STRONGLY NONLINEAR POTENTIAL THEORY ON METRIC SPACES

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We define Orlicz-Sobolev spaces on an arbitrary metric space with a Borel regular outer measure, and we develop a capacity theory based on these spaces. We study basic properties of capacity and several convergence results. We prove that each Orlicz-Sobolev function has a quasi-continuous representative. We give estimates for the capacity of balls when the measure is doubling. Under additional regularity assumption on the measure, we establish some relations between capacity and Hausdorff measures.

1. Introduction

The introduction and the extensive study of Sobolev spaces on arbitrary metric spaces by Franchi et al. [9], Hajłasz [12], Hajłasz and Koskela [13], Hajłasz and Martio [14], and others, have given a great impulse to several developments in geometric analysis on metric measure spaces. Important examples are the substantial progress of various domains such as fractals, partial differential equations, Carnot-Carathéodory geometries, stochastic process, and so forth.

The nonlinear potential theory on metric spaces has seen a great jump since the development of the capacity theory in these spaces by Kilpeläinen et al. [16], Kinnunen and Martio [17], and others.

For Orlicz and Orlicz-Sobolev spaces in the Euclidean space, we have developed in [2, 3, 4, 5, 6, 7] a potential theory, called strongly nonlinear potential theory. It is natural to develop this theory in the setting of metric spaces. It is the object of this paper.

A Lipschitz characterization of Orlicz-Sobolev spaces in Euclidean case is given. Since a density argument and the Hardy-Littlewood maximal function are involved, we must suppose the Orlicz space to be reflexive. This characterization is used to introduce a definition of Orlicz-Sobolev spaces on an arbitrary metric measure space. We prove an approximation theorem of Orlicz-Sobolev

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functions by Lipschitz functions, both in Lusin and in norm sense. This generalizes a result by Hajłasz in [12] relative to the Sobolev case.

Then we develop a capacity theory based on the definition of Orlicz-Sobolev spaces on an arbitrary metric measure space. Basic properties of capacity and several convergence results are studied. Moreover, we prove that each Orlicz-Sobolev function has a quasi-continuous representative and we give estimates for the capacity of balls when the measure is doubling. Some relations between capacity and Hausdorff measures are established under additional regularity assumption on the measure.

This paper is organized as follows. In Section 2, we list the prerequisites from the Orlicz theory. Section 3 is dedicated to Lipschitz characterization of Orlicz-Sobolev spaces in the Euclidean case, to the study of Orlicz-Sobolev spaces on metric spaces and to establish an approximation theorem of Orlicz-Sobolev functions by Lipschitz functions. Section 4 is reserved to establish important properties of capacity on metric spaces. Section 5 deals with the comparison between capacity and measure in the metric space, and between capacity and the Hausdorff measure.

2. Preliminaries

An N-function is a continuous convex and even function Φ defined on $\mathbb{R}$, verifying $\Phi(t) > 0$ for $t > 0$, $\lim_{t\to 0}\Phi(t)/t = 0$, and $\lim_{t\to+\infty}\Phi(t)/t = +\infty$.

We have the representation $\Phi(t) = \int_0^{|t|}\varphi(x)\,d\mathfrak{L}(x)$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, right continuous, with $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$, $\lim_{t\to 0^+}\varphi(t) = 0$, and $\lim_{t\to+\infty}\varphi(t) = +\infty$. Here $\mathfrak{L}$ stands for the Lebesgue measure. We put in the sequel, as usual, $dx = d\mathfrak{L}(x)$.

The N-function Φ^* *conjugate* to Φ is defined by $\Phi^*(t) = \int_0^{|t|}\varphi^*(x)\,dx$, where φ^* is given by $\varphi^*(s) = \sup\{t : \varphi(t) \le s\}$.

Let Φ be an N-function. We say that Φ *verifies the* Δ_2 *condition* if there is a constant $C > 0$ such that $\Phi(2t) \le C\Phi(t)$ for all $t \ge 0$. We denote by $C(\Phi)$ the smallest of such constants. That is, $C(\Phi) = \sup_{t>0}\Phi(2t)/\Phi(t)$.

The Δ_2 condition for Φ can be formulated in the following equivalent way: for every $C > 0$ there exists $C' > 0$ such that $\Phi(Ct) \le C'\Phi(t)$ for all $t \ge 0$.

Let (X,Γ,μ) be a measure space and Φ an N-function. The *Orlicz class* $\mathcal{L}_{\Phi,\mu}(X)$ is defined by

$$\mathcal{L}_{\Phi,\mu}(X) = \left\{f : X \longrightarrow \mathbb{R} \text{ measurable} : \int_X \Phi(f(x))\,d\mu(x) < \infty\right\}. \tag{2.1}$$

We define the *Orlicz space* $\mathbf{L}_{\Phi,\mu}(X)$ by

$$\mathbf{L}_{\Phi,\mu}(X) = \left\{f : X \longrightarrow \mathbb{R} \text{ measurable} : \int_X \Phi(\alpha f(x))\,d\mu(x) < \infty \text{ for some } \alpha > 0\right\}. \tag{2.2}$$

We always have $\mathscr{L}_{\Phi,\mu}(X) \subset \mathbf{L}_{\Phi,\mu}(X)$. The equality $\mathscr{L}_{\Phi,\mu}(X) = \mathbf{L}_{\Phi,\mu}(X)$ occurs if Φ verifies the Δ_2 condition.

The Orlicz space $\mathbf{L}_{\Phi,\mu}(X)$ is a Banach space with the following norm, called the *Luxemburg norm*,

$$|||f|||_{\Phi,\mu,X} = \inf\left\{r > 0 : \int_X \Phi\left(\frac{f(x)}{r}\right) d\mu(x) \leq 1\right\}. \tag{2.3}$$

Recall that $\mathbf{L}_{\Phi,\mu}(X)$ is reflexive if Φ and Φ^* verify the Δ_2 condition.

Note that if Φ verifies the Δ_2 condition, then $\int \Phi(f_i(x))\,d\mu \to 0$ as $i \to \infty$ if and only if $|||f_i|||_{\Phi,\mu,X} \to 0$ as $i \to \infty$.

Let Ω be an open set in $\mathbb{R}^N$, $\mathbf{C}^\infty(\Omega)$ be the space of functions which, together with all their partial derivatives of any order, are continuous on Ω, and $\mathbf{C}_0^\infty(\mathbb{R}^N) = \mathbf{C}_0^\infty$ stands for all functions in $\mathbf{C}^\infty(\mathbb{R}^N)$ which have compact support in $\mathbb{R}^N$. The space $\mathbf{C}^k(\Omega)$ stands for the space of functions having all derivatives of order $\leq k$ continuous on Ω, and $\mathbf{C}(\Omega)$ is the space of continuous functions on Ω.

