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Research Article

Nonlinear Stability of Oblate Infinitesimal in Elliptic Restricted Three-Body Problem Influenced by the Oblate and Radiating Primaries

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This work deals with the nonlinear stability of the elliptical restricted three-body problem with oblate and radiating primaries and the oblate infinitesimal. The stability has been analyzed for the resonance cases around $\omega_1 = 2\omega_2$ and $\omega_1 = 3\omega_2$ and also the nonresonance cases. It was observed that the motion of the infinitesimal in this system shows instable behavior when considered in the third order resonance. However, for the fourth order resonance the stability is shown for some mass parameters. The motion in the case of nonresonance was found to be unstable. The problem has been numerically applied to study the movement of the infinitesimal around two binary systems, Luyten-726 and Sirius.

1. Introduction

The study of equilibrium points and their stability in restricted three-body problems has attracted the attention of many researchers in the past century, as the stable and unstable resonant motions explain many of the celestial phenomena. The nonlinear stability in circular and elliptical restricted three-body problem was studied in detail by many authors. Markeev [1] employed numerical and analytic methods to study the stability of equilibrium points and periodic motions of nonlinear Hamiltonian systems in cases of resonance. Gyorgyey [2] studied the nonlinear stability of motions around the triangular equilibrium point L_5. The work was further elaborately studied by various authors ([3–6], et al.) taking into account various other perturbing forces. Ferraz-Mello [7] used the averaging of the elliptic asteroidal problem to study the first order resonance. Henrard & Caranicolas [8] and Henrard [9] used the perturbation method to study the resonance. Further [10–16] and many others extended the work and explored various aspects of the problem.

In order to investigate the stability of the triangular liberation points the Hamiltonian is simplified by applying Birkhoff's transformation. The normalization method adopted is outlined as follows:

(i) The quadratic form H_2 should be reduced so that it corresponds to the normal oscillations modes. This transformation is performed by means of real, linear, and canonical changes of variables.

(ii) After the quadratic part H_2 has been reduced to normal form, a nonlinear 2π periodic Birkhoff transformation is required to suppress the third-degree term H_3.

(iii) The final step is obtaining a Hamiltonian function normalized to fourth order terms obtained by simplifying H_4 by means of a canonical Birkhoff transformation.

If H_2 is a function of definite sign, then by the virtue of Liapunov's theorem the equilibrium is stable. Otherwise, if H_2 is not a function of definite sign, then the stability is investigated by means of Arnold's theorem given by the following.

Let the Hamiltonian satisfy the three conditions:

(1) The characteristic equation of the linearized system has pure imaginary roots are $\pm\iota\omega_1, \pm\iota\omega_2$.

(2) The frequencies ω_1, ω_2 satisfy the inequalities $k_1\omega_1 + k_2\omega_2 \neq 0$ for $0 < |k_1| + |k_2| \leq 4$, where k_1 and k_2 are integers.

(3) The inequality $C_{20}\omega_1^2 + C_{11}\omega_1\omega_2 + C_{02}\omega_1^2 \neq 0$ is fulfilled.

If the above three conditions are satisfied, then the equilibrium points are stable.

The above-mentioned methodology has been used to investigate the nonlinear stability of the elliptic restricted three-body problem with bigger and smaller primaries and infinitesimal as oblate spheroid and also both the primaries as source of radiation. The paper is divided into following sections. Section 1 gives general introduction. The equations of motion are presented in Section 2, and also the triangular equilibrium points are obtained. Existence of resonance in circular case is briefly discussed in Section 3. The normalization of the Hamiltonian is done in Section 4. The second order terms are normalized retaining the third and fourth order terms by using a linear canonical transformation of variables. The stability in third and fourth order resonances is analyzed in Sections 5 and 6, respectively. The stability in nonresonance case is dealt in Section 7 of this paper. The stability of the system has been analyzed using the KAM theorem. The equations used in the intermediate calculation in the sections are given in the Appendix. Numerical applications are presented in Section 8. The discussion and conclusion are drawn in Section 9.

2. Equation of Motion and Existence of Triangular Points

The differential equation governing the motion of the oblate infinitesimal mass under the radiation and oblateness of the primaries is represented as follows [17]:

$$x'' - 2y' = \phi(\mathrm{e}, \mathrm{f}) \frac{\partial U}{\partial x}$$
$$y'' + 2x' = \phi(\mathrm{e}, \mathrm{f}) \frac{\partial U}{\partial y} \quad (1)$$

where

$$\mathrm{U} = \frac{x^2 + y^2}{2} + \frac{1}{n^2} \left\{ q_1 (1-\mu) \left(\frac{1}{r_1} + \frac{A_1}{2r_1^3} \right) + q_2\mu \left(\frac{1}{r_2} + \frac{A_1}{2r_2^3} \right) + \frac{A_3}{2} \left(\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3} \right) \right\} \quad (2)$$

$$\mathrm{r}_1^2 = (x+\mu)^2 + y^2$$
$$\mathrm{r}_2^2 = (x+\mu-1)^2 + y^2 \quad (3)$$

$$\phi(\mathrm{e}, \mathrm{f}) = \frac{1}{1 + e\cos f} \quad (4)$$

$$n^2 = \frac{1}{a^3} \left(1 + \frac{3}{2} \left(A_1 + A_2 + e^2 \right) \right) \quad (5)$$

Here prime ($'$) denotes the differentiation with respect to the true anomaly f. U_x and U_y denote the partial differentiation of U with respect to x and y, respectively. a and e represent the semimajor axis and eccentricity of the elliptic path followed by the two primaries. A_1, A_2, and A_3 are the oblateness parameter of the primaries and infinitesimal, respectively. q_1, q_2 are the radiation factors of the primaries, respectively.

The coordinates of the triangular equilibrium points (u, v) in linear terms of the perturbing forces are given as follows:

$$u = \frac{1}{2} - \mu + \left(\frac{1}{2} - 2\delta \right) A_2 - A_3 + \left(\frac{-1}{3} + \frac{2\delta}{3} \right) \beta_1 + \left(\frac{1}{3} - \frac{2\delta}{3} \right) \beta_2$$
$$v = \frac{\sqrt{3}}{2} (1 - \frac{5}{3}\delta + \frac{2}{3}e^2 (-1 + 2\delta) - \frac{A_1}{3} + \frac{A_2}{3} + \left(-\frac{1}{3} + \frac{10}{9}\delta \right) \beta_1 + \left(-\frac{2}{9} + \frac{4}{9}\delta \right) \beta_2 \quad (6)$$

Here $\delta = 1 - a$, $\beta_i = 1 - q_i$, $i = 1, 2$.

The Lagrangian equation of motion of the problem is written as follows:

$$L = \frac{1}{2} \left(\dot{x}^2 + \dot{y}^2 \right) + \dot{y}x - \dot{x}y + \frac{1}{1 + e\cos f} \left[(1-\mu) \cdot \left\{ \frac{r_1^2}{2} + \frac{1}{n^2} \left(\frac{q_1}{r_1} + \frac{q_1 A_1 + A_3}{2r_1^3} \right) \right\} + \mu \left\{ \frac{r_2^2}{2} + \frac{1}{n^2} \left(\frac{q_2}{r_2} + \frac{q_2 A_2 + A_3}{2r_2^3} \right) \right\} \right] \quad (7)$$

Hence, the perturbed Hamiltonian function of the problem is given by

$$H = -\frac{{p_x}^2 + {p_y}^2}{2} - p_y x + p_x y + \frac{x^2 + y^2}{2} - \frac{1}{1 + e\cos f} \left[(1-\mu) \right.$$

$$\cdot\left\{\frac{r_1^2}{2}+\frac{1}{n^2}\left(\frac{q_1}{r_1}+\frac{q_1A_1+A_3}{2r_1^3}\right)\right\} + \mu\left\{\frac{r_2^2}{2}+\frac{1}{n^2}\left(\frac{q_2}{r_2}+\frac{q_2A_2+A_3}{2r_2^3}\right)\right\}\Bigg] \tag{8}$$

where p_x and p_y are the generalized components of momentum. The nature of motion near the two equilibrium points will be the same as the two triangular equilibrium solutions are symmetrical to each other. Hence, we consider the motion near the equilibrium point L_4 for further calculations. To study the stability near this equilibrium point, we shift the origin to L_4 by the change of variables given by

$$\begin{aligned} x &= u + q_1; \\ y &= v + q_2; \\ p_x &= p_u + p_1 \\ \text{and } p_y &= p_v + p_2 \end{aligned} \tag{9}$$

where (u, v) denotes the triangular equilibrium point L_4 and

$$\begin{aligned} p_u &= -v, \\ p_v &= u \end{aligned} \tag{10}$$

3. Characteristic Roots and Existence of Resonance in Circular Case

Restricting the Hamiltonian to H_2 alone, the characteristic equation is obtained as [18]

$$\lambda^4 + (4 - A^* - C^*)\lambda^2 + A^*C^* - B^{*2} = 0, \tag{11}$$

where

$$\begin{aligned} A^* &= \frac{3}{4} - 2\Bigg(\frac{7e^2}{8}(1-2\mu) + \frac{\beta_1}{4}(1-3\mu) \\ &\quad - \frac{\beta_2}{4}(2-3\mu) - \frac{3A_1}{4}\left(1-\frac{7}{4}\mu\right) + \frac{3A_2}{4}\left(1-\frac{7}{4}\mu\right) \\ &\quad + \frac{\delta}{32} - 140 + \frac{9A_3}{8}\left(1-\frac{7}{3}\mu\right)\Bigg), \\ B^* &= \frac{3\sqrt{3}}{4}(1-2\mu) - \sqrt{3}\Bigg[-\frac{11}{4}e^2\left(1-\frac{20}{11}\mu\right) \\ &\quad + \frac{\beta_1}{6}(1+\mu) - \frac{\beta_2}{6}(2-\mu) - \frac{5\delta}{3}(1-2\mu) \\ &\quad - \frac{A_1}{2}\left(7-\frac{59}{4}\mu\right) - A_2\left(2-\frac{29}{8}\mu\right) \\ &\quad - \frac{9A_3}{4}\left(1-\frac{5}{3}\mu\right) - \frac{29\delta}{16}\left(1-\frac{38}{29}\mu\right)\Bigg] \\ C^* &= \frac{9}{4} + 2\Bigg(\frac{e^2}{8}(23-22\mu) + \frac{\beta_1}{4}(1-3\mu) \\ &\quad - \frac{\beta_2}{4}(2-3\mu) + \frac{3A_1}{4}\left(3+\frac{11}{4}\mu\right) \\ &\quad + \frac{3A_2}{4}\left(3-\frac{11}{4}\mu\right) - \frac{\delta}{32}(95-220\mu) \\ &\quad - \frac{33A_3}{8}(1-\mu)\Bigg). \end{aligned} \tag{12}$$

