SOLVABILITY OF A MULTI-POINT BOUNDARY VALUE PROBLEM OF NEUMANN TYPE

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Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying Carathéodory's conditions and $e(t) \in L^1[0,1]$. Let $\xi_i \in (0,1)$, $a_i \in \mathbb{R}$, $i = 1,2,\dots,m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ be given. This paper is concerned with the problem of existence of a solution for the m-point boundary value problem $x''(t) = f(t,x(t),x'(t)) + e(t)$, $0 < t < 1$; $x(0) = 0$, $x'(1) = \sum_{i=1}^{m-2} a_i x'(\xi_i)$. This paper gives conditions for the existence of a solution for this boundary value problem using some new Poincaré type a priori estimates. This problem was studied earlier by Gupta, Ntouyas, and Tsamatos (1994) when all of the $a_i \in \mathbb{R}$, $i = 1,2,\dots,m-2$, had the same sign. The results of this paper give considerably better existence conditions even in the case when all of the $a_i \in \mathbb{R}$, $i = 1,2,\dots,m-2$, have the same sign. Some examples are given to illustrate this point.

1. Introduction

Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying Carathéodory's conditions and $e : [0,1] \mapsto \mathbb{R}$ be a function in $L^1[0,1]$, $a_i \in \mathbb{R}$, $\xi_i \in (0,1)$, $i = 1,2,\dots,m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$. We study the problem of existence of solutions for the m-point boundary value problem

$$\begin{gathered} x''(t) = f\big(t,x(t),x'(t)\big) + e(t), \quad 0 < t < 1, \\ x(0) = 0, \qquad x'(1) = \sum_{i=1}^{m-2} a_i x'\big(\xi_i\big). \end{gathered} \tag{1.1}$$

This problem was studied earlier by Gupta, Ntouyas, and Tsamatos in [1] when all of the $a_i \in \mathbb{R}$, $i = 1,2,\dots,m-2$, have the same sign. Gupta, Ntouyas, and Tsamatos have studied problem (1.1) by first studying the three-point boundary value problem, for a given $\alpha \in \mathbb{R}$, $\alpha \neq 1$, $\eta \in (0,1)$,

$$\begin{gathered} x''(t) = f\big(t,x(t),x'(t)\big) + e(t), \quad 0 < t < 1, \\ x(0) = 0, \qquad x'(1) = \alpha x'(\eta). \end{gathered} \tag{1.2}$$

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The purpose of this paper is to obtain conditions for the existence of a solution for the boundary value problem (1.1), using new estimates and inequalities involving a function $x(t)$ and its derivative $x'(t)$. These results are motivated by the so-called *nonlocal* boundary value problem studied by Il'in and Moiseev in [5].

We use the classical spaces $C[0,1]$, $C^k[0,1]$, $L^k[0,1]$, and $L^\infty[0,1]$ of continuous, k-times continuously differentiable, measurable real-valued functions whose kth power of the absolute value is Lebesgue integrable on $[0,1]$, or measurable functions that are essentially bounded on $[0,1]$. We also use the Sobolev spaces $W^{2,k}(0,1)$, $k=1,2$ defined by

$$W^{2,k}(0,1)=\left\{x:[0,1]\longrightarrow\mathbb{R}\mid x,x' \text{ absolutely continuous on } [0,1] \text{ with } x''\in L^k[0,1]\right\} \tag{1.3}$$

with its usual norm. We denote the norm in $L^k[0,1]$ by $\|\cdot\|_k$, and the norm in $L^\infty[0,1]$ by $\|\cdot\|_\infty$.

2. A priori estimates

Let $a_i\in\mathbb{R}$, $\xi_i\in(0,1)$, $i=1,2,\dots,m-2$, $0<\xi_1<\xi_2<\cdots<\xi_{m-2}<1$, with $\alpha=\sum_{i=1}^{m-2}a_i\neq 1$ be given. Let $x(t)\in W^{2,1}(0,1)$ be such that $x(0)=0$, $x'(1)=\sum_{i=1}^{m-2}a_ix'(\xi_i)$ be given. We are interested in obtaining a priori estimates of the form $\|x'\|_\infty\le C\|x''\|_1$. The following theorem gives such an estimate. We recall that for $a\in\mathbb{R}$, $a_+=\max\{a,0\}$, $a_-=\max\{-a,0\}$ so that $a=a_+-a_-$ and $|a|=a_++a_-$.

