EXISTENCE OF POSITIVE ENTIRE SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS ON $\mathbb{R}^N$

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1. INTRODUCTION

In the present paper we are concerned with positive solutions of the following problem:

$$(P) \quad \begin{cases} -\Delta u + u = g(x, u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), & N \geq 3, \end{cases}$$

where $g : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous mapping. Recently, the existence of positive solutions of the semilinear elliptic problem

$$(P_Q) \quad \begin{cases} -\Delta u + u = Q(x) |u|^{p-1} u, & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), & N \geq 2, \end{cases}$$

has been studied by several authors, where $1 < p$ for $N = 2$, $1 < p < (N+2)/(N-2)$ for $N \geq 3$ and $Q(x)$ is a positive bounded continuous function. If $Q(x)$ is a radial function, we can find infinity many solutions of problem (P_Q) by restricting our attention to the radial functions (cf. [2, 5]). If $Q(x)$ is nonradial, we encounter a difficulty caused by the lack of a compact embedding of Sobolev type. To overcome this kind of difficulty, P. L. Lions developed the concentrate compactness method [8, 9], and established the following result: Assume that $\lim_{|x| \rightarrow \infty} Q(x) = \bar{Q} (> 0)$ and $Q(x) \geq \bar{Q}$ on $\mathbb{R}^N$. Then the problem (P_Q) has a positive solution. This result is based on the observation that the ground state level c_Q of the functional

$$I_Q(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) dx - \frac{1}{p+1} \int_{\mathbb{R}^N} Q(x) |u|^{p+1} dx$$

1991 *Mathematics Subject Classification*. Primary: 35J60, 35J65.

Key words and phrases. Positive solutions, concentrate compactness method, homology groups.

Received: January 10, 1996.

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is lower than that of

$$I^\infty(u) = \frac{1}{2} \int_{R^N} (|\nabla u|^2 + |u|^2) dx - \frac{1}{p+1} \int_{R^N} |u|^{p+1} dx,$$

then, under additional conditions on g , there exists a positive solution of (P) (cf. Ding and Ni [5], Stuart [14]). In [3], Cao proved the existence of a positive solution of (P_Q) for the case $c_Q \leq c_{\overline{Q}}$ under the hypothesis that $\lim_{|x| \rightarrow \infty} Q(x) = \overline{Q}$ and $Q(x) \geq 2^{(1-p)/2} \overline{Q}$ on R^N . The difficulty in treating the case $c_Q = c_{\overline{Q}}$ is caused by the fact that we can not apply the concentrate compactness method directly. The argument in [3] is based on Lagrange's method of indeterminate coefficients. That is, if we find a solution u of the minimizing problem

$$\inf \left\{ \{I_Q(u) : u \in V_\lambda\}, \right. \\ \left. V_\lambda = \left\{ \{u \in H^1(R^N), u > 0, \int_{R^N} Q(x) |u|^{p+1} dx = 1\} \right\} \right\},$$

then cu is a solution of (P_Q) for some $c > 0$. Lagrange's method does not work if g is not the form $Q(x)t^p$. Our purpose in this paper is to consider the existence of a positive solution of (P) for g satisfying $\lim_{|x| \rightarrow \infty} g(x, t) = |t|^{p-1} t$. Our method employed here is based on the singular homology theory.

Throughout this paper, we assume that $g \in C^1(R) \cap C^2(R \setminus \{0\})$ and we impose the following conditions on g :

(g1) There exists a positive number $d < 1$ such that

$$-dt + (1-d)|t|^{p-1}t \leq g(x, t) \leq dt + (1+d)|t|^{p-1}t$$

for all $(x, t) \in R^N \times [0, \infty)$;

(g2) there exists a positive number C such that

$$|g_t(x, 0)| < 1 \quad \text{and} \quad 0 < t^3 g_{tt}(x, t) < C(1 + |t|^{p+1})$$

for all $(x, t) \in R^N \times (0, \infty)$;

(g3)

$$\lim_{|x| \rightarrow \infty} g(x, t) = |t|^{p-1} t$$

uniformly on bounded intervals in $[0, \infty)$,

where $1 < p < (N+2)/(N-2)$ and $g_t(\cdot, \cdot)$ stands for the derivative of g with respect to the second variable.

Remark 1. (1) Throughout the rest of this paper, we assume for the simplicity of the proofs that $g(x, -t) = -g(x, t)$ for $(x, t) \in R^N \times [0, \infty)$. Since we are concerned with positive solutions, this assumption does not effect our result. By this assumption, the functional I is even and if u is a critical point of I , $-u$ is also a critical point of I . (2) Functions of the form $g(x, t) = \sum_{i=1}^m q_i(x)t^i + q_p(x)t^p$ satisfy (g1) and (g2) if m is a positive integer with $m < p$, $q_i(x)$ ($1 \leq i \leq m$) are sufficiently small and $|q_p(x) - 1| < 1 + d$. (g3) is satisfied if $\lim_{|x| \rightarrow \infty} q_i(x) = 0$ for $1 \leq i \leq p - 1$ and $\lim_{|x| \rightarrow \infty} q_p(x) = 1$.

Theorem. *Suppose that (g2) and (g3) hold. Then there exists $d_0 > 0$ such that if (g1) holds with $d < d_0$, then the problem (P) has a positive solution.*

2. PRELIMINARIES

Throughout the rest of this paper, we assume that (g2) and (g3) hold. We put $H = H^1(R^N)$. Then H is a Hilbert space with norm

$$\|u\| = \left(\int_{R^N} (|\nabla u|^2 + |u|^2) dx \right)^{1/2}.$$

The norm of the dual space $H^{-1}(R^N)$ of H is also denoted by $\|\cdot\|$. B_r stands for the open ball centered at 0 with radius r . For subsets A, B of H with $A \subset B$, we denote by $\text{int}_B A$ and $\partial_B A$ the relative interior of A in B and the relative boundary of A in B , respectively. For subsets A, B of H , we write $A \cong B$ when A and B have the same homotopy type. The norm and inner product of $L^2(R^N)$ are denoted by $|\cdot|_{L^2}$ and $\langle \cdot, \cdot \rangle$, respectively. For each $x \in R^N$ and $u \in H$, we set $\tau_x u = u(\cdot + x)$. For each functional F on H and $a \in R$, we set

$$F_a = \{u \in H : F(u) \leq a\} \quad \text{and} \quad \dot{F}_a = \{u \in H : F(u) < a\}.$$

We put

$$M = \left\{ u \in H \setminus \{0\} : \|u\|^2 = \int_{R^N} u g(x, u) dx \right\},$$

$$M^\infty = \left\{ u \in H \setminus \{0\} : \|u\|^2 = \int_{R^N} u^{p+1} dx \right\}.$$

From the assumption (g2), we find that for each $u \in H \setminus \{0\}$,

$$\frac{dI(tu)}{dt}(0) = 0, \quad \frac{d^2 I(tu)}{dt^2}(0) = |\nabla u|_{L^2}^2 + |u|_{L^2}^2 - \langle g_t(x, 0)u, u \rangle > 0,$$

and

$$(2.1) \quad \frac{d^3 I(tu)}{dt^3}(t) = -\langle g_{tt}(x, tu)u^2, u \rangle < 0 \quad \text{for } t > 0.$$

