# Noncollinear Equilibrium Points in CRTBP with Yukawa-Like Corrections to Newtonian Potential under an Oblate Primary Model 

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#### Abstract

This study is about the effects of Yukawa-like corrections to Newtonian potential on the existence and stability of noncollinear equilibrium points in a circular restricted three-body problem when bigger primary is an oblate spheroid. It is observed that $\partial x_{0} /$ $\partial \lambda=0=\partial y_{0} / \partial \lambda$ at $\lambda_{0}=1 / 2$, so we have a critical point $\lambda_{0}=1 / 2$ at which the maximum and minimum values of $x_{0}$ and $y_{0}$ can be obtained, where $\lambda \in(0, \infty)$ is the range of Yukawa force and $\left(x_{0}, y_{0}\right)$ are the coordinates of noncollinear equilibrium points. It is found that $x_{0}$ and $y_{0}$ are increasing functions in $\lambda$ in the interval $0<\lambda<\lambda_{0}$ and decreasing functions in $\lambda$ in the interval $\lambda_{0}<\lambda<\infty$ for all $\alpha \in \alpha^{+}$. On the other hand, $x_{0}$ and $y_{0}$ are decreasing functions in $\lambda$ in the interval $0<\lambda<\lambda_{0}$ and increasing functions in $\lambda$ in the interval $\lambda_{0}<\lambda<\infty$ for all $\alpha \in \alpha^{-}$, where $\alpha \in(-1,1)$ is the coupling constant of Yukawa force to gravitational force. The noncollinear equilibrium points are found linearly stable for the critical mass parameter $\beta_{0}$, and it is noticed that $\partial \beta_{0} / \partial \lambda=0$ at $\lambda^{*}=1 / 3$; thus, we got another critical point which gives the maximum and minimum values of $\beta_{0}$. Also, $\partial \beta_{0} / \partial \lambda>0$ if $0<\lambda<\lambda^{*}$ and $\partial \beta_{0} / \partial \lambda<0$ if $\lambda^{*}<\lambda<\infty$ for all $\alpha \in \alpha^{-}$, and $\partial \beta_{0} / \partial \lambda<0$ if $0<\lambda<\lambda^{*}$ and $\partial \beta_{0} / \partial \lambda>0$ if $\lambda^{*}<\lambda<\infty$ for all $\alpha \in \alpha^{+}$. Thus, the local minima for $\beta_{0}$ in the interval $0<\lambda<\lambda^{*}$ can also be obtained.


## 1. Introduction

The general three-body problem deals with the motion of three arbitrary spherically symmetric bodies considered as a point mass. The motions of these bodies are related to the Newtonian force of gravity, which are superimposed on each other and have no specific path. The closed form of analytical solution to the general three-body problem is yet to be determined.

The restricted three-body problem (R3BP) is an approximation of the general three-body problem in which one body is treated as having an infinitesimal mass compared to the other two bodies. The bigger bodies are called primaries which revolve around their common center of mass in circular or elliptical orbits in a rotating coordinate system in which the infinitesimal mass also moves without disturbing the motion of
the primaries. The restricted three-body problem has five equilibrium points, three collinear, and two noncollinear or triangular. The collinear equilibria are unstable for all values of mass parameter but triangular equilibria are stable for a critical mass parameter $\mu_{0}=0.03852$ [1].

The restricted three-body problem has been studied by many researchers in last two decades in different aspects. In the classical restricted three-body problem, the primaries are assumed as spherical in shape, but in real situation, several heavenly bodies such as Earth, Saturn, and Jupiter are sufficiently oblate. The oblateness effect in the restricted three-body problem has been investigated by El-Shaboury [2]; Khanna and Bhatnagar [3]; Raheem and Singh [4]; Ammar et al. [5]; Idrisi and Taqvi [6, 7], Singh and Umar [8]; Bury and McMahon [9]; Saeed and Zotos [10]; Alrebdi et al. [11] etc.

New theories in the contemporary world predict improvements to the theory of gravity. The Yukawa potential was first proposed by Yukawa [12] to modify the Newtonian potential. The strong interactions between particles are well described by the Yukawa potential, a nonrelativistic
potential. In a two-body problem, the modified potential energy may be used to express the gravity effects on the secondary primary $m$ in the presence of the Yukawa correction [13] as

$$
\begin{equation*}
V(r)=-\frac{G M m}{r}\left(1+\alpha e^{-(r / \lambda)}\right)=-\frac{G M m}{r}-\frac{G M m}{r} \alpha e^{-(r / \lambda)}=V_{N}(r)+V_{Y}(r), \tag{1}
\end{equation*}
$$

where $V_{N}(r)$ is the Newtonian potential between the two bodies $m$ and $M, V_{Y}(r)$ is the Yukawa correction to the Newtonian potential, $r$ is the distance between $m$ and $M, G$ is the Newtonian gravitational constant, $\alpha \epsilon(-1,1)$ is the coupling constant of the Yukawa force to the Gravitational force, and $\lambda \epsilon(0, \infty)$ is the range of the Yukawa force [14]. Therefore, the corresponding force between $m$ and $M$ can be expressed as

$$
\begin{equation*}
\vec{F}(r)=-\frac{G M m}{r^{2}}\left\{1+\alpha\left(1+\frac{r}{\lambda}\right) e^{-(r / \lambda)}\right\} \vec{r} \tag{2}
\end{equation*}
$$

As $\alpha \longrightarrow 0$, the Newtonian gravitational force can be obtained.