THEOREM 2.1. *Let $a_i\in\mathbb{R}$, $\xi_i\in(0,1)$, $i=1,2,\dots,m-2$, $0<\xi_1<\xi_2<\cdots<\xi_{m-2}<1$, with $\alpha=\sum_{i=1}^{m-2}a_i\neq 1$ be given. Then for $x(t)\in W^{2,1}(0,1)$ with $x(0)=0, x'(1)=\sum_{i=1}^{m-2}a_ix'(\xi_i)$ we have*

$$\|x'\|_\infty\le\frac{1}{1-\tau}\|x''\|_1, \tag{2.1}$$

where

$$\tau=\min\left\{\frac{\sum_{i=1}^{m-2}(a_i)_+}{\sum_{i=1}^{m-2}(a_i)_-+1},\frac{\sum_{i=1}^{m-2}(a_i)_-+1}{\sum_{i=1}^{m-2}(a_i)_+}\right\}. \tag{2.2}$$

Proof. We see that the assumption $x'(1)=\sum_{i=1}^{m-2}a_ix'(\xi_i)$ implies

$$x'(1)+\sum_{i=1}^{m-2}(a_i)_-x'(\xi_i)=\sum_{i=1}^{m-2}(a_i)_+x'(\xi_i) \tag{2.3}$$

and thus there exist $\lambda_1,\lambda_2\in[0,1]$ such that

$$\left(1+\sum_{i=1}^{m-2}(a_i)_-\right)x'(\lambda_1)=\sum_{i=1}^{m-2}(a_i)_+x'(\lambda_2). \tag{2.4}$$

If, now, either $x'(\lambda_1)=0$ or $x'(\lambda_2)=0$, then we clearly have

$$\|x'\|_\infty\le\|x''\|_1. \tag{2.5}$$

Suppose, now, that $x'(\lambda_1) \neq 0$ and $x'(\lambda_2) \neq 0$. Then it follows easily from (2.4) that $x'(\lambda_1) \neq x'(\lambda_2)$, in view of the assumption $\alpha = \sum_{i=1}^{m-2} a_i \neq 1$. Then it follows from (2.4), the estimate (2.5), and the equations

$$x'(t) = x'(\lambda_1) + \int_{\lambda_1}^{t} x''(s)\,ds, \qquad x'(t) = x'(\lambda_2) + \int_{\lambda_2}^{t} x''(s)\,ds, \tag{2.6}$$

that

$$\|x'\|_\infty \leq \frac{1}{1-\tau}\|x''\|_1 \tag{2.7}$$

with

$$\tau = \min\left\{\frac{\sum_{i=1}^{m-2}(a_i)_+}{\sum_{i=1}^{m-2}(a_i)_- + 1}, \frac{\sum_{i=1}^{m-2}(a_i)_- + 1}{\sum_{i=1}^{m-2}(a_i)_+}\right\}. \tag{2.8}$$

This completes the proof of the theorem. □

Remark 2.2. We note that if $a_i \leq 0$ for every $i = 1, 2, \ldots, m-2$, then $\tau = 0$ and if $a_i \geq 0$ for every $i = 1, 2, \ldots, m-2$ so that $\alpha = \sum_{i=1}^{m-2} a_i = \sum_{i=1}^{m-2}(a_i)_+ \geq 0$, then $\tau = \min\{\alpha, 1/\alpha\} \in [0, 1)$ since $\alpha \neq 1$, by assumption.

The following theorem gives a better estimate for the three-point boundary value in the case of the L^2-norm.

THEOREM 2.3. *Let $\alpha \in \mathbb{R}$, $\alpha \neq 1$, and $\eta \in (0, 1)$ be given. Let $x(t) \in W^{2,2}(0, 1)$ be such that $x'(1) = \alpha x'(\eta)$. Then*

$$\|x'\|_2 \leq C(\alpha, \eta)\|x''\|_2, \tag{2.9}$$

where

$$C(\alpha, \eta) = \begin{cases} \min\left\{\sqrt{F(\alpha, \eta)}, \dfrac{2}{\pi}\right\} & \text{if } \alpha \leq 0, \\ \sqrt{F(\alpha, \eta)} & \text{if } \alpha > 0, \end{cases} \tag{2.10}$$

$$F(\alpha, \eta) = \frac{1}{2(\alpha-1)^2}\left[\alpha^2(1-\eta)^2 + (\alpha^2 - 2\alpha)\eta^2 + 1\right].$$

