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Research Article

The Predictions of Noncollinear Equilibria Positions in ER3BP with Yukawa-Like Corrections

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The existence and stability of noncollinear equilibrium points in the elliptic restricted three-body problem under the consideration of Yukawa correction to Newtonian potential are studied in this paper. The effects of various parameters (μ, $\hat{e}$, α, and λ) on the noncollinear equilibrium points are discussed briefly, and it is found that only ordinate of noncollinear equilibria $E_{4,5}$ is affected by Yukawa correction while abscissa is affected by only mass parameter μ. The noncollinear equilibria was found linearly stable for a critical mass parameter μ_c. A critical point $\lambda = ½$ is also obtained for the critical mass parameter μ_c, and at this point, the critical mass parameter μ_c has maximum or minimum values according to $\alpha < 0$ or $\alpha > 0$, respectively.

1. Introduction

The Yukawa potential is the modified Newtonian potential proposed by Yukawa [1]. It is an effective nonrelativistic potential describing the strong interactions between particles. Fischbach et al. [2] have described the modified potential energy in two-body problem under the consideration of Yukawa correction to Newtonian potential as

$$V(r) = -\frac{GMm}{r} - \frac{GMm}{r}\alpha e^{-(r/\lambda)}, \tag{1}$$

where M and m are the masses of massive primary and secondary body, respectively, $V_N(r) = -(GMm/r)$ is the Newtonian potential between the two bodies M and m, $V_Y(r) = -(GMm/r)\alpha e^{-(r/\lambda)}$ is the Yukawa correction to the Newtonian potential, r is the distance between m and M, G is the Newtonian gravitational constant, $\alpha \in (-1, 1)$ is the coupling constant of the Yukawa force to the gravitational force, and $\lambda \in (0, \infty)$ is the range of the Yukawa force.

Therefore, the corresponding force between M and m can be expressed as

$$\overrightarrow{F}(r) = \frac{GMm}{r^2}\left\{1 + \alpha\left(1 + \frac{r}{\lambda}\right)e^{-(r/\lambda)}\right\}\hat{r}. \tag{2}$$

On substituting $\alpha = 0$ in the abovementioned expression, the Newtonian gravitational force can be obtained.

The restricted three-body problem under the consideration of Yukawa correction is studied by Kokubun [3]. The main aim of this research was to study the effect of modified potential on the important aspects of restricted three-body problem. Kolosnitsyn and Melnikov [4] have shown that the minimum value of Yukawa coupling constant is $\alpha_{\min} = 1.38 \times 10^{-11}$ for $\lambda = 6.081 \times 10^6$ m for the artificial Earth satellites LAGEOS and LAGEOS II. A gravitational solution to the pioneer 10/11 anomaly is given by Brownstein and Moffat [5]. The constraints on the range of lambda in Yukawa-type modifications to the Newtonian gravitation from solar system planetary motions are discussed in detail by Iorio [6]. Massa [7] has found a link between the mean mass density and the radius of the observable universe using Sciama linear approach to Mach's principle and Yukawa corrections to Newtonian potential. Haranas and Ragos [8] have discussed the Yukawa effects in satellite dynamics.

Pricopi [9] has worked on the stability of celestial orbits in the two-body problem taking into account the Yukawa correction to Newtonian potential. The circular and elliptical orbits of Earth under the consideration of Yukawa potential combined with the Poynting–Robertson effect have been studied by Haranas et al. [10]. The dynamics of the two-body problem with Yukawa corrections is discussed by Cavan et al. [11]. Idrisi et al. [12, 13] have studied the existence and stability of triangular equilibria in the restricted three-body problem under the consideration of Yukawa corrections.

In the present work, an elliptical restricted three-body problem with Yukawa correction to Newtonian potential has been considered. The effects of various parameters (μ, $\hat{e}$, α, and λ) on the noncollinear equilibrium points have been discussed briefly. The manuscript is arranged as follows: the equations of motion of the infinitesimal mass in terms of pulsating coordinates are given in Section 2. The noncollinear equilibrium points are derived in Section 3, and the effects of various parameters (μ, $\hat{e}$, α, and λ) on the noncollinear equilibrium points are discussed in subsection 3.1. The linear stability of noncollinear equilibrium points is discussed in Section 4. In the last section, the conclusions of the problem are drawn.