In the restricted three-body problem, Kokubun [14] has included Yukawa-like corrections to Newtonian potential. His findings differed significantly from the purely Newtonian case. Reference [15] provides the minimal values of the Yukawa coupling constant for the artificial satellites LAGEOS and LAGEOS II. Massa [16] investigated Mach's principle and Yukawa potential within the Sciama linear approach framework. Haranas and Ragos [17] investigated satellite dynamics while taking Yukawa-like corrections into account. Pricopi [18] has investigated the stability of celestial
orbits under the effect of the Yukawa potential in the twobody problem. Reference [19] has analyzed the elliptical and circular orbits of the Earth while taking into account the Yukawa potential and Poynting-Robertson effect. The dynamics and stability of the two-body problem were examined by Cavan et al. [20] while taking the Yukawa corrections to Newtonian potential into account. Idrisi et al. [21] have investigated the triangular equilibria in the framework of Yukawa correction to Newtonian potential in the circular restricted three-body problem.

The dynamics surrounding noncollinear equilibrium points in a circular restricted three-body problem with a Yukawa-like adjustment to Newtonian potential under an oblate primary model piqued our attention. The existence and linear stability of noncollinear equilibrium points under an oblate primary model with Yukawa like-corrections to Newtonian potential have been examined in this study.

## 2. Yukawa Correction to Newtonian Potential

The modified potential between two bodies $M$ and $m$ can be described as follows:

$$
\begin{equation*}
V(r)=-\frac{G M m}{r}\left(1+\alpha e^{-(r / \lambda)}\right)=-\frac{G M m}{r}-\frac{G M m}{r} \alpha e^{-(r / \lambda)}=V_{N}(r)+V_{Y}(r) \tag{3}
\end{equation*}
$$

where $V_{N}(r)=$ Newtonian potential between the two bodies $M$ and $m, V_{Y}(r)=$ Yukawa correction to the Newtonian potential, $r=$ distance between $m$ and $M, G=$ Newtonian gravitational constant, $\alpha \in(-1,1)$ is the coupling constant of

Yukawa force to the gravitational force, and $\lambda \in(0, \infty)$ is the range of Yukawa force [14].

Therefore, the corresponding force between $M$ and $m$ can be expressed as

$$
\begin{equation*}
F(r)=\frac{G M m}{r^{2}}\left\{1+\alpha\left(1+\frac{r}{\lambda}\right) e^{-(r / \lambda)}\right\}=\frac{G M m}{r^{2}}+\frac{G M m}{r^{2}} \alpha\left(1+\frac{r}{\lambda}\right) e^{-(r / \lambda)}=F_{N}(r)+F_{Y}(r) \tag{4}
\end{equation*}
$$

where $F_{N}(r)=$ Newtonian gravitational force between $M$ and $m$ and $F_{Y}(r)=$ Yukawa correction to Newtonian gravitational force between $M$ and $m$.

From (4), as $\alpha \longrightarrow 0$ or $\lambda \longrightarrow 0$, the term $F_{Y}(r)$ vanishes and $F(r)=F_{N}(r)$. If $\alpha<0, F(r)<F_{N}(r)$ and for $\alpha>0, F(r)$ $>F_{N}(r)$, Figure 1 . Thus, as $\alpha$ increases in the interval $(-1,1)$, the force between $m$ and $M$ also increases and vice-versa.

But as $\lambda \longrightarrow \infty$ the force between $M$ and $m$ is given by

$$
\begin{equation*}
F_{\infty}(r)=\frac{G M m}{r^{2}}(1+\alpha) . \tag{5}
\end{equation*}
$$

From (5), it is clear that as $\alpha \longrightarrow-1, F_{\infty}(r) \longrightarrow 0$, i.e., the force between $m$ and $M$ reduces as $\alpha$ reduces. For $\alpha \longrightarrow 0$, $F_{\infty}(r) \longrightarrow F_{N}(r)$ and the Newtonian gravitational force can


Figure 1: The force function $F(r)$ with respect to $\lambda$.

