A NOTE ON THE DIFFERENCE SCHEMES FOR HYPERBOLIC EQUATIONS

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The initial value problem for hyperbolic equations $d^2u(t)/dt^2 + Au(t) = f(t)$ $(0 \le t \le 1)$, $u(0) = \varphi$, $u'(0) = \psi$, in a Hilbert space H is considered. The first and second order accuracy difference schemes generated by the integer power of A approximately solving this initial value problem are presented. The stability estimates for the solution of these difference schemes are obtained.

1. Introduction

We consider the initial value problem

$$\begin{gathered} \frac{d^2u(t)}{dt^2} + Au(t) = f(t) \quad (0 \le t \le 1), \\ u(0) = \varphi, \qquad u'(0) = \psi, \end{gathered} \tag{1.1}$$

for a differential equation in a Hilbert space H with unbounded linear selfadjoint and positive definite operator $A = A^* \ge \delta I$ $(\delta > 0)$ with dense domain $\bar{D}(A) = H$. It is known (cf. [3]) that various initial boundary value problems for the hyperbolic equations can be reduced to problem (1.1). A study of discretization, over time only, of the initial value problem also permits one to include general difference schemes in applications, if the differential operator in space variables, A, is replaced by the difference operators A_h that act in the Hilbert spaces and are uniformly positive definite and selfadjoint in h for $0 < h \le h_0$. In the paper [4], the following first order accuracy difference scheme for approximately solving problem (1.1)

$$\begin{gathered} \tau^{-2}\left(u_{k+1} - 2u_k + u_{k-1}\right) + Au_{k+1} = f_k, \quad f_k = f\left(t_k\right),\ t_k = k\tau, \\ 1 \le k \le N-1, \qquad N\tau = 1, \\ \tau^{-1}\left(u_1 - u_0\right) + iA^{1/2}u_1 = iA^{1/2}u_0 + \psi, \quad u_0 = \varphi, \end{gathered} \tag{1.2}$$

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was considered. The stability estimates for the solution of the difference scheme (1.2) were obtained. The proof of these statements is based on the transform of second order difference equations to equivalent system of first order difference equations. Application of this approach in [1, 2] with similar results for the solutions of the second order accuracy of the following difference schemes

$$
\begin{gathered}
\tau^{-2}\left(u_{k+1}-2u_k+u_{k-1}\right)+Au_k+\frac{\tau^2}{4}A^2u_{k+1}=f_k,\\
f_k=f\left(t_k\right),\quad 1\le k\le N-1,\qquad N\tau=1,\\
\tau^{-1}\left(u_1-u_0\right)+iA^{1/2}\left(I+\frac{i\tau}{2}A^{1/2}\right)u_1=z_1,\\
z_1=\left(I+i\tau A^{1/2}\right)\psi+\frac{\tau}{2}f_0+\left(iA^{1/2}-\tau A\right)u_0,\quad f_0=f(0),\ u_0=\varphi,\\
\tau^{-2}\left(u_{k+1}-2u_k+u_{k-1}\right)+\frac{1}{4}A\left(u_{k+1}+2u_k+u_{k-1}\right)=f_k,\\
f_k=f\left(t_k\right),\quad 1\le k\le N-1,\qquad N\tau=1,\\
\tau^{-1}\left(u_1-u_0\right)+\frac{i}{2}A^{1/2}\left(u_1+u_0\right)=z_1,\\
z_1=\left(I+\frac{i\tau}{2}A^{1/2}\right)\psi+\frac{\tau}{2}f_0+\frac{1}{2}\left(iA^{1/2}-\tau A\right)u_0,\quad f_0=f(0),\ u_0=\varphi
\end{gathered}
\tag{1.3}
$$

for approximately solving the initial value problem (1.1) were obtained. However, for practical realization of these difference schemes it is necessary to first construct an operator $A^{1/2}$. This action is very difficult for a realization. Therefore, in spite of theoretical results the role of their application to a numerical solution for an initial value problem is not great.

In the present paper, first and second order accuracy difference schemes for approximate solutions of problem (1.1) are constructed using the integer powers of the operator A, and the stability estimates for the solution of these difference schemes are obtained.

