrm{cp,cv}}(E)$ be two multivalued operators satisfying that*
 (i) *A is a multivalued contraction,*
 (ii) *B is completely continuous.*
Then either
 (a) *the operator inclusion $x \in Ax + Bx$ has a solution in $B[0,r]$, or*
 (b) *there exists a $u \in E$ with $\|u\| = r$ such that $\lambda u \in Au + Bu$ for some $\lambda > 1$.*

12.2. Existence results

Consider the linear periodic problem with some given impulses, $\theta_j \in \mathbb{R}$, $j = 1,2,\dots,p$,

$$\begin{aligned} x'(t) + kx(t) &= \sigma(t), \quad \text{a.e. } t \in J', \\ x(t_j^+) - x(t_j^-) &= \theta_j, \quad j = 1,2,\dots,p, \\ x(0) &= x(T), \end{aligned} \tag{12.6}$$

where $k > 0$, and $\sigma \in L^1(J)$. The solution of (12.6) is given by (see [199, Lemma 2.1])

$$x(t) = \int_0^T g_k(t,s)\sigma(s)ds + \sum_{j=1}^{p} g_k(t,t_j)\theta_j, \tag{12.7}$$

where

$$g_k(t,s) = \begin{cases} \dfrac{e^{-k(t-s)}}{1-e^{-kT}}, & 0 \le s \le t \le T, \\ \dfrac{e^{-k(T+t-s)}}{1-e^{-kT}}, & 0 \le t < s \le T. \end{cases} \tag{12.8}$$

Clearly the function $g_k(t,s)$ is discontinuous and nonnegative on $J \times J$ and has a jump at $t = s$.

Let

$$M_k := \max\{|g_k(t,s)| : t,s \in [0,T]\} = \frac{1}{1-e^{-kT}}. \tag{12.9}$$

Now $x \in Y_T$ is a solution of (12.1)–(12.3) if and only if

$$x(t) \in B_k^1 x(t) + B_k^2 x(t), \quad t \in J, \tag{12.10}$$

where the multivalued operators B_k^1 and B_k^2 are defined by

$$\mathcal{B}_k^1 x(t) = \int_0^T g_k(t,s)F(s,x(s))ds, \tag{12.11}$$

$$\mathcal{B}_k^2 x(t) = \int_0^T g_k(t,s)[kx(s)+G(s,x(s))]ds + \sum_{j=1}^{p} g(t,t_j)I_j(x(t_j^-)). \tag{12.12}$$

Definition 12.2. A multifunction $\beta : J\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is called impulsive Carathéodory if

(i) $\beta(\cdot,x)$ is measurable for every $x\in\mathbb{R}$,

(ii) $\beta(t,\cdot)$ is upper semicontinuous a.e. on J.

Further the impulsive Carathéodory multifunction β is called impulsive L^1-Carathéodory if

(iii) for every $r>0$, there exists a function $h_r\in L^1(J)$ such that

$$||\beta(t,x)|| = \sup\{|u| : u\in\beta(t,x)\} \le h_r(t) \quad \text{a.e. } t\in J, \tag{12.13}$$

for all $x\in\mathbb{R}$ with $|x|\le r$.

Denote

$$S_\beta^1(x) = \{v\in L^1(J,\mathbb{R}) : v(t)\in\beta(t,x) \text{ a.e. } t\in J\}. \tag{12.14}$$

It is known (see Lasota and Opial [186]) that if E is a Banach space with $\dim(E)<\infty$ and $\beta : J\times E\to\mathcal{P}_{b,\mathrm{cl}}(E)$ is L^1-Carathéodory, then $S_\beta^1(x)\neq\emptyset$ for each $x\in E$.

Definition 12.3. A measurable multivalued function $F : J\to\mathcal{P}_{\mathrm{cp}}(\mathbb{R})$ is said to be integrably bounded if there exists a function $h\in L^1(J,\mathbb{R})$ such that $|v|\le h(t)$ a.e. $t\in J$ for all $v\in F(t)$.

Remark 12.4. It is known that if $F : J\to\mathbb{R}$ is an integrably bounded multifunction, then the set S_F^1 of all Lebesgue integrable selections of F is closed and nonempty, see Covitz and Nadler [123].

We now introduce some assumptions.

(H1) The functions $I_j : \mathbb{R}\to\mathbb{R}$, $j=1,2,\ldots,p$, are continuous, and there exist $c_j\in\mathbb{R}$, $j=1,2,\ldots,p$, such that $|I_j(x)|\le c_j$, $j=1,2,\ldots,p$, for every $x\in\mathbb{R}$.

