

PERIODIC SOLUTIONS OF A CLASS OF NON-AUTONOMOUS SECOND-ORDER DIFFERENTIAL INCLUSIONS SYSTEMS

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Using an abstract framework due to Clarke (1999), we prove the existence of periodic solutions for second-order differential inclusions systems.

1. Introduction

Consider the second-order system

$$\begin{aligned} \ddot{u}(t) &= \nabla F(t, u(t)) \quad \text{a.e. } t \in [0, T], \\ u(0) - u(T) &= \dot{u}(0) - \dot{u}(T) = 0, \end{aligned} \tag{1.1}$$

where $T > 0$ and $F : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies the following assumption:

(A) $F(t, x)$ is measurable in t for each $x \in \mathbb{R}^n$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T; \mathbb{R}^+)$ such that

$$\begin{aligned} |F(t, x)| &\leq a(\|x\|)b(t), \\ \|\nabla F(t, x)\| &\leq a(\|x\|)b(t), \end{aligned} \tag{1.2}$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0, T]$.

Wu and Tang in [4] proved the existence of solutions for problem (1.1) when $F = F_1 + F_2$ and F_1, F_2 satisfy some assumptions. Now we will consider problem (1.1) in a more general sense. More precisely, our results represent the extensions to systems with discontinuity (we consider the generalized gradients unlike continuously gradient in classical results).

2. Main results

Consider the second-order differential inclusions systems

$$\begin{aligned} \ddot{u}(t) &\in \partial F(t, u(t)) \quad \text{a.e. } t \in [0, T], \\ u(0) - u(T) &= \dot{u}(0) - \dot{u}(T) = 0, \end{aligned} \tag{2.1}$$

where $T > 0$, $F : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ and ∂ denotes the Clarke subdifferential.

We suppose that $F = F_1 + F_2$ and F_1, F_2 satisfy the following assumption:

(A') F_1, F_2 are measurable in t for each $x \in \mathbb{R}^n$, at least F_1 or F_2 are strictly differentiable in x and there exist $k_1 \in L^2(0, T; \mathbb{R})$ and $k_2 \in L^2(0, T; \mathbb{R})$ such that

$$\begin{aligned} |F_1(t, x_1) - F_1(t, x_2)| &\leq k_1(t) \|x_1 - x_2\|, \\ |F_2(t, x_1) - F_2(t, x_2)| &\leq k_2(t) \|x_1 - x_2\|, \end{aligned} \tag{2.2}$$

for all $x_1, x_2 \in \mathbb{R}^n$ and all $t \in [0, T]$.

THEOREM 2.1. *Assume that $F = F_1 + F_2$, where F_1, F_2 satisfy assumption (A') and the following conditions:*

- (i) $F_1(t, \cdot)$ is (λ, μ) -subconvex with $\lambda > 1/2$ and $\mu < 2\lambda^2$ for a.e. $t \in [0, T]$;
- (ii) there exist $c_1, c_2 > 0$ and $\alpha \in [0, 1)$ such that

$$\zeta \in \partial F_2(t, x) \implies \|\zeta\| \leq c_1 \|x\|^\alpha + c_2, \tag{2.3}$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0, T]$;

(iii)

$$\frac{1}{\|x\|^{2\alpha}} \left[\frac{1}{\mu} \int_0^T F_1(t, \lambda x) dt + \int_0^T F_2(t, x) dt \right] \longrightarrow \infty, \quad \text{as } \|x\| \longrightarrow \infty. \tag{2.4}$$

Then problem (2.1) has at least one solution which minimizes φ on H_T^1 .

Remark 2.2. Theorem 2.1 generalizes [3, Theorem 1]. In fact, [3, Theorem 1] follows from Theorem 2.1 letting $F_1 = 0$.