## 3. Model Description and Equations of Motion

Let us consider two primaries $P_{1}$ and $P_{2}$ having masses $m_{1}$ and $m_{2}\left(m_{1}>m_{2}\right)$ moving around their common center of mass in circular orbits. The more massive primary $m_{1}$ is considered to be an oblate body while less massive primary $m_{2}$ is spherical in shape. The equations of motion of the infinitesimal mass in a barycentric synodic co-ordinate system ( $x, y$ ) and dimensionless variables are

$$
\begin{equation*}
\ddot{x}-2 n \dot{y}=U_{x}, \ddot{y}+2 n \dot{x}=U_{y}, \tag{6}
\end{equation*}
$$

and the potential function $U$ can be expressed as
be obtained. But as $\alpha \longrightarrow 1, F_{\infty}(r) \longrightarrow 2 F_{N}(r)$, i.e., the force acting between $m$ and $M$ is twice of the Newtonian gravitational force, as shown in Figure 2.

$$
\begin{equation*}
U=\frac{n^{2}}{2}\left(x^{2}+y^{2}\right)+\frac{m_{1}}{r_{1}}\left(1+\frac{\sigma}{2 r_{1}^{2}}\right)\left(1+\alpha e^{-r_{1} / \lambda}\right)+\frac{m_{2}}{r_{2}}\left(1+\alpha e^{-r_{2} / \lambda}\right) \tag{7}
\end{equation*}
$$

$\sigma=\left(r_{\mathrm{e}}^{2}-r_{\mathrm{p}}^{2}\right) / 5 r^{2}$ is the oblateness factor due to bigger primary $m_{1}, r_{e}$ and $r_{p}$ are the equatorial and polar radii respectively of $m_{1}, r$ is the distance between $m_{1}$ and $m_{2}$ considered as unity, $n$ is the mean-motion of the primaries, and defined as

$$
\begin{equation*}
n^{2}=\left(1+\frac{3 \sigma}{2}\right)\left[1+\alpha\left(1+\frac{1}{\lambda}\right) e^{-1 / \lambda}\right] \tag{8}
\end{equation*}
$$

$|\alpha|<1$ is the coupling constant of Yukawa force to gravitational force, $\lambda \in(0, \infty)$ is the range of Yukawa force.

We can define a mass parameter $\beta>0$ as

$$
\begin{equation*}
\beta=\frac{m_{2}}{m_{1}+m_{2}}<\frac{1}{2} \Rightarrow m_{1}=1-\beta ; \quad m_{2}=\beta . \tag{9}
\end{equation*}
$$

Therefore, the distances of infinitesimal mass from the primaries $P_{1}$ and $P_{2}$, are given by

$$
\begin{equation*}
r_{1}=\left|(x-\beta)^{2}+y^{2}\right| \text { and } r_{2}=\left|(x+1-\beta)^{2}+y^{2}\right| \tag{10}
\end{equation*}
$$

The Jacobi integral associated to the problem is given by

$$
\begin{equation*}
v^{2}=2 U-C \tag{11}
\end{equation*}
$$

where $v$ is the velocity of infinitesimal mass and $C$ is Jacobi constant.

## 4. Noncollinear Equilibrium Points

The noncollinear equilibrium points are the solution of the equations $U_{x}=0$ and $U_{y}=0, y \neq 0$, i.e.,

$$
\begin{gather*}
n^{2} x-\frac{(1-\beta)(x-\beta)}{r_{1}^{3}}\left(1+\frac{3 \sigma}{2 r_{1}^{2}}\right)\left[1+\alpha\left(1+\frac{r_{1}}{\lambda}\right) e^{-\left(r_{1} / \lambda\right)}\right]-\frac{\beta(x+1-\beta)}{r_{2}^{3}}\left[1+\alpha\left(1+\frac{r_{2}}{\lambda}\right) e^{-\left(r_{2} / \lambda\right)}\right]=0  \tag{12}\\
n^{2}-\frac{(1-\beta)}{r_{1}^{3}}\left(1+\frac{3 \sigma}{2 r_{1}^{2}}\right)\left[1+\alpha\left(1+\frac{r_{1}}{\lambda}\right) e^{-r_{1} / \lambda}\right]-\frac{\beta}{r_{2}^{3}}\left[1+\alpha\left(1+\frac{r_{2}}{\lambda}\right) e^{-r_{2} / \lambda}\right]=0 \tag{13}
\end{gather*}
$$



Figure 2: The force function $F_{\infty}(r)$ with respect to $\alpha$.

