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Research Article

Artificial Controlling of the Collinear Liberation Points Using Lorentz Force in the Restricted Three-Body Problem

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This work studies the possibility of generating artificial collinear liberation points for the planar circular restricted three-body problem using Lorentz force affecting a charged spacecraft due to the magnetic field of a planet. It is considered to be a magnetic dipole inclined by angle α with the spin axis of the planet. The acceleration components for Lorentz force are first derived in an inertial planet-center coordinate system. Then, they are transformed into the rotating coordinate system of the three-body system, with the planet naturally the smaller primary in a planet-Sun system. The equations for the liberation points are derived including the charge per unit mass as the controlling parameter. Finally, the values of the charge per unit mass required for controlling the collinear liberation point positions are derived. A numerical application for the Sun-Jupiter system is introduced and the relation between the position of the artificial liberation point and the charge per unit mass is presented graphically.

1. Introduction

The history of the restricted problem begins in 1772 by Euler and Lagrange, continues in 1836 by Jacobi and Hill [1], and is followed by Pioncare [2], Levi-Civita, Birkhoff, and then Szebhley [3]. The problem plays an important role in studying the motion of artificial satellites. Eckstien et al. [4] used the method of multiple variable expansion to estimate the motion of a close satellite of the smaller primary in the restricted three-body problem. Not only the orbits of close satellites but also the orbits of distant satellites are determined by Kogan [5]. Zhuravlev et al. [6] investigated the motion of artificial satellites in the restricted problem when the planet has a definite shape. Cordeiro et al. [7] investigate some dynamical properties of the phenomena of satellite capture; moreover, a numerical study of this phenomenon has been performed by Neto [8].

In the frame of studying the stability of Lagrangian points in the RTBP, great work was done. For literature reviewing, Arnol'd [9], Moser [10], Leontovich [11], Arnol'd [12], and Deprit and Deprit-Bartholomé [13] have recently shown that the equilateral triangular solution will indeed be stable for nearly all values of the mass parameter μ in the range $0<\mu<0.0385208$. Later, Markeev [14] proved that in the plane problem stability will hold for all values of μ in the previous range except the two special values $\mu_1 = 0.0242938$ and $\mu_2 = 0.0135160$, for which the motion will be unstable.

The literature on the RTBP is rich with articles studying equilibrium points in the planar case other than the well-known five points. These can exist when other forces than gravity are included in the problem whether natural or artificial [15–18]. Recently, many articles studied the three-body problem from different aspects. For example, Zotos [19] studied basins of convergence of the equilibrium points in the pseudo-Newtonian circular restricted three-body problem in a plane, and Suraj et al. [20, 21] studied the topology of the basins of convergence in the three-body problem. Zotos [22] studied the fractal basins of attraction with oblateness and radiation pressure. Suraj et al. [23] studied the perturbations effects of the Coriolis and centrifugal forces on the existence of the liberation points. Mittal et al. [24] studied the periodic orbits generated by Lagrangian solution when the primaries are not spherical.

Mostafa et al. [25] studied the artificial triangular points in the planar RTBP using Lorentz Force on a charged satellite due to the planet's magnetic field, neglecting the tilt angle of the magnetic field with the normal to the plane of motion. In this work, the collinear points are studied with the tilt angle considered. The acceleration due to Lorentz force generated by the second primary, in the restricted three-body problem will be driven in the planet-centric coordinate system. Then, the acceleration vector will be transformed into the rotating frame of the RTBP. Finally, the effect of that force on the position of the liberation points using the charge per unit mass as a controller parameter is studied.

2. Equation of Motion

Let the center of the coordinate system be located at the center of mass ($C.M$) of the primaries in the rotating coordinate system, XYZ, which is rotating with constant unit angular speed where the XY plane is the plane of motion of the two primaries,m_1 and m_2.

The X-axis is the line joining the bigger primary, m_1, and the smaller one m_2, Y-axis is normal to X-axis in their orbital plane while Z-axis is perpendicular to the XY plane (as shown in Figure 1). Let the separation distance between the primaries is l which will be the unit of distances.

Let (X, Y, Z) are the coordinates of the negligible mass$m$, in this coordinate system as shown in Figure 1.

The dimensionless equations of motion of the negligible mass, m, in the framework of the circular restricted three-body problem (CRTBP) are as follows (Fitzpatrick [26] and Murray and Dermott [27]):

$$\ddot{X} - 2\dot{Y} = X - \mu_1 \frac{(X + \mu_2)}{r_1^3} - \mu_2 \frac{(X - \mu_1)}{r^3}, \tag{1}$$

$$\ddot{Y} + 2\dot{X} = Y - \left(\mu_1 \frac{Y}{r_1^3} + \mu_2 \frac{Y}{r^3} \right), \tag{2}$$

$$\ddot{Z} = -\mu_1 \frac{Z}{r_1^3} - \mu_2 \frac{Z}{r^3}, \tag{3}$$

where μ_1 is the distance of the mass m_2 from the center of mass (CM) and is equal to Gm_1 while μ_2 is the distance of the mass m_1 from CM and also equal to Gm_2, with G is the gravitational constant. $\bar{r}_1$ and $\bar{r}$ are the position vectors of the mass m from m_1 and m_2, respectively.