THEOREM 2.3. *Assume that $F = F_1 + F_2$, where F_1, F_2 satisfy assumption (A') and the following conditions:*

- (iv) $F_1(t, \cdot)$ is (λ, μ) -subconvex for a.e. $t \in [0, T]$, and there exists $\gamma \in L^1(0, T; \mathbb{R})$, $h \in L^1(0, T; \mathbb{R}^n)$ with $\int_0^T h(t) dt = 0$ such that

$$F_1(t, x) \geq \langle h(t), x \rangle + \gamma(t), \tag{2.5}$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0, T]$;

- (v) there exist $c_1 > 0$, $c_0 \in \mathbb{R}$ such that

$$\zeta \in \partial F_2(t, x) \implies \|\zeta\| \leq c_1, \tag{2.6}$$

for all $x \in \mathbb{R}^n$ and all $t \in [0, T]$, and

$$\int_0^T F_2(t, x) dt \geq c_0, \tag{2.7}$$

for all $x \in \mathbb{R}^n$;

(vi)

$$\frac{1}{\mu} \int_0^T F_1(t, \lambda x) dt + \int_0^T F_2(t, x) dt \longrightarrow \infty, \quad \text{as } \|x\| \longrightarrow \infty. \tag{2.8}$$

Then problem (2.1) has at least one solution which minimizes φ on H_T^1 .

THEOREM 2.4. Assume that $F = F_1 + F_2$, where F_1, F_2 satisfy assumption (A') and the following conditions:

(vii) $F_1(t, \cdot)$ is (λ, μ) -subconvex for a.e. $t \in [0, T]$, and there exists $\gamma \in L^1(0, T; \mathbb{R})$, $h \in L^1(0, T; \mathbb{R}^n)$ with $\int_0^T h(t) dt = 0$ such that

$$F_1(t, x) \geq \langle h(t), x \rangle + \gamma(t), \tag{2.9}$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0, T]$;

(viii) there exist $c_1, c_2 > 0$ and $\alpha \in [0, 1)$ such that

$$\zeta \in \partial F_2(t, x) \implies \|\zeta\| \leq c_1 \|x\|^\alpha + c_2, \tag{2.10}$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0, T]$;

(ix)

$$\frac{1}{\|x\|^{2\alpha}} \int_0^T F_2(t, x) dt \longrightarrow \infty, \quad \text{as } \|x\| \longrightarrow \infty. \tag{2.11}$$

Then problem (2.1) has at least one solution which minimizes φ on H_T^1 .

3. Preliminary results

We introduce some functional spaces. Let $[0, T]$ be a fixed real interval ($0 < T < \infty$) and $1 < p < \infty$. We denote by $W_T^{1,p}$ the Sobolev space of functions $u \in L^p(0, T; \mathbb{R}^n)$ having a weak derivative $\dot{u} \in L^p(0, T; \mathbb{R}^n)$. The norm over $W_T^{1,p}$ is defined by

$$\|u\|_{W_T^{1,p}} = \left(\int_0^T \|u(t)\|^p dt + \int_0^T \|\dot{u}(t)\|^p dt \right)^{1/p}. \tag{3.1}$$

We denote by H_T^1 the Hilbert space $W_T^{1,2}$. We recall that

$$\|u\|_{L^p} = \left(\int_0^T \|u(t)\|^p dt \right)^{1/p}, \quad \|u\|_\infty = \max_{t \in [0, T]} \|u(t)\|. \tag{3.2}$$

For our aims, it is necessary to recall some very well-known results (for proof and details see [2]):

PROPOSITION 3.1. *If $u \in W_T^{1,p}$ then*

$$\|u\|_\infty \leq c \|u\|_{W_T^{1,p}}. \quad (3.3)$$

If $u \in W_T^{1,p}$ and $\int_0^T u(t)dt = 0$ then

$$\|u\|_\infty \leq c \|\dot{u}\|_{L^p}. \quad (3.4)$$

If $u \in H_T^1$ and $\int_0^T u(t)dt = 0$ then

$$\|u\|_{L^2} \leq \frac{T}{2\pi} \|\dot{u}\|_{L^2} \quad (\text{Wirtinger's inequality}), \quad (3.5)$$

$$\|u\|_\infty^2 \leq \frac{T}{12} \|\dot{u}\|_{L^2}^2 \quad (\text{Sobolev inequality}).$$

PROPOSITION 3.2. *If the sequence $(u_k)_k$ converges weakly to u in $W_T^{1,p}$, then $(u_k)_k$ converges uniformly to u on $[0, T]$.*

Let X be a Banach space. Now, following [1], for each $x, v \in X$, we define the *generalized directional derivative* at x in the direction v of a given $f \in \text{Lip}_{\text{loc}}(X, \mathbb{R})$ as

$$f^0(x; v) = \limsup_{y \rightarrow x, \lambda \searrow 0} \frac{f(y + \lambda v) - f(y)}{\lambda} \quad (3.6)$$

and denote x by

$$\partial f(x) = \{x^* \in X^* : f^0(x; v) \geq \langle x^*, v \rangle, \forall v \in X\} \quad (3.7)$$

the *generalized gradient* of f at x (the Clarke subdifferential).