The (weak) partial derivative of f of order $|\beta|$ is denoted by

$$D^\beta f = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdot \partial x_2^{\beta_2} \cdot \cdots \cdot \partial x_N^{\beta_N}} f. \tag{2.4}$$

Let Φ be an N-function and $m \in \mathbb{N}$. We say that a function $f : \mathbb{R}^N \to \mathbb{R}$ has a distributional (weak partial) derivative of order m, denoted by $D^\beta f$, $|\beta| = m$, if

$$\int f D^\beta \theta\, dx = (-1)^{|\beta|} \int (D^\beta f)\theta\, dx, \quad \forall \theta \in \mathbf{C}_0^\infty. \tag{2.5}$$

Let Ω be an open set in $\mathbb{R}^N$ and denote $\mathbf{L}_{\Phi,\mathfrak{L}}(\Omega)$ by $\mathbf{L}_\Phi(\Omega)$. The *Orlicz-Sobolev space* $W^m\mathbf{L}_\Phi(\Omega)$ is the space of real functions f, such that f and its distributional derivatives up to the order m, are in $\mathbf{L}_\Phi(\Omega)$.

The space $W^m\mathbf{L}_\Phi(\Omega)$ is a Banach space equipped with the norm

$$|||f|||_{m,\Phi} = \sum_{0 \leq |\beta| \leq m} |||D^\beta f|||_\Phi, \quad f \in W^m\mathbf{L}_\Phi(\Omega), \tag{2.6}$$

where $|||D^\beta f|||_\Phi = |||D^\beta f|||_{\Phi,\mathfrak{L},\Omega}$.

Recall that if Φ verifies the Δ_2 condition, then $\mathbf{C}^\infty(\Omega) \cap W^m\mathbf{L}_\Phi(\Omega)$ is dense in $W^m\mathbf{L}_\Phi(\Omega)$, and $\mathbf{C}_0^\infty(\mathbb{R}^N)$ is dense in $W^m\mathbf{L}_\Phi(\mathbb{R}^N)$.

For more details on the theory of Orlicz spaces, see [1, 18, 19, 20, 21].

3. Orlicz-Sobolev spaces on metric spaces

3.1. The Euclidean case. We begin by two lemmas which lead to a Lipschitz characterization of Orlicz-Sobolev spaces.

Lemma 3.1. *Let Φ be an N-function satisfying the Δ_2 condition, Q a cube in $\mathbb{R}^N$, and $f \in W^1\mathbf{L}_\Phi(Q)$. Then, for almost all $x \in Q$,*

$$|f(x) - f_Q| \le C\int_Q \frac{|\nabla f(y)|}{|x-y|^{N-1}}\,dy, \tag{3.1}$$

where $f_Q = 1/\mu(Q)\int_Q f\,d\mu$, and the constant C depends only on N and Q.

Proof. It is enough to establish (3.1) for $\mathbf{C}^1(Q)$. But in this case the proof can be found in [11, Lemma 7.16]. □

We omit the proof of the following lemma (Hedberg's inequality) since it is exactly the same as the one in [15] or in [23, Lemma 2.8.3].

Lemma 3.2. *Let Φ be an N-function and $f \in W^1\mathbf{L}_\Phi(\mathbb{R}^N)$. Then there is a constant C depending only on N such that for all $x \in \mathbb{R}^N$,*

$$\int_{B(x,R)} \frac{|\nabla f(y)|}{|x-y|^{N-1}}\,dy \le CR\mathcal{M}_R(|\nabla f|)(x), \tag{3.2}$$

where $\mathcal{M}_R(h)(x) = \sup_{r<R}(1/|B(x,r)|)\int_{B(x,r)}|h(y)|\,dy$ and $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$ on $\mathbb{R}^N$.

Now we give a Lipschitz characterization of Orlicz-Sobolev spaces.

Theorem 3.3. *Let Φ be an N-function such that Φ and Φ^* satisfy the Δ_2 condition. Then $f \in W^1\mathbf{L}_\Phi(\mathbb{R}^N)$ if and only if there exists a nonnegative function $g \in \mathbf{L}_\Phi(\mathbb{R}^N)$ such that*

$$|f(x) - f(y)| \le |x-y|[g(x)+g(y)], \tag{3.3}$$

for all $x, y \in \mathbb{R}^N \setminus F$, $|F| = 0$.

Proof. Let Q be a cube in $\mathbb{R}^N$ and $x, y \in Q$. Then we can find a subcube Q^* with $x, y \in Q^*$ and $\operatorname{diam} Q^* \approx |x-y|$. Here $A \approx B$ if there is a constant C such that $C^{-1} \le A \le CB$. Let $f \in W^1\mathbf{L}_\Phi(\mathbb{R}^N)$. Then by (3.1) and (3.2) we get

$$\begin{aligned} |f(x) - f(y)| &\le |f(x) - f_{Q^*}| + |f(y) - f_{Q^*}| \\ &\le C|x-y|\big(\mathcal{M}_{|x-y|}(|\nabla f|)(x) + \mathcal{M}_{|x-y|}(|\nabla f|)(y)\big) \\ &\le C|x-y|\big(\mathcal{M}(|\nabla f|)(x) + \mathcal{M}(|\nabla f|)(y)\big), \end{aligned} \tag{3.4}$$

where $\mathcal{M}(h)(x) = \sup_{0<r}(1/|B(x,r)|)\int_{B(x,r)}|h(y)|\,dy$ is the Hardy-Littlewood maximal function. Since Φ^* verifies the Δ_2 condition, by [10] the maximal operator is bounded in $\mathbf{L}_\Phi(\mathbb{R}^N)$. Hence, there is a nonnegative function $g \in \mathbf{L}_\Phi(\mathbb{R}^N)$ such that

$$|f(x) - f(y)| \le |x-y|[g(x)+g(y)] \quad \text{a.e.} \tag{3.5}$$

For the reverse implication, it suffices to show that, due to Riesz representation and Radon-Nikodym theorem, there is a nonnegative function $h \in \mathbf{L}_\Phi(\mathbb{R}^N)$ such that

$$\left| \frac{\partial f}{\partial x_i}[\phi] \right| \overset{\text{def}}{=} \left| -\int f \frac{\partial \phi}{\partial x_i}\, dx \right| \le \int |\phi| h\, dx \tag{3.6}$$

for all $\phi \in \mathbf{C}_0^\infty$. Now, integrating (3.3) twice over a ball $B(x,\varepsilon)$, we get

$$\frac{1}{|B(x,\varepsilon)|} \int_{B(x,\varepsilon)} |f(x) - f_{Q^*}|\, dx \le C\varepsilon \frac{1}{|B(x,\varepsilon)|} \int_{B(x,\varepsilon)} g(x)\, dx. \tag{3.7}$$