Assume the frequencies ω_1 and ω_2, are given by the relation $\omega_1^2 = -\{\lambda_{1,2}^{(0)}\}^2$ and $\omega_2^2 = -\{\lambda_{3,4}^{(0)}\}^2$, where $\lambda_{1,2,3,4}^{(0)}$ are the roots of the characteristic equation (11), when $e = 0$. The values are obtained as

$$\begin{aligned} (\omega_{1,2})^2 &= \frac{1}{2}\Bigg[1 \pm \left\{1 - 27\mu(1-\mu)\left(1+\frac{2}{9}\beta_1+\frac{2}{9}\beta_2+\frac{94}{9}\delta+\frac{119}{6}A_1+\frac{61}{6}A_2+17A_3\right)\right\}^{1/2} \\ &\quad \times \left(1-\frac{13}{4}\delta-6A_1-3A_2-6A_3\right)\Bigg]. \end{aligned} \tag{13}$$

Figures 1–3 show the correlation between ω and μ for varying values of the oblateness of the infinitesimal. For the figures, the following values of the perturbing factors are taken: $\beta_1 = \beta_2 = 0.01$, $A_1 = A_2 = 0.0001$, and $\delta = 0.001$.

In order to discuss the existence of resonance, firstly we consider the case when $\omega_1 = \omega_2$. Solving (13) for the case, we obtain

$$1 - 27\mu(1-\mu)\left(1+\frac{2}{9}\beta_1+\frac{2}{9}\beta_2+\frac{94}{9}\delta+\frac{119}{6}A_1 + \frac{61}{6}A_2+17A_3\right) = 0 \tag{14}$$

Solving the above equation for value of $\mu < 1/2$, we get that the region of stability defined by first approximation is

$$0 < \mu < \frac{1}{2} - \frac{\sqrt{69}}{18}\left(1+\frac{4}{9}\beta_1+\frac{4}{9}\beta_2+\frac{188}{9}\delta+\frac{119}{3}A_1 + \frac{61}{3}A_2+34A_3\right) \tag{15}$$

Thus, the value of μ admisible for stable equilibrium point for the case $\omega_1 = \omega_2$, when $e = 0$, is given as

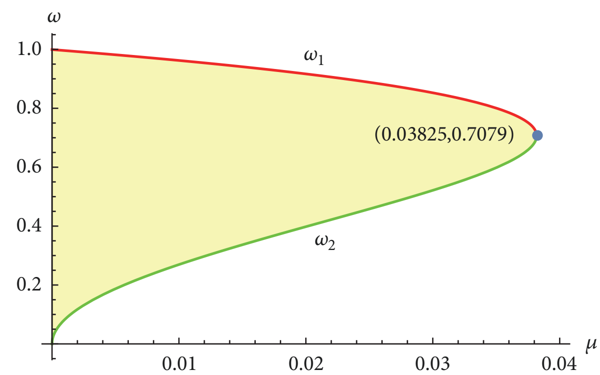

Figure 1: Correlation between frequency and mass ratio for $A_3 = 0$.

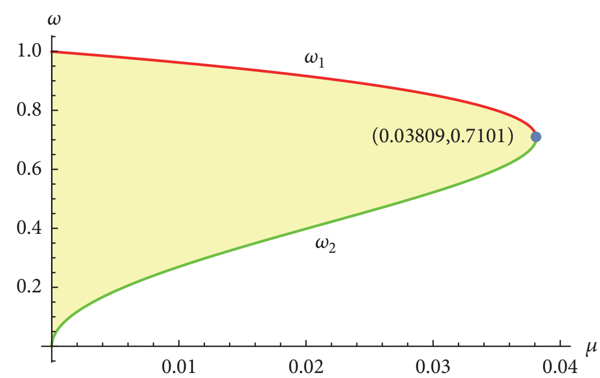

Figure 2: Correlation between frequency and mass ratio for $A_3 = 0.0001$.

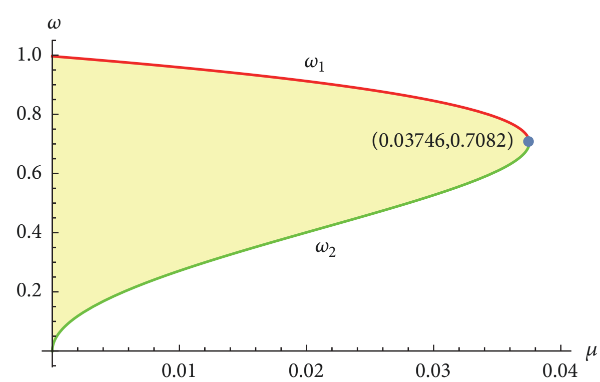

Figure 3: Correlation between frequency and mass ratio for $A_3 = 0.01$.

$$\begin{aligned}\mu^{(0)} &= 0.0385209 - 0.4191210\delta - 0.795884A_1 \\ &\quad - 0.407974A_2 - 0.682187A_3 - 0.00891747\beta_1 \\ &\quad - 0.00891747\beta_2\end{aligned} \tag{16}$$

Following the similar procedure, we obtain the critical value of μ when $\omega_1 = 2\omega_2$ and $\omega_1 = 3\omega_2$ as follows:

$$\begin{aligned}\mu^{(02)} &= 0.0242939 + 0.4941323\mathrm{A}_1 - 0.2532947\mathrm{A}_2 \\ &\quad - 0.4235419\mathrm{A}_3 - 0.2602153\delta \\ &\quad - 0.0055365\beta_1 + 0.0055365\beta_2\end{aligned} \tag{17}$$

$$\begin{aligned}\mu^{(03)} &= 0.013516 + 0.2717915\mathrm{A}_1 - 0.1393217\mathrm{A}_2 \\ &\quad - 0.23296416\mathrm{A}_3 - 0.1431283\delta \\ &\quad - 0.0030453\beta_1 + 0.0030453\beta_2\end{aligned} \tag{18}$$

4. Normalization of the Hamiltonian

The Hamiltonian given by (8) is expanded about the Lagrangian point given by (6). Neglecting the terms independent of p_i and q_i, we get the following representation of Hamiltonian:

$$H = \frac{q_1^2 + q_2^2}{2} - p_2 q_1 + p_1 q_2 + \frac{p_1^2 + p_2^2}{2} - \frac{1}{1 + e\cos f}\left[(1-\mu) \cdot \left\{\frac{r_1^2}{2} + \frac{1}{n^2}\left(\frac{q_1}{r_1} + \frac{q_1 A_1 + A_3}{2r_1^3}\right)\right\} + \mu\left\{\frac{r_2^2}{2} + \frac{1}{n^2}\left(\frac{q_2}{r_2} + \frac{q_2 A_2 + A_3}{2r_2^3}\right)\right\}\right] \quad (19)$$

Now, expanding the Hamiltonian function given by (8) in the powers of p_i and q_i $1 \le i \le 2$, we obtain

$$H = \sum_{k=0}^{\infty} H_K$$
$$H = H_0 + H_1 + H_2 + H_4 + H_5 + \cdots . \quad (20)$$

Here, $H_0 = f(u, v, p_u, p_v)$= *constant*, $H_1 = 0$. H_2, H_3, and H_4 are expression in second, third, and forth order terms of p_i and q_i.

Now, consider the canonical transformation $[q_1, q_2, p_1, p_2]$ which transform into $[q_1', q_2', p_1', p_2']$.

That is,

$$[q_1, q_2, p_1, p_2] = [q_1', q_2', p_1', p_2']\, N \quad (21)$$

$$N = \begin{vmatrix} a_1 & a_1 c_1 & -a_1 c_1 & a_1\left(1 - \omega_1^2 b_1\right) \\ a_2 & a_2 c_2 & -a_2 c_2 & a_1\left(1 - \omega_2^2 b_2\right) \\ 0 & a_1 b_1 & a_1(1 - b_1) & a_1 c_1 \\ 0 & -a_2 b_2 & -a_2(1 - b_2) & -a_2 c_2 \end{vmatrix} \quad (22)$$

Using the canonical transformation, the Hamiltonian in the variables will be of the following form:

$$H = \frac{1}{2}\left(p_1'^2 + \omega_1^2 q_1'^2\right) - \frac{1}{2}\left(p_1'^2 + \omega_2^2 q_2'^2\right) + \frac{e\cos f}{1 + e\cos f}\left[a q_2'^2 + b p_2'^2 + c p_2' q_2' + \cdots\right]$$
$$\text{or } H = \frac{1}{2}\left(p_1'^2 + \omega_1^2 q_1'^2\right) - \frac{1}{2}\left(p_1'^2 + \omega_2^2 q_2'^2\right) + \sum_{\alpha+\gamma=3}^{\infty} h_{\alpha_1\alpha_2\gamma_1\gamma_2} q_2'^{\alpha_2} q_1'^{\alpha_1} p_2'^{\gamma_2} p_1'^{\gamma_1} \quad (23)$$

where $\alpha = \alpha_1 + \alpha_2, \gamma = \gamma_1 + \gamma_2$.