Let $\mu = (\mu_1 + \mu_2)$ be the unit of distance. Then, $\mu_1 + \mu_2 = 1$, which means that μ_2 and μ_1 are dependent parameters, so only one parameter can be chosen to describe the problem. If μ is chosen to be the dimensionless mass of the smaller primary then, $(1 - \mu)$ will be the dimensionless mass of the other primary.

Let us change the center of the coordinate system to be in the smaller primary, m_2, in the new coordinate system x, y, and z such that the x-axis is pointing from the bigger primary to the smaller one, the y-axis is normal to it in the primaries orbital plane, and the z-axis is normal to the primaries orbital plan.

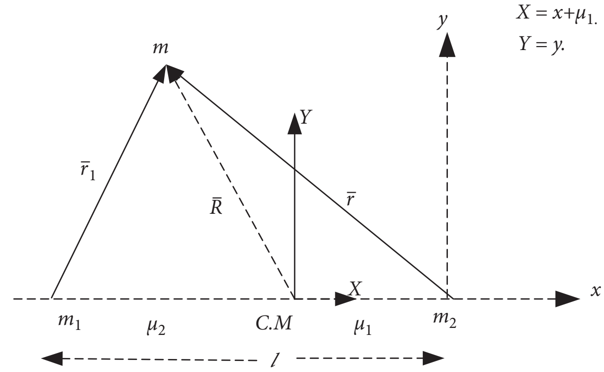

Figure 1: The kinematics of circular planer three body.

Then, the dimensionless equations of motion, in the smaller primary coordinate system, will be

$$\ddot{x} - 2\dot{y} = (x + 1 - \mu) - (1 - \mu) \frac{(x + 1)}{r_1^3} - \frac{\mu x}{r^3}, \tag{4}$$

$$\ddot{y} + 2\dot{x} = y - (1 - \mu) \frac{y}{r_1^3} - \frac{\mu y}{r^3}, \tag{5}$$

$$\ddot{z} = -(1 - \mu) \frac{z}{r_1^3} - \frac{\mu z}{r^3}. \tag{6}$$

With

$$r_1 = \sqrt{(x + l)^2 + y^2 + z^2}, \tag{7}$$

$$r = \sqrt{x^2 + y^2 + z^2}. \tag{8}$$

The coordinate of the small body in terms of the distances can be given by

$$x = \frac{1}{2l} \left(r_1^2 - r^2 - l^2 \right), \tag{9}$$

$$y = \pm \frac{1}{2l} \sqrt{4l^2 r^2 - \left(r_1^2 - r^2 - l \right)^2}. \tag{10}$$

To control the motion described by Equations (4)–(6), additional acceleration must add as a controller for the motion.

In our study, we will examine the Lorentz force generated by the magnetic field of the planets. Let this acceleration be denoted by

$$a_L = \left[\, a_{LX} \;\; a_{LY} \;\; a_{LY} \, \right]^T. \tag{11}$$

3. Lorentz Acceleration

If the small body with mass m experienced by a charge q (Coulombs) moving through a planet magnetic field $\overline{B}$, in

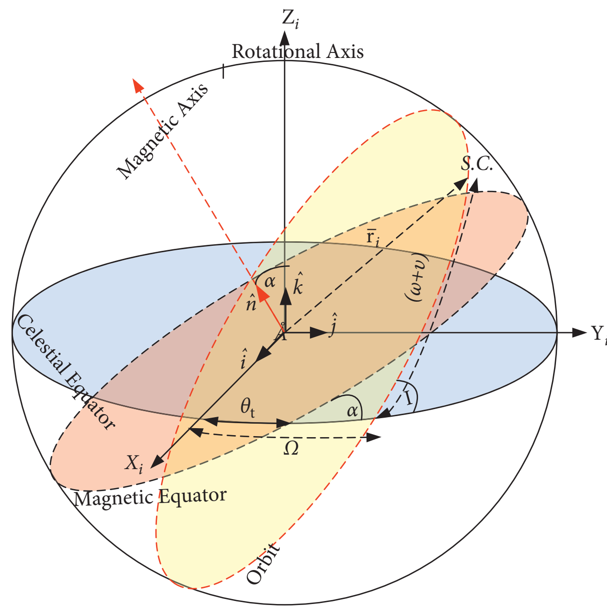

FIGURE 2: The apparent magnetic axis, rotational axis, and SC orbit in the inertial planet-centric coordinate system on the celestial sphere.

planet-center inertial coordinate system $\hat{i}$, $\hat{j}$, and $\hat{k}$, as shown in Figure 2.