Let $\psi \in \mathbf{C}_0^\infty(B(0,1))$ be such that $\int \psi(x)\, dx = 1$. Put $\psi_\varepsilon(x) = \varepsilon^{-N}\psi(x/\varepsilon)$. We have

$$\int f \frac{\partial \phi}{\partial x_i}\, dx = \lim_{\varepsilon \to 0} \int \frac{\partial \phi}{\partial x_i} (\psi_\varepsilon * f)(x)\, dx = -\lim_{\varepsilon \to 0} \int \phi(x) \left(\frac{\partial \psi_\varepsilon}{\partial x_i} * f \right)(x)\, dx. \tag{3.8}$$

Since $\int (\partial \psi_\varepsilon / \partial x_i)(x)\, dx = 0$, we get

$$\frac{\partial \psi_\varepsilon}{\partial x_i} * f = (f - f_{B(x,\varepsilon)}) * \frac{\partial \psi_\varepsilon}{\partial x_i}. \tag{3.9}$$

Hence

$$\begin{aligned} \left| \frac{\partial \psi_\varepsilon}{\partial x_i} * f \right|(x) &\le C\varepsilon^{-N-1} \int_{B(x,\varepsilon)} |f - f_{B(x,\varepsilon)}|(x)\, dx \\ &\le C \frac{1}{|B(x,\varepsilon)|} \int_{B(x,\varepsilon)} g(x)\, dx. \end{aligned} \tag{3.10}$$

Thus

$$\left| \int f \frac{\partial \phi}{\partial x_i}\, dx \right| \le \int |\phi| \mathcal{M}(g)(x)\, dx. \tag{3.11}$$

Now by [10], $\mathcal{M}(g) \in \mathbf{L}_\Phi(\mathbb{R}^N)$ since Φ^* satisfies the Δ_2 condition. The proof is complete. □

3.2. The metric case. Let (X,d) be a metric space and let μ be a nonnegative Borel regular outer measure on X. The triplet (X,d,μ) will be fixed in the sequel and will be denoted by X.

Let $u : X \to [-\infty, +\infty]$ be a μ-measurable function defined on X. We denote by $D(u)$ the set of all μ-measurable functions $g : X \to [0,+\infty]$ such that

$$|u(x) - u(y)| \le d(x,y)(g(x) + g(y)) \tag{3.12}$$

for every $x, y \in X \setminus F$, $x \neq y$, with $\mu(F) = 0$. The set F is called the exceptional set for g.

Note that the right-hand side of (3.12) is always defined for $x \neq y$. For the points $x, y \in X$, $x \neq y$ such that the left-hand side of (3.12) is undefined, we may assume that the left-hand side is $+\infty$.

Let Φ be an N-function. The Dirichlet-Orlicz space $\mathbf{L}^1_{\Phi,\mu}(X)$ is the space of all μ-measurable functions u such that $D(u) \cap \mathbf{L}_{\Phi,\mu}(X) \neq \emptyset$. This space is equipped with the seminorm

$$|||u|||_{\mathbf{L}^1_{\Phi,\mu}(X)} = \inf\{|||g|||_{\Phi,\mu,X} : g \in D(u) \cap \mathbf{L}_{\Phi,\mu}(X)\}. \tag{3.13}$$

The Orlicz-Sobolev space is $M^1_{\Phi,\mu}(X) = \mathbf{L}_{\Phi,\mu}(X) \cap \mathbf{L}^1_{\Phi,\mu}(X)$ equipped with the norm

$$|||u|||_{M^1_{\Phi,\mu}(X)} = |||u|||_{\Phi,\mu,X} + |||u|||_{\mathbf{L}^1_{\Phi,\mu}(X)}. \tag{3.14}$$

Lemma 3.4. *Let $g_1 \in D(u_1)$, $g_2 \in D(u_2)$, and $\alpha, \beta \in \mathbb{R}$. If $g \geq |\alpha| g_1 + |\beta| g_2$ almost everywhere, then $g \in D(\alpha u_1 + \beta u_2)$.*

Proof. The proof is a simple verification. □

Lemma 3.5. *Let Φ be an N-function. Let $(u_i)_i$ and $(g_i)_i$ be two sequences of functions such that, for all $i, g_i \in D(u_i)$. If $u_i \to u$ in $\mathbf{L}_{\Phi,\mu}(X)$ and $g_i \to g$ in $\mathbf{L}_{\Phi,\mu}(X)$, then $g \in D(u)$.*

Proof. There are two subsequences of $(u_i)_i$ and $(g_i)_i$, which we denote again by $(u_i)_i$ and $(g_i)_i$, such that $u_i \to u$ and $g_i \to g$, μ-a.e. Let, for $i = 1, 2, \dots$, F_i be the exceptional set for g_i and let G be a set of measure zero such that $u_i \to u$ and $g_i \to g$ on ${}^cG$. We set $F = G \cup (\bigcup_{i=1}^{\infty} F_i)$. It is clear that $\mu(F) = 0$ and $|u(x) - u(y)| \leq d(x, y)(g(x) + g(y))$, for all $x, y \in X \setminus F$. This implies that $g \in D(u)$. The proof is complete. □

Theorem 3.6. *Let Φ be an N-function. Then $M^1_{\Phi,\mu}(X)$, equipped with the norm defined by (3.14), is a Banach space.*

Proof. It is clear that $M^1_{\Phi,\mu}(X)$ is a vector space. It remains to prove that $M^1_{\Phi,\mu}(X)$ is complete for the norm defined by (3.14). Let $(u_i)_i$ be an arbitrary Cauchy sequence in $M^1_{\Phi,\mu}(X)$. Taking if necessary a subsequence, we may assume that $|||(u_i - u_{i+1})|||_{M^1_{\Phi,\mu}(X)} \leq 2^{-i}$. Set $v_j = u_j - u_{j+1}$. There exists $h_j \in D(v_j) \cap \mathbf{L}_{\Phi,\mu}(X)$ such that $|||h_j|||_{M^1_{\Phi,\mu}(X)} \leq 2^{-i}$. Let $g_1 \in D(u_1)$ be arbitrary and set, for $k \geq 2$, $g_k = g_1 + \sum_{j=1}^{k-1} h_j$. From the identity $u_k = u_1 - \sum_{j=1}^{k-1} v_j$ and Lemma 3.4, it follows that $g_k \in D(u)$, for all $k \in N$. Now $(u_k)_k$ and $(g_k)_k$ are Cauchy sequences in $\mathbf{L}_{\Phi,\mu}(X)$. Thus there are limit functions $u = \lim u_k$ in $\mathbf{L}_{\Phi,\mu}(X)$ and $g = \lim g_k$ in $\mathbf{L}_{\Phi,\mu}(X)$. Lemma 3.5 implies that $g \in D(u)$ and therefore the sequence $(u_k)_k$ has a limit in $M^1_{\Phi,\mu}(X)$. The proof is complete. □

The previous results lead to the following characterization of $M^1_{\Phi,\mu}(X)$.