Here $\omega_1^2 = -\{\lambda_{1,2}^{(0)}\}^2$ and $\omega_2^2 = -\{\lambda_{3,4}^{(0)}\}^2$ are the frequencies of the linear system with Hamiltonian H_2 and given by the relation:

$$\lambda^4 - \lambda^2\left(-1 - \frac{\alpha}{4}(13 - 20\mu) + 6A_1 + \frac{3\mu A_1}{2} + 3A_2 - \frac{3\mu A_2}{2} + 6A_3 - 3\mu A_3\right) + \frac{27}{4}\mu(1-\mu)\left[1 + \frac{2}{9}\beta_1 + \frac{2}{9}\beta_2 + \frac{71}{18}\alpha + \frac{47}{6}A_1 + \frac{25}{6}A_2 + \frac{15}{3}A_3 - \frac{1}{\mu}\left(\frac{26}{18}\alpha + \frac{8}{6}A_1 + \frac{8}{6}A_2 + \frac{4}{3}A_3\right)\right] \quad (24)$$

Equating the similar coefficients of $h_{\alpha_1\alpha_2\gamma_1\gamma_2}$ and $H_{\alpha_1\alpha_2\gamma_1\gamma_2}$ upto the third order terms, the value can be evaluated in terms of p_i', q_i', which are given in the Appendix.

The next transformation is obtained by making the substitution of variables:

$$q_1' = \frac{1}{2} q_1'' + \frac{\iota}{\omega_1} p_1''$$
$$p_1' = \frac{1}{2}\omega_1 q_1'' + p_1''$$
$$q_2' = \frac{1}{2} q_2'' + \frac{\iota}{\omega_2} p_2''$$
$$p_2' = \frac{1}{2}\omega_2 q_2'' + i p_2'' \quad (25)$$

So that the Hamiltonian of (23) got converted to the form:

$$H = \iota\omega_1 q_1'' p_1'' + \iota\omega_2 q_2'' p_2'' + \sum_{\alpha+\beta=3}^{\infty} h'_{\alpha_1\alpha_2\gamma_1\gamma_2} q_1''^{\alpha_1} q_2''^{\alpha_2} p_1''^{\gamma_1} p_2''^{\gamma_2} \quad (26)$$

where the coefficient of third order terms of $h'_{\alpha_1\alpha_2\gamma_1\gamma_2}$ depends on $\iota\omega_1$, $(i = 1, 2)$ and $h_{\alpha_1\alpha_2\gamma_1\gamma_2}$ which are given in the Appendix. Finally, we apply the Birkhoff's transformation of the form (q_j'', p_j'') to (Q_j, P_j) and nullify all the third-degree terms except those giving rise to resonance of the form $\omega_1 = 2\omega_2$. For this, take the generating function of the form:

$$S = q_1'' P_1 + q_2'' P_2 + \varepsilon S_3 + \varepsilon^2 S_4 + \cdots . \quad (27)$$

Choose S_3 in such a way so that

$$\overline{H_3} = H_3 + \sum_j \left(\frac{\partial S_3}{\partial Q_j}\frac{\partial H_2}{\partial P_j} - \frac{\partial S_3}{\partial P_j}\frac{\partial H_2}{\partial Q_j}\right) = 0 \quad (28)$$

$$H_2(Q_j, P_j) = \iota\omega_1 Q_1 P_1 + \iota\omega_2 Q_2 P_2 \quad (29)$$

Let

$$S_3 = \sum_{\alpha+\gamma=3} g_{\alpha_1\alpha_2\gamma_1\gamma_2} Q_1^{\alpha_1} Q_2^{\alpha_2} P_1^{\gamma_1} P_2^{\gamma_2} \quad (30)$$

where $g_{\alpha_1\alpha_2\gamma_1\gamma_2}$, $(\alpha + \gamma = 3)$, $\alpha = \alpha_1 + \alpha_2$, $\gamma = \gamma_1 + \gamma_2$ are to be determined satisfying (28).

Let

$$H_3 = \sum_{\alpha+\gamma=3} h'_{\alpha_1\alpha_2\gamma_1\gamma_2} q_1^{''\alpha_1} q_2^{''\alpha_2} p_1^{''\gamma_1} p_2^{''\gamma_2} \tag{31}$$

Now substituting the values of H_2, H_3 in (28) and (29) and equating the coefficients of like powers of the powers of the variable we obtain

$$g_{\alpha_1\alpha_2\gamma_1\gamma_2} = \frac{\iota h'_{\alpha_1\alpha_2\gamma_1\gamma_2}}{(\alpha_1 - \gamma_1)\,\omega_1 + (\alpha_2 - \gamma_2)\,\omega_2} \tag{32}$$

5. Stability in Third Order Resonance

From (32), it can be observed that substituting different values of α_i, γ_i $(i = 1, 2)$, where $\alpha + \gamma = 3$ and $\omega_1 = 2\omega_2$, the denomination of R.H.S. of (32) vanishes giving rise to resonances for two set of values of α and γ rendering S_3 in determinate. Let

$$D^r = (\alpha_1 - \gamma_1)\,\omega_1 + (\alpha_2 - \gamma_2)\,\omega_2 \quad \text{then for } \omega_1 = 2\omega_2 \tag{33}$$

Case 1. When $\alpha_1 = 1, \alpha_2 = 0, \gamma_1 = 0, \gamma_2 = 2$, we have $D^r = 0$.

Case 2. When $\alpha_1 = 0, \alpha_2 = 2, \gamma_1 = 1, \gamma_2 = 0$, again we have $D^r = 0$.

Thus, in resonant case $\omega_1 = 2\omega_2$ using Birkhoff's transformation, it is not possible to cancel H_3 of the Hamiltonian. In this case H_3 retain two resonant terms with coefficients h'_{1002} and h'_{0210}. Thus, Hamiltonian reduces to the following form:

$$H = \iota\omega_1 Q_1 P_1 + \iota\omega_2 Q_2 P_2 + h'_{1002} Q_1 P_2^2 + h'_{0210} P_1 Q_2^2, \tag{34}$$

where $h'_{1002} = x_{1002} + \iota y_{1002}$ and $h'_{0210} = (-\omega_2^2/2\omega_1)(y_{1002} + \iota x_{1002})$

Applying canonical change of variables

$$\begin{aligned} Q_1 &= \frac{1}{(\omega_1)^{1/2}}\left(Q_1^0 - \iota P_1^0\right) \\ Q_2 &= \frac{1}{(\omega_2)^{1/2}}\left(Q_2^0 - \iota P_2^0\right) \\ P_1 &= \frac{(\omega_1)^{1/2}}{2}\left(-\iota Q_1^0 - P_1^0\right) \\ P_2 &= \frac{(\omega_2)^{1/2}}{2}\left(Q_2^0 - \iota P_2^0\right) \end{aligned} \tag{35}$$

the Hamiltonian equation (34) becomes

$$\begin{aligned} H^0 = {} & \frac{\omega_1}{2}\left(Q_1^{02} + P_1^{02}\right) - \frac{\omega_2}{2}\left(Q_2^{02} + P_2^{02}\right) \\ & + \Bigg[\frac{\omega_2}{2(\omega_2)^{1/2}}\Big\{x_{1002}\left(Q_1^0 Q_2^{02} - Q_1^0 P_2^{02} - 2P_2^0 Q_2^0\right) \\ & + y_{1002}\left(2P_2^0 Q_2^0 Q_1^0 + P_1^0 Q_2^{02} - P_1^0 P_2^{02}\right)\Big\}\Bigg] \end{aligned} \tag{36}$$

If $x_{1002}^2 + y_{1002}^2 \neq 0$, then the canonical transformation in polar coordinates is given by

$$\begin{aligned} Q_1^0 &= (2r_1)^{1/2} \sin(\phi_1 - \theta_1); \\ P_1^0 &= (2r_1)^{1/2} \cos(\phi_1 - \theta_1); \\ Q_2^0 &= (2r_1)^{1/2} \sin(\phi_2); \\ P_2^0 &= (2r_1)^{1/2} \cos(\phi_2), \end{aligned} \tag{37}$$

where θ_1 is given as

$$\begin{aligned} \sin\theta_1 &= \frac{y_{1002}}{\left(x_{1002}^2 + y_{1002}^2\right)^{1/2}}; \\ \sin\theta_1 &= \frac{x_{1002}}{\left(x_{1002}^2 + y_{1002}^2\right)^{1/2}}. \end{aligned} \tag{38}$$

Thus, the Hamiltonian will of the form

$$\begin{aligned} H &= 2\omega_2 r_1 - \omega_2 r_2 \\ &\quad - \left[\omega_2\left(x_{1002}^2 + y_{1002}^2\right)^{1/2} r_2 (r_1)^{1/2} \sin(\phi_1 + 2\phi_2)\right] \\ &\quad + H_4^0\left(r_j, \phi_j\right) + \cdots \end{aligned} \tag{39}$$