Proof. If $\alpha \leq 0$, we note from $x'(1) = \alpha x'(\eta)$ that there exists an $\xi \in (\eta, 1)$ such that $x'(\xi) = 0$. It follows from the Wirtinger's inequality (see [4, Theorem 256]) that

$$\|x'\|_2 \leq \frac{2}{\pi}\|x''\|_2. \tag{2.11}$$

Next, we note, again, from $x'(1) = \alpha x'(\eta)$ that

$$x'(t) = \int_0^t x''(s)\,ds + \frac{\alpha}{1-\alpha}\int_0^\eta x''(s)\,ds - \frac{1}{1-\alpha}\int_0^1 x''(s)\,ds \quad \text{for } 0 < t < 1. \tag{2.12}$$

Accordingly, we have for $t \in [0, \eta]$

$$\begin{aligned} x'(t) &= \int_0^t x''(s)\,ds + \frac{\alpha}{1-\alpha}\int_0^\eta x''(s)\,ds - \frac{1}{1-\alpha}\int_0^1 x''(s)\,ds \\ &= \int_0^t \left(1+\frac{\alpha}{1-\alpha}-\frac{1}{1-\alpha}\right)x''(s)ds + \int_t^\eta \left(\frac{\alpha}{1-\alpha}-\frac{1}{1-\alpha}\right)x''(s)\,ds - \frac{1}{1-\alpha}\int_\eta^1 x''(s)\,ds \\ &= -\int_t^\eta x''(s)\,ds - \frac{1}{1-\alpha}\int_\eta^1 x''(s)\,ds, \end{aligned} \tag{2.13}$$

and for $t \in [\eta, 1]$

$$\begin{aligned} x(t) &= \int_0^t x''(s)\,ds + \frac{\alpha}{1-\alpha}\int_0^\eta x''(s)\,ds - \frac{1}{1-\alpha}\int_0^1 x''(s)\,ds \\ &= \int_0^\eta \left(1+\frac{\alpha}{1-\alpha}-\frac{1}{1-\alpha}\right)x''(s)\,ds + \int_\eta^t \left(1-\frac{1}{1-\alpha}\right)x''(s)\,ds - \frac{1}{1-\alpha}\int_t^1 x''(s)\,ds \\ &= -\int_\eta^t \frac{\alpha}{1-\alpha}x''(s)\,ds - \frac{1}{1-\alpha}\int_t^1 x''(s)\,ds. \end{aligned} \tag{2.14}$$

We now define a function $K : [0,1]\times[0,1] \mapsto \mathbb{R}$ by

$$K(t,s) = \begin{cases} -\chi_{[t,\eta]}(s) - \dfrac{1}{1-\alpha}\chi_{[\eta,1]}(s) & \text{for } t \in [0,\eta],\ s \in [0,1], \\ -\dfrac{\alpha}{1-\alpha}\chi_{[\eta,t]}(s) - \dfrac{1}{1-\alpha}\chi_{[t,1]}(s) & \text{for } t \in [\eta,1],\ s \in [0,1]. \end{cases} \tag{2.15}$$

Now, we see from (2.13) and (2.14) that

$$x'(t) = \int_0^1 K(t,s)x''(s)\,ds \quad \text{for } t \in [0,1], \tag{2.16}$$

$$\|x'\|_2^2 \le \left(\int_0^1\int_0^1 \big(K(t,s)\big)^2\,ds\,dt\right)\|x''\|_2^2. \tag{2.17}$$

Now, it is easy to see that

$$\int_0^1\int_0^1 \big(K(t,s)\big)^2\,ds\,dt = \frac{1}{2(\alpha-1)^2}\big[\alpha^2(1-\eta)^2 + \big(\alpha^2-2\alpha\big)\eta^2 + 1\big]. \tag{2.18}$$

For $\alpha \le 0$ the estimate (2.9) is now immediate from (2.11), (2.17), and (2.18) and for $\alpha > 0$, $\alpha \ne 1$, by (2.17) and (2.18). This completes the proof of the theorem. □

Remark 2.4. It is easy to see that $C(-0.1,\eta) = 2/\pi$, for all $\eta \in (0,1)$, indeed, $\sqrt{F(-0.1,\eta)} \ge 0.648986183$ and $2/\pi \approx 0.6366197724$. Also $C(-2,1/3) = \sqrt{11/54}$ and $C(-2,15/16) = 2/\pi$, since $\sqrt{F(-2,15/16)} = \sqrt{1030}/48 > 2/\pi$.