2. First order difference schemes

We consider the first order accuracy difference scheme for approximately solving the initial value problem (1.1)

$$
\begin{gathered}
\tau^{-2}\left(u_{k+1}-2u_k+u_{k-1}\right)+Au_{k+1}=f_k,\quad f_k=f\left(t_k\right),\ t_k=k\tau,\\
1\le k\le N-1,\qquad N\tau=1,\\
\tau^{-1}\left(u_1-u_0\right)=\psi,\quad u_0=\varphi.
\end{gathered}
\tag{2.1}
$$

THEOREM 2.1. *Let $\varphi \in D(A)$, $\psi \in D(A)$. Then for the solution of the difference scheme (2.1) the following stability inequalities, for $2 \le k \le N$, hold*

$$
\begin{aligned}
\|u_k\|_H &\le \tau \sum_{s=1}^{k-1} \|A^{-1/2} f_s\|_H + \|A^{-1/2}\psi\|_H + \|\varphi\|_H, \\
\|A^{1/2}u_k\|_H &\le \tau \sum_{s=1}^{k-1} \|f_s\|_H + \|A^{1/2}\varphi\|_H + \|\psi\|_H, \\
\|Au_k\|_H &\le 2\sum_{s=2}^{k-1} \|f_s - f_{s-1}\|_H + \|f_1\|_H + \|A^{1/2}\psi\|_H + \|A\varphi\|_H, \\
\|u_1\|_H &\le \|\varphi\|_H + \|(I + i\tau A^{1/2})A^{-1/2}\psi\|_H, \\
\|A^{1/2}u_1\|_H &\le \|A^{1/2}\varphi\|_H + \|(I + i\tau A^{1/2})\psi\|_H, \\
\|Au_1\|_H &\le \|A\varphi\|_H + \|(I + i\tau A^{1/2})A^{1/2}\psi\|_H.
\end{aligned}
\tag{2.2}
$$

The proof of this theorem uses the method of [4] and is based on the following formulas:

$$
\begin{aligned}
u_1 &= \varphi + \tau\psi, \\
u_k &= \frac{1}{2}[R^{k-1} + \tilde{R}^{k-1}]\varphi + (R - \tilde{R})^{-1}\tau R(R^k - \tilde{R}^k)\psi - \sum_{s=1}^{k-1} \frac{\tau}{2i} A^{-1/2}[R^{k-s} - \tilde{R}^{k-s}] f_s \\
&= \frac{1}{2}[R^{k-1} + \tilde{R}^{k-1}]\varphi + (R - \tilde{R})^{-1}\tau R(R^k - \tilde{R}^k)\psi \\
&\quad + A^{-1}\frac{1}{2}\sum_{s=2}^{k-1}[R^{k-s} + \tilde{R}^{k-s}](f_{s-1} - f_s) + 2f_{k-1} - [R^{k-1} + \tilde{R}^{k-1}]f_1,
\end{aligned}
\tag{2.3}
$$

for $2 \le k \le N$, where $R = (I + i\tau A^{1/2})^{-1}$, $\tilde{R} = (I - i\tau A^{1/2})^{-1}$ and on the estimates

$$
\begin{aligned}
\|R\|_{H\mapsto H} &\le 1, & \|\tilde{R}\|_{H\mapsto H} &\le 1, \\
\|R\tilde{R}^{-1}\|_{H\mapsto H} &\le 1, & \|\tilde{R}R^{-1}\|_{H\mapsto H} &\le 1, \\
\|\tau A^{1/2}R\|_{H\mapsto H} &\le 1, & \|\tau A^{1/2}\tilde{R}\|_{H\mapsto H} &\le 1.
\end{aligned}
\tag{2.4}
$$

Note that formulas (2.3) are generated by the operator $A^{1/2}$ and are used to prove stability estimates for the solutions of the difference scheme (2.1).

However, for the practical realization of this difference scheme (2.1) the operator $A^{1/2}$ as in [1, 2, 4] is not used. Note also that these stability inequalities in the case $k = 1$ are weaker than the respective inequalities in the cases $k = 2, \dots, N$. However, obtaining this type of inequalities is important for applications. We denote by $a^\tau = (a_k)$ the mesh function of approximation. Then $\|(I + i\tau A^{1/2})a_1\|_H \sim \|a_1\|_H = o(\tau)$ if we assume that $\tau\|Aa_1\|_H$ tends to 0 as τ tends to 0 not slower than $\|a_1\|_H$. It takes place in applications by supplementary restriction on the smoothness property of the data in the space variables.

It is clear that the estimate

$$\|u_1\|_H \le \|\varphi\|_H + \|A^{-1/2}\psi\|_H \tag{2.5}$$

is absent. However, estimates for the solution of first order accuracy modification difference scheme for approximately solving the initial value problem (1.1)

$$\begin{gathered}\tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_{k+1} = f_k, \quad f_k = f(t_k),\ t_k = k\tau,\\ 1 \le k \le N-1, \qquad N\tau = 1,\\ (I + \tau^2 A)\tau^{-1}(u_1 - u_0) = \psi, \quad u_0 = \varphi,\end{gathered} \tag{2.6}$$

are better than the estimates for the solution of the difference scheme (2.1).