2. Equations of Motion

Let m_1 and m_2 ($m_1 > m_2$) be the masses of two particles P_1 and P_2, respectively, moving in elliptical orbits around their common center of mass. It is also assumed that an infinitesimal mass m_3 is moving in the orbital plane of motion of P_1 and P_2. The equations of motion of the infinitesimal mass m_3 in the elliptic restricted three-body problem in terms of pulsating coordinates (ξ, η) and dimensionless variables are given by

$$\begin{aligned} \xi'' - 2\eta' &= \Omega_\xi, \\ \eta'' + 2\xi' &= \Omega_\eta, \end{aligned} \tag{3}$$

where Ω is the potential function which can be expressed as

$$\Omega = \frac{1}{\sqrt{1-\hat{e}^2}}\left[\frac{1}{2}\left(\xi^2+\eta^2\right) + \frac{1}{n^2}\left\{\frac{1-\mu}{r_1}\left(1+\alpha e^{-(r_1/\lambda)}\right) + \frac{\mu}{r_2}\left(1+\alpha e^{-(r_2/\lambda)}\right)\right\}\right], \tag{4}$$

where $|\alpha| < 1$ is the coupling constant of Yukawa force to gravitational force, $\lambda \in (0, \infty)$ is the range of Yukawa force, $\mu \in (0, 1/2]$ is the mass parameter and defined as $\mu = m_2/(m_1 + m_2)$, and r_1 and r_2 are the distances of m_3 from the m_1 and m_2, respectively,

$$\begin{aligned} r_1 &= \sqrt{(\xi-\mu)^2+\eta^2}, \\ r_2 &= \sqrt{(\xi+1-\mu)^2+\eta^2}, \end{aligned} \tag{5}$$

where n is the mean-motion of the primaries which is defined as

$$n^2 = \left(1+\frac{3}{2}\hat{e}^2\right)\left[1+\alpha\left(1+\frac{1}{\lambda}\right)e^{-(1/\lambda)}\right], \tag{6}$$

where $\hat{e}$ is the eccentricity of the orbits of the primaries around their common center of mass.

Thus, the Jacobi integral associated to the problem is

$$v^2 = 2\Omega - C, \tag{7}$$

where v is the velocity of m_3. As $v^2 = 2\Omega - C \geq 0$, thus the possible regions of motion of m_3 must satisfy the inequality $\Omega \geq C/2$, C is a well-known Jacobi constant.

3. Noncollinear Equilibrium Points

The noncollinear equilibrium points can be obtained by setting $\Omega_\xi = 0$ and $\Omega_\eta = 0$ for $\eta \neq 0$, i.e.,

$$\xi - \frac{1}{n^2}\left[\begin{aligned} &\frac{(1-\mu)(\xi-\mu)}{r_1^3}\left\{1+\alpha\left(1+\frac{r_1}{\lambda}\right)e^{-(r_1/\lambda)}\right\} \\ &+\frac{\mu(\xi+1-\mu)}{r_2^3}\left\{1+\alpha\left(1+\frac{r_2}{\lambda}\right)e^{-(r_2/\lambda)}\right\}\end{aligned}\right] = 0, \tag{8}$$

and

$$1 - \frac{1}{n^2}\left[\frac{(1-\mu)}{r_1^3}\left\{1+\alpha\left(1+\frac{r_1}{\lambda}\right)e^{-(r_1/\lambda)}\right\} + \frac{\mu}{r_2^3}\left\{1+\alpha\left(1+\frac{r_2}{\lambda}\right)e^{-(r_2/\lambda)}\right\}\right] = 0. \tag{9}$$

On simplifying equations (8) and (9) , we have

$$n^2 = \frac{1}{r_i^3}\left[1+\alpha\left(1+\frac{r_i}{\lambda}\right)e^{-(r_i/\lambda)}\right], i = 1, 2. \tag{10}$$