3. Existence theorems

Definition 3.1. A function $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies Carathéodory's conditions if

(i) for each $(x, y) \in \mathbb{R}^2$, the function $t \in [0,1] \to f(t, x, y) \in \mathbb{R}$ is measurable on $[0,1]$,

(ii) for a.e. $t \in [0,1]$, the function $(x, y) \in \mathbb{R}^2 \to f(t, x, y) \in \mathbb{R}$ is continuous on $\mathbb{R}^2$,

(iii) for each $r > 0$, there exists $\alpha_r(t) \in L^1[0,1]$ such that $|f(t, x, y)| \le \alpha_r(t)$ for a.e. $t \in [0,1]$ and all $(x, y) \in \mathbb{R}^2$ with $\sqrt{x^2+y^2} \le r$.

THEOREM 3.2. *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying Carathéodory's conditions. Assume that there exist functions $p(t)$, $q(t)$, and $r(t)$ in $L^1(0,1)$ such that*

$$\left|f\left(t, x_1, x_2\right)\right| \le p(t)|x_1| + q(t)|x_2| + r(t) \tag{3.1}$$

for a.e. $t \in [0,1]$ and all $(x_1, x_2) \in \mathbb{R}^2$. Also let $a_i \in \mathbb{R}$, $\xi_i \in (0,1)$, $i = 1, 2, \dots, m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, with $\alpha = \sum_{i=1}^{m-2} a_i \ne 1$ be given. Then the boundary value problem (1.1) has at least one solution in $C^1[0,1]$ provided

$$\|tp(t)\|_1 + \|q(t)\|_1 + \tau < 1, \tag{3.2}$$

where τ is as defined in Theorem 2.1.

Proof. Let X denote the Banach space $C^1[0,1]$ and Y denote the Banach space $L^1(0,1)$ with their usual norms. We define a linear mapping $L : D(L) \subset X \to Y$ by setting

$$D(L) = \left\{ x \in W^{2,1}(0,1) \,\middle|\, x(0) = 0,\ x'(1) = \sum_{i=1}^{m-2} a_i x'\left(\xi_i\right) \right\}, \tag{3.3}$$

and for $x \in D(L)$,

$$Lx = x''. \tag{3.4}$$

We also define a nonlinear mapping $N : X \to Y$ by setting

$$(Nx)(t) = f\left(t, x(t), x'(t)\right), \quad t \in [0,1]. \tag{3.5}$$

We note that N is a bounded mapping from X into Y. Next, it is easy to see that the linear mapping $L : D(L) \subset X \to Y$, is a one-to-one mapping. Next, the linear mapping $K : Y \to X$, defined for $y \in Y$ by

$$(Ky)(t) = \int_0^t (t-s)y(s)\,ds + At, \tag{3.6}$$

where A is given by,

$$A\left(1 - \sum_{i=1}^{m-2} a_i\right) = \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} y(s)\,ds - \int_0^1 y(s)\,ds, \tag{3.7}$$

is such that for $y \in Y$, $Ky \in D(L)$, and $LKy = y$; and for $u \in D(L)$, $KLu = u$. Furthermore, it follows easily using the Arzela-Ascoli theorem that KN maps a bounded subset of X into a relatively compact subset of X. Hence $KN : X \to X$ is a compact mapping.

We, next, note that $x \in C^1[0,1]$ is a solution of the boundary value problem (1.2) if and only if x is a solution to the operator equation

$$Lx = Nx + e. \tag{3.8}$$

Now, the operator equation $Lx = Nx + e$ is equivalent to the equation

$$x = KNx + Ke. \tag{3.9}$$

We apply the Leray-Schauder continuation theorem (cf. [6, Corollary IV.7]) to obtain the existence of a solution for $x = KNx + Ke$ or equivalently to the boundary value problem (1.2).

To do this, it suffices to verify that the set of all possible solutions of the family of equations

$$\begin{aligned} x''(t) &= \lambda f\big(t, x(t), x'(t)\big) + \lambda e(t), \quad 0 < t < 1, \\ x(0) &= 0, \qquad x'(1) = \sum_{i=1}^{m-2} a_i x'\big(\xi_i\big), \end{aligned} \tag{3.10}$$

is, a priori, bounded in $C^1[0,1]$ by a constant independent of $\lambda \in [0,1]$.