Now, to find the value of H_4, let us assume $x_{1002}^2 + y_{1002}^2 = 0$. Then the normalized form of H_4 is given as

$$\begin{aligned} H_4 = {} & C_{20}(Q_1, P_1)^2 + C_{11}(Q_1, P_1)(Q_2, P_2) \\ & - C_{02}(Q_2, P_2)^2 \end{aligned} \tag{40}$$

with the help of generating function S_4 chosen, so that it satisfies

$$\sum_j \left(\frac{\partial S_4}{\partial Q_j}\frac{\partial H_2}{\partial P_j} - \frac{\partial S_4}{\partial P_j}\frac{\partial H_2}{\partial Q_j}\right) + K_4 = 0 \tag{41}$$

where K_4 is the nonhomogenous part of (40):

$$\begin{aligned} & H_4(Q_i, P_i) + \sum_{i,j} \frac{\partial S_3}{\partial P_i}\frac{\partial S_3}{\partial Q_j}\frac{\partial^2 H_2}{\partial P_i \partial Q_j} \\ & + \sum_{i,j} \frac{\partial S_3}{\partial Q_j}\frac{\partial^2 S_3}{\partial P_i \partial P_j}\frac{\partial H_2}{\partial Q_i} - \sum_{i,j} \frac{\partial S_3}{\partial Q_j}\frac{\partial^2 S_3}{\partial Q_i \partial P_j}\frac{\partial H_2}{\partial P_i} \end{aligned} \tag{42}$$

The coefficients C_{20}, C_{11}, and C_{02} are given as

$$
\begin{aligned}
C_{20} &= \frac{3}{2}\omega_1^2 a_1^4 b_1^4 H_{0400} + \frac{3a_1^4}{2\omega_1^2}\left(H_{4000} + c_1 H_{3100}\right.\\
&\quad \left. + c_1^2 H_{2200} + c_1^3 H_{1300} + 6c_1^4 H_{0400}\right) + \frac{a_1^4 b_1^2}{2}\left(H_{2200}\right.\\
&\quad \left. + 3c_1 H_{1300} + 6c_1^2 H_{0400}\right) - \frac{3}{8}\omega_1^2\left(y_{0030}^2 + x_{0030}^2\right)\\
&\quad - \frac{3}{2}\left(y_{1020}^2 + x_{1020}^2\right) - \frac{\omega_1^2}{8\left(2\omega_1 - \omega_2\right)}\left(y_{0120}^2 + x_{0120}^2\right)\\
&\quad + \frac{1}{2}\left(y_{1011}^2 + x_{1011}^2\right) + \frac{\omega_2\omega_1^2}{8\left(2\omega_1 + \omega_2\right)}\left(y_{0021}^2 + x_{0021}^2\right)\\
C_{11} &= 6\omega_1\omega_2 a_1^2 a_2^2 b_1^2 b_2^2 H_{0400} + \frac{a_1^2 a_2^2}{\omega_1\omega_2}\left\{6H_{4000} + 3\left(c_1\right.\right.\\
&\quad \left. + c_2\right) H_{3100} + \left(c_1^2 + 4c_1c_2 + c_2^2\right) H_{2200} + 3c_1c_2\left(c_1 + c_2\right)\\
&\quad \left. \cdot H_{1300} + 6c_1^2c_2^2 H_{0400}\right) + \frac{\omega_1 a_1^2 a_2^2 b_1^2}{\omega_2}\left(H_{2200}\right.\\
&\quad \left. + 3c_2 H_{1300} + 6c_2^2 H_{0400}\right) + \frac{\omega_2 a_1^2 a_2^2 b_2^2}{\omega_1}\left(H_{2200}\right.\\
&\quad \left. + 3c_1 H_{1300} + 6c_1^2 H_{0400}\right) - \frac{2\omega_2^2}{\left(\omega_1 - 2\omega_2\right)}\left(x_{1002}^2\right.\\
&\quad \left. + y_{1002}^2\right) + \frac{\omega_1\omega_2^2}{2\left(\omega_1 + 2\omega_2\right)}\left(x_{0012}^2 + y_{0012}^2\right)\\
&\quad - \frac{\omega_2\omega_1^2}{2\left(2\omega_1 + \omega_2\right)}\left(x_{0021}^2 + y_{0021}^2\right)\\
&\quad - \frac{2\omega_1^2}{\left(2\omega_1 - \omega_2\right)}\left(x_{0120}^2 + y_{0120}^2\right) + 2\left(x_{0111}x_{1020}\right.\\
&\quad \left. + y_{0111}y_{1020}\right) - \frac{4}{\omega_2}\left(x_{0201}y_{1011} + x_{1011}y_{0201}\right)\\
C_{02} &= \frac{3}{2}\omega_2^2 a_2^4 b_2^4 H_{0400} + \frac{3a_2^4}{2\omega_2^2}\left(H_{4000} + c_2 H_{3100}\right.\\
&\quad \left. + c_2^2 H_{2200} + c_2^3 H_{1300} + c_2^4 H_{0400}\right) + \frac{a_2^4 b_2^2}{2}\left(H_{2200}\right.\\
&\quad \left. + 3c_2 H_{1300} + 6c_2^2 H_{0400}\right) + \frac{3}{8}\omega_2^2\left(y_{0003}^2 + x_{0003}^2\right)\\
&\quad + \frac{6}{\omega_2^2}\left(x_{0201}^2 + y_{0201}^2\right) - \frac{\omega_2^2}{2\omega_1\left(\omega_1 - 2\omega_2\right)}\left(y_{1002}^2\right.\\
&\quad \left. + x_{1002}^2\right) - \frac{1}{2}\left(x_{0111}^2 + y_{0111}^2\right)\\
&\quad - \frac{\omega_1\omega_2^2}{8\left(\omega_1 + 2\omega_2\right)}\left(x_{0012}^2 + y_{0012}^2\right)
\end{aligned}
\tag{43}
$$

Consequently, the Hamiltonian of the dynamical system reduces to the form as

$$
\begin{aligned}
H &= \iota\omega_1 Q_1 P_1 + \iota\omega_2 Q_2 P_2 - C_{20}\left(Q_1P_1\right)^2\\
&\quad + C_{11}\left(Q_1P_1\right)\left(Q_2P_2\right) - C_{02}\left(Q_2P_2\right)^2 + O\left|Q\right|^5
\end{aligned}
\tag{44}
$$

If $x_{1002}^2 + y_{1002}^2 = 0$ and $C_{20} + 2C_{11} + 4C_{02} \neq 0$ then by virtue of Markeev's theorem (Markeev 1967) the equilibrium is stable.

6. Stability in Fourth Order Resonance

The Hamiltonian H in this case will be written as

$$
\begin{aligned}
H &= \iota\omega_1 q_1'' p_1'' + \iota\omega_2 q_2'' p_2''\\
&\quad + \sum_{\alpha+\gamma=4} h'_{\alpha_1\alpha_2\gamma_1\gamma_2} q_1''^{\alpha_1} q_2''^{\alpha_2} p_1''^{\gamma_1} p_2''^{\gamma_2} + O\left|q''\right|^5
\end{aligned}
\tag{45}
$$

where $|q''| = (q_1'' + q_2'' + p_1'' + p_2'')^{1/2}$ and $h'_{\alpha_1\alpha_2\gamma_1\gamma_2}$ depend on ω_i and $h_{\alpha_1\alpha_2\gamma_1\gamma_2}$. Now, using Birkhoff's transformation by means of generating function S where $S = S_2 + S_3 + S_4 \ldots$, choose S_4 such that $\overline{H_4}$ takes the normalized form which is given as follows:

$$
\begin{aligned}
\overline{H_4} &= -C_{20}\left(Q_1, P_1\right)^2 + C_{11}\left(Q_1, P_1\right)\left(Q_2, P_2\right)\\
&\quad - C_{02}\left(Q_2, P_2\right)^2
\end{aligned}
\tag{46}
$$

where

$$
\begin{aligned}
\overline{H_4} &= H_4 + \sum_j\left(\frac{\partial S_4}{\partial Q_j}\frac{\partial H_2}{\partial P_j} - \frac{\partial S_4}{\partial P_j}\frac{\partial H_2}{\partial Q_j}\right)\\
&\quad + \sum_{i,j}\frac{\partial S_3}{\partial P_i}\frac{\partial^2 H_2}{\partial P_i \partial Q_j}\frac{\partial S_3}{\partial Q_j} + \sum_{i,j}\frac{\partial S_3}{\partial Q_j}\frac{\partial^2 S_3}{\partial P_i \partial P_j}\frac{\partial H_2}{\partial Q_i}\\
&\quad - \sum_{i,j}\frac{\partial S_3}{\partial Q_j}\frac{\partial^2 S_3}{\partial Q_i \partial P_j}\frac{\partial H_2}{\partial P_i}
\end{aligned}
\tag{47}
$$

That is,

$$
\sum_j\left(\frac{\partial S_4}{\partial Q_j}\frac{\partial H_2}{\partial P_j} - \frac{\partial S_4}{\partial P_j}\frac{\partial H_2}{\partial Q_j}\right) + K_4 = 0 \tag{48}
$$

where K_4 is the nonhomogeneous part of (45), where homogeneity is considered in terms of product Q_iP_i.