For $\hat{e} = 0$, $r_i = 1$ is the solution of equation (10). Thus, for $\hat{e} \neq 0$, let us assume that $r_i = 1 + \epsilon_i$, $\epsilon_i \ll 1$, $i = 1$, and 2 be the solution of equation (10). On substituting $r_i = 1 + \epsilon_i$ in equation (10) and considering only linear terms of ϵ_i and α, we have

$$\begin{aligned} \epsilon_i &= -\frac{\hat{e}^2}{2}\left(1-\frac{\alpha}{3}f(\lambda)\right), \\ f(\lambda) &= \frac{e^{-(1/\lambda)}}{\lambda^2}, i = 1,2. \end{aligned} \tag{11}$$

Thus, $r_i = 1 - (\hat{e}^2/2)(1 - (\alpha/3)f(\lambda))$ are the solutions of equation (10), and hence, the noncollinear equilibrium points form an isosceles triangle with the primaries. The coordinates of noncollinear equilibrium points $E_{4,5}$ (ξ, η) are given by

$$\begin{aligned} \xi &= \mu - \frac{1}{2}, \\ \eta &= \pm\frac{\sqrt{3}}{2}\sqrt{1-\frac{4}{3}\hat{e}^2\left(1-\frac{\alpha}{3}f(\lambda)\right)}. \end{aligned} \tag{12}$$

3.1. Effect of Various Parameters on Noncollinear Equilibrium Points. In this section, we consider few cases to discuss the effect of various parameters (μ, $\hat{e}$, α, and λ) on the noncollinear equilibrium points $E_{4,5}$ (ξ, η).

Case 1. Effect of μ on $E_{4,5}$:

The effects of the mass parameter μ on the abscissa of $E_{4,5}$ can be clearly observed from equation (12). As μ approaches 0, the abscissa ξ tends towards $-1/2$, while as μ approaches $1/2$, ξ tends to 0. This suggests that an increase in the mass parameter μ results in an increase in the abscissa ξ for both E_4 and E_5. By varying μ, it is possible to observe that $E_{4,5}$ form an isosceles triangle with the primaries and move horizontally toward η-axis. This phenomenon is illustrated in Figure 1 where for different values of μ, the equilibrium points $E_{4,5}$ are located graphically.

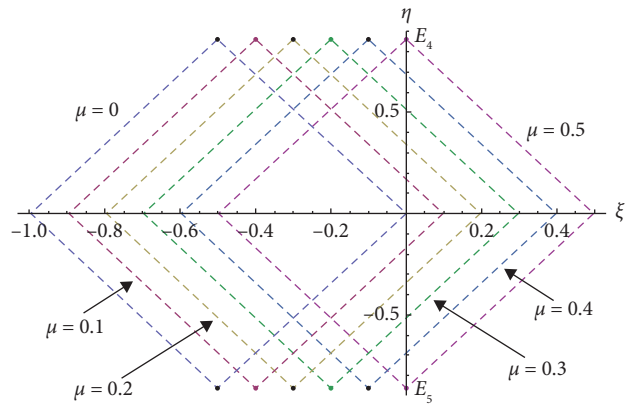

Figure 1: $E_{4,5}$ versus μ.

Case 2. Effect of $\hat{e}$ on $E_{4,5}$:

The eccentricity $\hat{e}$ of the orbits of the primaries affects only the ordinate of noncollinear equilibrium points $E_{4,5}$. For $\hat{e} = 0$, the classical case of the restricted three-body problem has been obtained in which the noncollinear equilibrium points form an equilateral triangle with the primaries. However, when $\hat{e}$ is greater than zero, these values will change due to the perturbations caused by eccentricity. As the parameter $\hat{e}$ approaches the value of unity, the noncollinear equilibrium points E_4 and E_5 in the $\xi\eta$-plane will move vertically in opposite directions and form isosceles triangle with the primaries. The point E_4 moves downwards while point E_5 moves upwards (Figure 2). This motion is due to the change in eccentricity from zero to unity.

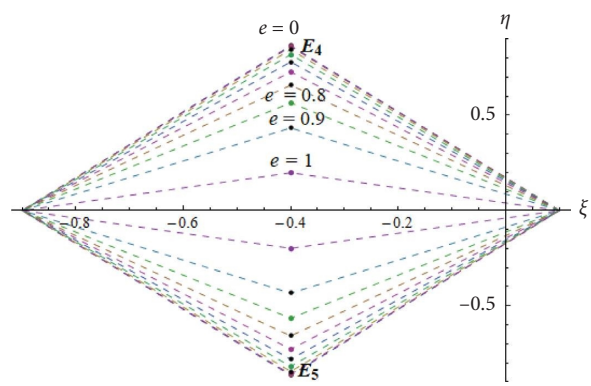

Figure 2: $E_{4,5}$ versus $\hat{e}$.