We observe that if $x \in W^{2,1}(0,1)$, with $x(0) = 0$, $x'(1) = \sum_{i=1}^{m-2} a_i x'(\xi_i)$, then $x(t) = \int_0^t x'(s)ds$. Hence, $|x(t)| \le t\|x'\|_\infty$ for $t \in [0,1]$ and $\|x'\|_\infty \le (1/(1-\tau))\|x''\|_1$, where τ is as defined in Theorem 2.1.

Let $x(t)$ be a solution of (3.10) for some $\lambda \in [0,1]$, so that $x \in W^{2,1}(0,1)$ with $x(0) = 0$, $x'(1) = \sum_{i=1}^{m-2} a_i x'(\xi_i)$. We then get from the equation in (3.10) and Theorem 2.1 that

$$\begin{aligned} \|x'\|_\infty &\le \frac{\lambda}{1-\tau} \big\| f\big(t, x(t), x'(t)\big) + e(t) \big\|_1 \\ &\le \frac{1}{1-\tau} \big(\big\| p(t)|x(t)| + q(t)\big|x'(t)\big| + r(t) \big\|_1 + \|e(t)\|_1 \big) \\ &\le \frac{1}{1-\tau} \big(\big\| tp(t)\|x'\|_\infty + q(t)\big|x'(t)\big| + r(t) \big\|_1 + \|e(t)\|_1 \big) \\ &\le \frac{1}{1-\tau} \big(\|tp(t)\|_1 + \|q(t)\|_1 \big) \|x'\|_\infty + \frac{1}{1-\tau} \big(\|r(t)\|_1 + \|e(t)\|_1 \big). \end{aligned} \tag{3.11}$$

It follows from assumption (3.2) that there is a constant c, independent of $\lambda \in [0,1]$, such that

$$\|x\|_\infty \le \|x'\|_\infty \le c. \tag{3.12}$$

It is now immediate that the set of solutions of the family of equations (3.10) is, a priori, bounded in $C^1[0,1]$ by a constant, independent of $\lambda \in [0,1]$.

This completes the proof of the theorem. □

THEOREM 3.3. *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying Carathéodory's conditions. Assume that there exist functions $p(t)$, $q(t)$, and $r(t)$ in $L^2(0,1)$ such that*

$$\left|f\left(t, x_1, x_2\right)\right| \le p(t)|x_1| + q(t)|x_2| + r(t) \tag{3.13}$$

for a.e. $t \in [0,1]$ and all $(x_1, x_2) \in \mathbb{R}^2$. Also let $\alpha \neq 1$, and $\eta \in (0,1)$ be given. Then for any given $e(t)$ in $L^2(0,1)$ the boundary value problem (1.2) has at least one solution in $C^1[0,1]$ provided

$$C(\alpha,\eta)\left(\frac{2}{\pi}\|p\|_2 + \|q\|_2\right) < 1, \tag{3.14}$$

where $C(\alpha,\eta)$ is as in Theorem 2.3.

Proof. As in the proof of Theorem 3.2 it suffices to prove that the set of all possible solutions of the family of equations

$$\begin{aligned} x''(t) &= \lambda f\left(t, x(t), x'(t)\right) + \lambda e(t), \quad 0 < t < 1, \\ x(0) &= 0, \qquad x'(1) = \alpha x'(\eta), \end{aligned} \tag{3.15}$$

is, a priori, bounded in $C^1[0,1]$ by a constant independent of $\lambda \in [0,1]$. For $x \in W^{2,2}(0,1)$, with $x(0) = 0$, and $x'(1) = \alpha x'(\eta)$ we use the Wirtinger's inequality (see [4, Theorem 256]) and Theorem 2.3, above, to note that

$$\|x\|_2 \le \frac{2}{\pi}\left\|x'\right\|_2 \qquad \text{and} \qquad \left\|x'\right\|_2 \le C(\alpha,\eta)\left\|x''\right\|_2. \tag{3.16}$$