Lemma 3.7. *Let Φ be an N-function. The function $u \in M^1_{\Phi,\mu}(X)$ if and only if $u \in \mathbf{L}_{\Phi,\mu}(X)$ and there are functions $u_i \in \mathbf{L}_{\Phi,\mu}(X)$, $i = 1,2,\dots$, such that $u_i \to u$ μ-a.e. and $g_i \in D(u_i) \cap \mathbf{L}_{\Phi,\mu}(X)$, $i = 1,2,\dots$, such that $g_i \to g$ μ-a.e. for some $g \in \mathbf{L}_{\Phi,\mu}(X)$.*

Proof. The proof is an immediate consequence of Lemma 3.5 and Theorem 3.6. □

Moreover, $M^1_{\Phi,\mu}(X)$ satisfies the following lattice property.

Lemma 3.8. *Let Φ be an N-function. Let $u_1, u_2 \in M^1_{\Phi,\mu}(X)$. If $g_1 \in D(u_1)$ and $g_2 \in D(u_2)$, then*

(i) $u = \max(u_1, u_2) \in M^1_{\Phi,\mu}(X)$ *and* $\max(g_1, g_2) \in D(u) \cap \mathbf{L}_{\Phi,\mu}(X)$;
(ii) $v = \min(u_1, u_2) \in M^1_{\Phi,\mu}(X)$ *and* $\max(g_1, g_2) \in D(v) \cap \mathbf{L}_{\Phi,\mu}(X)$.

Proof. We prove the case (i) only, the proof of (ii) is similar. Let $g = \max(g_1, g_2)$ and suppose that F_1 and F_2 are the exceptional sets for u_1 and u_2 in (3.12), respectively. It is evident that $u, g \in \mathbf{L}_{\Phi,\mu}(X)$. It remains to show that $g \in D(u)$. Let $U = \{x \in X \setminus (F_1 \cup F_2) : u_1(x) \geq u_2(x)\}$. Let $x, y \in U$. Then

$$|u(x) - u(y)| = |u_1(x) - u_1(y)| \leq d(x,y)\big(g_1(x) + g_1(y)\big). \tag{3.15}$$

By the same manner we obtain, for $x, y \in X \setminus U$,

$$|u(x) - u(y)| \leq d(x,y)\big(g_2(x) + g_2(y)\big). \tag{3.16}$$

For the remaining cases, let $x \in U$ and $y \in X \setminus U$. If $u_1(x) \geq u_2(y)$, then

$$|u(x) - u(y)| = u_1(x) - u_2(y) \leq u_1(x) - u_1(y) \leq d(x,y)\big(g_1(x) + g_1(y)\big). \tag{3.17}$$

If $u_1(x) < u_2(y)$, then

$$|u(x) - u(y)| = -u_1(x) + u_2(y) \leq -u_2(x) + u_2(y) \leq d(x,y)\big(g_2(x) + g_2(y)\big). \tag{3.18}$$

The case $x \in X \setminus U$ and $y \in U$ follows by symmetry. Hence, for all $x, y \in X \setminus (F_1 \cup F_2)$ with $\mu(F_1 \cup F_2) = 0$, $|u(x) - u(y)| \leq d(x,y)(g(x) + g(y))$. □

Next, we prove the following important Poincaré inequality for Orlicz-Sobolev functions.

Proposition 3.9. *Let Φ be an N-function. If $u \in M^1_{\Phi,\mu}(X)$ and $E \subset X$ is μ-measurable with $0 < \mu(E) < \infty$, then for every $g \in D(u) \cap \mathbf{L}_{\Phi,\mu}(X)$,*

$$\| |u - u_E| \|_{\mathbf{L}^1_{\Phi,\mu}(E)} \leq 2\operatorname{diam}(E) \| |g| \|_{\mathbf{L}^1_{\Phi,\mu}(E)}, \tag{3.19}$$

where $u_E = 1/\mu(E) \int_E f \, d\mu$.

Proof. Put $C = 2\operatorname{diam}(E)\| |g| \|_{\mathbf{L}^1_{\Phi,\mu}(E)}$ and remark that

$$(u - u_E)(x) = \frac{1}{\mu(E)} \int_E (u(x) - u(y))\, d\mu(y). \tag{3.20}$$

By the Jensen inequality

$$\begin{aligned} \int_E \Phi\left(\frac{u-u_E}{C}\right)(x)\, d\mu(x) &= \int_E \Phi\left[\frac{1}{\mu(E)} \int_E \frac{u(x)-u(y)}{C}\, d\mu(y)\right](x)\, d\mu(x) \\ &\leq \int_E \left[\int_E \frac{1}{\mu(E)} \Phi\left(\frac{u(x)-u(y)}{C}\right) d\mu(y)\right] d\mu(x). \end{aligned} \tag{3.21}$$

On the other hand, using the definition of $D(u)$ and the convexity of Φ, we get

$$\begin{aligned} \Phi\left(\frac{u(x)-u(y)}{C}\right) &\leq \Phi\left(\frac{g(x)-g(y)}{2\| |g| \|_{\mathbf{L}^1_{\Phi,\mu}(E)}}\right) \\ &\leq \frac{1}{2}\left[\Phi\left(\frac{g(x)}{\| |g| \|_{\mathbf{L}^1_{\Phi,\mu}(E)}}\right) + \Phi\left(\frac{g(y)}{\| |g| \|_{\mathbf{L}^1_{\Phi,\mu}(E)}}\right)\right]. \end{aligned} \tag{3.22}$$

Now we use the fact that $\int_E \Phi(g(x)/\| |g| \|_{\mathbf{L}^1_{\Phi,\mu}(E)})\, d\mu(x) \leq 1$ to deduce that

$$\begin{aligned} &\int_E \Phi\left(\frac{u-u_E}{C}\right)(x)\, d\mu(x) \\ &\quad \leq \frac{1}{2\mu(E)} \int_E \int_E \left[\Phi\left(\frac{g(x)}{\| |g| \|_{\mathbf{L}^1_{\Phi,\mu}(E)}}\right) + \Phi\left(\frac{g(y)}{\| |g| \|_{\mathbf{L}^1_{\Phi,\mu}(E)}}\right)\right] d\mu(y)\, d\mu(x) \\ &\quad \leq \frac{1}{2\mu(E)} \int_E \left[1 + \Phi\left(\frac{g(x)}{\| |g| \|_{\mathbf{L}^1_{\Phi,\mu}(E)}}\right)\mu(E)\right] d\mu(x) \leq 1. \end{aligned} \tag{3.23}$$

The proof is complete. □

Now, we prove an approximation theorem of Orlicz-Sobolev functions by Lipschitz functions, both in Lusin sense and in norm. This generalizes a result in [12] relative to the Sobolev case.

Theorem 3.10. *Let Φ be an N-function satisfying the Δ_2 condition and $u \in M^1_{\Phi,\mu}(X)$. Then for every $\varepsilon > 0$, there is a Lipschitz function h such that*

(1) $\mu(\{x : u(x) \neq h(x)\}) < \varepsilon$;

(2) $\| |u - h| \|_{M^1_{\Phi,\mu}(X)} < \varepsilon$.