Then, the Lorentz force vector, in the inertial coordinate system $\overline{F}_{Li}$, will affect the motion of such a body. Lorentz acceleration, $\overline{a}_{Li}$, can be given by Streetman [28]:

$$\overline{a}_{Li} = \frac{\overline{F}_{Li}}{m} = \tilde{q}\left(\overline{V}_i - \vartheta\hat{k} \times \overline{r}_i\right) \times \overline{B}_i, \tag{12}$$

where $\overline{V}_i$ is the inertial velocity while $\overline{\vartheta}$ is the angular velocity of the source of the magnetic field which is rotating with the spin speed ϑ, of the planet and can be written as $\overline{\vartheta} = \vartheta\hat{k}$, with $\tilde{q} = q/m$ is the charge-to-mass ratio of the small body (specific charge) in Coulombs per kilogram (C/kg).

If we assume the source is a magnetic dipole, then, general vector model of a dipole magnetic field is (Rothwell [29])

$$\overline{B}_i = \frac{B_0}{r_i^3}\left[3\left(\hat{n}_i \cdot \hat{r}_i\right)\hat{r}_i - \hat{n}_i\right]. \tag{13}$$

Note that, the subscript "i" refers to "inertial system." $\hat{n}_i$ is unit vector along the north magnetic pole, and B_0 is the strength of the field in Weber-meters. For a tilted magnetic dipole, the angle α between $\hat{n}_i$ and $\hat{k}$, is the angle between the rotational axis and the magnetic axis, as shown in Figure 2. In the Earth case, the angle α is 11.7° and is independent of time.

From Figure 2, we can express the direction of the magnetic field in the inertial reference frame, X_i Y_i Z_i, as

$$\hat{n}_i = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}. \tag{14}$$

With,

$$\begin{aligned} n_1 &= \mathrm{Sin}\alpha\mathrm{Cos}\alpha_t, \\ n_2 &= \mathrm{Sin}\alpha\mathrm{Sin}\theta_t, \\ n_3 &= \mathrm{Cos}\alpha. \end{aligned} \tag{15}$$

Finally, we can express the magnetic field described by Equation (13) as

$$\begin{aligned} \overline{B}_i = \frac{B_0}{r_i^3}\Bigg\{&\left[\frac{3}{r_i^2}\left(n_1X_i + n_2Y_i + n_3Z_i\right)X_i - n_1\right]\hat{i} \\ &+\left[\frac{3}{r_i^2}\left(n_1X_i + n_2Y_i + n_3Z_i\right)Y_i - n_2\right]\hat{j} \\ &+\left[\frac{3}{r_i^2}\left(n_1X_i + n_2Y_i + n_3Z_i\right)Z_i - n_3\right]\hat{k}\Bigg\}, \end{aligned} \tag{16}$$

where X_i, Y_i, and Z_i are the coordinates of the position vector $\overline{r}_i$ in the planet-centric inertial coordinate system.

The acceleration components due to the planet magnetic field, in the planet-centric inertial coordinate system, using Equation (12), can give by

$$
\begin{aligned}
\begin{bmatrix} a_{LiX} \\ a_{LiY} \\ a_{LiZ} \end{bmatrix}^T &= \tilde{q}\frac{B_0}{r_i^3}\left\{\begin{bmatrix} \dot{X}_i \\ \dot{Y}_i \\ \dot{Z}_i \end{bmatrix}^T - \vartheta\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ X_i & Y_i & Z_i \end{bmatrix}\right\} \times \left\{\cdot\begin{bmatrix} \frac{3}{r_i^2}\left(n_1X_i + n_2Y_i + n_3Z_i\right)X_i - n_1 \\ \frac{3}{r_i^2}\left(n_1X_i + n_2Y_i + n_3Z_i\right)Y_i - n_2 \\ \frac{3}{r_i^2}\left(n_1X_i + n_2Y_i + n_3Z_i\right)Z_i - n_3 \end{bmatrix}^T\right\} \\
&= \tilde{q}\frac{B_0}{r_i^3}\left\{\begin{bmatrix} \dot{X}_i \\ \dot{Y}_i \\ \dot{Z}_i \end{bmatrix}^T - \vartheta\begin{bmatrix} -Y_i \\ X_i \\ 0 \end{bmatrix}^T\right\} \times \left\{\cdot\begin{bmatrix} \frac{3}{r_i^2}\left(n_1X_i + n_2Y_i + n_3Z_i\right)X_i - n_1 \\ \frac{3}{r_i^2}\left(n_1X_i + n_2Y_i + n_3Z_i\right)Y_i - n_2 \\ \frac{3}{r_i^2}\left(n_1X_i + n_2Y_i + n_3Z_i\right)Z_i - n_3 \end{bmatrix}^T\right\} \\
&= \tilde{q}\frac{B_0}{r_i^3}\left\{\begin{bmatrix} \dot{X}_i + \vartheta Y_i \\ \dot{Y}_i - \vartheta X_i \\ \dot{Z}_i \end{bmatrix}^T\right\} \times \left\{\cdot\begin{bmatrix} \frac{3}{r_i^2}\left(n_1X_i + n_2Y_i + n_3Z_i\right)X_i - n_1 \\ \frac{3}{r_i^2}\left(n_1X_i + n_2Y_i + n_3Z_i\right)Y_i - n_2 \\ \frac{3}{r_i^2}\left(n_1X_i + n_2Y_i + n_3Z_i\right)Z_i - n_3 \end{bmatrix}^T\right\} \\
&= \tilde{q}\frac{B_0}{r_i^3}\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \left(\dot{X}_i + \vartheta Y_i\right) & \left(\dot{Y}_i - \vartheta X_i\right) & \dot{Z}_i \\ \frac{3}{r_i^2}\left(n_1X_i + n_2Y_i + n_3Z_i\right)X_i - n_1 & \frac{3}{r_i^2}\left(n_1X_i + n_2Y_i + n_3Z_i\right)Y_i - n_2 & \frac{3}{r_i^2}\left(n_1X_i + n_2Y_i + n_3Z_i\right)Z_i - n_3 \end{bmatrix}.
\end{aligned} \tag{17}
$$