Now, for a solution x of the family of equations (3.15) for some $\lambda \in [0,1]$ we have

$$\begin{aligned} \left\|x''\right\|_2 &\le \lambda\left\|f\left(t, x(t), x'(t)\right) + e(t)\right\|_2 \\ &\le \left\|p(t)|x(t)| + q(t)\left|x'(t)\right| + r(t)\right\|_2 + \|e\|_2 \\ &\le \|p\|_2\|x\|_2 + \|q\|_2\left\|x'\right\|_2 + \|r\|_2 + \|e\|_2 \\ &\le \left(\frac{2}{\pi}\|p\|_2 + \|q\|_2\right)\left\|x'\right\|_2 + \|r(t)\|_2 + \|e\|_2 \\ &\le C(\alpha,\eta)\left(\frac{2}{\pi}\|p\|_2 + \|q\|_2\right)\left\|x''\right\|_2 + \|r(t)\|_2 + \|e\|_2, \end{aligned} \tag{3.17}$$

in view of estimate (3.16), for a solution x of the family of equations (3.15) for some $\lambda \in [0,1]$. It then follows from (3.14) that there is a constant c independent of $\lambda \in [0,1]$ such that

$$\left\|x''\right\|_2 \le c, \tag{3.18}$$

for a solution x of the family of equations (3.15) for some $\lambda \in [0,1]$. Finally, we see, using Theorem 2.1 that $\|x\|_\infty \le \|x'\|_\infty \le (1/(1-\tau))\|x''\|_1 \le (1/(1-\tau))\|x''\|_2$ and accordingly, the set of solutions of the family of equations (3.15) is, a priori, bounded in $C^1[0,1]$ by a constant independent of $\lambda \in [0,1]$. This completes the proof of Theorem 3.3. □

We next give an existence condition independent of α and η for the three-point boundary value problem (1.2).

Let $p(t), q(t)$ be given functions in $L^1(0,1)$. For, a given measurable function $x(t)$ on $[0,1]$, we define for $t \in [0,1]$,

$$\begin{gathered} P(t) = \int_t^1 p(u)\,du, \qquad (Vx)(t) = \int_t^1 q(s)x(s)\,ds, \\ (Sx)(t) = P(t)\int_0^t x(u)\,du + \int_t^1 P(u)x(u)\,du; \end{gathered} \tag{3.19}$$

provided that the integrals in (3.19) exist. We, further, suppose that the operator $M : L^2(0,1) \mapsto L^2(0,1)$ defined for $x(t) \in L^2(0,1)$ by

$$(Mx)(t) = (Sx)(t) + (Vx)(t), \quad 0 < t < 1; \tag{3.20}$$

maps $L^2(0,1)$ into itself and is continuous.

THEOREM 3.4. *Let $p(t), q(t)$, and M be as above. Let $f : [0,1] \times \mathbb{R}^2 \mapsto \mathbb{R}$ be a given function satisfying Carathéodory conditions. Suppose that $p(t), q(t) \in L^1(0,1)$ and $r(t) \in L^2(0,1)$ be such that*

$$|f(t,x,y)| \le p(t)|x| + q(t)|y| + r(t) \quad \text{for } t \in [0,1],\ x, y \in \mathbb{R}. \tag{3.21}$$

Then, given $\alpha \in \mathbb{R}$, $\alpha \le 0$, and $\eta \in (0,1)$, the three-point boundary value problem

$$\begin{gathered} x''(t) = f\big(t, x(t), x'(t)\big), \quad 0 < t < 1, \\ x(0) = 0, \qquad x'(1) = \alpha x'(\eta), \end{gathered} \tag{3.22}$$

has at least one solution if the spectral radius, $r(M)$ of the operator M is less than one.

Proof. Let $x(t)$ be a solution of the boundary value problem (3.22), so that $x(0) = 0$, $x'(1) = \alpha x'(\eta)$. It is then easy to see that there exists a $\mu \in (0,1)$ such that $x'(\mu) = 0$. The rest of the proof is identical to the proof of Theorem 5 of [2] and is omitted. □

COROLLARY 3.5. *Let $p(t), q(t)$ in Theorem 3.4 be such that $p(t), q^2(t) \in L^1(\sigma,1)$ for every $\sigma > 0$, and $\sqrt{t}\int_t^1 q^2(s)ds \in L^2(0,1)$. Suppose, further, that*

$$\|\sqrt{2t}P(t)\|_2 + \left\|\sqrt{2t}\int_t^1 q^2(s)\,ds\right\|_2^{1/2} < 1. \tag{3.23}$$

Then, given $\alpha \in \mathbb{R}$, $\alpha \le 0$, and $\eta \in (0,1)$, the boundary value problem (3.22) has at least one solution.