Then, noting that $(dI(tu)/dt)(\lambda) = 0$ if $\lambda u \in M$, we can see that there exists a positive number $\lambda_0(u)$ such that $I_u = \{\lambda u : \lambda > 0\}$ intersects M at exactly one point $\lambda_0(u)u$. Similarly, we can define a positive number $\lambda_\infty(u)$ by $\lambda_\infty(u)u \in M^\infty$. For simplicity, we write $\lambda_0 u$ and $\lambda_\infty u$ instead of $\lambda_0(u)u$ and $\lambda_\infty(u)u$ respectively, when it is clear in the context what it means. It also follows from the definition of M^∞ that for each $u \in M^\infty$,

$$(2.2) \quad \begin{aligned} I^\infty(u) &= \frac{p-1}{2(p+1)} \int_{R^N} (|\nabla u|^2 + |u|^2) dx \\ &= \frac{p-1}{2(p+1)} \int_{R^N} |\nabla u|^{p+1} dx. \end{aligned}$$

It is known that there exists a positive radial solution u_∞ of problem

$$(P_\infty) \quad \begin{cases} -\Delta u + u = |u|^{p-1} u, & x \in R^N \\ u \in H^1(R^N), \end{cases}$$

such that $c = I^\infty(u_\infty) = \min\{I^\infty(u) : u \in M^\infty\}$. In [6], Kwong showed that u_∞ is the unique positive solution up to the translation. It then follows as a direct consequence of the concentrate compactness lemma(cf. Lions [8]) that the second critical level of I^∞ is $2c$. That is,

Lemma 2.1. *For each $0 < \epsilon < c$, $\inf\{\|\nabla I^\infty(u)\| : u \in I_{2c-\epsilon} \setminus \dot{I}_{c+\epsilon}\} > 0$.*

We put $c_1 = \inf\{I(u) : u \in M\}$. It then follows from the definition of I and M that if $u \in M$ satisfies $c_1 = I(u)$, then u is a solution of (P). It also follows that u is positive. In fact, if $u^+ = \max\{u, 0\} \not\equiv 0$ and $u^- = -\min\{u, 0\} \not\equiv 0$, then $u^\pm \in M$ and therefore $I(u) = I(u^+) + I(u^-) \geq 2c_1$. This is a contradiction. Then to find a positive solution of problem (P), we will find a critical point of M with critical level c_1 . We can see from (g3) that $\lim_{|x| \rightarrow \infty} I(u_\infty(\cdot + x)) = c$. Therefore we have that $c_1 \leq c$. Moreover we have

Proposition 2.2. *Suppose that (g1) holds with $d \leq \tilde{d}_0$, where $\tilde{d}_0$ is a positive number such that*

$$\delta = \inf \left\{ \frac{1-d}{2} - \frac{(1+d)^2}{(1-d)(p+1)} : 0 \leq d \leq \tilde{d}_0 \right\} > 0.$$

If $c_1 < c$, then there exists a positive solution of problem (P).

Proof. Let $u \in H$. Then by (g1), we have

$$\begin{aligned} I(u) &= \frac{1}{2} \int_{R^N} (|\nabla u|^2 + |u|^2) dx - \int_{R^N} \int_0^{u(x)} g(x, t) dt dx \\ &\geq \int_{R^N} \left(\frac{1}{2} (|\nabla u|^2 + (1-d)|u|^2) - \frac{1+d}{p+1} |u|^{p+1} \right) dx. \end{aligned}$$

Suppose that $u \in M$. Then again by (g1), we have

$$\|u\|^2 = \int_{R^N} u g(x, u) dx \geq \int_{R^N} (-d|u|^2 + (1-d)|u|^{p+1}) dx.$$

Combining the inequalities above, we have

$$\begin{aligned} I(u) &\geq \int_{R^N} \left(\frac{1}{2} - \frac{1+d}{(1-d)(p+1)} |\nabla u|^2 \right. \\ (2.3) \quad &\quad \left. + \left(\frac{1-d}{2} - \frac{(1+d)^2}{(1-d)(p+1)} \right) |u|^2 \right) dx \\ &\geq \delta \int_{R^N} (|\nabla u|^2 + |u|^2) dx. \end{aligned}$$

Let $\{u_n\} \subset M$ be a sequence such that $\lim_{n \rightarrow \infty} I(u_n) = c_1$ and $\lim_{n \rightarrow \infty} \nabla I(u_n) = 0$. It then follows from (2.3) that $\{u_n\}$ is bounded in H . Then by a parallel argument as in the proof of theorem I.2 of Lions [9], we can see that $\{u_n\}$ converges to $u \in H$ and $\nabla I(u) = 0$ and this completes the proof. $\blacksquare$

By Proposition 2.2, it is sufficient to consider the case that $c_1 = c$. In the sequel, we assume that $c_1 = c$. We prove Theorem by contradiction, that is, we assume in the following that the functional I does not have nontrivial critical points. Our purpose in the rest of this section is to prove the following Proposition.

Proposition 2.3. *There exists a positive number $d_0 < \tilde{d}_0$ such that if (g1) holds with $d \leq d_0$, then for each $0 < \epsilon < c$,*

$$H_*(I_{c+\epsilon}^\infty, I_\epsilon^\infty) = H_*(I_{c+\epsilon}, I_\epsilon)$$

where $H_*(A, B)$ denotes the singular homology group for a pair (A, B) of topological spaces (cf. Spanier [11]).

In the following we denote by $M^{0,\infty}$ and M_α ($\alpha > 0$) the sets defined by $M^{0,\infty} = \{t\lambda_0 u + (1-t)\lambda_\infty u : u \in H \setminus \{0\}, t \in [0, 1], \lambda_0 u \in M, \lambda_\infty u \in M^\infty\}$ and

$$(2.4) \quad M_\alpha = \{(1+\tau)u : u \in M^{0,\infty}, \tau \in (-R(u), R(u))\}$$

where

$$\begin{aligned} (2.5) \quad R(u) &= \sup \left\{ t > 0 : \max \left\{ \frac{I(u)}{I((1+\tau)u)}, \frac{I^\infty((1+\tau)u)}{I^\infty(u)} \right\} < 1 + \alpha \right. \\ &\quad \left. \text{for all } \tau \in [-t, t] \right\}. \end{aligned}$$

From the definition, $M^\infty, M \subset M^{0,\infty}$ and M_α is an open neighborhood of $M^{0,\infty}$.