Here,

$$
\begin{aligned}
S_3 &= \sum_{\alpha+\gamma=3} g_{\alpha_1\alpha_2\gamma_1\gamma_2} Q_1^{\alpha_1} Q_2^{\alpha_2} P_1^{\gamma_1} P_2^{\gamma_2}\\
H_2 &= \overline{H_2} = \iota\omega_1 Q_1 P_1 + \iota\omega_2 Q_2 P_2\\
\overline{H_3} &= 0
\end{aligned}
\tag{49}
$$

Let $H_4 = \sum_{\alpha+\gamma=4} h_{\alpha_1\alpha_2\gamma_1\gamma_2} Q_1^{\alpha_1} Q_2^{\alpha_2} P_1^{\gamma_1} P_2^{\gamma_2}$ and $S_4 = \sum_{\alpha+\gamma=4} g_{\alpha_1\alpha_2\gamma_1\gamma_2} Q_1^{\alpha_1} Q_2^{\alpha_2} P_1^{\gamma_1} P_2^{\gamma_2}$ where $g_{\alpha_1\alpha_2\gamma_1\gamma_2}$ are to be determined satisfying (48). Substituting the values in (48) and equating the coefficient of similar powers and different nonhomogeneous terms to zero, we have

$$g_{\alpha_1\alpha_2\gamma_1\gamma_2} = \frac{\iota(\textit{cofficient of } K_4)}{(\alpha_1-\gamma_1)\,\omega_1+(\alpha_2-\gamma_2)\,\omega_2} \tag{50}$$

In the above equation, when substituting different values (α_i,γ_i), $i = 1,2$ and $\omega_1 = 3\omega_2$ the denominator of (50) vanishes for two sets of values of α and γ giving rise to resonant terms $l_{1003}Q_1P_2^3$ and $l_{0310}P_1Q_2^3$. Thus, the new Hamiltonian is obtained as

$$\begin{aligned}\overline{H} &= \iota\omega_1Q_1P_1+\iota\omega_2Q_2P_2+\left\{l_{1003}Q_1P_2^3+l_{0310}P_1Q_2^2\right.\\ &\quad -C_{20}\left(Q_1P_1\right)^2+C_{11}\left(Q_1P_1\right)\left(Q_2P_2\right)\\ &\quad \left.-C_{02}\left(Q_2P_2\right)^2\right\}\end{aligned} \tag{51}$$

where l_{1003} and l_{0310} are given by

$$\begin{aligned} l_{1003} &= \left\{\frac{\omega_1}{2}h_{0013}+\frac{h_{1300}}{2\omega_2^3}-\frac{h_{1102}}{2\omega_2}-\frac{\omega_1h_{0211}}{2\omega_2^2}\right\}\\ &\quad -\frac{2h'_{2001}h'_{0012}}{\iota(2\omega_1-\omega_2)}-\frac{3h'_{0003}h'_{1101}}{\iota\omega_1}\\ &\quad +\frac{2h'_{1002}h'_{0102}}{\iota\omega_2}-\frac{h'_{1011}h'_{1002}}{\iota(\omega_1-2\omega_2)}\\ l_{0310} &= \left\{-\frac{\omega_{1h_{0112}}}{2\omega_2}-\frac{h_{1003}}{2}+\frac{h_{1201}}{2\omega_2^2}-\frac{\omega_1h_{0310}}{2\omega_2^3}\right\}\\ &\quad -\frac{2h'_{0120}h'_{1200}}{\iota(\omega_1+2\omega_2)}-\frac{h'_{1110}h'_{0210}}{\iota\omega_2}\\ &\quad +\frac{2h'_{0210}h'_{0201}}{\iota(\omega_1-2\omega_2)}-\frac{h'_{0300}h'_{0111}}{\iota\omega_2}\end{aligned} \tag{52}$$

where the values $h_{\alpha_1\alpha_2\gamma_1\gamma_2}$ are given in the Appendix. Let $l_{1003}=x_{1003}+\iota\,y_{1003}$, and

$$l_{1003} = -\frac{\omega_1^2}{12}\left(x_{1003}-\iota\,y_{1003}\right) \tag{53}$$

in which

$$\begin{aligned} x_{1003} &= -6\omega_2a_1a_2^3a_1a_2^3H_{0400}+\frac{a_1a_2^3}{2\omega_2^3}\left\{4H_{4000}\right.\\ &\quad +\left(c_1+3c_2\right)H_{0400}+2c_2\left(c_1+c_2\right)H_{2200}\\ &\quad \left.+c_2^2\left(3c_1+c_2\right)H_{1300}\right\}-\frac{a_1b_2^2a_2^3}{2\omega_2}\left\{2H_{2200}\right.\\ &\quad \left.+\left(c_1+c_2\right)H_{1300}+12c_1c_2H_{0400}\right\}\\ &\quad +\frac{3a_1b_1^2b_2^2a_2^3}{2\omega_2}\left\{H_{2200}+3c_2H_{1300}+6c_2^2H_{0400}\right\}\\ &\quad -\frac{9}{5}\left(x_{0120}x_{0012}+y_{0120}y_{0012}\right)-\frac{1}{\omega_2}\left(x_{1002}y_{1011}\right.\\ &\quad \left.+x_{1011}y_{1002}\right)+\frac{4}{\omega_2^2}\left(x_{1002}x_{0201}+y_{1002}y_{0201}\right)\\ &\quad +\frac{3}{2}\left(x_{0003}x_{0111}+y_{0003}y_{0111}\right)\\ y_{1003} &= -\frac{9}{2}a_1a_2^3b_1b_2^2\left(H_{1300}+4c_2H_{0400}\right)\\ &\quad +\frac{b_2^3a_1a_2^3}{2}\left(H_{1300}+4c_1H_{0400}\right)-\frac{a_1b_2a_2^3}{2\omega_2}\left\{3H_{3100}\right.\\ &\quad +2\left(c_1+2c_2\right)H_{2200}+3c_2\left(2c_1+c_2\right)H_{1300}\\ &\quad \left.+12c_1c_2^2H_{0400}\right\}+\frac{3a_1b_1a_2^3}{2\omega_2^2}\left\{H_{3100}+2c_2H_{2200}\right.\\ &\quad \left.+3c_2^2H_{1300}+4c_2^3H_{0400}\right\}-\frac{9}{5}\left(x_{0120}y_{0012}\right.\\ &\quad \left.+x_{0012}y_{0120}\right)-\frac{1}{\omega_2}\left(y_{1011}y_{1002}-x_{1011}y_{1002}\right)\\ &\quad +\frac{4}{\omega_2^2}\left(x_{0201}y_{1002}-x_{1002}y_{0201}\right)+\frac{3}{2}\left(x_{0111}y_{0003}\right.\\ &\quad \left.-x_{0003}y_{0111}\right)\end{aligned} \tag{54}$$

Now, using the transformation (35) and assuming that $x_{1003}^2+y_{1003}^2\neq 0$, the Hamiltonian reduces to the form

$$\begin{aligned} H &= \frac{3}{2}\omega_2\left(Q_1^{02}+P_1^{02}\right)-\frac{\omega_2}{2}\left(Q_2^{02}+P_2^{02}\right)\left\{\frac{1}{4}C_{02}\left(Q_1^{02}\right.\right.\\ &\quad \left.+P_1^{02}\right)+\frac{c_{11}}{4}\left(Q_1^{02}+P_1^{02}\right)\left(Q_2^{02}+P_2^{02}\right)\\ &\quad +\frac{\omega_2\sqrt{3}}{12}\left\{P_2^0\left(P_2^{02}-3Q_2^{02}\right)\left(x_{1003}P_1^0-y_{1003}Q_1^0\right)\right.\\ &\quad +Q_2^0\left(Q_2^{02}-3P_2^{02}\right)\left(y_{1003}P_1^0+x_{0003}Q_1^0\right)\\ &\quad \left.\left.+O\left|Q\right|^5\right)\right\}\end{aligned} \tag{55}$$

Applying transformation in polar coordinates given by (39) where $\theta_2=0$ and θ_1 is given by the relations:

$$\begin{aligned}\sin\theta_1 &= \frac{x_{1003}}{\left(x_{1003}^2+y_{1003}^2\right)^{1/2}},\\ \cos\theta_1 &= \frac{-y_{1003}}{\left(x_{1003}^2+y_{1003}^2\right)^{1/2}}.\end{aligned} \tag{56}$$

Hence, the normalized Hamiltonian in the polar form is given by

$$H = 3\omega_2 r_1 - \omega_2 r_2 + \Big[C_{20} r_1^2 + C_{11} r_1 r_2 C_{02} r_2^2 \cdot \frac{\omega_2}{3}\left\{3\left(x_{1003}^2 + y_{1003}^2\right)\right\}^{1/2} \cdot r_2 \left(r_1 r_2\right)^{1/2} . \sin\left(\emptyset_1 + 3\emptyset_2\right) + O\left\{\left(r_1 + r_2\right)^{5/2}\right\}\Big] \quad (57)$$

Assume

$$a = \left|C_{20} + 3C_{11} + 9C_{02}\right| \quad \text{and } d = \left|3\omega_2 \left(x_{1003}^2 + y_{1003}^2\right)^{1/2}\right| \quad (58)$$

Now, for the Hamiltonian of the form equation (63), the stability is decided based on the following theorem:

(1) If for a Hamiltonian of perturbed motion, the inequality

$$\left(x_{1003}^2 + y_{1003}^2\right)^{1/2} \neq 0, \quad \text{and d} > a \quad (59)$$

is simultaneously satisfied, then the equilibrium point is unstable.

If the inequality signs in (59) change its position and the Hamiltonian contains no terms of the order higher than the fourth, then the equilibrium point is stable.

(2) If the conditions $(x_{1003}^2 + y_{1003}^2)^{1/2} = 0, C_{20} + 3C_{11} + 9C_{02} = 0$ are simultaneously satisfied, then the equilibrium point is stable. But if $(x_{1003}^2 + y_{1003}^2)^{1/2} = 0, C_{20} + 3C_{11} + 9C_{02} \neq 0$, then the stability will be decided by higher order terms than the fourth involving further resonances. It needs separate investigation.