We recall the *Lebourg's mean value theorem* (see [1, Theorem 2.3.7]). *Let x and y be points in X , and suppose that f is Lipschitz on an open set containing the line segment $[x, y]$. Then there exists a point u in (x, y) such that*

$$f(y) - f(x) \in \langle \partial f(u), y - x \rangle. \quad (3.8)$$

Clarke considered in [1] the following abstract framework:

- let (T, \mathcal{T}, μ) be a positive complete measure space with $\mu(T) < \infty$, and let Y be a separable Banach space;
- let Z be a closed subspace of $L^p(T; Y)$ (for some p in $[1, \infty)$), where $L^p(T; Y)$ is the space of p -integrable functions from T to Y ;
- we define a functional f on Z via

$$f(x) = \int_T f_t(x(t)) \mu(dt), \quad (3.9)$$

where $f_t : Y \rightarrow \mathbb{R}$, $(t \in T)$ is a given family of functions;

- we suppose that for each y in Y the function $t \rightarrow f_t(y)$ is measurable, and that x is a point at which $f(x)$ is defined (finitely).

Hypothesis 3.3. There is a function k in $L^q(T, \mathbb{R})$, $(1/p + 1/q = 1)$ such that, for all $t \in T$,

$$|f_t(y_1) - f_t(y_2)| \leq k(t) \|y_1 - y_2\|_Y \quad \forall y_1, y_2 \in Y. \quad (3.10)$$

Hypothesis 3.4. Each function f_t is Lipschitz (of some rank) near each point of Y , and for some constant c , for all $t \in T$, $y \in Y$, one has

$$\zeta \in \partial f_t(y) \implies \|\zeta\|_{Y^*} \leq c \{1 + \|y\|_Y^{p-1}\}. \quad (3.11)$$

Under the conditions described above Clarke proved (see [1, Theorem 2.7.5]):

THEOREM 3.5. *Under either of Hypotheses 3.3 or 3.4, f is uniformly Lipschitz on bounded subsets of Z , and there is*

$$\partial f(x) \subset \int_T \partial f_t(x(t)) \mu(dt). \quad (3.12)$$

Further, if each f_t is regular at $x(t)$ then f is regular at x and equality holds.

Remark 3.6. The function f is globally Lipschitz on Z when [Hypothesis 3.3](#) holds.

Now we can prove the following result.

THEOREM 3.7. *Let $F : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that $F = F_1 + F_2$ where F_1, F_2 are measurable in t for each $x \in \mathbb{R}^n$, and there exist $k_1 \in L^2(0, T; \mathbb{R})$, $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T; \mathbb{R}^+)$, $c_1, c_2 > 0$, and $\alpha \in [0, 1)$ such that*

$$|F_1(t, x_1) - F_1(t, x_2)| \leq k_1(t) \|x_1 - x_2\|, \quad (3.13)$$

$$|F_2(t, x)| \leq a(\|x\|)b(t), \quad (3.14)$$

$$\zeta \in \partial F_2(t, x) \implies \|\zeta\| \leq c_1 \|x\|^\alpha + c_2, \quad (3.15)$$

for all $t \in [0, T]$ and all $x, x_1, x_2 \in \mathbb{R}^n$. We suppose that $L : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is given by $L(t, x, y) = (1/2)\|y\|^2 + F(t, x)$.