On eliminating $r_{1}$ and $r_{2}$ from the (12) and (13), respectively, we have

$$
\begin{align*}
& r_{2}^{3}=\frac{1}{n^{2}}\left[1+\alpha\left(1+\frac{r_{2}}{\lambda}\right) e^{-r_{2} / \lambda}\right]  \tag{14}\\
& r_{1}^{3}=\frac{1}{n^{2}}\left(1+\frac{3 \sigma}{2 r_{1}^{2}}\right)\left[1+\alpha\left(1+\frac{r_{1}}{\lambda}\right) e^{-r_{1} / \lambda}\right] \tag{15}
\end{align*}
$$

The solution of (15) is $r_{1}=1$. To solve (14), we assume that $r_{2}=1+\delta, \delta \ll 1$. On substituting $r_{2}=1+\delta$ in (14) and considering only linear terms in $\alpha$ and $\delta$ and then solving it for $\delta$, we obtain

$$
\begin{equation*}
\delta=-\frac{\sigma}{2} f(\alpha, \lambda) ; f(\alpha, \lambda)=1-\frac{\alpha}{3} \frac{e^{-1 / \lambda}}{\lambda^{2}} \tag{16}
\end{equation*}
$$

$$
\text { Thus } r_{2}=1-\frac{\sigma}{2} f(\alpha, \lambda)
$$

Now, solving $r_{1}=1$ and (17), we have the coordinates of noncollinear equilibrium points $E_{4,5}\left(x_{0}, y_{0}\right)$, i.e.,

$$
\begin{equation*}
x_{0}=\beta-\frac{1}{2}-\frac{\sigma}{2} f(\alpha, \lambda), y_{0}= \pm \frac{\sqrt{3}}{2}\left(1-\frac{\sigma}{3} f(\alpha, \lambda)\right) \tag{17}
\end{equation*}
$$

For a nonoblate case, i.e., $\sigma=0$ we obtain $r_{i}=1$ which is the classic case of restricted three-body problem [1], and hence in the nonoblate case, the noncollinear equilibrium points are not affected by the Yukawa force [21]. For $\alpha=0$, the results are agreed with [22].

As shown in Figure 3, $f(\alpha, \lambda)$ is a continuous function for $|\alpha|<1$ and $\lambda \in(0, \infty)$, and $\lim _{\lambda \longrightarrow \infty} f(\alpha, \lambda)=1=\lim _{\lambda \longrightarrow 0}$ $f(\alpha, \lambda)$. Thus, for very small and large values of $\lambda$, the term
$f(\alpha, \lambda)$ becomes unity and the noncollinear equilibrium points $E_{4,5}\left(x^{*}, y^{*}\right)$ in this case are given by

$$
\begin{equation*}
x^{*}=\beta-\frac{1}{2}-\frac{\sigma}{2}, y^{*}= \pm \frac{\sqrt{3}}{2}\left(1-\frac{\sigma}{3}\right) . \tag{18}
\end{equation*}
$$

Since $\partial x_{0} / \partial \lambda=0=\partial y_{0} / \partial \lambda$ at $\lambda=1 / 2$. So we have a critical point $\lambda=\lambda_{0}=1 / 2$ at which the maximum and minimum values of $x_{0}$ and $y_{0}$ can be obtained. As it has been examined that $\partial x_{0} / \partial \lambda, \partial y_{0} / \partial \lambda>0$ for $0<\lambda<\lambda_{0}$ and $\partial x_{0} / \partial \lambda, \partial y_{0} / \partial \lambda<0$ for $\lambda_{0}<\lambda<\infty$ for all $\alpha \in \alpha^{+}$. Similarly, $\partial x_{0} / \partial \lambda, \partial y_{0} / \partial \lambda<0$ for $0<\lambda<\lambda_{0}$ and $\partial x_{0} / \partial \lambda, \partial y_{0} / \partial \lambda>0$ for $\lambda_{0}<\lambda<\infty$ for all $\alpha \in \alpha^{-}$. Thus, it is concluded that $x_{0}$ and $y_{0}$ are increasing functions in $\lambda$ in the interval $0<\lambda<\lambda_{0}$ and decreasing functions in $\lambda$ in the interval $\lambda_{0}<\lambda<\infty$ for all $\alpha \in \alpha^{+}$. On the other hand, $x_{0}$ and $y_{0}$ are decreasing functions in $\lambda$ in the interval $0<\lambda<\lambda_{0}$ and increasing functions in $\lambda$ in the interval $\lambda_{0}<\lambda<\infty$ for all $\alpha \in \alpha^{-}$.

For $\alpha \in \alpha^{+}$, as $\lambda$ increases in the interval $0<\lambda<\lambda_{0}$, the abscissa $x_{0}$ of $E_{4}$ moves toward the center of mass of the system and the ordinate $y_{0}$ moves vertically upward and vice-versa. In the interval $\lambda_{0}<\lambda<\infty, x_{0}$ and $y_{0}$ decrease and hence approach to $x^{*}$ and $y^{*}$, respectively, as $\lambda$ increases and vice versa. For $\alpha \in \alpha^{-}$, as $\lambda$ increases in the interval $0<\lambda<\lambda_{0}$, the abscissa $x_{0}$ moves away from $x^{*}$ and $y_{0}$ moves vertically downward and vice-versa. In the interval $\lambda_{0}<\lambda<\infty, x_{0}$ and $y_{0}$ increase and hence approach to $x^{*}$ and $y^{*}$, respectively, as $\lambda$ increases and vice-versa (Figures 4 and 5).