Case 3. Effect of λ on $E_{4,5}$:

The term $f(\lambda)$ in equation (12) is a function of the parameter λ, which tends to zero both as λ approaches 0 and as λ approaches infinity. A graph of the function $f(\lambda)$ (Figure 3) reveals that it is increasing in the open interval $(0, 1/2)$, while decreasing after that point. It can be seen that $\lambda = ½$ is a critical point of the function, where its rate of change is at its maximum value. In particular, for very small and large values of λ, the Yukawa force vanishes and the noncollinear equilibrium points are affected by the eccentricity of the orbits of the primaries $\hat{e}$ and mass parameter μ, and therefore,

$$r_i = 1 - \frac{\hat{e}^2}{2} \Rightarrow \xi = \mu - \frac{1}{2}, \qquad \eta = \pm \frac{\sqrt{3}}{2}\sqrt{1 - \frac{4}{3}\hat{e}^2}, 0 < \hat{e} < \frac{\sqrt{3}}{2}. \tag{13}$$

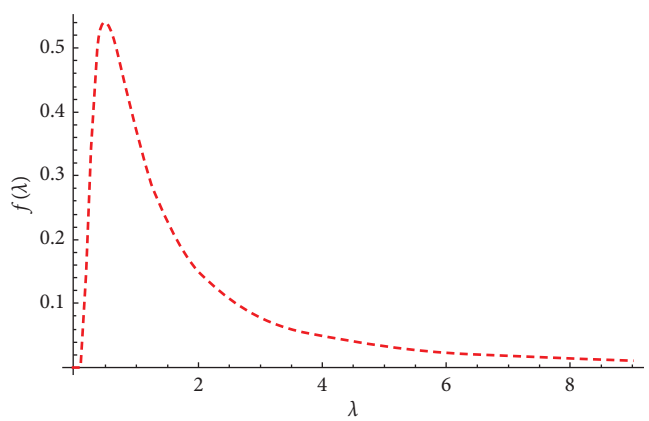

Figure 3: $f(\lambda)$ versus λ.

Case 4. Effect of α on $E_{4,5}$:

The coupling constant $\alpha \in (-1,1)$, and thus we have the following cases:

(i) If $\alpha = 0$, then the ordinate of noncollinear equilibrium points $E_{4,5}$ is given by $\eta = \pm(\sqrt{3}/2)\sqrt{1 - (4/3)\hat{e}^2}$, which is the classic case of the elliptic restricted three-body problem (Figure 4)

(ii) If $\alpha < 0$, then $\eta < \pm(\sqrt{3}/2)\sqrt{1 - (4/3)\hat{e}^2(1 + (\alpha/3)f(\lambda))}$

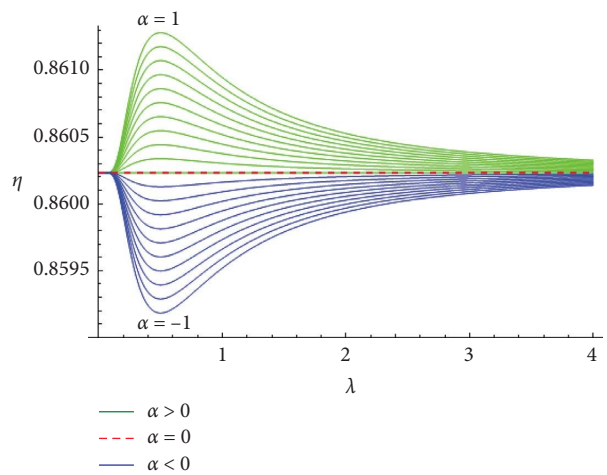

Figure 4: η for different values of α.

(iii) If $\alpha > 0$, then $\eta > \pm(\sqrt{3}/2)\sqrt{1-(4/3)\hat{e}^2(1-(\alpha/3)f(\lambda))}$.