Then, the functional $f : Z \rightarrow \mathbb{R}$, where

$$Z = \left\{ (u, v) \in L^2(0, T; Y) : u(t) = \int_0^t v(s) ds + c, c \in \mathbb{R}^n \right\} \quad (3.16)$$

given by $f(u, v) = \int_0^T L(t, u(t), v(t)) dt$, is uniformly Lipschitz on bounded subsets of Z and

$$\partial f(u, v) \subset \int_0^T \{ \partial F_1(t, u(t)) + \partial F_2(t, u(t)) \} \times \{ v(t) \} dt. \quad (3.17)$$

Proof. Let $L_1(t, x, y) = F_1(t, x)$, $L_2(t, x, y) = (1/2)\|y\|^2 + F_2(t, x)$, and $f_1, f_2 : Z \rightarrow \mathbb{R}$ given by $f_1(u, v) = \int_0^T L_1(t, u(t), v(t))dt$, $f_2(u, v) = \int_0^T L_2(t, u(t), v(t))dt$. For f_1 we can apply [Theorem 3.5](#) under [Hypothesis 3.3](#), with the following cast of characters:

- $(T, \mathcal{F}, \mu) = [0, T]$ with Lebesgue measure, $Y = \mathbb{R}^n \times \mathbb{R}^n$ is the Hilbert product space (hence is separable);
- $p = 2$ and

$$Z = \left\{ (u, v) \in L^2(0, T; Y) : u(t) = \int_0^t v(s)ds + c, c \in \mathbb{R}^n \right\} \quad (3.18)$$

is a closed subspace of $L^2(0, T; Y)$;

- $f_t(x, y) = L_1(t, x, y) = F_1(t, x)$; in our assumptions it results that the integrand $L_1(t, x, y)$ is measurable in t for a given element (x, y) of Y and there exists $k \in L^2(0, T; \mathbb{R})$ such that

$$\begin{aligned} |L_1(t, x_1, y_1) - L_1(t, x_2, y_2)| &= |F_1(t, x_1) - F_1(t, x_2)| \\ &\leq k_1(t) \|x_1 - x_2\| \\ &\leq k_1(t) (\|x_1 - x_2\| + \|y_1 - y_2\|) \\ &= k_1(t) \|(x_1, y_1) - (x_2, y_2)\|_Y, \end{aligned} \quad (3.19)$$

for all $t \in [0, T]$ and all $(x_1, y_1), (x_2, y_2) \in Y$. Hence f_1 is uniformly Lipschitz on bounded subsets of Z and one has

$$\partial f_1(u, v) \subset \int_0^T \partial L_1(t, u(t), v(t))dt. \quad (3.20)$$

For f_2 we can apply [Theorem 3.5](#) under [Hypothesis 3.4](#) with the same cast of characters, but now $f_t(x, y) = L_2(t, x, y) = (1/2)\|y\|^2 + F_2(t, x)$. In our assumptions, it results that the integrand $L_2(t, x, y)$ is measurable in t for a given element (x, y) of Y and locally Lipschitz in (x, y) for each $t \in [0, T]$.

Proposition 2.3.15 in [1] implies

$$\partial L_2(t, x, y) \subset \partial_x L_2(t, x, y) \times \partial_y L_2(t, x, y) = \partial F_2(t, x) \times y. \quad (3.21)$$

Using (3.15) and (3.21), if $\zeta = (\zeta_1, \zeta_2) \in \partial L_2(t, x, y)$ then $\zeta_1 \in \partial F_2(t, x)$ and $\zeta_2 = y$, and hence

$$\|\zeta\| = \|\zeta_1\| + \|\zeta_2\| \leq c_1 \|x\|^\alpha + c_2 + \|y\| \leq \tilde{c} \{1 + \|(x, y)\|\}, \quad (3.22)$$

for each $t \in [0, T]$. Hence f_2 is uniformly Lipschitz on bounded subsets of Z and one has

$$\partial f_2(u, v) \subset \int_0^T \partial L_2(t, u(t), v(t))dt. \quad (3.23)$$

It follows that $f = f_1 + f_2$ is uniformly Lipschitz on the bounded subsets of Z .