The noncollinear equilibrium points $E_{4,5}$ at the critical point $\lambda=1 / 2$ have maximum or minimum values according to $\alpha \in \alpha^{+}$or $\alpha \in \alpha^{-}$, respectively, are given as

$$
\begin{equation*}
x_{c}=\beta-\frac{1}{2}-\frac{\sigma}{2}\left(1-\frac{4}{3} \alpha e^{-2}\right), y_{c}= \pm \frac{\sqrt{3}}{2}\left[1-\frac{\sigma}{3}\left(1-\frac{4}{3} \alpha e^{-2}\right)\right] . \tag{19}
\end{equation*}
$$



Figure 3: Curves of $f(\alpha, \lambda)$ with respect to $\lambda$.


Figure 4: Curves of $x_{0}$ with respect to $\lambda$.


Figure 5: Curves of $y_{0}$ with respect to $\lambda$.


Figure 6: Critical mass parameter $\beta_{0}$ versus $\lambda$.

## 5. Stability of Equilibrium Points

The variational equations of motion can be obtained by perturbing the equilibrium point $\left(x_{0}, y_{0}\right)$ to a small displacement $\left(\delta_{1}, \delta_{2}\right), \delta_{i} \ll 1, i=1,2$. Therefore, on substituting $x=x_{\mathrm{o}}+\delta_{1}$ and $y=y_{o}+\delta_{2}$ in Equation (3), we have

$$
\left.\begin{array}{l}
\ddot{\delta}_{1}-2 n \dot{\delta}_{2}=\delta_{1} \stackrel{o}{U}_{x x}+\delta_{2} \stackrel{o}{U}_{x y}  \tag{20}\\
\ddot{\delta}_{2}+2 n \dot{\delta}_{1}=\delta_{1} \stackrel{o}{U}_{x y}+\delta_{2} \stackrel{o}{U}_{y y}
\end{array}\right\}
$$

$$
\begin{align*}
& \stackrel{o}{U}_{x x}=\left.\frac{\partial^{2} U}{\partial x^{2}}\right|_{\left(x_{0}, y_{0}\right)}=\frac{3}{4}+\frac{\left(3 \lambda^{2}+3 \lambda+1\right)}{4 \lambda^{2}} \alpha e^{-1 / \lambda}-3\left(\beta-\frac{9}{8}\right) \sigma \\
& \stackrel{o}{U}_{x y}=\left.\frac{\partial^{2} U}{\partial x \partial y}\right|_{\left(x_{0}, y_{0}\right)}=\frac{3 \sqrt{3}}{2}\left(\beta-\frac{1}{2}\right)+\frac{\sqrt{3}(2 \beta-1)\left(3 \lambda^{2}+3 \lambda+1\right)}{4 \lambda^{2}} \alpha e^{-1 / \lambda}+\frac{13 \sqrt{3}}{4}\left(\beta-\frac{19}{26}\right) \sigma,  \tag{22}\\
& \stackrel{o}{U}_{y y}=\left.\frac{\partial^{2} U}{\partial y^{2}}\right|_{\left(x_{0}, y_{0}\right)}=\frac{9}{4}+\frac{3\left(3 \lambda^{2}+3 \lambda+1\right)}{4 \lambda^{2}} \alpha e^{-1 / \lambda}+\frac{33}{8} \sigma .
\end{align*}
$$

The quadratic equation corresponding to (22) is given by

$$
\begin{equation*}
\Upsilon^{2}+p_{1} \Upsilon+p_{2}=0 \tag{23}
\end{equation*}
$$

where $\Upsilon=\Gamma^{2}, p_{1}=4 n^{2}-\stackrel{o}{U}_{x x}-\stackrel{o}{U}_{y y}, p_{2}=\stackrel{o}{U}_{x x} \stackrel{o}{U}_{y y}-\left(\stackrel{o}{U}_{x y}\right)^{2}$.
The roots of (24) are

$$
\begin{equation*}
\Upsilon_{1,2}=1 / 2\left(-p_{1} \pm \sqrt{p_{1}^{2}-4 p_{2}}\right) \tag{24}
\end{equation*}
$$

The motion near the equilibrium point $\left(x_{0}, y_{0}\right)$ is said to be bounded if $p_{1}^{2}-4 p_{2} \geq 0$, i.e.,