(H2) $G : J\times\mathbb{R}\to\mathcal{P}_{\mathrm{cp,cv}}(\mathbb{R})$ is an impulsive Carathéodory multifunction.

(H3) There exist a real number $k>0$ and a Carathéodory function $\omega : J\times\mathbb{R}_+\to\mathbb{R}_+$ which is nondecreasing with respect to its second argument such that

$$\|G(t,x)+kx\| = \sup\{|v| : v \in G(t,x)+kx\} \leq \omega(t,|x|) \tag{12.15}$$

a.e. $t \in J'$, $x \in \mathbb{R}$.

(H4) The multifunction $t \mapsto F(t,x)$ is measurable and integrally bounded for each $x \in \mathbb{R}$.

(H5) The multifunction $F(t,x)$ is $F : J \times \mathbb{R} \to \mathcal{P}_{\mathrm{cl,cv},bd}(\mathbb{R})$, and there exists a function $\ell \in L^1(J,\mathbb{R})$ such that

$$H(F(t,x),F(t,y)) \leq \ell(t)|x-y| \quad \text{a.e. } t \in J, \tag{12.16}$$

for all $x, y \in \mathbb{R}$.

Lemma 12.5. *Assume that (H2)-(H3) hold. Then the operator $S^1_{k+G} : Y_T \to \mathcal{P}(L^1(J,\mathbb{R}))$ defined by*

$$S^1_{k+G}(x) := \{v \in L^1(J,\mathbb{R}) : v(t) \in kx(t)+G(t,x(t)) \text{ a.e. } t \in J\} \tag{12.17}$$

is well defined, u.s.c., closed and convex-valued, and sends bounded subsets of Y_T into bounded subsets of $L^1(J,\mathbb{R})$.

Proof. Since (H2) holds, $S^1_{k+G}(x) \neq \emptyset$ for each $x \in Y_T$. Below, we show that S^1_{k+G} has the desired properties on Y_T.

Step 1. First we show that S^1_{k+G} has closed values on Y_T. Let $x \in Y_T$ be arbitrary and let $\{\omega_n\}$ be a sequence in $S^1_{k+G}(x) \subset L^1(J,\mathbb{R})$ such that $\omega_n \to \omega$. Then $\omega_n \to \omega$ in measure. So there exists a subset S of positive integers such that $\omega_n \to \omega$ a.e. $n \to \infty$ through S. Since the hypothesis (H2) holds, we have $\omega \in S^1_{k+G}(x)$. Therefore $S^1_{k+G}(x)$ is a closed set in $L^1(J,\mathbb{R})$. Thus, for each $x \in Y_T$, $S^1_{k+G}(x)$ is a nonempty, closed subset of $L^1(J,\mathbb{R})$, and consequently S^1_{k+G} has nonempty and closed values on Y_T.

Step 2. Next we show that $S^1_{k+G}(x)$ is a convex subset of $L^1(J,\mathbb{R})$ for each $x \in Y_T$. Let $v_1, v_2 \in S^1_{k+G}(x)$ and let $\lambda \in [0,1]$. Then there exist functions $f_1, f_2 \in S^1_{k+G}(x)$ such that

$$v_1(t) = kx(t)+f_1(t), \qquad v_2(t) = kx(t)+f_2(t) \tag{12.18}$$

for $t \in J$. Therefore we have

$$\begin{aligned}\lambda v_1(t)+(1-\lambda)v_2(t) &= \lambda[kx(t)+f_1(t)]+(1-\lambda)[kx(t)+f_2(t)]\\ &= \lambda kx(t)+(1-\lambda)kx(t)+\lambda f_1(t)+(1-\lambda)f_2(t)\\ &= kx(t)+f_3(t),\end{aligned} \tag{12.19}$$

where $f_3(t) = \lambda f_1(t)+(1-\lambda)f_2(t)$ for all $t \in J$. Since $G(t,x)$ is convex for each

$x \in \mathbb{R}$, one has $f_3(t) \in G(t, x(t))$ for all $t \in J$. Therefore

$$\lambda v_1(t) + (1-\lambda)v_2(t) \in kx(t) + G(t, x(t)) \tag{12.20}$$

for all $t \in J$, and consequently $\lambda v_1 + (1-\lambda)v_2 \in S^1_{k+G}(x)$. As a result, $S^1_{k+G}(x)$ is a convex subset of $L^1(J, \mathbb{R})$.