Lemma 2.4. *There exist positive numbers d_1 and α_0 such that if (g1) holds with $d \leq d_1$, then for each positive number $\alpha < \alpha_0$,*

- (1) $I_{(7/6)c}^\infty \subset I_{(4/3)c} \cup (M_\alpha)^c,$
- (2) $I_{(4/3)c} \subset I_{(5/3)c}^\infty \cup (M_\alpha)^c,$
- (3) $I_{(5/3)c}^\infty \subset I_{(11/6)c} \cup (M_\alpha)^c.$

Proof. The assertions (1), (2) and (3) can be proved by parallel arguments. We give only the proof of (2). Let $d_1 > 0$ such that

$$\frac{4}{5} < \rho = \min \left\{ \left(\frac{(1-d)^2}{2(1+d)} - \frac{1+d}{p+1} \right) \frac{2(p+1)}{p-1}, \right. \\ \left. \left(\frac{1-d}{1+d} \right)^{2/(p-1)} \left(\frac{2(p+1)}{p-1} \right) \left(\frac{1-d}{2} - \frac{(1+d)}{p+1} \right) \right\}, \text{ for } 0 \leq d \leq d_0.$$

We assume that (g1) holds with $d \leq d_1$. Fix $u \in H \setminus \{0\}$. Then we have from the definitions of M and M^∞ that

$$(2.6) \quad \|\lambda_0 u\|^2 = \int_{R^N} \lambda_0 u g(x, \lambda_0 u) dx \quad \text{and} \quad \|\lambda_\infty u\|^2 = \int_{R^N} |\lambda_\infty u|^{p+1} dx.$$

By (g1) and (2.6), we have

$$\begin{aligned} \frac{1-d}{1+d} \int_{R^N} |\lambda_0 u|^{p+1} dx &\leq \frac{1}{1+d} \int_{R^N} (\lambda_0 u g(x, \lambda_0 u) + d |\lambda_0 u|^2) dx \\ &\leq \frac{1}{1+d} \int_{R^N} (|\nabla \lambda_0 u|^2 dx + (1+d) |\lambda_0 u|^2) dx \\ &\leq \int_{R^N} (|\nabla \lambda_0 u|^2 dx + |\lambda_0 u|^2) dx \\ &\leq \frac{1}{1-d} \int_{R^N} (|\nabla \lambda_0 u|^2 dx + (1-d) |\lambda_0 u|^2) dx \\ &\leq \frac{1}{1-d} \int_{R^N} (\lambda_0 u g(x, \lambda_0 u) - d |\lambda_0 u|^2) dx \\ &\leq \frac{1+d}{1-d} \int_{R^N} |\lambda_0 u|^{p+1} dx. \end{aligned}$$

That is, we have

$$(2.7) \quad \begin{aligned} \frac{1-d}{1+d} \int_{R^N} |\lambda_0 u|^{p+1} dx &\leq \int_{R^N} (|\nabla \lambda_0 u|^2 dx + |\lambda_0 u|^2) dx \\ &\leq \frac{1+d}{1-d} \int_{R^N} |\lambda_0 u|^{p+1} dx. \end{aligned}$$

We find from the second equality of (2.6) and (2.7) that

$$(2.8) \quad \frac{1-d}{1+d} \lambda_0^{p-1} \leq \lambda_\infty^{p-1} \leq \frac{1+d}{1-d} \lambda_0^{p-1}.$$

To prove the assertion, we will show that for $0 < \alpha < \alpha_0$,

$$I_{(4/3)c} \cap M_\alpha \subset I_{(5/3)c}^\infty.$$

Now let $u \in M^{0,\infty}$. From the definition of $M^{0,\infty}$, we have that $\lambda_0 \leq 1 \leq \lambda_\infty$ or $\lambda_\infty \leq 1 \leq \lambda_0$ holds. We first consider the case that $\lambda_\infty \leq 1 \leq \lambda_0$. Since $\lambda_\infty \leq 1$, we have that

$$\|u\|^2 = \int_{R^N} (|\nabla u|^2 + |u|^2) dx \leq \int_{R^N} |u|^{p+1} dx.$$

Then we find that

$$(2.9) \quad I^\infty(u) \leq \frac{p-1}{2(p+1)} \int_{R^N} |u|^{p+1} dx.$$

On the other hand, recalling that the second equality of (2.6) holds, we obtain from (g1), (2.9) and (2.8) that

$$\begin{aligned} (2.10) \quad I(u) &\geq \frac{1-d}{2} \int_{R^N} (|\nabla u|^2 + |u|^2) dx - \frac{1+d}{p+1} \int_{R^N} |u|^{p+1} dx \\ &= \left(\frac{1-d}{2} \lambda_\infty^{p-1} - \frac{1+d}{p+1} \right) \int_{R^N} |u|^{p+1} dx \\ &\geq \left(\frac{(1-d)^2}{2(1+d)} - \frac{1+d}{p+1} \right) \frac{2(p+1)}{p-1} I^\infty(u) \\ &\geq \rho I^\infty(u). \end{aligned}$$

We choose a positive number $\alpha_1 < 1$ such that $4/5 < \rho/(1+\alpha_1)^2$. Now suppose that $(1+\tau)u \in M_{\alpha_1}$, $\tau \in R$. Then, by (2.10), we have

$$\begin{aligned} I((1+\tau)u) &\geq (1/(1+\alpha_1))I(u) \geq (\rho/(1+\alpha_1))I^\infty(u) \\ &\geq (\rho/(1+\alpha_1)^2)I^\infty((1+\tau)u). \end{aligned}$$

Assume that $(1+\tau)u \in I_{(4/3)c}$. Then it follows from the inequalities above that

$$I^\infty((1+\tau)u) \leq (4/3)c(1+\alpha_1)^2/\rho \leq (5/3)c.$$

We next assume that $\lambda_0 \leq 1 \leq \lambda_\infty$. Then by (2.2),

$$(2.11) \quad I^\infty(u) \leq I^\infty(\lambda_\infty u) = \frac{p-1}{2(p+1)} \lambda_\infty^2 \int_{R^N} (|\nabla u|^2 + |u|^2) dx.$$

On the other hand, we have by (2.8) that

$$\lambda_\infty \leq \left(\frac{1+d}{1-d} \right)^{1/(p-1)}.$$

Then, noting that $\lambda_\infty^{-(p-1)} \leq 1$, we have from (g1) and (2.11) that

$$\begin{aligned} I(u) &\geq \frac{1-d}{2} \int_{R^N} (|\nabla u|^2 + |u|^2) dx - \frac{1+d}{p+1} \int_{R^N} |u|^{p+1} dx \\ &\geq \left(\frac{1-d}{2} - \frac{1+d}{p+1} \lambda_\infty^{-(p-1)} \right) \int_{R^N} (|\nabla u|^2 + |u|^2) dx \\ (2.12) \quad &= \lambda_\infty^{-2} \frac{2(p+1)}{p-1} \left(\frac{1-d}{2} - \lambda_\infty^{-(p-1)} \frac{(1+d)}{p+1} \right) I^\infty(u) \\ &\geq \left(\frac{1-d}{1+d} \right)^{2/(p-1)} \frac{2(p+1)}{p-1} \left(\frac{1-d}{2} - \frac{(1+d)}{p+1} \right) I^\infty(u) \\ &\geq \rho I^\infty(u). \end{aligned}$$

Then we have that there exists $\alpha_2 > 0$ such that for all $u \in M_{\alpha_2}$ with $I(u) \leq (4/3)c$, $I^\infty(u) \leq (5/3)c$. Thus we obtain that the assertion holds with $\alpha_0 = \min\{\alpha_1, \alpha_2\}$. $\blacksquare$

Throughout the rest of this section we fix the positive number $\alpha < \alpha_0$.