Proof. Let $u \in M^1_{\Phi,\mu}(X)$ and let g be taken from the definition of $\| |u| \|_{\mathbf{L}^1_{\Phi,\mu}(X)}$. Let $X_n = \{x \in X : |u(x)| \leq n \text{ and } g(x) \leq n\}$. Since $u, g \in \mathbf{L}_{\Phi,\mu}(X)$ and Φ satisfies the Δ_2 condition, we conclude that $u, g \in \mathscr{L}_{\Phi\mu}(X)$. This implies that $\lim_{n\to\infty} \Phi(n)\mu({}^cX_n) = 0$, and $\lim_{n\to\infty} \mu({}^cX_n) = 0$ because $\Phi(n) \to \infty$ when $n \to \infty$. The restriction $u|_{X_n}$

is Lipschitz with the constant $2n$ and can be extended to the Lipschitz function u' on X with the same constant (see [8, Section 2.10.4] and [22, Theorem 5.1]). We put $u_n = (\operatorname{sgn} u')\min(|u'|, n)$. It is clear that u_n is Lipschitz with the constant $2n$, $u_n|_{X_n} = u|_{X_n}$, $|u_n| \le n$ and $\mu(\{x : u(x) \ne u_n(x)\}) \le \mu({}^cX_n) \xrightarrow{n\to\infty} 0$.

We must simply prove that $u_n \to u$ in $M^1_{\Phi,\mu}(X)$ when $n \to \infty$. We have

$$\begin{aligned}\int \Phi(u-u_n)\,d\mu &= \int_{{}^cX_n} \Phi(u-u_n)\,d\mu \\ &\le \int_{{}^cX_n} \Phi(|u|+|u_n|)\,d\mu \\ &\le \frac{C_\Phi}{2}\left[\int_{{}^cX_n} \Phi(|u|)\,d\mu + \int_{{}^cX_n} \Phi(|u_n|)\,d\mu\right] \\ &\le \frac{C_\Phi}{2}\left[\int_{{}^cX_n} \Phi(|u|)\,d\mu + \Phi(n)\mu({}^cX_n)\right] \xrightarrow{n\to\infty} 0.\end{aligned} \tag{3.24}$$

Since Φ satisfies the Δ_2 condition, $\||u-u_n|\|_{\mathbf{L}^1_{\Phi,\mu}(X)} \xrightarrow{n\to\infty} 0$. It remains to estimate the gradient. Let

$$g_n = \begin{cases} 0 & \text{for } x \in X_n, \\ g(x)+3n & \text{for } x \in^c X_n. \end{cases} \tag{3.25}$$

We have $|(u-u_n)(x) - (u-u_n)(y)| \le d(x,y)(g_n(x)+g_n(y))$. Since $\int \Phi(g_n)\,d\mu = \int_{{}^cX_n} \Phi(g_n)\,d\mu \to 0$ as $n \to \infty$, we conclude that $\||g_n|\|_{\mathbf{L}^1_{\Phi,\mu}(X)} \to 0$ when $n \to \infty$ because Φ satisfies the Δ_2 condition. The theorem follows. □

4. Capacity on metric spaces

Definition 4.1. For a set $E \subset X$, define $C_{\Phi,\mu}(E)$ by

$$C_{\Phi,\mu}(E) = \inf\{\||u|\|_{M^1_{\Phi,\mu}(X)} : u \in B(E)\}, \tag{4.1}$$

where $B(E) = \{u \in M^1_{\Phi,\mu}(X) : u \ge 1 \text{ on a neighborhood of } E\}$.

If $B(E) = \emptyset$, we set $C_{\Phi,\mu}(E) = \infty$. Functions belonging to $B(E)$ are called admissible functions for E.

Remark 4.2. In the definition of $C_{\Phi,\mu}(E)$, we can restrict ourselves to those admissible functions u such that $0 \le u \le 1$.

Proof. Let $B'(E) = \{u \in B(E) : 0 \le u \le 1\}$, then $B'(E) \subset B(E)$ implies that

$$C_{\Phi,\mu}(E) \le \inf\{\||u|\|_{M^1_{\Phi,\mu}(X)} : u \in B'(E)\}. \tag{4.2}$$

On the other hand, let $\varepsilon > 0$ and take $u \in B(E)$ such that $|||u|||_{M^1_{\Phi,\mu}(X)} \le C_\Phi(E) + \varepsilon$. Then $v = \max(0, \min(u,1)) \in B'(E)$, and by Lemma 3.8 we get $D(u) \subset D(v)$. Hence,

$$\inf\{|||u|||_{M^1_{\Phi,\mu}(X)} : u \in B'(E)\} \le |||v|||_{M^1_{\Phi,\mu}(X)} \le |||u|||_{M^1_{\Phi,\mu}(X)} \le C_{\Phi,\mu}(E) + \varepsilon. \tag{4.3}$$

This completes the proof. □

We define a *capacity* as an increasing positive set function C given on a σ-additive class of sets Γ, which contains compact sets and such that $C(\emptyset) = 0$ and $C(\bigcup_{i\ge1} X_i) \le \sum_{i\ge1} C(X_i)$ for $X_i \in \Gamma$, $i = 1,2,\ldots$. C is called outer capacity if for every $X \in \Gamma$,

$$C(X) = \inf\{C(O) : O \text{ open}, X \subset O\}. \tag{4.4}$$

Theorem 4.3. *Let Φ be an N-function. The set function $C_{\Phi,\mu}$ is an outer capacity.*

Proof. It is obvious that $C_{\Phi,\mu}(\emptyset) = 0$ and that $C_{\Phi,\mu}$ is increasing. For countable subadditivity, let E_i, $i = 1,2,\ldots$ be subsets of X and let $\varepsilon > 0$. We may assume that $\sum_{i=1}^\infty C_{\Phi,\mu}(E_i) < \infty$. We choose $u_i \in B(E_i)$ and $g_{u_i} \in D(u_i) \cap \mathbf{L}_{\Phi,\mu}(X)$ so that for $i = 1,2,\ldots,$

$$|||u_i|||_{\Phi,\mu,X} + |||g_{u_i}|||_{\Phi,\mu,X} \le C_\Phi(E_i) + \varepsilon 2^{-i}. \tag{4.5}$$

We show that $v = \sup_i u_i$ is admissible for $\bigcup_{i=1}^\infty E_i$ and $g = \sup_i g_{u_i} \in D(v) \cap \mathbf{L}_{\Phi,\mu}(X)$.

Observe that $v, g \in \mathbf{L}_{\Phi,\mu}(X)$. Define $v_k = \max_{1\le i\le k} u_i$. By Lemma 3.8 the function $g_{v_k} = \max_{1\le i\le k} g_{u_i} \in D(v_k) \cap \mathbf{L}_{\Phi,\mu}(X)$. Since $v_k \to v$ μ-a.e., Lemma 3.7 gives $v \in M^1_{\Phi,\mu}(X)$. Clearly $v \ge 1$ in a neighborhood of $\bigcup_{i=1}^\infty E_i$. Hence $C_{\Phi,\mu}(\bigcup_{i=1}^\infty E_i) \le |||v|||_{M^1_{\Phi,\mu}(X)}$.