As $\delta_{i} \ll 1$ and $|\alpha|<1$, therefore we consider only linear terms in $\delta_{1}, \delta_{2}$, and $\alpha$, and the characteristic equation corresponding to (21) is given by

$$
\begin{equation*}
\Gamma^{4}+\left(4 n^{2}-\stackrel{o}{U}_{x x}-\stackrel{o}{U}_{y y}\right) \Gamma^{2}+\stackrel{o}{U}_{x x} \stackrel{o}{U}_{y y}-\left(\stackrel{o}{U}_{x y}\right)^{2}=0 \tag{21}
\end{equation*}
$$

where

$$
\begin{equation*}
27 \beta^{2}-27 \beta+(1-3 \sigma+Q \alpha) \geq 0 ; \quad Q=\frac{2\left(\lambda^{2}+\lambda-1\right)}{\lambda^{2}} e^{-1 / \lambda} \tag{25}
\end{equation*}
$$

On solving the inequality (19), we get

$$
\begin{equation*}
\beta \leq \beta_{0}=\mu_{0}-\frac{\sigma}{\sqrt{69}}+\frac{Q \alpha}{3 \sqrt{69}}, \tag{26}
\end{equation*}
$$

where $\mu_{0}=0.0385209 \ldots$. For $\alpha=0, \beta_{0}=\beta^{*}=\mu_{0}-\sigma / \sqrt{ } 69$ is the critical mass parameter in the circular restricted threebody problem when bigger primary is an oblate body [22]. For $\alpha=0$ and $\sigma=0, \beta_{0}=\mu_{0}=0.0385209 \ldots$ is the critical mass


Figure 7: Stability surface for noncollinear equilibrium points.


Figure 8: Continued.


Figure 8: Continued.

(g)

Figure 8: Stability region for noncollinear equilibrium points for various values of $\alpha$. (a) $\alpha=-0.8$. (b) $\alpha=-0.5$. (c) $\alpha=-0.2$. (d) $\alpha=0$. (e) $\alpha=0.2$. (f) $\alpha=0.5$. (g) $\alpha=0.8$.
parameter in the classical circular restricted three-body problem [1]. For $\sigma=0$, all results are in conformity with those of Idrisi et al. [21]. Thus, the noncollinear equilibrium points obtained in the proposed model are linearly stable for the critical mass parameter $\beta_{0}$ defined in (26).

The third term in (26) vanishes either for $\alpha=0$ or $Q=0$, i.e., $\lambda=\lambda_{1}=\sqrt{5}-1 / 2$. Thus, $\beta_{0}>\beta^{*}$ in the interval $0<\lambda<\lambda_{1}$ and $\beta_{0}<\beta^{*}$ in the interval $\lambda_{1}<\lambda<\infty$ for all $\alpha \in(-1,0)$. Similarly, $\beta_{0}<\beta^{*}$ when $0<\lambda<\lambda_{1}$ and $\beta_{0}>\beta^{*}$ in the interval $\lambda_{1}<\lambda<\infty$ for all $\alpha \in(0,1)$, Figure 6.

From (26), we have the following observations: $\left(\partial \beta_{0} / \partial \lambda\right)=0$ at $\lambda=1 / 3$. Thus, $\lambda=\lambda^{*}=1 / 3$ is a critical point which gives the maximum and minimum values of $\beta_{0}$. Also, $\left(\partial \beta_{0} / \partial \lambda\right)>0$ if $0<\lambda<\lambda^{*}$ and $\left(\partial \beta_{0} / \partial \lambda\right)<0$ if $\lambda^{*}<\lambda<\infty$ for all $\alpha \in(-1,0)$, and $\left(\partial \beta_{0} / \partial \lambda\right)<0$ if $0<\lambda<\lambda^{*}$ and $\left(\partial \beta_{0} / \partial \lambda\right)>0$ if $\lambda^{*}<\lambda<\infty$ for all $\alpha \in(0,1)$. Thus, we have the local minima in the interval $0<\lambda<\lambda^{*}$. The local maximum and minimum values of $\beta_{0}$ at the critical point $\lambda=\lambda^{*}$ are given by

$$
\beta_{0}=\mu_{0}-\frac{\sigma}{\sqrt{69}}+\frac{10 \alpha e^{-3}}{3 \sqrt{69}}=\left\{\begin{array}{ll}
\beta_{0 \max }, & -1<\alpha<0  \tag{27}\\
\beta_{0 \min }, & 0<\alpha<1
\end{array} .\right.
$$

In Figure 7, the stability surface is plotted, and it can be seen that when $\alpha$ rises, the stability surface does too and vice versa. As a result, the noncollinear equilibrium points are on the surface are stable and unstable otherwise.

The shaded region in Figure 8 corresponds to stable region for the noncollinear equilibrium points, and it is seen that as alpha increases the stability region also increases and vice-versa.