THEOREM 2.2. *Let $\varphi \in D(A)$, $\psi \in D(A^{1/2})$. Then for the solution of the difference scheme (2.6), the following stability inequalities, for $1 \le k \le N$, hold*

$$\begin{aligned}\|u_k\|_H &\le \tau\sum_{s=1}^{k-1}\|A^{-1/2}f_s\|_H + \|A^{-1/2}\psi\|_H + \|\varphi\|_H,\\ \|A^{1/2}u_k\|_H &\le \tau\sum_{s=1}^{k-1}\|f_s\|_H + \|A^{1/2}\varphi\|_H + \|\psi\|_H,\\ \|Au_k\|_H &\le 2\sum_{s=2}^{k-1}\|f_s - f_{s-1}\|_H + \|f_1\|_H + \|A^{1/2}\psi\|_H + \|A\varphi\|_H.\end{aligned} \tag{2.7}$$

The proof of this theorem is based on the following formulas:

$$\begin{aligned}u_1 &= \varphi + \tau R\tilde{R}\psi,\\ u_k &= \frac{1}{2}\left[R^{k-1} + \tilde{R}^{k-1}\right]\varphi + \left(R - \tilde{R}\right)^{-1}\tau R\left(R^k - \tilde{R}^k\right)R\tilde{R}\psi\\ &\quad - \sum_{s=1}^{k-1}\frac{\tau}{2i}A^{-1/2}\left[R^{k-s} - \tilde{R}^{k-s}\right]f_s\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left[R^{k-1}+\tilde{R}^{k-1}\right]\varphi+\left(R-\tilde{R}\right)^{-1}\tau R\left(R^{k}-\tilde{R}^{k}\right)R\tilde{R}\psi \\
&\quad + A^{-1}\frac{1}{2}\sum_{s=2}^{k-1}\left[R^{k-s}+\tilde{R}^{k-s}\right]\left(f_{s-1}-f_{s}\right) \\
&\quad + 2f_{k-1}-\left[R^{k-1}+\tilde{R}^{k-1}\right]f_{1}, \quad 2\le k\le N,
\end{aligned}
\tag{2.8}
$$

and on the estimates (2.4).

3. Second order difference schemes

We consider the second order accuracy difference schemes for approximate solutions of the initial value problem (1.1)

$$
\begin{gathered}
\tau^{-2}\left(u_{k+1}-2u_{k}+u_{k-1}\right)+Au_{k}+\frac{\tau^{2}}{4}A^{2}u_{k+1}=f_{k}, \\
f_{k}=f\left(t_{k}\right), \quad t_{k}=k\tau,\ 1\le k\le N-1, \qquad N\tau=1, \\
\left(I+\tau^{2}A\right)\tau^{-1}\left(u_{1}-u_{0}\right)=\frac{\tau}{2}\left(f_{0}-Au_{0}\right)+\psi, \quad f_{0}=f(0),\ u_{0}=\varphi,
\end{gathered}
\tag{3.1}
$$

$$
\begin{gathered}
\tau^{-2}\left(u_{k+1}-2u_{k}+u_{k-1}\right)+\frac{1}{2}Au_{k}+\frac{1}{4}A\left(u_{k+1}+u_{k-1}\right)=f_{k}, \\
f_{k}=f\left(t_{k}\right), \quad t_{k}=k\tau,\ 1\le k\le N-1, \qquad N\tau=1, \\
\left(I+\tau^{2}A\right)\tau^{-1}\left(u_{1}-u_{0}\right)=\frac{\tau}{2}\left(f_{0}-Au_{0}\right)+\psi, \quad f_{0}=f(0),\ u_{0}=\varphi.
\end{gathered}
\tag{3.2}
$$

The stability estimates for the solution of these difference schemes are obtained.