4. Stability of Noncollinear Equilibrium Points

In this section, we study the linear stability of noncollinear equilibria $E_{4,5}$ by displacing the infinitesimal mass m_3 to the points $(\xi_0+\delta_1, \eta_0+\delta_2)$, where (ξ_0, η_0) are the coordinates of E_4 and $\delta_i \ll 1$, $i=1, 2$. Therefore, linearizing system (1), we have the variational equations of motion as

$$\left.\begin{aligned} \ddot{\delta}_1 - 2\dot{\delta}_2 &= \delta_1 \overset{o}{\Omega}_{\xi\xi} + \delta_2 \overset{o}{\Omega}_{\xi\eta}, \\ \ddot{\delta}_2 + 2\dot{\delta}_1 &= \delta_1 \overset{o}{\Omega}_{\xi\eta} + \delta_2 \overset{o}{\Omega}_{\eta\eta}. \end{aligned}\right\} \tag{14}$$

Now, considering only linear terms in δ_i and α, the characteristic equation corresponding to system (14) is given by

$$\kappa^4 + \left(4 - \overset{o}{\Omega}_{\xi\xi} - \overset{o}{\Omega}_{\eta\eta}\right)\kappa^2 + \overset{o}{\Omega}_{\xi\xi}\overset{o}{\Omega}_{\eta\eta} - \left(\overset{o}{\Omega}_{\xi\eta}\right)^2 = 0, \tag{15}$$

where

$$\begin{aligned} \overset{o}{\Omega}_{\xi\xi} &= \frac{3}{4}\left(1+\hat{e}^2\right) + \frac{\alpha}{4} f(\lambda)\left[1 - \left(1 - \frac{1}{2\lambda}\right)\hat{e}^2\right], \\ \overset{o}{\Omega}_{\xi\eta} &= \frac{\sqrt{3}}{2}\left(\mu - \frac{1}{2}\right)\left(3+\hat{e}^2\right) + \frac{\sqrt{3}}{2}\left(\mu - \frac{1}{2}\right)\alpha f(\lambda)\left[1 - \left(1 - \frac{1}{2\lambda}\right)\hat{e}^2\right], \\ \overset{o}{\Omega}_{\eta\eta} &= \frac{3}{4}\left(3-\hat{e}^2\right) + \frac{3\alpha}{4} f(\lambda)\left[1 - \left(1 - \frac{1}{2\lambda}\right)\hat{e}^2\right]. \end{aligned} \tag{16}$$

Let $\kappa^2 = \Lambda$, therefore equation (15) reduces to quadratic equation in Λ as

$$\Lambda^2 + p_1\Lambda + p_2 = 0, \tag{17}$$

where $p_1 = 4 - \overset{o}{\Omega}_{\xi\xi} - \overset{o}{\Omega}_{\eta\eta}$ and $p_2 = \overset{o}{\Omega}_{\xi\xi}\overset{o}{\Omega}_{\eta\eta} - (\overset{o}{\Omega}_{\xi\eta})^2$.

The roots of equation (17) are

$$\Lambda_{1,2} = \frac{1}{2}\left(-p_1 \pm \sqrt{p_1^2 - 4p_2}\right). \tag{18}$$

The motion around the equilibrium point $E_4(\xi_0, \eta_0)$ will be bounded if the discriminant of equation (18) is either zero or greater than zero, i.e., $p_1^2 - 4p_2 \geq 0$ gives

$$\mu \leq \mu_c = \mu_0 - 0.0267524\,\hat{e}^2 - \alpha f(\lambda)\left[0.10701 - \left(0.19347 - \frac{0.0535048}{\lambda}\right)\hat{e}^2\right], \tag{19}$$