Propositions 2.3.3 and 2.3.15 in [1] imply that

$$\begin{aligned}
 \partial f(u, v) &\subset \partial f_1(u, v) + \partial f_2(u, v) \\
 &\subset \int_0^T [\partial L_1(t, u(t), v(t)) + \partial L_2(t, u(t), v(t))] dt \\
 &\subset \int_0^T [(\partial_x L_1(t, u(t), v(t)) \times \partial_y L_1(t, u(t), v(t))) \\
 &\quad + (\partial_x L_2(t, u(t), v(t)) \times \partial_y L_2(t, u(t), v(t)))] dt \quad (3.24) \\
 &\subset \int_0^T [(\partial_x L_1(t, u(t), v(t)) + \partial_x L_2(t, u(t), v(t))) \\
 &\quad \times (\partial_y L_1(t, u(t), v(t)) + \partial_y L_2(t, u(t), v(t)))] dt \\
 &= \int_0^T (\partial F_1(t, u(t)) + \partial F_2(t, u(t))) \times \{v(t)\} dt.
 \end{aligned}$$

Moreover, Corollary 1 of Proposition 2.3.3 in [1] implies that, if at least one of the functions F_1, F_2 is strictly differentiable in x for all $t \in [0, T]$ then

$$\partial f(u, v) \subset \int_0^T \partial F(t, u(t)) \times \{v(t)\} dt. \quad (3.25)$$

□

Remark 3.8. The interpretation of expression (3.25) is that if (u_0, v_0) is an element of Z (so that $v_0 = \dot{u}_0$) and if $\zeta \in \partial f(u_0, v_0)$, we deduce the existence of a measurable function $(q(t), p(t))$ such that

$$q(t) \in \partial F(t, u_0(t)), \quad p(t) = v_0(t) \quad \text{a.e. on } [0, T] \quad (3.26)$$

and for any (u, v) in Z , one has

$$\langle \zeta, (u, v) \rangle = \int_0^T \{\langle q(t), u(t) \rangle + \langle p(t), v(t) \rangle\} dt. \quad (3.27)$$

In particular, if $\zeta = 0$ (so that u_0 is a critical point for $\varphi(u) = \int_0^T [(1/2)\|\dot{u}(t)\|^2 + F(t, u(t))] dt$), it then follows easily that $q(t) = \dot{p}(t)$ a.e., or taking into account (3.26)

$$\ddot{u}_0(t) \in \partial F(t, u_0(t)) \quad \text{a.e. on } [0, T], \quad (3.28)$$

so that u_0 satisfies the inclusions system (2.1).

Remark 3.9. Of course, if F is continuously differentiable in x , then system (2.1) becomes system (1.1).

4. Proofs of the theorems

Proof of Theorem 2.1. From assumption (A') it follows immediately that there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T; \mathbb{R}^+)$ such that

$$|F_1(t, x)| \leq a(\|x\|)b(t), \quad (4.1)$$

for all $x \in \mathbb{R}^n$ and all $t \in [0, T]$. Like, in [4], we obtain

$$F_1(t, x) \leq (2\mu\|x\|^\beta + 1)a_0b(t), \quad (4.2)$$

for all $x \in \mathbb{R}^n$ and all $t \in [0, T]$, where $\beta < 2$ and $a_0 = \max_{0 \leq s \leq 1} a(s)$.

For $u \in H_T^1$, let $\bar{u} = (1/T) \int_0^T u(t) dt$ and $\tilde{u} = u - \bar{u}$. From Lebourg's mean value theorem it follows that for each $t \in [0, T]$ there exist $z(t)$ in $(\bar{u}, u(t))$ and $\zeta \in \partial F_2(t, z(t))$ such that $F_2(t, u(t)) - F_2(t, \bar{u}) = \langle \zeta, \tilde{u}(t) \rangle$. It follows from (2.3) and Sobolev's inequality that

$$\begin{aligned} & \left| \int_0^T [F_2(t, u(t)) - F_2(t, \bar{u})] dt \right| \\ & \leq \int_0^T |F_2(t, u(t)) - F_2(t, \bar{u})| dt \leq \int_0^T \|\zeta\| \|\tilde{u}(t)\| dt \\ & \leq \int_0^T [2c_1(\|\bar{u}\|^\alpha + \|\tilde{u}(t)\|^\alpha) + c_2] \|\tilde{u}(t)\| dt \\ & \leq 2c_1 T \|\tilde{u}\|_\infty \|\bar{u}\|^\alpha + 2c_1 T \|\tilde{u}\|_\infty^{\alpha+1} + c_2 T \|\tilde{u}\|_\infty \\ & \leq \frac{3}{T} \|\tilde{u}\|_\infty^2 + \frac{T^3}{3} c_1^2 \|\bar{u}\|^{2\alpha} + 2c_1 T \|\tilde{u}\|_\infty^{\alpha+1} + c_2 T \|\tilde{u}\|_\infty \\ & \leq \frac{1}{4} \|\dot{u}\|_{L^2}^2 + C_1 \|\dot{u}\|_{L^2}^{\alpha+1} + C_2 \|\dot{u}\|_{L^2} + C_3 \|\bar{u}\|^{2\alpha}, \end{aligned} \quad (4.3)$$