By [6, Lemma 2],

$$|||v|||_{M^1_{\Phi,\mu}(X)} \le \sum_{i=1}^\infty \left(|||u_i|||_{\Phi,\mu,X} + |||g_{u_i}|||_{\Phi,\mu,X}\right) \le \sum_{i=1}^\infty C_{\Phi,\mu}(E_i) + \varepsilon. \tag{4.6}$$

Since ε is arbitrary, we deduce that $C_{\Phi,\mu}(\bigcup_{i=1}^\infty E_i) \le \sum_{i=1}^\infty C_{\Phi,\mu}(E_i)$. Hence $C_{\Phi,\mu}$ is a capacity.

It remains to prove that $C_{\Phi,\mu}$ is outer, that is,

$$C_{\Phi,\mu}(E) = \inf\{C_{\Phi,\mu}(O) : E \subset O,\ O \text{ open}\}. \tag{4.7}$$

By monotonicity, $C_{\Phi,\mu}(E) \le \inf\{C_{\Phi,\mu}(O) : E \subset O,\ O \text{ open}\}$. For the reverse inequality, let $\varepsilon > 0$ and let $u \in B(E)$ be such that $|||u|||_{M^1_{\Phi,\mu}(X)} \le C_{\Phi,\mu}(E) + \varepsilon$. Since

$u \in B(E)$, there is an open set O, $E \subset O$, such that $u \geq 1$ on O. This implies that $C_{\Phi,\mu}(O) \leq \||u|\|_{M^1_{\Phi,\mu}(X)} \leq C_{\Phi,\mu}(E) + \varepsilon$.

Since ε is arbitrary, we obtain the claim, and the proof is complete. □

Corollary 4.4. *Let Φ be an N-function. Let $(K_i)_{i\geq 1}$ be a decreasing sequence of compact sets in X and let $K = \bigcap_{i=1}^{\infty} K_i$. Then*

$$C_{\Phi,\mu}(K) = \lim_{i\to\infty} C_{\Phi,\mu}(K_i). \tag{4.8}$$

Proof. This is a direct consequence of the fact that $C_{\Phi,\mu}$ is an outer capacity. □

Theorem 4.5. *Let Φ be a uniformly convex N-function such that Φ satisfies the Δ_2 condition. Let $(O_i)_{i\geq 1}$ be an increasing sequence of open sets in X and let $O = \bigcup_{i=1}^{\infty} O_i$. Then*

$$C_{\Phi,\mu}(O) = \lim_{i\to\infty} C_{\Phi,\mu}(O_i). \tag{4.9}$$

Proof. The hypothesis implies that $\mathbf{L}_{\Phi,\mu}(X)$ is uniformly convex. By monotonicity, $\lim_{i\to\infty} C_{\Phi,\mu}(O_i) \leq C_{\Phi,\mu}(O)$. To prove the reverse inequality, we may assume that $\lim_{i\to\infty} C_{\Phi,\mu}(O_i) < \infty$. Let $\varepsilon > 0$ and for $i = 1, 2, \ldots$, $u_i \in B(O_i)$, and $g_{u_i} \in D(u_i) \cap \mathbf{L}_{\Phi,\mu}(X)$ be such that

$$\||u_i|\|_{\Phi,\mu,X} + \||g_{u_i}|\|_{\Phi,\mu,X} \leq C_{\Phi,\mu}(O_i) + \varepsilon. \tag{4.10}$$

The sequence $(u_i)_i$ is bounded in $\mathbf{L}_{\Phi,\mu}(X)$ and, hence, it possesses a weakly convergent subsequence, which we denote again by $(u_i)_i$. The sequence $(g_{u_i})_i$ is also bounded in $\mathbf{L}_{\Phi,\mu}(X)$ and, by passing to a subsequence, we may assume that $u_i \to u$ weakly in $\mathbf{L}_{\Phi,\mu}(X)$ and $g_{u_i} \to g$ weakly in $\mathbf{L}_{\Phi,\mu}(X)$. We use the Banach-Saks theorem to deduce that the sequence defined by $v_j = j^{-1}\sum_{i=1}^{j} u_i$ converges to u in $\mathbf{L}_{\Phi,\mu}(X)$ and $g_{v_j} = j^{-1}\sum_{i=1}^{j} g_{u_i}$ converges to g in $\mathbf{L}_{\Phi,\mu}(X)$.

Now there is a subsequence of the sequence $(v_j)_j$ so that $v_j \to u$ μ-a.e. and $g_{v_j} \to g$ μ-a.e. By Lemma 3.7, $u \in M^1_{\Phi,\mu}(X)$. On the other hand, $v_j \to 1$ μ-a.e. in O and hence $u \geq 1$ μ-a.e. in O. Thus $u \in B(O)$. By the weak lower semicontinuity of norms, we get

$$\begin{aligned} C_{\Phi,\mu}(O) &\leq \||u|\|_{\Phi,\mu,X} + \||g|\|_{\Phi,\mu,X} \\ &\leq \liminf_{i\to\infty} \left(\||u_i|\|_{\Phi,\mu,X} + \||g_{u_i}|\|_{\Phi,\mu,X}\right) \\ &\leq \lim_{i\to\infty} C_{\Phi,\mu}(O_i) + \varepsilon. \end{aligned} \tag{4.11}$$

By letting $\varepsilon \to 0$, we obtain the result. □

The set function $C_{\Phi,\mu}$ is called the Φ-capacity. If a statement holds except on a set E where $C_{\Phi,\mu}(E) = 0$, then we say that the statement holds Φ-quasi-everywhere (abbreviated Φ-q.e.).