The proof of the corollary is identical to the proof of Theorem 3 of [3] and is omitted.

Example 3.6. Let $\alpha \le 0$ and $\eta \in (0,1)$ be given and $A \in \mathbb{R}$. For $e(t) \in L^1(0,1)$, we consider the three-point boundary value problem

$$\begin{gathered} x''(t) = t^{-1/2}|x(t)| + At\left|x'(t)\right| + e(t), \quad 0 < t < 1, \\ x(0) = 0, \qquad x'(1) = \alpha x'(\eta). \end{gathered} \tag{3.24}$$

We apply Theorem 3.2 to obtain a condition for the existence of a solution for the three-point boundary value problem (3.24). Here $p(t) = t^{-1/2}$, $q(t) = At$, and $\tau = 0$. Now, $\|tp(t)\|_1 = 2/3$ and $\|q(t)\|_1 = (1/2)|A|$. Now, if

$$\frac{2}{3} + \frac{1}{2}|A| < 1, \tag{3.25}$$

or, equivalently

$$|A| < \frac{2}{3}, \tag{3.26}$$

then Theorem 3.2 implies the existence of a solution for the three-point boundary value problem (3.24).

Example 3.7. Let $\alpha = -2$, $\eta = 1/3$, and $A \in \mathbb{R}$. For $e(t) \in L^2(0,1)$, we, next, consider the three-point boundary value problem

$$\begin{gathered} x''(t) = t^{-1/4}|x(t)| + At^{-1/4}\left|x'(t)\right| + e(t), \quad 0 < t < 1, \\ x(0) = 0, \qquad x'(1) = \alpha x'(\eta). \end{gathered} \tag{3.27}$$

We apply Theorem 3.3 to obtain a condition for the existence of a solution for the three-point boundary value problem (3.27). Here $p(t) = t^{-1/4}$, $q(t) = At^{-1/4}$. Now, $\|p(t)\|_2 = \sqrt{2}$ and $\|q(t)\|_2 = \sqrt{2}|A|$. Now the existence condition required to apply Theorem 3.3 is

$$C(\alpha,\eta)\left(\frac{2\sqrt{2}}{\pi} + \sqrt{2}|A|\right) < 1. \tag{3.28}$$

Since we have $C(-2,1/3) = \sqrt{11/54}$, we get from (3.28)

$$\frac{2\sqrt{22}}{\sqrt{54}\pi} + \sqrt{\frac{22}{54}}|A| < 1. \tag{3.29}$$

Accordingly, we see from Theorem 3.3 that a solution for the three-point boundary value problem (3.27) exists if $|A| < \sqrt{54/22}(1 - 2\sqrt{22}/(\sqrt{54}\pi)) = 0.930079132$. Next, we apply Corollary 3.5 to the three-point boundary value problem (3.27). Now, we see that $P(t) = \int_t^1 u^{-1/4}du = 4/3 - 4/3(\sqrt[4]{t})^3$, so that

$$\begin{gathered} \left\|\sqrt{2t}P(t)\right\|_2^2 = \int_0^1 \left(\sqrt{2t}\left(\frac{4}{3} - \frac{4}{3}\left(\sqrt[4]{t}\right)^3\right)\right)^2 dt = 0.20779, \\ \left\|\sqrt{2t}\int_t^1 q^2(s)\,ds\right\|_2^2 = 8A^4\int_0^1 t\left(1-\sqrt{t}\right)^2 dt = \frac{4}{15}A^4, \end{gathered} \tag{3.30}$$

so that a solution to the three-point boundary value problem (3.27) exists if

$$\sqrt{0.20779}+\left(\frac{4}{15}\right)^{0.25}|A|<1 \tag{3.31}$$

or equivalently, if $|A| < (15/4)^{0.25}(1-\sqrt{0.20779}) = 0.7572417038$ for every $\eta \in (0,1)$. So we see that Corollary 3.5 does not give a better result than Theorem 3.3. On the other hand, if we apply Theorem 3.3 when $\alpha = -0.1$, $\eta \in (0,1)$ so that $C(-0.1,\eta) = 2/\pi$ we see that a solution to the three-point boundary value problem (3.27) exists if $|A| < 0.4741009622$, which is not as good as that given by Corollary 3.5.