Then,

$$
\begin{bmatrix} a_{LiX} \\ a_{LiY} \\ a_{LiZ} \end{bmatrix}^T = \tilde{q}\frac{B_0}{r_i^3}\begin{bmatrix} \left(\dot{Y}_i - \vartheta X_i\right)\left[\frac{3}{r_i^2}\left(n_1X_i + n_2Y_i + n_3Z_i\right)Z_i - n_3\right] - \left[\frac{3}{r_i^2}\left(n_1X_i + n_2Y_i + n_3Z_i\right)Y_i - n_2\right]\dot{Z}_i \\ -\left(\dot{X}_i + \vartheta Y_i\right)\left[\frac{3}{r_i^2}\left(n_1X_i + n_2Y_i + n_3Z_i\right)Z_i - n_3\right] + \left[\frac{3}{r_i^2}\left(n_1X_i + n_2Y_i + n_3Z_i\right)X_i - n_1\right]\dot{Z}_i \\ \left(\dot{X}_i + \vartheta Y_i\right)\left[\frac{3}{r_i^2}\left(n_1X_i + n_2Y_i + n_3Z_i\right)Y_i - n_2\right] - \left(\dot{Y}_i - \vartheta X_i\right)\left[\frac{3}{r_i^2}\left(n_1X_i + n_2Y_i + n_3Z_i\right)X_i - n_1\right] \end{bmatrix}^T. \tag{18}
$$

Finally, the Lorenz acceleration components will be:

$$a_{LiX} = \tilde{q}\frac{B_0}{r_i^3}\left\{\frac{3}{r_i^2}\left[\left(\dot{Y}_i - \vartheta X_i\right)Z_i - Y_i\dot{Z}_i\right](n_1X_i + n_2Y_i + n_3Z_i) - \left[\left(\dot{Y}_i - \vartheta X_i\right)n_3 - n_2\dot{Z}_i\right]\right\}, \tag{19}$$

$$a_{LiY} = \tilde{q}\frac{B_0}{r_i^3}\left\{-\frac{3}{r_i^2}\left[\left(\dot{X}_i + \vartheta Y_i\right)Z_i - X_i\dot{Z}_i\right](n_1X_i + n_2Y_i + n_3Z_i) + \left[\left(\dot{X}_i + \vartheta Y_i\right)n_3 - n_1\dot{Z}_i\right]\right\}, \tag{20}$$

$$a_{LiZ} = \tilde{q}\frac{B_0}{r_i^3}\left\{\frac{3}{r_i^2}\left[\left(\dot{X}_i + \vartheta Y_i\right)Y_i - \left(\dot{Y}_i - \vartheta X_i\right)X_i\right](n_1X_i + n_2Y_i + n_3Z_i) - \left[\left(\dot{X}_i + \vartheta Y_i\right)n_2 - \left(\dot{Y}_i - \vartheta X_i\right)n_1\right]\right\}. \tag{21}$$

Equations (19)–(21) represent the acceleration experienced by spacecraft due to an inclined magnetic field by an angle α with the rotational axis of the central planet.

To substitute Lorenz acceleration, Equations (19)–(21), in the equations of motion for the three body, we need to transform Equations (19)–(21) to be in the rotational coordinate system (x, y, z) described by Equations (4)–(6).