Table 1: Noncollinear equilibrium points in the Earth-moon system.

| $\alpha$ | $x_{0}$ | $y_{0}$ | Stability |
| :--- | :---: | :---: | :---: |
| -1.0 | -0.4878457082878896 | $\pm 0.866025283529371$ | Stable |
| -0.9 | -0.4878457061037406 | $\pm 0.866025284790389$ | Stable |
| -0.8 | -0.4878457039195917 | $\pm 0.866025286051407$ | Stable |
| -0.7 | -0.4878457017354426 | $\pm 0.866025287312426$ | Stable |
| -0.6 | -0.4878456995512938 | $\pm 0.866025288573446$ | Stable |
| -0.5 | -0.4878456973671447 | $\pm 0.866025289834465$ | Stable |
| -0.4 | -0.4878456951829958 | $\pm 0.866025291095484$ | Stable |
| -0.3 | -0.4878456929988468 | $\pm 0.866025992356502$ | Stable |
| -0.2 | -0.4878456908146979 | $\pm 0.866025293617522$ | Stable |
| -0.1 | -0.4878456886305489 | $\pm 0.866025294878541$ | Stable |
| $\mathbf{0}$ | $-\mathbf{0 . 4 8 7 8 4 5 6 8 6 4 4 6 4}$ | $\pm \mathbf{0 . 8 6 6 0 2 5 2 9 6 1 3 9 5 5 9}$ | Stable |
| 0.1 | -0.487845684262251 | $\pm 0.866025297400578$ | Stable |
| 0.2 | -0.4878456820781021 | $\pm 0.866025298661598$ | Stable |
| 0.3 | -0.4878456798939531 | $\pm 0.866025299922616$ | Stable |
| 0.4 | -0.4878456777098041 | $\pm 0.866025301183635$ | Stable |
| 0.5 | -0.4878456755256552 | $\pm 0.866025302444654$ | Stable |
| 0.6 | -0.4878456733415063 | $\pm 0.866025303705673$ | Stable |
| 0.7 | -0.4878456711573572 | $\pm 0.866025304966692$ | Stable |
| 0.8 | -0.4878456689732083 | $\pm 0.866025306227711$ | Stable |
| 0.9 | -0.4878456667890594 | $\pm 0.866025307488731$ | Stable |
| 1.0 | -0.4878456646049104 | $\pm 0.866025308749749$ | Stable |
| It is a special case when $\alpha=0$, i.e., pure Newtonian case. |  |  |  |

## 6. Real Application to the Earth-Moon System

From astrophysical data [23], mass of Earth $=5.972 \times$ $10^{24} \mathrm{~kg}$, mass of moon $=7.348 \times 10^{22} \mathrm{~kg}$, axes of the Earth: $r_{e}=6378.140 \mathrm{~km}, r_{p}=6356.755 \mathrm{~km}$, and average distance between Earth and moon $=382500 \mathrm{~km}$.

In the Earth-moon system, $\lambda=400000 \mathrm{~km}[24]$.
In a dimensionless system, we have

$$
\begin{equation*}
\beta=0.0121545, \sigma=3.72893 \times 10^{-7} \text { and } \lambda=1.04575 \tag{28}
\end{equation*}
$$

Table 1 lists the numerical locations of noncollinear equilibrium points $E_{4,5}\left(x_{0}, y_{0}\right)$ for the aforementioned values of $\beta, \sigma$ and for $|\alpha|<1$. For all possible values of $\alpha$, it has been found that the numerical values of $x_{0}$ and $y_{0}$ are identical up to six decimal places.

## 7. Conclusion

We studied the dynamics around noncollinear equilibrium points in the circular restricted three-body problem under the considerations of oblateness of more massive primary and Yukawa-like corrections to Newtonian potential. The modified gravitational force between the two masses $M$ and $m$, therefore, can be written as $F(r)=F_{N}(r)+F_{Y}(r)$, where $F_{N}(r)$ is Newtonian gravitational force between $M$ and $m$, and $F_{Y}(r)$ is Yukawa correction to Newtonian gravitational force between $M$ and $m$. It is found that as $\alpha \longrightarrow 0$ or $\lambda \longrightarrow 0$, the term $F_{Y}(r)$ vanishes and $F(r)=F_{N}(r)$, where $\alpha \in(-1,1)$ is the coupling constant of Yukawa force to gravitational force and $\lambda \in(0, \infty)$ is the range of Yukawa force. If $\alpha<0, F(r)$ $<F_{N}(r)$ and for $\alpha>0, F(r)>F_{N}(r)$, Figure 1. Thus, as $\alpha$ increases in the interval $(-1,1)$, the force between $m$ and $M$ also increases and vice-versa. But as $\lambda \longrightarrow \infty$, the force between $M$ and $m$ is given by $F_{\infty}(r)$ and $F_{\infty}(r) \longrightarrow 0$ as $\alpha \longrightarrow-1$, i.e., the force between $m$ and $M$ reduces as $\alpha$ reduces. For $\alpha \longrightarrow 0, F_{\infty}(r) \longrightarrow F_{N}(r)$ and the Newtonian gravitational force can be obtained. But as $\alpha \longrightarrow 1$, $F_{\infty}(r) \longrightarrow 2 F_{N}(r)$, i.e., the force acting between $m$ and $M$ is twice of the Newtonian gravitational force, as shown in Figure 2.