Example 3.8. Let $\alpha = -2$, $\eta = 1/3$, and $A \in \mathbb{R}$. For $e(t) \in L^2(0,1)$, we, next, consider the three-point boundary value problem

$$\begin{gathered} x''(t) = t^{-15/32}|x(t)| + At\left|x'(t)\right| + e(t), \quad 0 < t < 1, \\ x(0) = 0, \qquad x'(1) = \alpha x'(\eta). \end{gathered} \tag{3.32}$$

We apply Theorem 3.3 to obtain a condition for the existence of a solution for the three-point boundary value problem (3.32). Here $p(t) = t^{-15/32}$, $q(t) = At$. Now, $\|p(t)\|_2 = 4$ and $\|q(t)\|_2 = (1/\sqrt{3})|A|$. Now the existence condition required to apply Theorem 3.3 is

$$C(\alpha,\eta)\left(\frac{8}{\pi}+\frac{1}{\sqrt{3}}|A|\right) < 1. \tag{3.33}$$

Since, $C(-2,1/3) = \sqrt{11/54}$ and we get from (3.33)

$$\frac{8\sqrt{11}}{\sqrt{54}\pi}+\sqrt{\frac{11}{162}}|A| < 1, \tag{3.34}$$

which is impossible. Now, to apply Theorem 3.2 we see that $\|tp(t)\|_1 = \int_0^1 t^{17/32}dt = 32/49$ and $\|q(t)\|_1 = (1/2)|A|$. Accordingly, we see using Theorem 3.2 a solution for the three-point boundary value problem (3.32) exists if

$$\frac{32}{49}+\frac{1}{2}|A| < 1, \tag{3.35}$$

or, equivalently, if

$$|A| < 2\left(1-\frac{32}{49}\right) = \frac{34}{49} = 0.69387751. \tag{3.36}$$

Next, we apply Corollary 3.5 to the three-point boundary value problem (3.32). Now, we see that $P(t) = \int_t^1 u^{-15/32}du = 32/17-(32/17)(\sqrt[32]{t})^{17}$, so that

$$\begin{gathered} \left\|\sqrt{2t}P(t)\right\|_2^2 = \int_0^1\left(\sqrt{2t}\left(\frac{32}{17}-\frac{32}{17}\left(\sqrt[32]{t}\right)^{17}\right)\right)^2 dt = 0.258, \\ \left\|\sqrt{2t}\int_t^1 q^2(s)\,ds\right\|_2^2 = \frac{2A^4}{9}\int_0^1 t\left(1-t^3\right)^2 dt = \frac{1}{20}A^4, \end{gathered} \tag{3.37}$$

so that a solution to the three-point boundary value problem (3.32) exists if

$$\sqrt{0.258}+\left(\frac{1}{20}\right)^{0.25}|A|<1 \tag{3.38}$$

or equivalently, if $|A| < (20)^{0.25}(1-\sqrt{0.258}) = 1.040586544$. Clearly, Corollary 3.5 gives a better result than Theorem 3.2.

References

[1] C. P. Gupta, S. K. Ntouyas, and P. C. Tsamatos, *Existence results for m-point boundary value problems*, Differential Equations Dynam. Systems **2** (1994), no. 4, 289–298. MR 97a:34045. Zbl 877.34019.

[2] C. P. Gupta and S. I. Trofimchuk, *Existence of a solution of a three-point boundary value problem and the spectral radius of a related linear operator*, Nonlinear Anal. **34** (1998), no. 4, 489–507. MR 99d:34029. Zbl 944.34009.

[3] _____, *A Wirtinger type inequality and a three point boundary value problem*, Dynam. Systems Appl. **8** (1999), no. 1, 127–132. MR 99m:26025. Zbl 939.26007.

[4] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, New York, 1988, Reprint of the 1952 edition. MR 89d:26016.

[5] V. A. Il'in and E. I. Moiseev, *Nonlocal boundary value problem of the second kind for the Sturm-Liouville operator*, Differential Equations **23** (1987), no. 8, 979–987, [translation from Differentsial'nye Uravneniya 23 (1987), no. 8, 1422–1431, 1471, MR 89e:34043]. Zbl 668.34024.

[6] J. Mawhin, *Topological Degree Methods in Nonlinear Boundary Value Problems*, CBMS Regional Conference Series in Mathematics, vol. 40, American Mathematical Society, Providence, R.I., 1979, Expository lectures from the CBMS Regional Conference held at Harvey Mudd College, Claremont, Calif., June 9–15, 1977. MR 80c:47055. Zbl 414.34025.

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