Lemma 2.5. *There exists a continuous mapping $\gamma_1 : [0, 1] \times (I_{(11/6)c} \cup M_\alpha^c) \rightarrow I_{(11/6)c} \cup M_\alpha^c$ such that*

- (i) $\gamma_1(0, x) = x$ for all $x \in I_{(11/6)c} \cup M_\alpha^c$,
- (ii) $\gamma_1(t, x) = x$ for all $(t, x) \in [0, 1] \times (I_{(4/3)c} \cup M_\alpha^c)$,
- (iii) $I(\gamma_1(t, x)) \leq I(\gamma_1(0, x))$ for all $(t, x) \in [0, 1] \times (I_{(11/6)c} \cup M_\alpha^c)$,
- (iv) $\gamma_1(1, I_{(11/6)c} \cup M_\alpha^c) \subset I_{(4/3)c} \cup M_\alpha^c$.

Proof. We set

$$M_o = \{\lambda u : u \in M, \lambda > 1\} \quad \text{and} \quad M_i = \{\lambda u : u \in M, \lambda < 1\}.$$

Let U be an open set such that

$$(M_\alpha)^c \subset U \quad \text{and} \quad U \cap M_{\alpha/2} = \emptyset.$$

Then since $M \subset M_{\alpha/2}$, we can see that

$$\langle \nabla I(v), v \rangle > 0 \quad \text{on } M_i \cap U \quad \text{and} \quad \langle \nabla I(v), v \rangle < 0 \quad \text{on } M_o \cap U.$$

Then by arguing standard way (cf. Lemma 1.6 of Rabinowitz [10]), we can construct a pseudo-gradient vector field $\tilde{V}$ associated with ∇I such that

- (a) $\|\tilde{V}(u)\| \leq 2 \|\nabla I(u)\|$, for $u \in H$;
- (b) $\langle \nabla I(u), \tilde{V}(u) \rangle \geq \|\nabla I(u)\|^2$, for $u \in H$;
- (c) $\langle \tilde{V}(v), v \rangle > 0$ on $M_i \cap U$;
- (d) $\langle \tilde{V}(v), v \rangle < 0$ on $M_o \cap U$.

We put

$$h_1(v) = \|v - M_\alpha^c\| / (\|v - U^c\| + \|v - M_\alpha^c\|) \quad \text{for } v \in H,$$

$$h_2(v) = \|v - U^c\| / (\|v - U^c\| + \|v - M_\alpha^c\|) \quad \text{for } v \in H$$

and

$$(2.13) \quad V(v) = h_1(v)\tilde{V}(v) + h_2(v) \operatorname{sgn}(\langle \tilde{V}(v), v \rangle) v \quad \text{for } v \in H.$$

Then V is Lipschitz continuous on $I_{(11/6)c} \cup (M_\alpha)^c$. Consider the ordinary differential equation

$$(2.14) \quad \frac{d\eta}{dt} = -V(\eta), \quad \eta(0, v) = v \quad \text{for } v \in I_{(11/6)c} \cup (M_\alpha)^c.$$

The solution $\eta : R^+ \times H \rightarrow H$ defines a semiflow on H . It follows from the definition of V that $\eta(t, v) \in (M_\alpha)^c$ for $(t, v) \in [0, \infty) \times (M_\alpha)^c$. In fact, if $v \in (M_\alpha)^c$, then for each $t > 0$, $\eta(t, v) = \lambda_t v$, where $\lambda_t \in R$ such that $\lambda_t v \in (M_\alpha)^c$. We also have from (a)-(c) and (2.13) that $\langle V(v), \nabla I(v) \rangle > 0$ on $U \cup I_{(11/6)c}$ and then

$$I(\eta(t, v)) < I(\eta(s, v)) \quad \text{for } t > s \text{ and } v \in U \cup I_{(11/6)c}.$$

Thus we find that $\eta(t, v) \in I_{(11/6)c} \cup (M_\alpha)^c$ for $(t, v) \in [0, \infty) \times I_{(11/6)c} \cup (M_\alpha)^c$. It follows from Lemma 2.1 that

$$\inf\{\|\nabla I(u)\| : u \in I_{(11/6)c} \setminus I_{(4/3)c}\} > 0.$$

Then we have

$$\inf\{\|V(u)\| : u \in (U \cup I_{(11/6)c}) \setminus I_{(4/3)c}\} > 0.$$

Therefore, there exists $T > 0$ such that

$$(2.15) \quad \begin{aligned} \eta(t, v) &\in \operatorname{int}(I_{(4/3)c} \cup (M_\alpha)^c) \text{ for all } t > T \\ &\text{and all } v \in I_{(11/6)c} \cup (M_\alpha)^c. \end{aligned}$$

Here we put

$$\gamma(t, v) = \eta(t_v \cdot t, v) \quad \text{for } (t, v) \in [0, 1] \times I_{(11/6)c} \cup (M_\alpha)^c,$$

where

$$t_v = \inf\{t \geq 0 : \eta(t, v) \in I_{(4/3)c} \cup (M_\alpha)^c\} \quad \text{for } v \in I_{(11/6)c} \cup (M_\alpha)^c.$$

Then, by (2.15), we have $\gamma_1 : [0, 1] \times I_{(11/6)c} \cup (M_\alpha)^c \rightarrow I_{(11/6)c} \cup (M_\alpha)^c$ satisfying the desired properties. $\blacksquare$

By a parallel argument as in the proof of Lemma 2.5, we have

Lemma 2.6. *There exists a continuous mapping $\gamma_2 : [0, 1] \times I_{(5/3)c}^\infty \cup M_\alpha^c \rightarrow I_{(5/3)c}^\infty \cup M_\alpha^c$ such that*

- (v) $\gamma_2(0, x) = x$ for all $x \in I_{(5/3)c}^\infty \cup M_\alpha^c$;
- (vi) $\gamma_2(t, x) = x$ for all $(t, x) \in [0, 1] \times (I_{(7/6)c}^\infty \cup M_\alpha^c)$;
- (vii) $I^\infty(\gamma_2(t, x)) \leq I^\infty(\gamma_2(0, x))$ for all $(t, x) \in [0, 1] \times (I_{(5/3)c}^\infty \cup M_\alpha^c)$;
- (viii) $\gamma_2(1, I_{(5/3)c}^\infty \cup M_\alpha^c) \subset I_{(7/6)c}^\infty \cup M_\alpha^c$.

Lemma 2.7. *For each $0 < \epsilon < c$, I_ϵ^∞ and I_ϵ have the same homotopy type.*

Proof. Let $0 < \epsilon < c$. Then we have by (2.1) that there exist continuous mappings $t_1 : H \setminus \{0\} \rightarrow R^+$ and $t_2 : H \setminus \{0\} \rightarrow R^+$ such that for each $u \in H \setminus \{0\}$, $t_1(u) < t_2(u)$ and

$$\{I(tu) : t \geq 0\} \cap I_\epsilon = \{tu : t \in [0, t_1(u)] \cup [t_2(u), \infty)\}$$

Similarly, there exist continuous mappings $t_1^\infty : H \setminus \{0\} \rightarrow R^+$ and $t_2^\infty : H \setminus \{0\} \rightarrow R^+$ such that for each $u \in H \setminus \{0\}$, $t_1^\infty(u) < t_2^\infty(u)$ and

$$\{I^\infty(tu) : t \geq 0\} \cap I_\epsilon^\infty = \{tu : t \in [0, t_1^\infty(u)] \cup [t_2^\infty(u), \infty)\}.$$

Then we find that I_ϵ^∞ and I_ϵ have the same homotopy type. ■

We can now prove Proposition 2.3.