The values of C_{20}, C_{11}, and C_{02} for $\omega_1 = 3\omega_2$ are given as

$$\begin{aligned}
C_{20} = {} & \frac{27}{2}\omega_2^2 a_1^4 b_1^4 H_{0400} + \frac{1}{6\omega_2^2}\left\{a_1^4\left(H_{4000} + c_1 H_{3100} + c_1^2 H_{2200} + c_1^3 H_{1300} + c_1^4 H_{0400}\right)\right\} + \frac{1}{2}\left\{a_1^4 b_1^2\left(H_{2200} + 3c_1 H_{1300} + 6c_1^2 H_{0400}\right)\right\} - \frac{27}{8}\omega_2^2\left(x_{0030}^2 + y_{0030}^2\right) \\
& - \frac{3}{2}\left(x_{1020}^2 + y_{1020}^2\right) - \frac{9}{10}\left(x_{0120}^2 + y_{0120}^2\right) + \frac{1}{2}\left(x_{1011}^2 + y_{1011}^2\right) + \frac{9\omega_2^2}{10}\left(x_{0021}^2 + y_{0201}^2\right), \\
C_{11} = {} & 18\omega_2^2 a_1^2 b_1^2 H_{0400} + \frac{a_1^2 a_2^2}{3\omega_2}\left\{6H_{4000} + 3\left(c_1 + c_2\right) \cdot H_{3100} + \left(c_1^2 + 4c_1 c_2 + c_2^2\right) H_{2200} + 3c_1 c_2\left(c_1 + c_2\right) \cdot H_{1300} + 6c_1^2 c_2^2 H_{0400}\right\} 3a_2^2 a_1^2 b_1^2\left(H_{2200} + 3c_2 H_{1300} + 6c_2^2 H_{0400}\right) + \frac{a_2^2 a_1^2 b_2^2}{3}\left\{H_{2200} + 3c_1 H_{1300} + 6c_1^2 H_{0400}\right\} \\
& - \frac{2}{3}\left(x_{1002}^2 + y_{1002}^2\right) + \frac{3\omega_2^2}{10}\left(x_{0012}^2 + y_{0012}^2\right) - \frac{9\omega_2}{14}\left(x_{0021}^2 + y_{0021}^2\right) - \frac{18}{5}\left(x_{0120}^2 + y_{0120}^2\right) + 2\left(x_{0111}x_{1020} + y_{0111}y_{1020}\right) - \frac{4}{\omega_2}\left(x_{0111}x_{1020} + y_{0111}y_{1020}\right), \\
C_{02} = {} & \frac{3}{2}\omega_2^2 a_2^2 b_2^2 H_{0400} + \frac{3a_2^2}{3\omega_2^2}\left(H_{4000} + c_2 H_{3100} + c_2^2 H_{2200} + c_2^3 H_{1300} + c_2^4 H_{0400}\right) + \frac{a_2^4 b_2^2}{2}\left(H_{2200} + 3c_2 H_{1300} + 6c_2^2 H_{0400}\right) + \frac{3\omega_2^2}{8}\left(x_{0003}^2 + y_{1002}^2\right) \\
& + \frac{6}{\omega_2}\left(x_{0201}^2 + y_{0201}^2\right) - \frac{\left(x_{1002}^2 + y_{1002}^2\right)}{6} - \frac{\left(x_{0111}^2 + y_{0111}^2\right)}{2} - \frac{3\omega_2^2}{40}\left(x_{0012}^2 + y_{0012}^2\right).
\end{aligned} \quad (60)$$

7. Stability in Nonresonance Case

Equation (30) gives the coefficient of S_3 in terms of coefficients of H_3 reducing $\overline{H_3} = 0$. Also S_4 in (42) and (44) is chosen, so that $\overline{H_4}$ retains only terms in normal form (34). Now S_3 can be expanded as

$$\begin{aligned}
S_3 = {} & g_{0003} P_2^3 + g_{0030} P_1^3 + g_{0300} Q_2^3 + g_{3000} Q_1^3 + q_{2100} Q_1^2 Q_2 + g_{2010} Q_1^2 P_1 + g_{2001} Q_2^2 P_2 \\
& + g_{1200} Q_2^2 Q_1 + g_{0210} Q_2^2 P_1 + g_{0201} Q_2^2 P_2 + g_{1020} P_1^2 Q_1 + g_{0120} P_1^2 Q_2 + g_{0021} P_1^2 P_2 \\
& + g_{1002} P_2^2 Q_1 + g_{0102} P_2^2 Q_2 + g_{0021} P_2^2 P_1 + g_{1110} Q_1 Q_2 P_1 + g_{1101} Q_1 Q_2 P_2 + g_{1011} Q_1 P_1 P_2 \\
& + g_{0111} Q_2 P_1 P_2.
\end{aligned} \quad (61)$$

Substituting the required values in (42), the Hamiltonian in the nonresonant case reduces to the form (57), where the coefficients C_{20}, C_{11}, C_{02} are given as

$$\begin{aligned}
C_{20} &= h^*_{2020} - \frac{3}{\iota\omega_1} h'_{3000} h'_{0030} - \frac{3}{\iota\omega_1} h'_{2010} h'_{1020} \\
&\quad + \frac{1}{\iota(2\omega_1 - \omega_2)} h'_{0120} h'_{2001} - \frac{1}{\iota\omega_2} h'_{1110} h'_{1011} \\
&\quad - \frac{1}{\iota(2\omega_1 + \omega_2)} h'_{2100} h'_{0021}, \\
C_{11} &= h^*_{1111} + \frac{4}{\iota(\omega_1 - 2\omega_2)} h'_{0210} h'_{1002} \\
&\quad - \frac{4}{\iota(\omega_1 + 2\omega_2)} h'_{1200} h'_{0012} \\
&\quad - \frac{4}{\iota(\omega_1 - 2\omega_2)} h'_{2100} h'_{0021} \\
&\quad - \frac{4}{\iota(2\omega_1 - \omega_2)} h'_{2001} h'_{0120} - \frac{2}{\iota\omega_1} h'_{2010} h'_{0111} \\
&\quad - \frac{2}{\iota\omega_1} h'_{1101} h'_{1020} - \frac{2}{\iota\omega_2} h'_{0201} h'_{1011} \\
&\quad - \frac{2}{\iota\omega_2} h'_{1110} h'_{0102}, \\
C_{02} &= h^*_{0202} - \frac{3}{\iota\omega_2} h'_{0300} h'_{0003} - \frac{3}{\iota\omega_2} h'_{0102} h'_{0201} \\
&\quad - \frac{1}{\iota\omega_1 - 2\omega_2} h'_{1002} h'_{0210} - \frac{1}{\iota\omega_1} h'_{1101} h'_{0111} \\
&\quad - \frac{1}{\iota\omega_1} h'_{1200} h'_{0012}.
\end{aligned} \tag{62}$$

Here,

$$\begin{aligned}
h^*_{2020} &= -\frac{1}{2} h'_{2020} - \frac{3}{2\omega_1^2} h'_{4000} - \frac{3}{2}\omega_1^2 h'_{0040}, \\
h^*_{1111} &= \omega_1\omega_2 h'_{0022} + \frac{1}{\omega_1\omega_2} h'_{2200} + \frac{\omega_1}{\omega_2} h'_{0220} + \frac{\omega_1}{\omega_2} h'_{2002}, \\
h^*_{0202} &= -\frac{3}{2\omega_2^2} h'_{0004} - \frac{3}{2\omega_2^2} h'_{0400} - \frac{1}{2} h'_{0202}.
\end{aligned} \tag{63}$$

The values of $h'_{\alpha_1\alpha_2\gamma_1\gamma_2}$ are defined in the Appendix. Now if we define

$$D = C_{20}\omega_2^2 + C_{11}\omega_1\omega_2 + C_{02}\omega_1^2 \tag{64}$$

the stability in this case is analyzed by applying the KAM for the normalized Hamiltonian (32). The first two conditions are satisfied except for the resonance cases, which is dealt with in a separate section.

8. Numerical Exploration

To numerically investigate the results obtained in the study in the previous three sections, the values $(x^2_{1002} + y^2_{1002})$ for the system with different values of μ is tabulated in Table 1. But it is observed that the value of $y^2_{1002} + x^2_{1002} \neq 0$ for the various values of perturbing factors considered. Hence, the motion is unstable for small values of eccentricity e in third order resonance. Similarly, for studying the fourth order resonance case, the values of a and d are presented in Table 2. It is found that the inequalities $d < a$ *and* $d > a$ are satisfied giving rise to unstable and stable motion depending on the values of μ *and* for small values of e. For verifying the third condition obtaining the values of D. It is clear from Table 3 that the value of $D \neq 0$ for all values of q_1, q_2, A_1, A_2, A_2 and A_3 and $D < 0$ consistently; that is, any possibility that within the assumed values D will vanish at any point does not arise. Hence, the equilibrium points are stable.

Motion of an infinitesimal of assumed oblateness around two binary systems, Luyten 726 and Sirius, has also been explored numerically, by evaluating the deciding factors discussed in the previous sections. The data related to the two binary systems used in the calculation are presented in Table 4.

For both the binary systems the oblateness of both the primaries are assumed to be 0.001, whereas radiation pressure $q_1 = 0.99$ and $q_2 = 0.98$ and values of all the deciding factors are given in Table 5.

9. Discussion and Conclusion

The nonlinear stability of the elliptical restricted three-body problem with radiating and oblate primaries and infinitesimal satellite has been analyzed. The character of motion is analyzed in the presence as well as in the absence of resonance. If $x^2_{1002}+y^2_{1002} = 0$ and $C_{20}+C_{11}+4C_{02} \neq 0$ then, by virtue of Markeev's theorem (Markeev, 1967), the equilibrium is stable for third order resonance corresponding to $\omega_1 = 2\omega_2$. But, it is observed that for no value of q_1, q_2, A_1, A_2, and A_3, the value of $x^2_{1002} + y^2_{1002} = 0$, which is clear from Table 2. Hence, the motion is unstable for small values of eccentricity "e" in third order resonance.

In the resonance cases of fourth order corresponding to $\omega_1 = 3\omega_2$, for different values of q_1, q_2, A_1, A_2, and A_3, the values of a and d defined by (58) have been calculated. It is found that the inequality $d < a$ is satisfied giving rise to stable motion depending on the values of q_1, q_2, A_1, A_2, and A_3 and for small values if e is as given in Table 3.