Finally, the Lorentz acceleration described in the sidereal planet-centered coordinate system can be derived by substituting the transformation into Equations (19)–(21), assuming the angle ε is constant:

$$a_{Lx} = \frac{Q}{2r^5}\left\{\begin{array}{c}\left(2x^3\vartheta + 2xy^2\vartheta - 4xz^2\vartheta - 2x^2\dot{y} - 2y^2\dot{y} + 4z^2\dot{y} - 6yz\dot{z}\right)\text{Cos}\alpha + (3yz\dot{y} - 3xyz\vartheta \\ +x^2\dot{z} - 2y^2\dot{z} + z^2\dot{z}\big)\text{Cos}(\alpha-\theta) + \left(3xyz\vartheta - 3yz\dot{y} - x^2\dot{z} + 2y^2\dot{z} - z^2\dot{z}\right)\text{Cos}(\alpha+\theta) \\ +3\left(xz\dot{y} - x^2z\vartheta - xy\dot{z}\right)\text{Sin}(\alpha-\theta) + 3\left(xz\dot{y} - x^2z\vartheta - xy\dot{z}\right)\text{Sin}(\alpha+\theta)\end{array}\right\}, \tag{22}$$

$$a_{Ly} = \frac{Q}{2r^5}\left\{\begin{array}{c}\left(2x^2y\vartheta + 2y^3\vartheta - 4yz^2\vartheta + 2x^2\dot{x} + 2y^2\dot{x} - 4z^2\dot{x} + 6xz\dot{z}\right)\text{Cos}\alpha + 3\left(xy\dot{z} - y^2z\vartheta - yz\dot{x}\right)\cdot \\ \text{Cos}(\alpha-\theta) + 3\left(y^2z\vartheta + yz\dot{x} - xy\dot{z}\right)\text{Cos}(\alpha+\theta) + \left(-3xyz\vartheta - 3xz\dot{x} + 2x^2\dot{z} - y^2\dot{z} - z^2\dot{z}\right)\cdot \\ \text{Sin}(\alpha-\theta) + \left(-3xyz\vartheta - 3xz\dot{x} + 2x^2\dot{z} - y^2\dot{z} - z^2\dot{z}\right)\text{Sin}(\alpha+\theta)\end{array}\right\}, \tag{23}$$

$$a_{Lz} = \frac{Q}{2r^5}\left\{\begin{array}{c}6\left(x^2z\vartheta + y^2z\vartheta + yz\dot{x} - xz\dot{y}\right)Cos\alpha + \left(2x^2y\vartheta + 2y^3\vartheta - yz^2\vartheta - x^2\dot{x} + y^2\dot{x} - z^2\dot{x} - 3xy\dot{y}\right) \\ Cos(\alpha-\theta) + \left(-2x^2y\vartheta - 2y^3\vartheta + yz^2\vartheta + x^2\dot{x} - y^2\dot{x} + z^2\dot{x} + 3xy\dot{y}\right)Cos(\alpha+\theta) \\ +\left(2x^3\vartheta + 2xy^2\vartheta - xz^2\vartheta + 3xy\dot{x} - 2x^2\dot{y} + y^2\dot{y} + z^2\dot{y}\right)Sin(\alpha-\theta)+ \\ \left(2x^3\vartheta + 2xy^2\vartheta - xz^2\vartheta + 3xy\dot{x} - x^2\dot{y} + y^2\dot{y} + z^2\dot{y}\right)Sin(\alpha+\theta)\end{array}\right\}, \tag{24}$$

where $Q = \tilde{q}B_0$.

4. Lorentz Three-Body Problem

The equation of motion for the three-body problem in existence of Lorenz acceleration, using Equations (22)–(24) into Equations (4)–(6), will be

$$\ddot{x} - 2\dot{y} = (x+1-\mu) - (1-\mu)\frac{(x+1)}{r_1^3} - \mu\frac{x}{r^3} + \frac{Q}{2r^5}\left\{\begin{array}{c} \left(2x^3\vartheta + 2xy^2\vartheta - 4xz^2\vartheta - 2x^2\dot{y} - 2y^2\dot{y} + 4z^2\dot{y} - 6yz\dot{z}\right) \\ \text{Cos}\alpha + \left(3yz\dot{y} - 3xyz\vartheta + x^2\dot{z} - 2y^2\dot{z} + z^2\dot{z}\right)\text{Cos}\,(\alpha-\theta) \\ +\left(3xyz\vartheta - 3yz\dot{y} - x^2\dot{z} + 2y^2\dot{z} - z^2\dot{z}\right)\text{Cos}\,(\alpha+\theta) + 3\left(xz\dot{y} - x^2z\vartheta - xy\dot{z}\right)\cdot \\ \text{Sin}\,(\alpha-\theta) + 3\left(xz\dot{y} - x^2z\vartheta - xy\dot{z}\right)\text{Sin}\,(\alpha+\theta) \end{array}\right\}, \tag{25}$$