Proof of Proposition 2.3. Let $0 < \epsilon < c$. Then $I_{c+\epsilon}^\infty$ and $I_{c+\epsilon}$ have the same homotopy types as $I_{(7/6)c}^\infty$ and $I_{(7/6)c}$, respectively. We also have that I_ϵ^∞ and I_ϵ have the same homotopy types with as $I_{(1/3)c}^\infty$ and $I_{(1/3)c}$, respectively. Then to prove the assertion, it is sufficient to show that

$$H_*(I_{(7/6)c}^\infty, I_{(1/3)c}^\infty) \cong H_*(I_{(7/6)c}, I_{(1/3)c}).$$

We first define a mapping $\tilde{\gamma} : [0, 1] \times (I_{(11/6)c} \cup (M_\alpha)^c) \rightarrow I_{(11/6)c} \cup (M_\alpha)^c$ by

$$\tilde{\gamma}(t, u) = \begin{cases} \gamma_1(2t, u), & \text{for } t \in [0, 1/2], \\ \gamma_2(2(t - 1/2), \gamma_1(1, u)), & \text{for } t \in (1/2, 1]. \end{cases}$$

Then from (iii), we have that

$$(2.16) \quad \tilde{\gamma}(t, u) \in I_{(11/6)c} \cup (M_\alpha)^c$$

for $(t, u) \in [0, 1/2] \times (I_{(11/6)c} \cup (M_\alpha)^c)$. On the other hand, we have, by combining (iv) and (vii) with (3) of Lemma 2.4, that (2.16) holds for $(t, u) \in [1/2, 1] \times (I_{(11/6)c} \cup (M_\alpha)^c)$. Thus we have that $\tilde{\gamma}$ is well defined and a strong deformation retraction from $I_{(11/6)c} \cup (M_\alpha)^c$ onto $I_{(7/6)c}^\infty \cup (M_\alpha)^c$. We next

define a mapping $\gamma_3 : [0, 1] \times (I_{(7/6)c}^\infty \cup M_\alpha^c) \rightarrow I_{(7/6)c}^\infty$. For each $u \in (M_\alpha)^c$ with $I^\infty(u) > (7/6)c$, we set

$$\begin{aligned}\tau_u^+ &= \min\{\tau > 1 : I^\infty(\tau u) \leq (7/6)c\}, \\ \tau_u^- &= \max\{\tau < 1 : I^\infty(\tau u) \leq (7/6)c\}, \\ M_o^\infty &= \{\lambda u : u \in M^\infty, \lambda > 1\}\end{aligned}$$

and

$$M_i^\infty = \{\lambda u : u \in M^\infty, \lambda < 1\}.$$

Then we put

$$\gamma_3(t, x) = \begin{cases} t\tau_u^+u + (1-t)u & \text{if } u \in M_o^\infty \setminus (I_{(7/6)c}^\infty \cup M_\alpha), \\ t\tau_u^-u + (1-t)u & \text{if } u \in M_i^\infty \setminus (I_{(7/6)c}^\infty \cup M_\alpha), \\ u & \text{if } u \in I_{(7/6)c}^\infty. \end{cases}$$

It then easy to see that γ_3 is a strong deformation retraction from $I_{(7/6)c}^\infty \cup (M_\alpha)^c$ to $I_{(7/6)c}^\infty$. Therefore we obtain that $I_{(7/6)c}^\infty$ is a strong deformation retract of $I_{(11/6)c} \cup (M_\alpha)^c$. It then follows that

$$(2.17) \quad H_*(I_{(7/6)c}^\infty, I_{(1/3)\epsilon}^\infty) = H_*(I_{(11/6)c} \cup (M_\alpha)^c, I_{(1/3)\epsilon}^\infty).$$

Then by Lemma 2.7,

$$(2.18) \quad H_*(I_{(11/6)c} \cup (M_\alpha)^c, I_{(1/3)\epsilon}^\infty) = H_*(I_{(11/6)c} \cup (M_\alpha)^c, I_{(1/3)\epsilon}).$$

On the other hand, we can see by a parallel argument as above that $I_{(7/6)c}$ is a strong deformation retract of $I_{(11/6)c} \cup (M_\alpha)^c$. Then from (2.17) and (2.18), we have $H_*(I_{(7/6)c}^\infty, I_{(1/3)c}^\infty) \cong H_*(I_{(7/6)c}, I_{(1/3)c})$, which completes the proof. $\blacksquare$

3. PROOF OF THE THEOREM

We start with the following proposition.

Proposition 3.1. *For each positive number $\epsilon < c$,*

$$H_q(I_{c+\epsilon}^\infty, I_\epsilon^\infty) = \begin{cases} 2 & \text{if } q = 0, \\ 0 & \text{if } q \neq 0. \end{cases}$$

Proposition 3.1 was proved in [6]. For completeness, we give the proof of it in the appendix. We next consider a triple $(U, K, \epsilon) \subset H \times H \times R^+$ satisfying the following conditions:

- (1) $U \cap (-U) = \emptyset$;
- (2) $\{\tau_x u_\infty : |x| \geq r\} \subset \text{int}K$ for some $r > 0$;
- (3) $cl(I_{c+\epsilon} \cap K) \subset \text{int}_{I_{c+\epsilon}}(I_{c+\epsilon} \cap U)$;
- (4) I_ϵ is a strong deformation retract of $I_{c+\epsilon} \setminus (K \cup (-K))$;
- (5) $H_{N-1}(I_{c+\epsilon} \cap U) = 1$, $H_1(I_{c+\epsilon} \cap U) = 0$;
- (6) $H_{N-1}((I_{c+\epsilon} \cap U) \setminus K) = 2$ or $H_0((I_{c+\epsilon} \cap U) \setminus K) = 1$ holds.

Proposition 3.2. *There exists a triple $(U, K, \epsilon) \subset H \times H \times R^+$ which satisfies (1) – (6).*

The proof of Proposition 3.2 is given in Section 4.

Lemma 3.3. *Suppose that there exist a triple $(U, K, \epsilon) \subset H \times H \times R^+$ satisfying (1) – (6). Suppose, in addition, that $H_{N-1}((I_{c+\epsilon} \cap U) \setminus K) \geq 2$. Then $H_N(I_{c+\epsilon}, I_\epsilon) \geq 2$.*

Proof. We put $\tilde{K} = K \cup (-K)$. Since I_ϵ is a strong deformation retract of $I_{c+\epsilon} \setminus \tilde{K}$, we find that

$$H_q(I_{c+\epsilon} \setminus \tilde{K}, I_\epsilon) \cong H_q(I_\epsilon, I_\epsilon) \cong 0.$$

Then we have from the exactness of the singular homology groups of the triple $(I_{c+\epsilon}, I_{c+\epsilon} \setminus \tilde{K}, I_\epsilon)$ that

$$0 \rightarrow H_q(I_{c+\epsilon}, I_\epsilon) \rightarrow H_q(I_{c+\epsilon}, I_{c+\epsilon} \setminus \tilde{K}) \rightarrow 0.$$

That is,

$$H_q(I_{c+\epsilon}, I_\epsilon) \cong H_q(I_{c+\epsilon}, I_{c+\epsilon} \setminus \tilde{K}).$$