On the other hand, when resonance is not present the values of term D, defined by (64), $D \neq 0$, which is clear from Table 1. Thus, it can be concluded that the motion is stable in nonresonance case by the use of KAM theorem.

It is observed that for both the binary systems the movement of infinitesimal in 1:2 resonance shows instable characteristic. However, system Luyten-726 shows stable behavior for 1:3 resonance whereas the values of a and d in the case of binary system Sirius suggest that the system will be instable even in the fourth order resonance. In case of nonresonant movement, it was found that on changing the

Table 1: Values of $x_{1002}^2 + y_{1002}^2$ for third order resonance case.

μ	e	A_1	A_2	A_3	β_1	β_2	α	x_{1002}	y_{1002}	$x_{1002}^2 + y_{1002}^2$
0.01	0.02	0.001	0.001	0.001	0.0002	0.0001	0.0001	-0.41720	-6.19915	38.6035
0.02	0.02	0.001	0.001	0.001	0.0002	0.0002	0.0001	-0.69421	-4.84432	23.9494
0.03	0.02	0.0001	0.0001	0.0001	0.002	0.001	0.001	-1.1825	-3.96232	17.0983
0.0001	0.016	0.001	0.001	0.001	0.0002	0.0001	0.001	-200.227	-975.62	991925
0.000004	0.04	0.001	0.001	0.001	0.0002	0.00001	0.0001	-2665.54	-10322.6	1.13661×10^8

Table 2: Values of *a and d* for fourth order resonance case.

μ	e	A_1	A_2	A_3	β_1	β_2	α	a	d	Nature
0.01	0.02	0.001	0.001	0.001	0.0002	0.0001	0.0001	1.15522×10^4	2.86723×10^3	Stable
0.02	0.02	0.001	0.001	0.001	0.0002	0.0002	0.0001	2.15443×10^6	2.22344×10^5	Stable
0.03	0.02	0.0001	0.0001	0.0001	0.002	0.001	0.001	2.33605×10^6	1.01561×10^6	Stable
0.0001	0.016	0.001	0.001	0.001	0.0002	0.0001	0.001	2.53384×10^3	1.26845×10^3	Stable

Table 3: Values of D for nonresonance case.

μ	e	A_1	A_2	A_3	β_1	β_2	α	ω_1	ω_2	D
0.01	0.02	0.001	0.001	0.001	0.0002	0.0001	0.0001	0.697033	0.208148	-109.123
0.02	0.02	0.001	0.001	0.001	0.0002	0.0002	0.0001	0.68135	0.205552	-49.9179
0.03	0.02	0.0001	0.0001	0.0001	0.002	0.001	0.001	0.568721	0.39769	-169.405
0.0001	0.016	0.001	0.001	0.001	0.0002	0.0001	0.001	0.664444	0.020441	-30022.702
0.000004	0.04	0.001	0.001	0.001	0.0002	0.00001	0.0001	0.677762	0.003967	-2.27342×10^4

Table 4: Data related to binary systems.

Binary System	$M_1(M_\odot)$	$M_2(M_\odot)$	$a(AU)$	e
Luyten-726	0.101	0.99	1.95	0.62
Sirius	2.15	1.05	7.5	0.59

Table 5

Binary System	A_3	$x_{1002}^2 + y_{1002}^2$	a	d	D
Luyten-726	0	1.33×10^8	4.30×10^{10}	1.80×10^{10}	1.05×10^8
	0.001	1.33×10^7	4.01×10^{10}	2.00×10^{10}	-8.48×10^{10}
	0.01	6.11×10^7	3.46×10^{10}	2.12×10^{10}	9.28×10^8
Sirius	0	5.03×10^8	4.67×10^8	1.81×10^{10}	-1.83×10^{10}
	0.001	6.45×10^{10}	3.34×10^8	2.00×10^{10}	1.04×10^9
	0.01	4.88×10^8	5.20×10^8	2.12×10^{10}	-1.94×10^{10}

value of A_3, there is a sign change for both the systems, so there may exist certain values of A_3 for which D vanishes. Other than those values, the system shows stable behavior in nonresonant case.

Appendix

The values of $h_{\alpha_1\alpha_2\gamma_1\gamma_2}$ in terms of (q_i, p_i) are given as

$$h_{0030} = a_1^3 b_1^3 H_{0300},$$

$$h_{3000} = a_1^3 \left(H_{3000} + c_1 H_{2100} + c_1^2 H_{1200} + c_1^3 H_{0300}\right),$$

$$h_{1020} = a_1^3 b_1^2 \left(H_{1200} + 3c_1 H_{0300}\right),$$

$$h_{2010} = a_1^3 b_1 \left(H_{2100} + 2c_1 H_{1200} + 3c_1^2 H_{0300}\right),$$

$$h_{2001} = -a_1^2 a_2 b_2 \left(H_{2100} + 2c_1 H_{1200} + 3c_1^2 H_{0300}\right),$$

$$h_{1011} = -2a_1^2 a_2 b_1 b_2 \left(H_{1200} + 3c_1 H_{0300}\right),$$

$$h_{1110} = 2a_1^2 a_2 b_1 \{H_{2100} + (c_1 + c_2) H_{1200} + 3c_1 c_2 H_{0300}\},$$

$$h_{0021} = -3a_1^2 b_1^2 a_2 b_2 H_{0300},$$

$$h_{2100} = a_1^2 a_2 \left\{3H_{3000} + (2c_1 + c_2) H_{2100} + c_1 (c_1 + c_2) H_{1200} + 3c_1^2 c_2 H_{0300}\right\},$$

$$h_{1002} = a_1 a_2^2 b_2^2 \left(H_{1200} + 3c_1 H_{0300}\right),$$
$$h_{0210} = a_1 a_2^2 b_1 \left\{H_{2100} + 2c_2 H_{1200} + 3c_2^2 H_{0300}\right\},$$
$$h_{0012} = 3a_1 a_2^2 b_1 b_2^2 H_{0300},$$
$$h_{1200} = a_1 a_2^2 \left\{3H_{3000} + \left(c_1 + 2c_2\right) H_{2100} + c_2 \left(2c_1 + 2c_2\right) H_{1200} + 3c_1 c_2^2 H_{0300}\right\},$$
$$h_{0120} = a_2 a_1^2 b_1^2 \left(H_{1200} + 3c_2 H_{0300}\right)$$
$$h_{0111} = -2a_2^2 a_1 b_1 b_2 \left(H_{1200} + 3c_2 H_{0300}\right),$$
$$h_{1101} = -2a_2^2 a_1 b_2 \left\{H_{2100} + \left(c_1 + c_2\right) H_{1200} + 3c_1 c_2 H_{0300}\right\},$$
$$h_{0201} = -a_2^3 b_2 \left(H_{2100} + 2c_2 H_{1200} + 3c_2^2 H_{0300}\right),$$
$$h_{0102} = a_2^3 b_2^2 \left(H_{1200} + 3c_2 H_{0300}\right)$$
$$h_{0003} = -a_2^3 b_2^3 H_{0300},$$
$$h_{0300} = a_2^3 \left(H_{3000} + c_2 H_{2100} + c_2^2 H_{1200} + c_2^3 H_{0300}\right),$$
$$h_{0040} = a_1^4 b_1^4 H_{0400},$$
$$h_{4000} = a_1^4 \left(H_{4000} + c_1 H_{3100} + c_1^2 H_{2200} + c_1^3 H_{1300} + c_1^4 H_{0400}\right),$$
$$h_{2020} = a_1^4 b_1^2 \left(H_{2200} + 3c_1 H_{1300} + 6c_1^2 H_{0400}\right),$$
$$h_{0022} = 6a_1^2 b_1^2 a_2^2 b_2^2 H_{0400}$$
$$h_{2200} = a_1^2 a_2^2 \left\{6H_{4000} + 3\left(c_1 + c_2\right) H_{3100} + \left(c_1^2 + 4c_1 c_2 + c_2^2\right) H_{2200} + 3c_1 c_2 \left(c_1 + c_2\right) H_{1300} + 6c_1^2 c_2^2 H_{0400}\right\},$$
$$h_{0220} = a_1^2 a_2^2 b_1^2 \left(H_{2200} + 3c_2 H_{1300} + 6c_2^2 H_{0400}\right),$$
$$h_{2002} = a_1^2 a_2^2 b_2^2 \left(H_{2200} + 3c_1 H_{1300} + 6c_1^2 H_{0400}\right),$$
$$h_{0004} = a_2^4 b_2^4 H_{0400},$$
$$h_{0400} = a_2^4 \left(H_{4000} + c_2 H_{3100} + c_2^2 H_{2200} + c_2^3 H_{1300} + c_2^4 H_{0400}\right),$$
$$h_{0202} = a_2^4 b_2^2 \left(H_{2200} + 3c_2 H_{3100} + 6c_2^2 H_{0400}\right),$$
$$h_{0013} = -4a_1 a_2^3 b_1 b_2^3 H_{0400},$$
$$h_{1300} = a_1 a_2^3 \left\{4H_{4000} + \left(c_1 + 3c_2\right) H_{3100} + 4c_1 c_2^3 H_{0400} + 2c_2 \left(c_1 + c_2\right) H_{2200} + c_2^2 \left(3c_1 + c_2\right) H_{1300}\right\},$$
$$h_{1102} = a_1 a_2^3 b_2^2 \left\{2H_{2200} + 3\left(c_1 + c_2\right) H_{1300} + 12c_1 c_2 H_{0400}\right\},$$
$$h_{0211} = -2a_1 a_2^3 b_1 b_2 \left(H_{2200} + 3c_2 H_{3100} + 6c_2^2 H_{0400}\right),$$
$$h_{0112} = 3a_1 a_2^3 b_1 b_2^2 \left(H_{1300} + 4c_2 H_{0400}\right)$$
$$h_{1003} = -a_1 a_2^3 b_2^3 \left(H_{1300} + 4c_1 H_{0400}\right),$$
$$h_{1102} = -a_1 a_2^3 b_2 \left\{3H_{3100} + 2\left(c_1 + c_2\right) H_{2200} + 3c_2 \left(2c_1 + c_2\right) H_{1300} + 12c_1 c_2^2 H_{0400}\right\},$$
$$h_{0310} = a_1 a_2^3 b_1 \left\{H_{3100} + 2c_2 H_{2200} + 3c_2^2 H_{1300} + 4c_2^3 H_{0400}\right\}, \tag{A.1}$$