$$\ddot{y} + 2\dot{x} = y - (1-\mu)\frac{y}{r_1^3} - \frac{\mu y}{r^3} + \frac{Q}{2r^5}\left\{\begin{array}{c} \left(2x^2y\vartheta + 2y^3\vartheta - 4yz^2\vartheta + 2x^2\dot{x} + 2y^2\dot{x} - 4z^2\dot{x} + 6xz\dot{z}\right)\cdot \\ Cos\alpha + 3\left(xy\dot{z} - y^2z\vartheta - yz\dot{x}\right)Cos\,(\alpha-\theta) + 3\left(y^2z\vartheta + yz\dot{x} - xy\dot{z}\right)Cos\,(\alpha+\theta) \\ +\left(-3xyz\vartheta - 3xz\dot{x} + 2x^2\dot{z} - y^2\dot{z} - z^2\dot{z}\right)Sin\,(\alpha-\theta) + \left(-3xyz\vartheta - 3xz\dot{x} + 2x^2\dot{z} - -y^2\dot{z} - z^2\dot{z}\right)Sin\,(\alpha+\theta) \end{array}\right\} \tag{26}$$

$$\ddot{z} = -(1-\mu)\frac{z}{r_1^3} - \mu\frac{z}{r^3} + \frac{Q}{2r^5}\left\{\begin{array}{c} 6\left(x^2z\vartheta + y^2\vartheta + yz\dot{x} - xz\dot{y}\right) \\ \text{Cos}\alpha + \left(2x^2y\vartheta + 2y^3\vartheta - yz^2\vartheta - x^2\dot{x} + y^2\dot{x} - z^2\dot{x} - 3xy\dot{y}\right)\text{Cos}\,(\alpha-\theta) + \\ \left(-2x^2y\vartheta - 2y^3\vartheta + yz^2\vartheta + x^2\dot{x} - y^2\dot{x} + \right. \\ \left. +z^2\dot{x} + 3xy\dot{y}\right)\text{Cos}\,(\alpha+\theta) + \\ \left(2x^3\vartheta + 2xy^2\vartheta - xz^2\vartheta + 3xy\dot{x} - 2x^2\dot{y} + y^2\dot{y} + z^2\dot{y}\right)\cdot \\ \text{Sin}\,(\alpha-\theta) + \left(2x^3\vartheta + 2xy^2\vartheta - xz^2\vartheta + 3xy\dot{x} - x^2\dot{y} + y^2\dot{y} + z^2\dot{y}\right)\text{Sin}\,(\alpha+\theta) \end{array}\right\}. \tag{27}$$

We note that the units of the quantity $Q = qB_0/m$ are $(c/M) \times (ML^3/cT)$ which is after simplifying $T^{-1}L^3$ where we use C for the dimension of electric charge, M for mass, T for time, and L for length.

Equations (25)–(27) are the dimensionless controlling equations of the motion of a small, charged body under the gravitational effect of two big bodies with one of them has a significant magnetic field. The equations for the planar problem, $z = 0$, are

$$\ddot{x} - 2\dot{y} = (x+1-\mu) - (1-\mu)\frac{(x+1)}{r_1^3} - \mu\frac{x}{r^3} + \frac{Q}{r3}[x\vartheta - \dot{y}]\text{Cos}\alpha, \tag{28}$$

$$\ddot{y} + 2\dot{x} = y - (1-\mu)\frac{y}{r_1^3} - \frac{\mu y}{r^3} + \frac{Q}{r3}[y\vartheta + \dot{x}]\text{Cos}\alpha. \tag{29}$$

5. Artificial Equilibrium Points

The liberation points are found by letting $\ddot{x} = \dot{x} = \ddot{y} = \dot{y} = 0$, thus equations (28) and (29) will give

$$(x+1-\mu) - (1-\mu)\frac{(x+1)}{r_1^3} - \mu\frac{x}{r^3} + \frac{Q\vartheta x}{r^3}\text{Cos}\alpha = 0, \tag{30}$$

$$y - (1-\mu)\frac{y}{r_1^3} - \frac{\mu y}{r^3} + \frac{Q\vartheta y}{r^3}\text{Cos}\alpha = 0. \tag{31}$$

Equations (30) and (31) represent the controlling equations depending on the position of the spacecraft's liberation points and the charge (per unit mass) required to generate Lorenz acceleration and the masses of the primaries.

From Equation (31), we have

$$y = 0, \tag{32}$$

or,

$$1 - (1-\mu)\frac{1}{r_1^3} - \frac{\mu}{r^3} + \frac{Q\vartheta}{r^3}\text{Cos}\alpha = 0. \tag{33}$$

Equation (32) leads to the collinear solution while Equation (33) gives us the equilateral triangular solution as a function of the charge per unit mass, the magnetic field strength of the smaller primary and the relative masses of the primaries.