From (1) and (3), we find

$$H_q(I_{c+\epsilon}, I_{c+\epsilon} \setminus \tilde{K}) \cong H_q(W, W \setminus K) \oplus H_q(-W, (-W) \setminus (-K)),$$

where $W = I_{c+\epsilon} \cap U$. Then since $H_{N-1}(W \setminus K) \geq 2$, we have from (5) and the exactness of the sequence

$$(3.1) \quad \begin{aligned} & \rightarrow H_q(W, W \setminus K) \rightarrow H_{q-1}(W \setminus K) \\ & \rightarrow H_{q-1}(W) \rightarrow H_{q-1}(W, W \setminus K) \rightarrow, \end{aligned}$$

with $q = N$, that $H_N(I_{c+\epsilon}, I_\epsilon) \cong H_N(W, W \setminus K) \oplus H_N(W, W \setminus K) \geq 2$. ■

Lemma 3.4. *Suppose that $(U, K, \epsilon) \subset H \times H \times R^+$ satisfies (1) – (6). Suppose in addition that $H_0(I_{c+\epsilon} \cap U) = H_0((I_{c+\epsilon} \cap U) \setminus K) = 1$. Then $H_1(I_{c+\epsilon}, I_\epsilon) = 0$ or $H_0(I_{c+\epsilon}, I_\epsilon) = 2$ holds.*

Proof. From the argument in the proof of Proposition 3.2, we have

$$H_1(I_{c+\epsilon}, I_\epsilon) \cong H_1(I_{c+\epsilon} \cap U, (I_{c+\epsilon} \cap U) \setminus K) \oplus H_1(I_{c+\epsilon} \cap U, (I_{c+\epsilon} \cap U) \setminus K).$$

Then since $H_1(I_{c+\epsilon} \cap U) = 0$ and $H_0(I_{c+\epsilon} \cap U) = H_0((I_{c+\epsilon} \cap U) \setminus K) = 1$, the assertion follows from the exactness of the sequence (3.1) with $q = 1$. ■

We can now prove the Theorem.

Proof of the Theorem. Let (U, K, ϵ) satisfy (1) – (6). We have by Proposition 2.3 and Proposition 3.1 that $H_1(I_{c+\epsilon}, I_\epsilon) = 2$ and $H_q(I_{c+\epsilon}, I_\epsilon) = 0$ for $q \neq 1$. Now suppose that $(I_{c+\epsilon} \cap U) \setminus K$ is disconnected. Then since $H_0((I_{c+\epsilon} \cap U) \setminus K) \geq 2$, we find by (6) that $H_{N-1}(I_{c+\epsilon}, I_\epsilon) = 2$. This is a contradiction. On the other hand, if $U \setminus K$ is connected, then $H_0((I_{c+\epsilon} \cap U) \setminus K) = 1$. Then by Lemma 3.4, we have $H_1(I_{c+\epsilon}, I_\epsilon) = 0$ or $H_0(I_{c+\epsilon}, I_\epsilon) = 2$. This is a contradiction. Thus we obtain that there exists a positive solution of (P). ■

4. PROOF OF PROPOSITION 3.2

We shall construct a triple (U, K, ϵ) satisfying (1) - (6). First we state the following lemma.

Lemma 4.1. *If $0 < \epsilon < c < d < 2c$ and $\{u_n\} \subset I_d \setminus I_\epsilon$ is a sequence such that $\nabla I(u_n) \rightarrow 0$, then $u_n \rightarrow \tau_{x_n} u_\infty$ where $\{x_n\} \subset \mathbb{R}^N$ with $\lim_{n \rightarrow \infty} |x_n| = \infty$.*

Since we are assuming that I has no critical point in $\dot{I}_{2c} \setminus I_c$, the assertion of Lemma 4.1 is a direct consequence of the arguments in [8, 9]. Thus, we omit the proof (cf. also [3]).

We fix a positive number $\rho < 1$. Recalling that the mappings $t \rightarrow I^\infty((\pm t + 1)u_\infty)$ are decreasing as t varies from 0 to ± 1 , we have $I_c^\infty \cap \{tu_\infty : t \in [-\rho + 1, \rho + 1]\} = \{u_\infty\}$. Then we can choose positive numbers r_0 and δ such that

$$(4.1) \quad \{tv : t \in [-\rho + 1, -\rho/2 + 1] \cup [\rho/2 + 1, \rho + 1], v \in S_0\} \subset I_{c-\delta}^\infty$$

where $S_0 = (u_\infty + B_{r_0}) \cap M^\infty$. We note that S_0 is a contractible neighborhood of u_∞ in M^∞ . We may choose r_0 so small that

$$(4.2) \quad S_0 \subset I_{(4/3)c}^\infty.$$

Next, we fix a contractible neighborhood $\tilde{S}_0$ of u_∞ in M^∞ such that $\tilde{S}_0 \subset \text{int}_{M^\infty} S_0$. We put

$$\begin{aligned} D_0 &= \{\tau_x v : v \in S_0, x \in \mathbb{R}^N \text{ with } |x| \geq R_0\}, \\ \tilde{D}_0 &= \{\tau_x v : v \in \tilde{S}_0, x \in \mathbb{R}^N \text{ with } |x| \geq 2R_0\}, \end{aligned}$$

where R_0 is a positive number. Then $\tilde{D}_0 \subset D_0 \subset M^\infty$. Now we define subsets U, K of H by

$$(4.3) \quad \begin{aligned} U &= \{tv : t \in [-\rho + 1, \rho + 1], v \in D_0\}, \\ K &= \{tv : t \in [-\rho/2 + 1, \rho/2 + 1], v \in \tilde{D}_0\}. \end{aligned}$$

Since $\{\tau_x u_\infty : x \in \mathbb{R}^N\} \cap \{\tau_x(-u_\infty) : x \in \mathbb{R}^N\} = \emptyset$, by choosing r_0 and ρ sufficiently small, we have that $U \cap (-U) = \emptyset$. That is, (1) holds. Since (4.1) holds and $\lim_{|x| \rightarrow \infty} I(\tau_x u_\infty) = c$, we can choose R_0 so large that

$$(4.4) \quad \{tv : t \in [-\rho + 1, -\rho/2 + 1] \cup [\rho/2 + 1, \rho + 1], v \in D_0\} \subset I_c.$$

We also have by (4.2) that R_0 can be chosen so large that $U \subset I_{(6/5)c}$. It follows from the definition of U and K that

$$(4.5) \quad \{\tau_x u_\infty : |x| \geq 3R_0\} \subset \text{int} K \subset K \subset \text{int} U.$$

That is, (2) holds with $r = 3R_0$. From the definition, it is obvious that (3) holds. As a direct consequence of (3) of Lemma 4.1 and (4.5), we have

$$(4.6) \quad \inf\{\|\nabla I(v)\| : v \in I_d \setminus (I_\epsilon \cup K \cup (-K))\} > 0$$

for all $0 < \epsilon < c < d < 2c$. Then by deformation lemma(cf. [3]), there exists $\epsilon_0 > 0$ such that for each $0 < \epsilon < \epsilon_0$, I_ϵ is a strong deformation retract of $I_{c+\epsilon} \setminus (K \cup (-K))$. That is, (4) holds for all $0 < \epsilon < \epsilon_0$.