for all $u \in H_T^1$ and some positive constants C_1 , C_2 , and C_3 . Hence we have

$$\begin{aligned} \varphi(u) & \geq \frac{1}{2} \int_0^T \|\dot{u}(t)\|^2 dt + \frac{1}{\mu} \int_0^T F_1(t, \lambda \bar{u}) dt - \int_0^T F_1(t, -\tilde{u}(t)) dt \\ & \quad + \int_0^T F_2(t, \bar{u}) dt + \int_0^T [F_2(t, u(t)) - F_2(t, \bar{u})] dt \\ & \geq \frac{1}{4} \|\dot{u}\|_{L^2}^2 - C_1 \|\dot{u}\|_{L^2}^{\alpha+1} - C_2 \|\dot{u}\|_{L^2} - C_3 \|\bar{u}\|^{2\alpha} - (2\mu\|\tilde{u}\|_\infty^\beta + 1) \int_0^T a_0 b(t) dt \\ & \quad + \frac{1}{\mu} \int_0^T F_1(t, \lambda \bar{u}) dt + \int_0^T F_2(t, \bar{u}) dt \end{aligned}$$

$$\begin{aligned} &\geq \frac{1}{4} \|\dot{u}\|_{L^2}^2 - C_1 \|\dot{u}\|_{L^2}^{\alpha+1} - C_2 \|\dot{u}\|_{L^2} - C_4 \|\dot{u}\|_{L^2}^\beta - C_5 \\ &\quad + \|\bar{u}\|^{2\alpha} \left\{ \frac{1}{\|\bar{u}\|^{2\alpha}} \left[\frac{1}{\mu} \int_0^T F_1(t, \lambda \bar{u}) dt + \int_0^T F_2(t, \bar{u}) dt \right] - C_3 \right\} \end{aligned} \quad (4.4)$$

for all $u \in H_T^1$, which implies that $\varphi(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$ by (2.4) because $\alpha < 1$, $\beta < 2$, and the norm $\|u\| = (\|\bar{u}\|^2 + \|\dot{u}\|_{L^2}^2)^{1/2}$ is an equivalent norm on H_T^1 . Now we write $\varphi(u) = \varphi_1(u) + \varphi_2(u)$ where

$$\varphi_1(u) = \frac{1}{2} \int_0^T \|\dot{u}(t)\|^2 dt, \quad \varphi_2(u) = \int_0^T F(t, u(t)) dt. \quad (4.5)$$

The function φ_1 is weakly lower semi-continuous (w.l.s.c.) on H_T^1 . From (i), (ii), and Theorem 3.5, taking into account Remark 3.6 and Proposition 3.2, it follows that φ_2 is w.l.s.c. on H_T^1 . By [2, Theorem 1.1], it follows that φ has a minimum u_0 on H_T^1 . Evidently, $Z \simeq H_T^1$ and $\varphi(u) = f(u, v)$ for all $(u, v) \in Z$. From Theorem 3.7, it results that f is uniformly Lipschitz on bounded subsets of Z , and therefore φ possesses the same properties relative to H_T^1 . Proposition 2.3.2 in [1] implies that $0 \in \partial\varphi(u_0)$ (so that u_0 is a critical point for φ). Now from Theorem 3.7 and Remark 3.8 it follows that problem (2.1) has at least one solution $u \in H_T^1$. \square