5.1. Controlling the Collinear Solution. To control the collinear solution, substituting $y = 0$ into Equation (30) to get

$$(x+1-\mu) - (1-\mu)\frac{(x+1)}{r_1^3} - \mu\frac{x}{r^3} + \frac{Q\vartheta x}{r^3}\text{Cos}\alpha = 0. \tag{34}$$

Remembering that $\overline{r}_1$ is the position vector between the body m_1 and the infinitesimal body. From Figure 1, $\overline{r}_1 = \overline{r} + \overline{\ell}$ and in the case of collinear, r_1 is $x + \ell$. Noting that the distance between m_1 andm_2, ℓ is the unit of distances. While

r is the distance between the body m_2 and the infinitesimal body is x in the collinear case, then Equation (34) will be:

$$(x+1-\mu)-\frac{(1-\mu)(x+1)}{|x+1|^3}-\frac{\mu x}{|x|^3}+\frac{Q\vartheta x}{|x|^3}\mathrm{Cos}\alpha=0. \tag{35}$$

Equation (35) is composed of the three scalar equations for the coordinate x of the collinear liberation points L_1, L_2, and L_3 as shown in Figure 3:

For L_1 $(x<-1)$,

$$\begin{aligned}&x^5+(3-\mu)x^4+(3-2\mu)x^3+(2-\mu-Q\vartheta\mathrm{Cos}\alpha)x^2\\&+2(\mu-Q\vartheta\mathrm{Cos}\alpha)x+\mu-Q\vartheta\mathrm{Cos}\alpha=0.\end{aligned} \tag{36}$$

For L_2 $(0>x>-1)$,

$$\begin{aligned}&x^5+(3-\mu)x^4+(3-2\mu)x^3+(\mu-Q\vartheta\mathrm{Cos}\alpha)x^2\\&+2(\mu-Q\vartheta\mathrm{Cos}\alpha)x+\mu-Q\vartheta\mathrm{Cos}\alpha=0.\end{aligned} \tag{37}$$

For L_3 $(x>0)$,

$$\begin{aligned}&x^5+(3-\mu)x^4+(3-2\mu)x^3-(\mu-Q\vartheta\mathrm{Cos}\alpha)x^2\\&-2(\mu-Q\vartheta\mathrm{Cos}\alpha)x-(\mu-Q\vartheta Cos\alpha)=0.\end{aligned} \tag{38}$$

Remembering that $Q=\tilde{q}B_0$ with $\tilde{q}$ is the charge per unit mass. Solving Equation (36)–(38), for different cases, for $\tilde{q}$, we get

For L_1,

$$\tilde{q}=\left[\frac{x^5+(3-\mu)x^4+(3-2\mu)x^3+(2-\mu)x^2+2\mu x+\mu}{B_0\vartheta(x+1)^2}\right]\mathrm{Sec}\alpha. \tag{39}$$

For L_2,

$$\tilde{q}=\left[\frac{x^5+(3-\mu)x^4+(3-2\mu)x^3+\mu x^2+2\mu x+\mu}{B_0\vartheta(x+1)^2}\right]\mathrm{Sec}\alpha. \tag{40}$$

For L_3,

$$\tilde{q}=-\left[\frac{x^5+(3-\mu)x^4+(3-2\mu)x^3-\mu x^2-2\mu x-\mu}{B_0\vartheta(x+1)^2}\right]\mathrm{Sec}\alpha. \tag{41}$$

From Equations (39)–(41), we can get the range of charge per unit mass to control the position of the liberation points L_1, L_2, or L_3.

6. Numerical Investigation for the Case of the Sun-Jupiter System

The magnetic field of Jupiter has the values $B_0=1.5812\times10^{20}$ Tesla·meter^3. The parameter $\mu=0.000954$, the mean motion $n=0.00006042$ hour^{-1}, the Sun-Jupiter distance $L=779\times10^6$ Km., and the angular velocity $\vartheta=0.63301$ hour^{-1} (Nasa Planetary data sheet) [30]. Thus, the dimensionless angular velocity will have the value $\omega=\vartheta/n=10476.829$, and the dimensionless quantity $\tilde{q}B_0/nL^3=5.80366\times10^{-9}\tilde{q}$. Figures 4–6 illustrate the relation between the specific charge

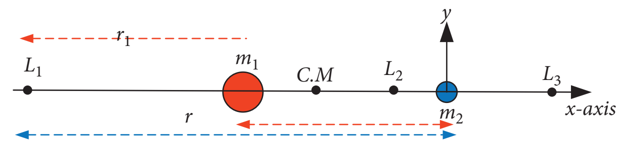

Figure 3: The position of collinear liberation points.