We will see that there exists $0 < \epsilon < \epsilon_0$ such that (U, K, ϵ) satisfies (5) and (6). Here we note that

$$(4.7) \quad c_2 = \inf\{I(\lambda_0(v)v) : v \in D_0 \setminus \tilde{D}_0\} > c_1$$

In fact, if $c_2 = c_1$, there exists a sequence $\{u_n\} \subset M$ such that $u_n = \lambda_0(v_n)v_n$, $v_n \in D_0 \setminus \tilde{D}_0$ for each $n \geq 1$ and that $\lim_{n \rightarrow \infty} I(u_n) = c$. This implies that $\nabla I(u_n) \rightarrow 0$ and then by Lemma 4.1, $u_n \rightarrow \tau_{x_n} u_\infty$, where $\{x_n\} \subset \mathbb{R}^N$ with $\lim |x_n| = \infty$. This implies that $v_n \rightarrow \tau_{x_n} u_\infty$ and this contradicts to the definition of $\{v_n\}$. Here we choose a positive number ϵ such that $\epsilon < c_2 - c$. Here we define subsets of M and H . Noting that

$$\lim_{|x| \rightarrow \infty} I(\tau_x u_\infty) = c$$

We can choose contractible neighborhoods S_1, S_2 of u_∞ in M^∞ and positive numbers R_1, R_2 such that $S_2 \subset \text{int}_{M^\infty} S_1 \subset S_0$, $R_1 < R_2$ and

$$U_i = \{t\tau_x v : t \in [-\rho + 1, \rho + 1], |x| \geq R_i, v \in S_i\} \subset I_{c+\epsilon}.$$

We also set

$$\begin{aligned} U_{1,+} &= \{tv : t \in [-\rho + 1, -\rho/2 + 1], v \in D_0\}, \\ U_{1,-} &= \{tv : t \in [\rho/2 + 1, \rho + 1], v \in D_0\} \end{aligned}$$

and

$$\begin{aligned} U_{2,+} &= \{tv : t \in [-\rho + 1, -\rho/4 + 1], v \in D_0\}, \\ U_{2,-} &= \{tv : t \in [\rho/4 + 1, \rho + 1], v \in D_0\}. \end{aligned}$$

Then from the definitions above and (4.2), we have that

$$\tilde{U}_2 = U_2 \cup U_{2,+} \cup U_{2,-} \subset \tilde{U}_1 = U_1 \cup U_{1,+} \cup U_{1,-} \subset I_{c+\epsilon},$$

and

$$(4.8) \quad \tilde{U}_1 \cong \tilde{U}_2 \cong \{\tau_x u_\infty : |x| \geq R_1\} \cong S^{N-1}.$$

Then we have that (5) holds, as a direct consequence of the following lemma 4.5.

Lemma 4.2. $\tilde{U}_1$ is a deformation retractor of $I_{c+\epsilon} \cap U$.

Proof. To prove the assertion it is sufficient to show the existence of a semiflow $\eta : [0, \infty) \times (I_{c+\epsilon} \cap U) \rightarrow I_{c+\epsilon} \cap U$ such that for each $v \in I_{c+\epsilon} \cap U$, there exists $t_v \geq 0$ satisfying $\eta(t, v) \in \text{int}_{I_{c+\epsilon} \cap U} \tilde{U}_1$ for all $t \geq t_v$. In fact, if there exists such a semiflow, we can construct a strong deformation retraction as in the proof of Lemma 2.5. By (4.7) and the definition of I ,

$$I(v) > c + \epsilon \quad \text{for } v \in \partial_{M^\infty} D_0,$$

and we have

$$D_2 = \{v \in D_0 : I(v) \leq c + \epsilon\} \subset \text{int}_{M^\infty} D_0,$$

Here we fix an open neighborhood D_1 of D_2 in M^∞ such that

$$D_2 \subset \text{int}_{M^\infty} D_1 \subset \text{cl}(D_1) \subset \text{int}_{M^\infty} D_0$$

and set

$$W_i = \{tv : t \in [-\rho + 1, \rho + 1], v \in D_i\}, \quad i = 1, 2.$$

Then

$$U_1 \subset W_2 \subset W_1 \subset I_{c+\epsilon} \cap U.$$

We note that

$$(4.9) \quad I(\lambda_0(v)v) > c + \epsilon \quad \text{for } v \in D_0 \setminus D_2.$$

Let V_1 be a Lipschitz continuous vector field associate with ∇I and V_2 be a vector field defined on $(I_{c+\epsilon} \cap U) \setminus W_2$ by

$$V_2(u) = \begin{cases} u & \text{if } \lambda_0(u) > 1 \\ -u & \text{if } \lambda_0(u) < 1. \end{cases}$$

Since $\lambda_0(u) \neq 1$ on $(I_{c+\epsilon} \cap U) \setminus W_2$ by (4.9), we can see that V_2 is well defined and continuous on $(U \cap I_{c_1+\bar{\epsilon}}) \setminus W_2$. We now set

$$V(u) = \|U_{2,-} \cup U_{2,+} - u\| (\|W_1^c - u\| V_1(u) + \|W_2 - u\| V_2(u))$$

Then V is a Lipschitz continuous vector field on $I_{c+\epsilon} \cap U$ and the solution η of (2.14) defines a semiflow. We shall see that

$$(4.10) \quad \eta(t, v) \in I_{c+\epsilon} \cap U \quad \text{for all } (t, v) \in [0, \infty) \times (I_{c+\epsilon} \cap U).$$

We first note that from the definition of V , $\langle \nabla I(v), V(v) \rangle > 0$ on $I_{c+\epsilon} \cap U$. Then it follows that $\eta(t, v) \leq \eta(s, v)$ for all $t > s \geq 0$ and $v \in I_{c+\epsilon} \cap U$. Since $W_1 \setminus (U_{1,-} \cup U_{1,+}) \subset \text{int}(I_{c+\epsilon} \cap U)$, to show (4.10), it is sufficient to show that (4.10) holds for all $v \in W_1^c \cap (I_{c+\epsilon} \cap U)$. If $v \in W_1^c \cap (I_{c+\epsilon} \cap U)$, then

from the definition of V , we can see that $\eta(t, v) \in W_1^c \cap (I_{c+\epsilon} \cap U)$ for $t \geq 0$ and then (4.10) holds. Moreover we have that for each $v \in W_1^c \cap (I_{c+\epsilon} \cap U)$, $\eta(t, v) \in U_{1,-} \cup U_{1,+}$ for t sufficiently large. On the other hand, it follows from the definition of V that

$$(4.11) \quad \inf\{\|V(u)\| : u \in (I_{c+\epsilon} \cap U) \setminus \tilde{U}_2\} > 0.$$

Then we can see that for any $v \in I_{c+\epsilon} \cap U$, there exists $t_v \geq 0$ such that $\eta(t, v) \in \tilde{U}_1$ for all $t \geq t_v$. This completes the proof. ■

We lastly show that (6) holds. (6) is a consequence of the following Lemma.