Step 3. Next we show that S^1_{k+G} is a u.s.c. multivalued operator on Y_T. Let $\{x_n\}$ be a sequence in Y_T such that $x_n \to x_*$, and let $\{y_n\}$ be a sequence such that $y_n \in S^1_{k+G}(x_n)$ and $y_n \to y_*$. To finish, it suffices to show that $y_* \in S^1_{k+G}(x_*)$. Since $y_n \in S^1_{k+G}(x_n)$, there is a function $f_n \in S^1_{k+G}(x_n)$ such that $y_n(t) = kx_n(t) + f_n(t)$ for all $t \in J$ and that $y_*(t) = kx_*(t) + f_*(t)$, where $f_n \to f_*$ as $n \to \infty$. Now the multifunction $G(t,x)$ is upper semicontinuous in x for all $t \in J$, and one has $f_*(t) \in G(t, x_*(t))$ for all $t \in J$. Hence it follows that $y_* \in S^1_{k+G}(x_*)$.

Step 4. Finally we show that S^1_{k+G} maps bounded sets of Y_T into bounded sets of $L^1(J,\mathbb{R})$. Let M be a bounded subset of Y_T. Then there is a real number $r > 0$ such that $\|x\| \le r$ for all $x \in M$. Let $y \in S^1_{k+G}(S)$ be arbitrary. Then there is an $x \in M$ such that $y \in S^1_{k+G}(x)$, and therefore $y(t) \in kx(t) + G(t, x(t))$ a.e. $t \in J$. Now, by (H3),

$$\begin{aligned} \|y\|_{L^1} &= \int_0^T |y(t)|\,dt \le \int_0^T \|kx(t) + G(t,x(t))\|dt \\ &\le \int_0^T \omega(t, |x(t)|)dt \le \int_0^T \omega(t,r)dt. \end{aligned} \tag{12.21}$$

Hence $S^1_{k+G}(S)$ is a bounded set in $L^1(J,\mathbb{R})$.

Thus the multivalued operator S^1_{k+G} is upper semicontinuous and has closed, convex values on Y_T. The proof is complete. □

Lemma 12.6. *Assume that* (H_1)–(H_3) *hold. The multivalued operator* $\mathcal{B}^2_k$ *defined by* (12.12) *is completely continuous and has convex, compact values on* Y_T.

Proof. Since S^1_{k+G} is upper semicontinuous and has closed and convex values and since (H1) holds, $\mathcal{B}^2_k$ is u.s.c. and has closed convex values on Y_T. To show that $\mathcal{B}^2_k$ is relatively compact, we use the Arzelá-Ascoli theorem. Let $M \subset B[0,r]$ be any set. Then $\|x\| \le r$ for all $x \in M$. First, we show that $\mathcal{B}^2_k(M)$ is uniformly bounded. Now, for any $x \in M$ and for any $y \in \mathcal{B}^2_k(x)$, one has

$$\begin{aligned} |y(t)| &\le \int_0^T |g_k(t,s)|\,\|[kx(s) + G(s,x(s))]\|ds + \sum_{j=1}^p |g_k(t,t_j)|\,|I_j(x(t_j^-))| \\ &\le \int_0^T M_k\omega(s, |x(s)|)ds + M_k\sum_{j=1}^p c_j \\ &\le M_k\int_0^T \omega(s,r)ds + M_k\sum_{j=1}^p c_j, \end{aligned} \tag{12.22}$$

where M_k is the bound of g_k on $[0,T]\times[0,T]$. Taking the supremum over t,

$$||\mathcal{B}_k^2 x|| \le M_k\left[\int_0^T \omega(s,r)ds + \sum_{j=1}^{p} c_j\right] \tag{12.23}$$

for all $x \in M$. Hence $\mathcal{B}_k^2(M)$ is a uniformly bounded set in Y_T. Next we prove the equicontinuity of the set $\mathcal{B}_k^2(M)$ in Y_T. Let $y \in B_k^2(M)$ be arbitrary. Then there is a $v \in S_{k+G}(x)$ such that

$$y(t) = \int_0^T g_k(t,s)v(s)ds + \sum_{j=1}^{p} g_k(t,t_j)I_j(x(t_j^-)), \quad t \in J, \tag{12.24}$$

for some $x \in M$.