THEOREM 3.1. *Let $\varphi\in D(A)$, $\psi\in D(A^{1/2})$. Then for the solution of the difference scheme (3.1), the following stability inequalities, for $1\le k\le N$, hold*

$$
\begin{gathered}
\left\|u_{k}\right\|_{H}\le\tau\sum_{s=0}^{k-1}\left\|A^{-1/2}f_{s}\right\|_{H}+\left\|A^{-1/2}\psi\right\|_{H}+\|\varphi\|_{H}, \\
\left\|A^{1/2}u_{k}\right\|_{H}\le\tau\sum_{s=0}^{k-1}\left\|f_{s}\right\|_{H}+\left\|A^{1/2}\varphi\right\|_{H}+\|\psi\|_{H}, \\
\left\|Au_{k}\right\|_{H}\le 2\sum_{s=1}^{k-1}\left\|f_{s}-f_{s-1}\right\|_{H}+\left\|f_{0}\right\|_{H}+\left\|A^{1/2}\psi\right\|_{H}+\|A\varphi\|_{H}.
\end{gathered}
\tag{3.3}
$$

The proof of this theorem is based on the following formulas:

$$
\begin{aligned}
u_1 &= \left(I+\tau^2 A^{-1}\right)\left[\left(I+\frac{\tau^2}{2}A\right)\varphi+\tau\psi+\frac{\tau^2}{2}f_0\right],\\
u_k &= \left[R^k+\tau R\left(R-\tilde{R}\right)^{-1}\left[R^k-\tilde{R}^k\right]\left(I+\tau^2 A\right)^{-1}\left(-\tau A+iA^{1/2}\right)\right]\varphi\\
&\quad+\tau R\left(R-\tilde{R}\right)^{-1}\left[R^k-\tilde{R}^k\right]\left(I+\tau^2 A\right)^{-1}\psi\\
&\quad+\frac{\tau^2}{2}R\left(R-\tilde{R}\right)^{-1}\left[R^k-\tilde{R}^k\right]\left(I+\tau^2 A\right)^{-1}f_0-\sum_{s=1}^{k-1}\frac{\tau}{2i}A^{-1/2}\left[R^{k-s}-\tilde{R}^{k-s}\right]f_s\\
&= \left[R^k+\tau R\left(R-\tilde{R}\right)^{-1}\left[R^k-\tilde{R}^k\right]\left(I+\tau^2 A\right)^{-1}\left(-\tau A+iA^{1/2}\right)\right]\varphi\\
&\quad+\tau R\left(R-\tilde{R}\right)^{-1}\left[R^k-\tilde{R}^k\right]\left(I+\tau^2 A\right)^{-1}\psi\\
&\quad+\frac{\tau^2}{2}R\left(R-\tilde{R}\right)^{-1}\left[R^k-\tilde{R}^k\right]\left(I+\tau^2 A\right)^{-1}f_0\\
&\quad+A^{-1}\frac{1}{2}\sum_{s=1}^{k-1}\left[\left(I+\frac{i\tau}{2}A^{1/2}\right)^{-1}R^{k-s}+\left(I-\frac{i\tau}{2}A^{1/2}\right)^{-1}\tilde{R}^{k-s}\right]\left(f_{s-1}-f_s\right)\\
&\quad+2\left(I+\frac{i\tau}{2}A^{1/2}\right)^{-1}\left(I-\frac{i\tau}{2}A^{1/2}\right)^{-1}f_{k-1}\\
&\quad-\left[\left(I+\frac{i\tau}{2}A^{1/2}\right)^{-1}R^{k-1}+\left(I-\frac{i\tau}{2}A^{1/2}\right)^{-1}\tilde{R}^{k-1}\right]f_0, \quad 2\le k\le N,
\end{aligned}
\tag{3.4}
$$

where $R=(I+i\tau A^{1/2}-(\tau^2/2)A)^{-1}$, $\tilde{R}=(I-i\tau A^{1/2}-(\tau^2/2)A)^{-1}$ and on the estimates

$$
\begin{gathered}
\|R\|_{H\mapsto H}\le 1, \qquad \left\|\tilde{R}\right\|_{H\mapsto H}\le 1, \qquad \left\|R\tilde{R}^{-1}\right\|_{H\mapsto H}\le 1,\\
\left\|\tilde{R}R^{-1}\right\|_{H\mapsto H}\le 1, \qquad \left\|\left(I\pm\frac{i\tau}{2}A^{1/2}\right)^{-1}\right\|_{H\mapsto H}\le 1,\\
\left\|\left(I\pm i\tau A^{1/2}\right)^{-1}\right\|_{H\mapsto H}\le 1, \qquad \left\|\tau A^{1/2}\left(I\pm i\tau A^{1/2}\right)^{-1}\right\|_{H\mapsto H}\le 1.
\end{gathered}
\tag{3.5}
$$