Proof of Theorem 2.3. Let (u_k) be a minimizing sequence of φ . It follows from (iv), (v), Lebourg’s mean value theorem, and Sobolev inequality, that

$$\begin{aligned} \varphi(u_k) &\geq \frac{1}{2} \|\dot{u}_k\|_{L^2}^2 + \int_0^T \langle h(t), u_k(t) \rangle dt + \int_0^T \gamma(t) dt \\ &\quad + \int_0^T F_2(t, \bar{u}_k) dt - \int_0^T \|\zeta\| \|\bar{u}_k(t)\| dt \\ &\geq \frac{1}{2} \|\dot{u}_k\|_{L^2}^2 - \|\bar{u}_k\|_\infty \int_0^T \|h(t)\| dt \\ &\quad + \int_0^T \gamma(t) dt - c_1 \|\bar{u}_k\|_\infty + c_0 \\ &\geq \frac{1}{2} \|\dot{u}_k\|_{L^2}^2 - c_2 \|\dot{u}_k\|_{L^2} - c_3, \end{aligned} \quad (4.6)$$

for all k and some constants c_2, c_3 , which implies that (\bar{u}_k) is bounded. On the other hand, in a way similar to the proof of Theorem 2.1, one has

$$\left| \int_0^T [F_2(t, u(t)) - F_2(t, \bar{u})] dt \right| \leq \frac{1}{4} \|\dot{u}\|_{L^2}^2 + C_1 \|\dot{u}\|_{L^2}, \quad (4.7)$$

for all k and some positive constant C_1 , which implies that

$$\begin{aligned}
 \varphi(u_k) &\geq \frac{1}{2} \|\dot{u}_k\|_{L^2}^2 + \frac{1}{\mu} \int_0^T F_1(t, \lambda \bar{u}_k) dt - \int_0^T F_1(t, -\bar{u}_k(t)) dt \\
 &\quad + \int_0^T F_2(t, \bar{u}_k) dt + \int_0^T [F_2(t, u(t)) - F_2(t, \bar{u}_k)] dt \\
 &\geq \frac{1}{4} \|\dot{u}_k\|_{L^2}^2 - a(\|\bar{u}_k\|_\infty) \int_0^T b(t) dt - C_1 \|\dot{u}_k\|_{L^2} \\
 &\quad + \frac{1}{\mu} \int_0^T F_1(t, \lambda \bar{u}_k) dt + \int_0^T F_2(t, \bar{u}_k) dt,
 \end{aligned} \tag{4.8}$$

for all k and some positive constant C_1 . It follows from (vi) and the boundedness of (\bar{u}_k) that (\bar{u}_k) is bounded. Hence φ has a bounded minimizing sequence (u_k) . This completes the proof. \square

Proof of Theorem 2.4. From (vii), (3.26), and Sobolev's inequality it follows that

$$\begin{aligned}
 \varphi(u) &\geq \frac{1}{2} \|\dot{u}\|_{L^2}^2 + \int_0^T \langle h(t), u(t) \rangle dt + \int_0^T \gamma(t) dt \\
 &\quad + \int_0^T F_2(t, \bar{u}) dt + \int_0^T [F_2(t, u(t)) - F_2(t, \bar{u})] dt \\
 &\geq \frac{1}{4} \|\dot{u}\|_{L^2}^2 - \|\bar{u}\|_\infty \int_0^T \|h(t)\| dt + \int_0^T \gamma(t) dt \\
 &\quad - C_1 \|\dot{u}\|_{L^2}^{\alpha+1} - C_2 \|\dot{u}\|_{L^2} + \int_0^T F_2(t, \bar{u}) dt - C_3 \|\bar{u}\|^{2\alpha} \\
 &\geq \frac{1}{4} \|\dot{u}\|_{L^2}^2 - C_1 \|\dot{u}\|_{L^2}^{\alpha+1} - C_4 (\|\dot{u}\|_{L^2} + 1) \\
 &\quad + \|\bar{u}\|^{2\alpha} \left[\frac{1}{\|\bar{u}\|^{2\alpha}} \int_0^T F_2(t, \bar{u}) dt - C_3 \right],
 \end{aligned} \tag{4.9}$$

for all $u \in H_T^1$ and some positive constants C_1 , C_3 , and C_4 . Now it follows like in the proof of Theorem 2.1 that φ is coercive by (ix), which completes the proof. \square

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