To finish, it is sufficient to show that y' is bounded on $[0,T]$. Now, for any $t \in [0,T]$,

$$\begin{aligned}
|y'(t)| &\le \left| \int_0^T \frac{\partial}{\partial t} g_k(t,s)v(s)ds + \sum_{j=1}^{p} \frac{\partial}{\partial t} g_k(t,t_k)I_j(y_j(t_j^-)) \right| \\
&= \left| \int_0^T (-k)g_k(t,s)v(s)ds + \sum_{j=1}^{p}(-k)g_k(t,t_k)I_j(y_j(t_j^-)) \right| \\
&\le kM_k \int_0^T \omega(s,r)ds + kM_k \sum_{j=1}^{p} c_j := c.
\end{aligned} \tag{12.25}$$

Hence, for any $t,\tau \in [0,T]$ and for all $y \in B_k^2(M)$, one has

$$|y(t) - y(\tau)| \le c|t-\tau| \longrightarrow 0 \quad \text{as } t \longrightarrow \tau. \tag{12.26}$$

This shows that $\mathcal{B}_k^2(M)$ is an equicontinuous set and consequently relatively compact in view of Arzelá-Ascoli theorem. Obviously, $\mathcal{B}_k^2(x) \subset \mathcal{B}_k^2(B[0,r])$ for each $x \in B[0,r]$. Since $\mathcal{B}_k^2(B[0,r])$ is relatively compact, $\mathcal{B}_k^2(x)$ is relatively compact, and hence is compact in view of hypothesis (H2). Hence $\mathcal{B}_k^2$ is a completely continuous multivalued operator on Y_T. The proof of the lemma is complete. □

Lemma 12.7. *Assume that the hypotheses (H4)-(H5) hold. Then the operator B_k^1 defined by* (12.11) *is a multivalued contraction operator on Y_T, provided that M_k* $\|\ell\|_{L^1} < 1$.

Proof. Define a mapping $\mathcal{B}_k^1 : Y_T \to Y_T$ by (12.11). We show that $\mathcal{B}_k^1$ is a multivalued contraction on Y_T. Let $x,y \in Y_T$ be arbitrary and let $u_1 \in \mathcal{B}_k^1(x)$. Then $u_1 \in Y_T$ and

$$u_1(t) = \int_0^T g_k(t,s)v_1(s)ds, \tag{12.27}$$

for some $v_1 \in S_F^1(x)$. Since $H(F(t,x(t)),F(t,y(t))) \leq \ell(t)|x(t)-y(t)|$, one obtains that there exists a $w \in F(t,y(t))$ such that

$$\left|v_1(t)-w\right| \leq \ell(t)\left|x(t)-y(t)\right|. \tag{12.28}$$

Thus the multivalued operator U defined by $U(t) = S_F^1(y)(t) \cap K(t)$, where

$$K(t) = \{w \mid \left|v_1(t)-w\right| \leq \ell(t)\left|x(t)-y(t)\right|\}, \tag{12.29}$$

has nonempty values and is measurable. Let v_2 be a measurable selection for U (which exists by Kuratowski-Ryll-Nardzewski's selection theorem [2]). Then $v_2 \in F(t,y(t))$ and

$$\left|v_1(t)-v_2(t)\right| \leq \ell(t)\left|x(t)-y(t)\right| \quad \text{a.e. } t \in J. \tag{12.30}$$

Define

$$u_2(t) = \int_0^T g_k(t,s)v_2(s)ds. \tag{12.31}$$

It follows that $u_2 \in \mathcal{B}_k^1(y)$ and

$$\begin{aligned}
\left|u_1(t)-u_2(t)\right| &\leq \left|\int_0^T g_k(t,s)v_1(s)ds - \int_0^T g_k(t,s)v_2(s)ds\right| \\
&\leq \int_0^T M_k\left|v_1(s)-v_2(s)\right|ds \\
&\leq \int_0^T M_k\ell(s)\left|x(s)-y(s)\right|ds \\
&\leq M_k\|\ell\|_{L^1}\|x-y\|.
\end{aligned} \tag{12.32}$$

Taking the supremum over t, we obtain

$$\left\|u_1-u_2\right\| \leq M_k\|\ell\|_{L^1}\|x-y\|. \tag{12.33}$$

By this and the analogous inequality obtained by interchanging the roles of x and y, we get that

$$H\left(\mathcal{B}_k^1(x),\mathcal{B}_k^1(y)\right) \leq \mu\|x-y\|, \tag{12.34}$$

for all $x, y \in Y_T$. This shows that $\mathcal{B}_k^1$ is a multivalued contraction, since $\mu = M_k\|\ell\|_{L^1} < 1$. □