THEOREM 3.2. *Let $\varphi \in D(A)$, $\psi \in D(A^{1/2})$. Then for the solution of the difference scheme (3.2), the following stability inequalities, for $1 \le k \le N$, hold*

$$\begin{aligned}
\|u_k\|_H &\le \tau \sum_{s=0}^{k-1} \|A^{-1/2} f_s\|_H + \|A^{-1/2}\psi\|_H + \|\varphi\|_H, \\
\|A^{1/2} u_k\|_H &\le \tau \sum_{s=0}^{k-1} \|f_s\|_H + \|A^{1/2}\varphi\|_H + \|\psi\|_H, \\
\left\| \frac{A(u_k + u_{k-1})}{2} \right\|_H &\le 2 \sum_{s=1}^{k-1} \|f_s - f_{s-1}\|_H + \|f_0\|_H + \|A^{1/2}\psi\|_H + \|A\varphi\|_H.
\end{aligned} \tag{3.6}$$

The proof of this theorem is based on the following formulas:

$$\begin{aligned}
u_1 &= (I + \tau^2 A)^{-1} \left[\left(I + \frac{\tau^2}{2} A \right) \varphi + \tau \psi + \frac{\tau^2}{2} f_0 \right], \\
u_k &= \Bigg[R^k + \frac{1}{2i} A^{-1/2} \left(I - \frac{i\tau A^{1/2}}{2} \right) [R^k - \tilde{R}^k] \\
&\quad \times \left(\left(I + \frac{i\tau A^{1/2}}{2} \right) \frac{\tau}{2} A - i\tau A^{1/2} (I + \tau^2 A) \right) (I + \tau^2 A)^{-1} \Bigg] \varphi \\
&\quad + \frac{i}{2} A^{-1/2} \left(I - \frac{i\tau A^{1/2}}{2} \right) [R^k - \tilde{R}^k] (I + \tau^2 A)^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right) \psi \\
&\quad + \frac{i}{2} A^{-1/2} \left(I - \frac{i\tau A^{1/2}}{2} \right) [R^k - \tilde{R}^k] (I + \tau^2 A)^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right) \frac{\tau}{2} f_0 \\
&\quad - \sum_{s=1}^{k-1} \frac{\tau}{2i} A^{-1/2} [R^{k-s} - \tilde{R}^{k-s}] f_s \\
&= \Bigg[R^k + \frac{1}{2i} A^{-1/2} \left(I - \frac{i\tau A^{1/2}}{2} \right) [R^k - \tilde{R}^k] \\
&\quad \times \left(\left(I + \frac{i\tau A^{1/2}}{2} \right) \frac{\tau}{2} A - i\tau A^{1/2} (I + \tau^2 A) \right) (I + \tau^2 A)^{-1} \Bigg] \varphi \\
&\quad + \frac{i}{2} A^{-1/2} \left(I - \frac{i\tau A^{1/2}}{2} \right) [R^k - \tilde{R}^k] (I + \tau^2 A)^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right) \psi \\
&\quad + \frac{i}{2} A^{-1/2} \left(I - \frac{i\tau A^{1/2}}{2} \right) [R^k - \tilde{R}^k] (I + \tau^2 A)^{-1} \left(I + \frac{i\tau A^{1/2}}{2} \right) \frac{\tau}{2} f_0
\end{aligned}$$

$$+A^{-1}\frac{1}{2}\sum_{s=1}^{k-1}\left[\left(I-\frac{i\tau A^{1/2}}{2}\right)R^{k-s}+\left(I+\frac{i\tau A^{1/2}}{2}\right)\tilde{R}^{k-s}\right](f_{s-1}-f_s)$$
$$+2f_{k-1}-\left[\left(I-\frac{i\tau A^{1/2}}{2}\right)R^{k-1}+\left(I+\frac{i\tau A^{1/2}}{2}\right)\tilde{R}^{k-1}\right]f_0, \quad 2\le k\le N, \tag{3.7}$$

where $R=(I-i\tau A^{1/2}/2)(I+i\tau A^{1/2}/2)^{-1}$, $\tilde{R}=(I+i\tau A^{1/2}/2)(I-i\tau A^{1/2}/2)^{-1}$ and on the estimates

$$\|R\|_{H\mapsto H}\le 1, \qquad \|\tilde{R}\|_{H\mapsto H}\le 1, \qquad \left\|\left(I\pm\frac{i\tau A^{1/2}}{2}\right)^{-1}\right\|_{H\mapsto H}\le 1,$$
$$\left\|\left(I\pm i\tau A^{1/2}\right)^{-1}\right\|_{H\mapsto H}\le 1, \qquad \left\|\tau A^{1/2}\left(I\pm i\tau A^{1/2}\right)^{-1}\right\|_{H\mapsto H}\le 1. \tag{3.8}$$

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