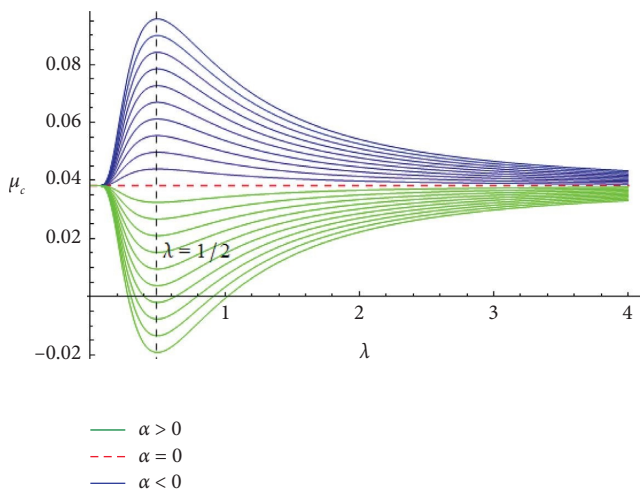

Figure 5: μ_c versus λ for different values of α.

where $\mu_0 = 0.0385209\ldots$ is the critical mass parameter in the classical circular restricted three-body problem [14]. Thus, the triangular equilibria $E_{4,5}$ are linearly stable for the critical mass parameter μ_c defined in equation (19). The third term in equation (19) vanishes as $\alpha \longrightarrow 0$ or $\lambda \longrightarrow 0$ or $\lambda \longrightarrow \infty$. In Figure 5, the minima and maxima of the critical mass parameter μ_c can be obtained at the critical point $\lambda = ½$. Thus, at $\lambda = ½$, $\mu_c = \mu_0 - 0.0276524\hat{e}^2 - 0.0574607\,\alpha$. It can be easily concluded that $\mu_c = \mu_0 - 0.0276524\hat{e}^2$ for $\alpha = 0$, $\mu_c > \mu_0 - 0.0276524\hat{e}^2$ for $\alpha < 0$ and $\mu_c < \mu_0 - 0.0276524\hat{e}^2$ for $\alpha > 0$. Also, μ_c is maximum if $\alpha < 0$ while it is minimum for $\alpha > 0$.

5. Conclusion

The existence and stability of noncollinear equilibrium points in the elliptic restricted three-body problem under the consideration of Yukawa correction to Newtonian potential is studied in this paper. The locations of noncollinear equilibria $E_{4,5}(\xi, \eta)$ can be obtained by equation (12). It is found that only ordinate of noncollinear equilibria $E_{4,5}$ is affected by Yukawa correction while abscissa is affected by only mass parameter μ. So, as $\mu \longrightarrow 0$, $\xi \longrightarrow -1/2$, and as $\mu \longrightarrow 1/2$, $\xi \longrightarrow 0$. Therefore, we can say that as the mass parameter μ increases the noncollinear equilibria $E_{4,5}$ move toward η-axis horizontally and vice-versa (Figure 1). As $\hat{e} \longrightarrow 0$, the noncollinear equilibria $E_{4,5}$ form equilateral triangles with the primaries and as $\hat{e} \longrightarrow 1$, $E_{4,5}$ moving toward ξ-axis vertically and vice-versa (Figure 2). The term $f(\lambda)$ in equation (12) tends to zero as $\lambda \longrightarrow 0$ and $\lambda \longrightarrow \infty$ (Figure 3). It is also observed that $\lambda = ½$ is the critical point of $f(\lambda)$. The function $f(\lambda)$ is increasing in the interval (0, ½) while decreasing in (½, ∞). Thus, for very small and large values of λ, the Yukawa force vanishes and the noncollinear equilibrium points are affected only by the eccentricity of the orbits of the primaries $\hat{e}$ and mass parameter μ. The noncollinear equilibria found linearly stable for a critical mass parameter μ_c defined in equation (16). A critical point $\lambda = ½$ is obtained for the critical mass parameter μ_c and it is observed that when $\lambda = ½$, $\mu_c = \mu_0 - 0.0276524\hat{e}^2 - 0.0574607\alpha$ and hence $\mu_c = \mu_0 - 0.0276524\hat{e}^2$ for $\alpha = 0$, $\mu_c > \mu_0 - 0.0276524\hat{e}^2$ for $\alpha < 0$ and $\mu_c < \mu_0 - 0.0276524\hat{e}^2$ for $\alpha > 0$. Also, μ_c is maximum or minimum according to $\alpha < 0$ or $\alpha > 0$, respectively.

Data Availability

The data used to support the findings of this study are included within the article. For simulation, we have used data from other research papers which are properly cited.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this manuscript.

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