Theorem 12.8. *Assume that (H1)–(H5) are satisfied. Further if there exists a real number $r > 0$ such that*

$$r > \frac{M_k \int_0^T \omega(s,r)ds + M_k F_0 + M_k \sum_{j=1}^p c_j}{1 - M_k \|\ell\|_{L^1}}, \tag{12.35}$$

where $M_k\|\ell\|_{L^1} < 1$ and $F_0 = \int_0^T \|F(s,0)\|ds$, then the problem IDI (12.1)–(12.3) *has at least one solution on J.*

Proof. Define an open ball $B(0,r)$ in Y_T, where the real number r satisfies the inequality given in condition (12.35). Define the multivalued operators $\mathcal{B}_k^1$ and $\mathcal{B}_k^2$ on Y_T by (12.11) and (12.12). We will show that the operators $\mathcal{B}_k^1$ and $\mathcal{B}_k^2$ satisfy all the conditions of Theorem 12.1.

Step 1. The assumptions (H2)-(H3) imply by Lemma 12.6 that $\mathcal{B}_k^2$ is a completely continuous multivalued operator on $B[0,r]$. Again since (H4)-(H5) hold, by Lemma 12.7, $\mathcal{B}_k^1$ is a multivalued contraction on Y_T with a contraction constant $\mu = M_k\|\ell\|_{L^1}$. Now an application of Theorem 12.1 yields that either the operator inclusion $x \in \mathcal{B}_k^1 x + \mathcal{B}_k^2 x$ has a solution in $B[0,r]$, or there exists a $u \in Y_T$ with $\|u\| = r$ satisfying that $\lambda u \in B_k^1 u + B_k^2 u$ for some $\lambda > 1$.

Step 2. Now we show that the second assertion of Theorem 12.1 is not true. Let $u \in Y_T$ be a possible solution of $\lambda u \in B_k^1 u + B_k^2 u$ for some real number $\lambda > 1$ with $\|u\| = r$. Then we have

$$\begin{aligned} u(t) \in{}& \lambda^{-1}\int_0^T g_k(t,s)F(s,u(s))ds + \lambda^{-1}\int_0^T g_k(t,s)[ku(s) + G(s,u(s))]ds \\ &+ \lambda^{-1}\sum_{j=1}^p g_k(t,t_j)I_j(u(t_j^-)). \end{aligned} \tag{12.36}$$

Hence, by (H3)–(H5),

$$\begin{aligned} |u(t)| \le{}& \int_0^T |g_k(t,s)|\,\omega(s,|u(s)|)ds + \int_0^T |g_k(t,s)|\,|\ell(s)|\,|u(s)|ds \\ &+ \int_0^T |g_k(t,s)|\,\|F(s,0)\|ds + \sum_{j=1}^p |g_k(t,s)|\,|I_j(u(t_j^-))| \\ \le{}& M_k\int_0^T \omega(s,\|u\|)ds + M_k\int_0^T |\ell(s)|\,\|u\|ds + M_kF_0 + M_k\sum_{j=1}^p c_j \\ \le{}& M_k\int_0^T \omega(s,\|u\|)ds + M_k\|\ell\|_{L^1}\|u\| + M_kF_0 + M_k\sum_{j=1}^p c_j. \end{aligned} \tag{12.37}$$

Taking the supremum over t, we get

$$\|u\| \leq M_k \int_0^T \omega(s, \|u\|)\,ds + M_k\|\ell\|_{L^1}\|u\| + M_kF_0 + M_k\sum_{j=1}^{p} c_j. \tag{12.38}$$

Substituting $\|u\| = r$ in the above inequality yields

$$r \leq \frac{M_k \int_0^T \omega(s,r)\,ds + M_kF_0 + M_k\sum_{j=1}^{p} c_j}{1 - M_k\|\ell\|_{L^1}}, \tag{12.39}$$

which is a contradiction to (12.35). Hence the operator inclusion $x \in \mathcal{B}_k^1 x + \mathcal{B}_k^2 x$ has a solution in $B[0,r]$. This further implies that the IDI (12.1)–(12.3) has a solution on J. The proof is complete. $\square$

12.3. Notes and remarks

The results of Chapter 12 are adapted from [128].

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