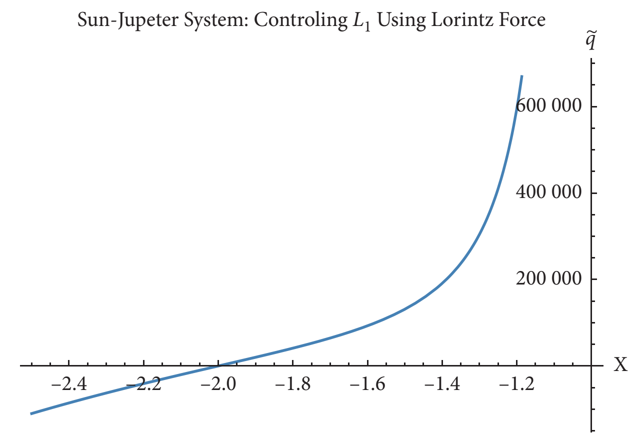

Figure 4: Specific charge to generate artificial liberation points about $L_1=-1.9994437087809749$.

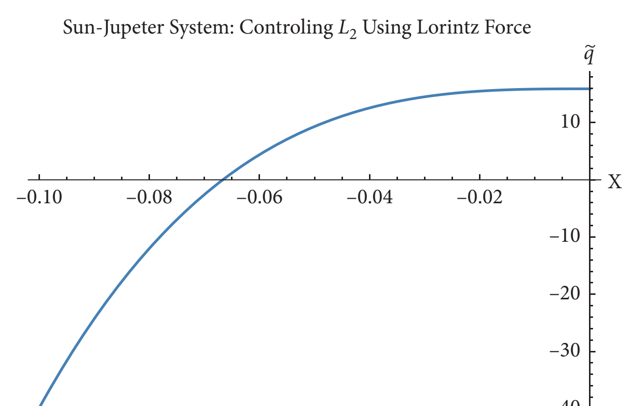

Figure 5: Specific charge to generate artificial liberation points about $L_2=-0.06667522806903911$.

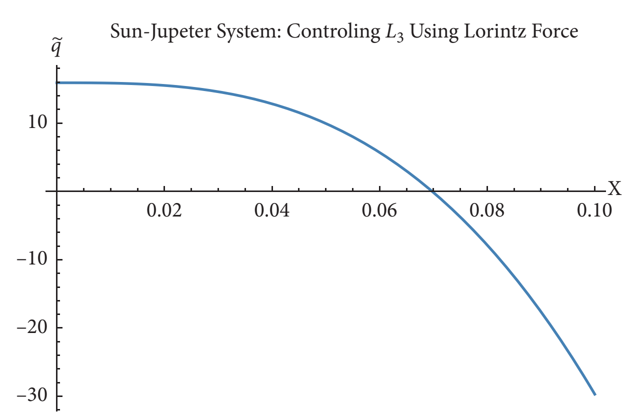

Figure 6: Specific charge to generate artificial liberation points about $L_3=0.06977858130630588$.

Table 1: The charge per unit mass required to change the liberation poits

Δx (dimensionless)	Δx *(km)*	$\tilde{q}$ (L_1)	$\tilde{q}$ (L_2)	$\tilde{q}$ (L_3)
−0.000002	−1556.6	−0.40031741	−0.001467102	0.001339323331
−0.000001	−778.3	−0.20015881	−0.00073353942	0.0006696708735
0.0	0.0	0.0	0.0	0.0
0.000001	778.3	0.2001584	0.0007335163	−0.0006696893
0.000002	1556.6	0.400317004	0.0014670095	−0.001339397

and the artificial liberation points for L_1, L_2, and L_3, respectively.

7. Conclusion and Discussions

In this work, we studied the possibility of generating artificial collinear liberation points for the planer circular restricted three-body problem using Lorentz force affecting a charged spacecraft due to the magnetic field of a planet. Although theoretically, we have a relation between the charge per unit mass and the required position of the collinear points, actually two points only can be controlled; those are L_2 and L_3 where L_1 is very far from being controlled using Lorentz force. This is expected since in the case of L_1 the bigger primary lies between the magnetic field of the smaller primary and the third body. The results show the possibility of controlling the positions of L_2 and L_3 with reasonable values of the charge per unit mass as shown in Figures 5 and 6.

The unity of distance was used as the Sun-Jupiter distance and is 778300000 km, so to change the liplation points by amount $x = 0.000001$ from the natural liberation points ($L_1 = -1.9999443708782$, $L_2 = -0.066675228069$, and $L_3 = 0.069778130631$), this means changing the points by actual distance (778.3 km). The charge per unit mass required to change the liberation points in hundreds of kilometres is shown in Table 1.

Concerning the stability of these points, it is well known that the classical collinear points are not stable; however, with the introduction of an artificial parameter of control, it is an interesting question to ask about the possibility of forcing artificial stability similar to generating artificial equilibrium points. This could be the subject of future work.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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