Lemma 4.6. *Let Φ be any N-function and let u be a function in $M^1_{\Phi,\mu}(X)$. If* $\mathrm{PL}_u = \{x \in X : \lim_{t\to x} u(t) = \infty\}$ *denotes the set of poles of u, then $C_{\Phi,\mu}(\mathrm{PL}_u) = 0$.*

Proof. If n is any positive integer, the function $u_n = n^{-1}\inf(u,n)$ is an admissible function for the Φ-capacity of the set PL_u. It is easily seen that $\| |u_n| \|_{M^1_{\Phi,\mu}(X)} \le n^{-1}\| |u| \|_{M^1_{\Phi,\mu}(X)}$. This implies that $\lim_{n\to\infty} \| |u_n| \|_{M^1_{\Phi,\mu}(X)} = 0$. Thus $C_{\Phi,\mu}(\mathrm{PL}_u) = 0$. □

Theorem 4.7. *Let Φ be an N-function and let E be any subset of X. Then $C_{\Phi,\mu}(E) = 0$ if and only if for any $\varepsilon > 0$, there exists a nonnegative function $u \in M^1_{\Phi,\mu}(X)$ such that $\lim_{t\to x} u(t) = \infty$ for any $x \in E$ and $\| |u| \|_{M^1_{\Phi,\mu}(X)} \le \varepsilon$.*

Proof. Let $E \subset X$ be such that $C_{\Phi,\mu}(E) = 0$, and let $\varepsilon > 0$. Then, by the definition of the Φ-capacity, there is a sequence of nonnegative functions, $(u_n)_n$, such that $u_n = 1$ in some neighborhood of E and $\| |u_n| \|_{M^1_{\Phi,\mu}(X)} \le 2^{-n}\varepsilon$, for any n. Then the function $u = \sum_n u_n$ belongs to $M^1_{\Phi,\mu}(X)$ and $\lim_{t\to x} u(t) = \infty$. Furthermore, it is evident that $\| |u| \|_{M^1_{\Phi,\mu}(X)} \le \varepsilon$.

The converse implication is an immediate consequence of Lemma 4.6. □

Theorem 4.8. *Let Φ be an N-function and $(u_i)_i$ be a Cauchy sequence of functions in $M^1_{\Phi,\mu}(X) \cap \mathbf{C}(X)$. Then, there is a subsequence $(u_i')_i$ of $(u_i)_i$ which converges pointwise Φ-q.e. in X. Moreover, the convergence is uniform outside a set of arbitrary small Φ-capacity.*

Proof. Since $(u_i)_i$ is a Cauchy sequence, there is a subsequence $(u_i')_i$ of $(u_i)_i$ such that

$$\sum_{i=1}^{\infty} 2^i \| |u_i' - u_{i+1}'| \|_{M^1_{\Phi,\mu}(X)} < \infty. \tag{4.12}$$

We set $X_i = \{x \in X : |u_i'(x) - u_{i+1}'(x)| > 2^{-i}\}$ for $i = 1,2,\ldots$, and $Y_j = \bigcup_{i=j}^{\infty} X_i$. Since the functions u_i' are continuous by hypothesis, X_i and Y_j are open. Then, for all i, $2^i(u_i' - u_{i+1}') \in B(X_i)$. Hence

$$C_{\Phi,\mu}(X_i) \le 2^i \| |u_i' - u_{i+1}'| \|_{M^1_{\Phi,\mu}(X)}. \tag{4.13}$$

By subadditivity of $C_{\Phi,\mu}$, we obtain

$$C_{\Phi,\mu}(Y_j) \le \sum_{i=j}^{\infty} C_{\Phi,\mu}(X_i) \le \sum_{i=j}^{\infty} 2^i \| |u_i' - u_{i+1}'| \|_{M^1_{\Phi,\mu}(X)}. \tag{4.14}$$

From the convergence of the sum (4.12) we conclude that

$$C_{\Phi,\mu}\left(\bigcap_{j=1}^{\infty} Y_j\right) \le \lim_{j\to\infty} C_{\Phi,\mu}(Y_j) = 0. \tag{4.15}$$

Thus $(u_i')_i$ converges in $X \setminus \bigcap_{j=1}^{\infty} Y_j$. Moreover, we have for any $x \in X \setminus Y_j$ and all $k > j$

$$|u_j'(x) - u_k'(x)| \le \sum_{i=j}^{k-1} |u_i'(x) - u_{i+1}'(x)| \le \sum_{i=j}^{k-1} 2^{-i} \le 2^{1-j}. \tag{4.16}$$

This implies that $(u_i')_i$ converges uniformly in $X \setminus Y_j$.

This completes the proof. □

Definition 4.9. Let Φ be an N-function. A function $u : X \to [-\infty, +\infty]$ is Φ-quasi-continuous in X if for every $\varepsilon > 0$, there is a set E such that $C_{\Phi,\mu}(E) < \varepsilon$ and the restriction of u to $X \setminus E$ is continuous. Since $C_{\Phi,\mu}$ is an outer capacity, we may assume that E is open.

Theorem 4.10. *Let Φ be an N-function satisfying the Δ_2 condition and $u \in M^1_{\Phi,\mu}(X)$. Then there is a function $v \in M^1_{\Phi,\mu}(X)$ such that $u = v$ μ-a.e. and v is Φ-quasi-continuous in X. The function v is called a Φ-quasi-continuous representative of u.*

Proof. We know that $M^1_{\Phi,\mu}(X)$ is a Banach space and by Theorem 3.10, $\mathbf{C}(X) \cap M^1_{\Phi,\mu}(X)$ is a dense subspace of $M^1_{\Phi,\mu}(X)$. Hence, completeness implies that $M^1_{\Phi,\mu}(X)$ can be characterized as the completion of $\mathbf{C}(X) \cap M^1_{\Phi,\mu}(X)$ in the norm of $M^1_{\Phi,\mu}(X)$. Thus there are sequences of functions $(u_i)_i \subset \mathbf{C}(X) \cap \mathbf{L}_{\Phi,\mu}(X)$ and $(g_i)_i \subset D(u_i - u)$ such that $u_i \to u$ and $g_i \to 0$ in $\mathbf{L}_{\Phi,\mu}(X)$. Hence for a subsequence, which we denote again by $(u_i)_i$, $u_i \to u$ μ-a.e. From Theorem 4.8, we deduce that the limit function v of the sequence $(u_i)_i$ is Φ-quasi-continuous in X. This completes the proof. □

5. Comparison between capacity and measures

5.1. Comparison between capacity and the measure μ. The sets of zero capacity are exceptional sets in the strongly nonlinear potential theory. We show in the next lemma that sets of vanishing capacity are also of measure zero.

Lemma 5.1. *Let Φ be any N-function. Then*

(1) *if $E \subset X$ is such that $C_{\Phi,\mu}(E) = 0$, then $\mu(E) = 0$;*

(2) *if $E \subset X$ is such that $\mu(E) \neq 0$, then $C_{\Phi,\mu}(E) \ge 1/\Phi^{-1}(1/\mu(E))$.*

Proof. (1) If n is a strictly positive integer, we can find $u_n \in B(E)$ such that $\| |nu_n| \|_{\Phi,\mu,X} \le \| |nu_n| \|_{M^1_{\Phi,\mu}(X)} \le 1$. Hence, there is an open set O_n such that $E \subset O_n$ and $u_n \ge 1$ in O_n. Then

$$\int_E \Phi(nu_n)\, d\mu \le \int_{O_n} \Phi(nu_n)\, d\mu \le \int_X \Phi(nu_n)\, d\mu \le 1. \tag{5.1}$$

By the inequality $n\Phi(u_n) \le \Phi(nu_n)$ and by the fact that $u_n \ge 1$ in E, we get $\Phi(1)\mu(E) \le 1/n$. This implies the claim.