The nonequilibrium points are the solutions of $r_{1}=1$ and (17). On solving these equations, we got two noncollinear equilibrium points $E_{4,5}\left(x_{0}, y_{0}\right)$ given in (18). For nonoblate case, i.e., $\sigma=0$ we obtain $r_{i}=1$ which is the classic case of restricted three-body problem [1], and hence in the nonoblate case, the noncollinear equilibrium points are not affected by the Yukawa force [21]. For $\alpha=0$, the results are agreed with [22]. It is observed that $\partial x_{0} / \partial \lambda=0=\partial y_{0} / \partial \lambda$ at $\lambda=1 / 2$. So, we have a critical point $\lambda=\lambda_{0}=1 / 2$ at which the maximum and minimum values of $x_{0}$ and $y_{0}$ can be obtained. As it has been examined that $\partial x_{0} / \partial \lambda, \partial y_{0} / \partial \lambda>0$ for $0<\lambda<\lambda_{0}$ and $\partial x_{0} / \partial \lambda, \partial y_{0} / \partial \lambda<0$ for $\lambda_{0}<\lambda<\infty$ for all $\alpha \in \alpha^{+}$. Similarly, $\partial x_{0} / \partial \lambda, \partial y_{0} / \partial \lambda<0$ for $0<\lambda<\lambda_{0}$ and $\partial x_{0} / \partial \lambda, \partial y_{0} /$ $\partial \lambda>0$ for $\lambda_{0}<\lambda<\infty$ for all $\alpha \in \alpha^{-}$. Thus, it is concluded that $x_{0}$ and $y_{0}$ are increasing functions in $\lambda$ in the interval $0<\lambda<\lambda_{0}$ and decreasing functions in $\lambda$ in the interval $\lambda_{0}<\lambda<\infty$ for all $\alpha \in \alpha^{+}$. On the other hand, $x_{0}$ and $y_{0}$ are decreasing functions in $\lambda$ in the interval $0<\lambda<\lambda_{0}$ and increasing functions in $\lambda$ in the interval $\lambda_{0}<\lambda<\infty$ for all $\alpha \in \alpha^{-}$. For $\alpha \in \alpha^{+}$, as $\lambda$ increases in the interval $0<\lambda<\lambda_{0}$, the abscissa $x_{0}$ of $E_{4}$ moves toward the center of mass of the system and the ordinate $y_{0}$ moves vertically upward and vice-versa. In the interval $\lambda_{0}<\lambda<\infty, x_{0}$ and $y_{0}$ decrease and hence approach to $x^{*}$ and $y^{*}$, respectively, as $\lambda$ increases and vice versa. For $\alpha \in \alpha^{-}$, as $\lambda$ increases in the interval $0<\lambda<\lambda_{0}$, the abscissa $x_{0}$ moves away from $x^{*}$ and $y_{0}$ moves vertically
downward and vice-versa. In the interval $\lambda_{0}<\lambda<\infty, x_{0}$ and $y_{0}$ increase and hence approach to $x^{*}$ and $y^{*}$, respectively, as $\lambda$ increases and vice-versa (Figures 4 and 5).

The noncollinear equilibrium points obtained in the proposed model are linearly stable for the critical mass parameter $\beta_{0}$ defined in (26). From (26), $\partial \beta_{0} / \partial \lambda=0$ at $\lambda^{*}=1 /$ 3 thus we got another critical point which gives the maximum and minimum values of $\beta_{0}$. Also, $\partial \beta_{0} / \partial \lambda>0$ if $0<\lambda<\lambda^{*}$ and $\partial \beta_{0} / \partial \lambda<0$ if $\lambda^{*}<\lambda<\infty$ for all $\alpha \in \alpha^{-}$, and $\partial \beta_{0} /$ $\partial \lambda<0$ if $0<\lambda<\lambda^{*}$ and $\partial \beta_{0} / \partial \lambda>0$ if $\lambda^{*}<\lambda<\infty$ for all $\alpha \in \alpha^{+}$. The local maximum and minimum values of $\beta_{0}$ at the critical point $\lambda=\lambda^{*}$ are given in (27).

## Data Availability

The data used to support the findings of this study are included within this research article. For simulation, we have used data from other research papers which are properly cited.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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