Lemma 4.3. *If $(I_{c+\epsilon} \cap U) \setminus K$ is disconnected, then $H_{N-1}((I_{c+\epsilon} \cap U) \setminus K) = 2$.*

Proof. Let $V_{\pm}$ be the components of $(I_{c+\epsilon} \cap U) \setminus K$ containing $U_{1,\pm}$, respectively. We will see that $(I_{c+\epsilon} \cap U) \setminus K$ consists of exactly two components $V_{\pm}$ and that $V_{\pm} \cong S^{N-1}$. Let $v \in D_0$. Then from the definition of M and U , we have that

$$(4.12) \quad \begin{aligned} \{tv : t \in [-\rho + 1, \rho + 1]\} \cap (I_{c+\epsilon} \setminus K) \\ = \{tv : t \in [-\rho + 1, t_1(v)] \cup [t_2(v), \rho + 1]\}, \end{aligned}$$

where $-\rho/2 + 1 \leq t_1(v) \leq t_2(v) \leq \rho/2 + 1$. This implies that if $t_1(v) = t_2(v)$ for some $v \in D_0$, then $(I_{c+\epsilon} \cap U) \setminus K$ is connected. Therefore $t_1(v) < t_2(v)$ for all $v \in D_0$. Then, again by (4.12), $(I_{c+\epsilon} \cap U) \setminus K \cong U_{1,+} \cup U_{1,-}$. Then since $U_{1,\pm} \cong S^{N-1}$, the assertion follows. ■

5. APPENDIX

We put $\mathcal{C} = \cup\{\tau_x u_{\infty} : x \in R^N\}$ and

$$T_{u_{\infty}}(\mathcal{C}) = \{\lim_{t \rightarrow 0} (u_{\infty}(\cdot + tx) - u_{\infty}(\cdot))/t : x \in R^N\}.$$

It is obvious that $\dim T_{u_{\infty}}(\mathcal{C}) = N$. We denote by $\tilde{H}$ the subspace such that $\tilde{H} \oplus T_{u_{\infty}}(\mathcal{C})$. Then $H = \tau_x \tilde{H} \oplus \tau_x T_{u_{\infty}}(\mathcal{C})$ for each $x \in R^N$. For each $r > 0$, we set $B_r^0 = B_r \cap \tilde{H}$. Since $\mathcal{C}$ is a smooth N -manifold, we have that there exists a positive number $r_0 < \|u_{\infty}\|/4$ such that for $x, y \in R^N$ with $x \neq y$,

$$(5.1) \quad \tau_x(u_{\infty} + B_{r_0}^0) \cap \tau_y(u_{\infty} + B_{r_0}^0) = \emptyset$$

We choose a closed contractible neighborhood S_0 of u_{∞} in $M^{\infty} \cap (u_{\infty} + B_{r_0}^0)$ and $0 < \rho < 1$ such that

$$(5.2) \quad \sup\{I^{\infty}((\pm\rho/2 + 1)v) : v \in S_0\} < c.$$

Since $I(v) > c$ for all $v \in S_0 \setminus \{u_\infty\}$, we have that

$$(5.3) \quad \inf\{I^\infty(v) : v \in \partial_{M^\infty \cap (u_\infty + B_{r_0}^0)} S_0\} > c.$$

Here we recall that mappings $t \rightarrow I^\infty((\pm t + 1)v)$ are decreasing as t varies from 0 to $\pm\rho$. Then from (5.2), we have

$$(5.4) \quad \begin{aligned} I_c^\infty \cap \{tv : t \in [-\rho + 1, \rho + 1]\} \\ = \{tv : t \in [-\rho + 1, \lambda_-(v)]\} \cup \{tv : t \in [\lambda_+(v), \rho + 1]\} \end{aligned}$$

where

$$\begin{cases} \lambda_-(v) < 1 < \lambda_+(v) & \text{for } v \in S_0 \setminus \{u_\infty\} \\ \lambda_-(v) = \lambda_+(v) = 1 & \text{for } v = u_\infty. \end{cases}$$

That is, for each $v \in S_0 \setminus \{u_\infty\}$, the set $I_c^\infty \cap \{tv : t \in [-\rho + 1, \rho + 1]\}$ consists of two intervals, and each interval has one end point in one of the sets

$$V_\pm = \{(\pm\rho + 1)v : v \in S_0\}.$$

Then noting that $\lambda_-(\cdot)$ and $\lambda_+(\cdot)$ are continuous and $V_\pm$ are contractible, we have from observations above that

$$(5.5) \quad I_c^\infty \cap (V \setminus \{u_\infty\}) \cong V_- \cup V_+ \cong \{0, 1\} \quad \text{and} \quad I_c^\infty \cap V \cong [0, 1]$$

Now let $0 < \epsilon < c$. First we note that

$$I^\infty(u) = \tau_x \cdot I^\infty(u) = I^\infty(\tau_x u) \quad \text{for all } x \in R^N \text{ and } u \in H.$$

Then we have that $I_c^\infty \cap (\cup\{\tau_x V : x \in R^N\})$ and $I_\epsilon^\infty \cap (\cup\{\tau_x V : x \in R^N\})$ have the same homotopy type with that of $I_c^\infty \cap V$ and $I_\epsilon^\infty \cap V$, respectively. On the other hand, by the same argument for the second deformation lemma in Chang [4], we have that I_c^∞ is a strong deformation retraction of $I_{c+\epsilon}^\infty$. Then we find

$$H_q(I_{c+\epsilon}^\infty, I_{c-\epsilon}^\infty) \cong H_q(I_c^\infty, I_{c-\epsilon}^\infty).$$

We also have by the deformation property that

$$H_q(I_c^\infty \setminus \mathcal{C}, I_{c-\epsilon}^\infty) \cong H_q(I_{c-\epsilon}^\infty, I_{c-\epsilon}^\infty) \cong 0.$$

From the exactness of the singular homology groups, we have

$$\begin{aligned} H_q(I_c^\infty \setminus \mathcal{C}, I_{c-\epsilon}^\infty) &\rightarrow H_q(I_c^\infty, I_{c-\epsilon}^\infty) \\ &\rightarrow H_q(I_c^\infty, I_c^\infty \setminus \mathcal{C}) \rightarrow H_{q-1}(I_c^\infty \setminus \mathcal{C}, I_{c-\epsilon}^\infty) \rightarrow \cdots \end{aligned}$$

and we find

$$0 \rightarrow H_q(I_c^\infty, I_{c-\epsilon}^\infty) \rightarrow H_q(I_c^\infty, I_c^\infty \setminus \mathcal{C}) \rightarrow 0.$$

That is,

$$H_q(I_c^\infty, I_{c-\epsilon}^\infty) \cong H_q(I_c^\infty, I_c^\infty \setminus \mathcal{C}).$$

Then from the excision property of homology groups and (5.5), we have

$$\begin{aligned} H_*(I_{c+\epsilon}^\infty, I_\epsilon^\infty) &\cong H_*(I_c^\infty, I_c^\infty \setminus \mathcal{C}) \\ &\cong H_*(I_c^\infty \cap (\cup_x \tau_x V), I_c^\infty \cap ((\cup_x \tau_x V) \setminus \mathcal{C})) \\ &\cong H_*(I_c^\infty \cap V, I_c^\infty \cap (V \setminus \{u_\infty\})) \\ &\cong H_*([0, 1], \{0, 1\}). \end{aligned}$$

This completes the proof. ■

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