The coefficients of $h'_{\alpha_1 \alpha_2 \gamma_1 \gamma_2}$ are given as

$$h'_{0030} = \left(h_{0030} - \frac{h_{2010}}{\omega_1^2}\right) + i\left(\frac{h_{1020}}{\omega_1} - \frac{h_{3000}}{\omega_1^3}\right), = x_{0030} + iy_{0030},$$
$$h'_{1020} = \left(-\frac{1}{3}\left(h_{1020} - \frac{3}{2}\frac{h_{3000}}{\omega_1^2}\right) + i\left(\frac{3\omega_1}{2}h_{0030} - \frac{1}{2\omega_1}h_{2011}\right), = x_{1020} + iy_{1020},$$
$$h'_{1020} = \left(-\frac{1}{3}\left(h_{1020} - \frac{3}{2}\frac{h_{3000}}{\omega_1^2}\right) + i\left(\frac{3\omega_1}{2}h_{0030} - \frac{1}{2\omega_1}h_{2011}\right), = x_{1020} + iy_{1020},$$
$$h'_{0120} = \left(-\frac{\omega_2}{2}\left(h_{0021} + \frac{h_{1110}}{2\omega_1} + \frac{\omega_2}{2\omega_1^2}h_{2001}\right) + i\left(-\frac{1}{2}h_{0120} - \frac{\omega_2}{2\omega_1}h_{1011} + \frac{h_{2100}}{\omega_1\omega_2}\right), = x_{1011} + iy_{1011},$$
$$h'_{0120} = \left(-\frac{\omega_2}{2}\left(h_{0021} + \frac{h_{1110}}{2\omega_1} + \frac{\omega_2}{2\omega_1^2}h_{2001}\right) + i\left(-\frac{1}{2}h_{0120} - \frac{\omega_2}{2\omega_1}h_{1011} + \frac{h_{2100}}{\omega_1\omega_2}\right) = x_{0120} + iy_{0120},$$

$$h'_{1110} = \left(-\omega_1 h_{0021} - \frac{h_{2001}}{\omega_1}\right) + i\left(\omega_1 \frac{h_{0120}}{\omega_2} + \frac{h_{2100}}{\omega_1\omega_2}\right)$$

$$= x_{1011} + i y_{1011},$$

$$h'_{0021} = \left(\frac{h_{0120}}{\omega_2} - \frac{h_{1011}}{\omega_1} - \frac{h_{2100}}{\omega_1^2\omega_2}\right) + i\left(h_{0021} + \frac{h_{1110}}{\omega_1\omega_2} - \frac{h_{2001}}{\omega_1^2}\right)$$

$$= x_{0021} + i y_{0021},$$

$$h'_{1002} = \left(-\frac{\omega_1}{2\omega_2}\left(h_{1011} - \frac{h_{1002}}{2} + \frac{h_{1200}}{2\omega_2^2}\right) + i\left(-\omega_1 \frac{h_{0012}}{2} - \frac{\omega_1}{2\omega_2^2}h_{0210} + \frac{h_{1101}}{2\omega_2}\right)\right.$$

$$= x_{1002} + i y_{1002},$$

$$h'_{0012} = \left(-h_{0012} + \frac{h_{0210}}{\omega_2^2} - \frac{h_{1101}}{\omega_1\omega_2}\right) + i\left(\frac{h_{0111}}{\omega_2} - \frac{h_{1002}}{\omega_1} + \frac{h_{1200}}{\omega_1\omega_2^2}\right)$$

$$= x_{0012} + i y_{0012}$$

$$h'_{0111} = \left(-\frac{\omega_2}{\omega_1}\left(h_{1002} + \frac{h_{1200}}{\omega_1\omega_2}\right) + i\left(-\omega_2 h_{0012} - \frac{h_{0210}}{\omega_2}\right)\right.$$

$$= x_{0111} + i y_{0111},$$

$$h'_{0210} = \left(-\frac{\omega_2}{4}\left(h_{0120} - \frac{3}{4\omega_2}h_{0300}\right) + i\left(\frac{3}{4}\omega_2^2 h_{0003} + \frac{h_{0201}}{4}\right)\right.$$

$$= x_{0201} + i y_{0201}$$

$$h'_{0003} = \left(-\frac{h_{0120}}{\omega_2} + \frac{h_{0300}}{\omega_2^3}\right) + i\left(-h_{0003} + \frac{h_{0201}}{\omega_2^2}\right)$$

$$= x_{0003} + i y_{0003}$$

$$h'_{3000} = -\frac{\omega_1^3}{8}\left[\left(\frac{h_{1020}}{\omega_1} - \frac{h_{3000}}{\omega_1^3}\right) + i\left(h_{0030} - \frac{h_{2010}}{\omega_1^2}\right)\right]$$

$$= -\frac{\omega_1^3}{8}\left[y_{0030} + i x_{0030}\right]$$

$$h'_{0300} = -\frac{\omega_2^3}{8}\left[\left(\frac{h_{0201}}{\omega_2^2} - h_{0003}\right) + i\left(\frac{h_{0300}}{\omega_2^3} - \frac{h_{0102}}{\omega_2}\right)\right]$$

$$= -\frac{\omega_2^3}{8}\left[y_{0003} + i x_{0003}\right]$$

$$h'_{2010} = -\frac{\omega_1}{2}\left[\left(\frac{3\omega_1}{2}h_{0030} + \frac{h_{0210}}{2\omega_1}\right) + i\left(-\frac{h_{1020}}{3} - \frac{3h_{3000}}{2\omega_1^2}\right)\right]$$

$$= \frac{\omega_1}{2}\left[y_{1020} + i x_{1020}\right]$$

$$h'_{2001} = -\frac{\omega_1^2}{2\omega_2}\left[\left(-\frac{h_{0120}}{2} - \frac{\omega_2}{2\omega_1}h_{1011} + \frac{1}{2\omega_1^2}h_{2100}\right) + i\left(-\frac{\omega_2}{2}h_{0021} + \frac{h_{1110}}{2\omega_1} + \frac{\omega_2}{2\omega_1^2}h_{2001}\right)\right]$$

$$= \frac{\omega_1^2}{2\omega_2}\left[y_{0120} + i x_{0120}\right]$$

$$h'_{2100} = -\frac{\omega_1^2\omega_2}{8}\left[\left(-h_{0021} + \frac{h_{1110}}{\omega_1\omega_2} - \frac{h_{2001}}{\omega_1^2}\right) + i\left(\frac{h_{0120}}{\omega_2} - \frac{h_{1011}}{\omega_1} - \frac{h_{2100}}{\omega_1^2\omega_2}\right)\right]$$

$$= \frac{\omega_1^2\omega_2}{8}\left[y_{0021} + i x_{0021}\right]$$

$$h'_{0210} = -\frac{\omega_2^2}{2\omega_1}\left[\left(-\frac{\omega_1}{2}h_{0012} + \frac{\omega_1}{2\omega_2^2}h_{0210} + \frac{h_{1101}}{2\omega_2}\right) + i\left(-\frac{\omega_1}{2\omega_2}h_{0111} - \frac{h_{1002}}{2} + \frac{h_{1200}}{2\omega_2^2}\right)\right]$$

$$= -\frac{\omega_2^2}{2\omega_1}\left[y_{0021} + i x_{0021}\right]$$

$$h'_{1200} = -\frac{\omega_1\omega_2^2}{8}\left[\left(\frac{h_{0111}}{\omega_2} - \frac{h_{1002}}{\omega_1} + \frac{h_{1200}}{\omega_1\omega_2^2}\right) + i\left(-h_{0012} + \frac{h_{0210}}{\omega_2^2} - \frac{h_{1101}}{\omega_1\omega_2}\right)\right]$$

$$= -\frac{\omega_1\omega_2^2}{8}\left[y_{0012} + i x_{0012}\right]$$

$$h'_{1101} = -\frac{\omega_1}{2}\left[\left(-\omega_2 h_{0012} - \frac{h_{0210}}{\omega_2}\right) + i\left(\frac{\omega_2}{\omega_1}h_{1002} + \frac{h_{1200}}{\omega_1\omega_2}\right)\right]$$

$$= \frac{\omega_1}{2}\left[y_{0111} + i x_{0111}\right]$$

$$
\begin{aligned}
h'_{1110} &= \frac{\omega_2}{2}\left[\left(\frac{\omega_1}{\omega_2}h_{0120} + \frac{h_{2100}}{\omega_1\omega_2}\right)\right.\\
&\quad \left. + i\left(-\omega_1 h_{0021} - \frac{h_{2001}}{\omega_1}\right)\right]\\
&= \frac{\omega_2}{2}\left[y_{1011} + i x_{1011}\right]\\
h'_{0120} &= \frac{2}{\omega_2}\left[\left(\frac{3}{4}\omega_2^2 h_{0003} + \frac{h_{0201}}{4}\right)\right.\\
&\quad \left. + i\left(\omega_2\frac{h_{0102}}{4} - \omega_2 h_{0300}\right)\right]\\
&= \frac{2}{\omega_2}\left[y_{0201} + i x_{0201}\right]
\end{aligned}
\tag{A.2}
$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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