(2) Let $u \in B(E)$, then there is an open set O such that $E \subset O$ and $u \geq 1$ in O. Hence

$$\int_X \Phi\left[u \cdot \Phi^{-1}\left(\frac{1}{\mu(E)}\right)\right] d\mu \geq \int_E \Phi\left[u \cdot \Phi^{-1}\left(\frac{1}{\mu(E)}\right)\right] d\mu \geq 1. \tag{5.2}$$

Thus

$$|||u|||_{M^1_{\Phi,\mu}(X)} \geq |||u|||_{\Phi,\mu,X} \geq \frac{1}{\Phi^{-1}(1/\mu(E))}. \tag{5.3}$$

We obtain the result by taking the infimum over all $u \in B(E)$. □

Definition 5.2. A measure μ is said to be doubling if there is a constant $C \geq 1$ such that

$$\mu(B(x,2r)) \leq C\mu(B(x,r)), \tag{5.4}$$

for every $x \in X$ and $r > 0$.

The smallest constant C in (5.4) is called the doubling constant and is denoted by C_d.

THEOREM 5.3. *Let Φ be any N-function, μ a doubling measure with the doubling constant C_d, $x_0 \in X$, and $0 < r \leq 1$. Then*

$$C_{\Phi,\mu}(B(x_0,r)) \leq 2C_d r^{-1} \frac{1}{\Phi^{-1}(1/\mu(B(x_0,r)))}. \tag{5.5}$$

Proof. We define

$$u(x) = \begin{cases} \dfrac{2r - d(x,x_0)}{r}, & x \in B(x_0,2r) \setminus B(x_0,r), \\ 1, & x \in B(x_0,r), \\ 0, & x \in X \setminus B(x_0,2r). \end{cases} \tag{5.6}$$

Define also

$$g(x) = \begin{cases} \dfrac{1}{r}, & x \in B(x_0,2r), \\ 0, & x \in X \setminus B(x_0,2r). \end{cases} \tag{5.7}$$

We show that $g \in D(u)$. Let $E = B(x_0,2r) \setminus B(x_0,r)$, and $x, y \in E$. Then

$$|u(x) - u(y)| = \frac{|d(x,x_0) - d(y,x_0)|}{r} \leq \frac{d(x,y)}{r}. \tag{5.8}$$

Hence (3.12) follows in this case. Let $x \in E$ and $y \in B(x_0,r)$. We have

$$|u(x) - u(y)| = 1 - u(x) = \frac{d(x,x_0) - r}{r}. \tag{5.9}$$

Since $d(y,x_0) < r \le d(x,x_0)$, we get

$$d(x,x_0) - r \le d(x,x_0) - d(y,x_0) \le d(x,y), \tag{5.10}$$

and hence (3.12) follows. The case $y \in E$ and $x \in B(x_0,r)$ is completely analogous. Now, if $x,y \in B(x_0,r)$ or $x,y \in X \setminus B(x_0,2r)$, then clearly (3.12) holds. On the other hand, let $x \in X \setminus B(x_0,2r)$ and $y \in B(x_0,r)$. Then

$$|u(x) - u(y)| = 1 \le \frac{d(x,y)}{r}, \tag{5.11}$$

which implies (3.12). Finally, if $x \in E$ and $y \in X \setminus B(x_0,2r)$, then

$$|u(x) - u(y)| = u(x) = \frac{2r - d(x,x_0)}{r} \tag{5.12}$$

and since $d(x,x_0) < 2r < d(y,x_0)$, we obtain

$$2r - d(x,x_0) \le d(y,x_0) - d(x,x_0) \le d(x,y) \tag{5.13}$$

and (3.12) holds. Thus $g \in D(u)$, $u \in B(B(x_0,r))$ and

$$C_{\Phi,\mu}(B(x_0,r)) \le |||u|||_{\Phi,\mu,X} + |||g|||_{\Phi,\mu,X}. \tag{5.14}$$

We have $u \le \chi_{B(x_0,2r)}$, $|||u|||_{\Phi,\mu,X} \le |||\chi_{B(x_0,2r)}|||_{\Phi,\mu,X}$. We know that

$$|||\chi_{B(x_0,2r)}|||_{\Phi,\mu,B(x_0,2r)} = \frac{1}{\Phi^{-1}(1/\mu(B(x_0,2r)))}, \tag{5.15}$$

where $\chi_{B(x_0,2r)}$ is the characteristic function of $B(x_0,2r)$.

On the other hand, $g \le r^{-1}\chi_{B(x_0,2r)}$, and hence

$$|||g|||_{\Phi,\mu,X} \le r^{-1}\frac{1}{\Phi^{-1}(1/\mu(B(x_0,2r)))}. \tag{5.16}$$

Thus

$$C_{\Phi,\mu}(B(x_0,r)) \le (1 + r^{-1})\frac{1}{\Phi^{-1}(1/\mu(B(x_0,2r)))}. \tag{5.17}$$

Now, since μ is doubling, $\mu(B(x_0,2r)) \le C_d\mu(B(x_0,r))$. Hence

$$\frac{1}{\Phi^{-1}(1/\mu(B(x_0,2r)))} \le \frac{1}{\Phi^{-1}(1/C_d\mu(B(x_0,r)))}. \tag{5.18}$$

It remains to evaluate $\Phi^{-1}(1/C_d\mu(B(x_0,r)))$. Recall that, $\Phi(\alpha x) \le \alpha\Phi(x)$, for all $0 \le \alpha \le 1$, and all x. This implies that $\alpha\Phi^{-1}(x) \le \Phi^{-1}(\alpha x)$, for all $0 \le \alpha \le 1$, and all x. Hence

$$\frac{1}{\Phi^{-1}(1/C_d\mu(B(x_0,r)))} \le C_d \frac{1}{\Phi^{-1}(1/\mu(B(x_0,r)))}. \tag{5.19}$$

Thus

$$\begin{aligned} C_{\Phi,\mu}(B(x_0,r)) &\le (1+r^{-1})C_d \frac{1}{\Phi^{-1}(1/\mu(B(x_0,r)))} \\ &\le 2r^{-1}C_d \frac{1}{\Phi^{-1}(1/\mu(B(x_0,r)))}. \end{aligned} \tag{5.20}$$

This completes the proof. □

5.2. Comparison between capacity and Hausdorff measures. We recall the definition of Hausdorff measures. Let $h : [0,+\infty[\to [0,+\infty[$ be an increasing function such that $\lim_{r\to 0} h(r) = 0$. For $0 < \delta \le \infty$ and $E \subset X$, we define

$$\mathbf{H}^h_\delta(E) = \inf\left\{\sum_{i=1}^\infty h(r_i) : E \subset \bigcup_{i=1}^\infty B(x_i,r_i),\ r_i \le \delta\right\}. \tag{5.21}$$

The h-Hausdorff measure of E is given by

$$\mathbf{H}^h(E) = \lim_{\delta\to 0}\mathbf{H}^h_\delta(E). \tag{5.22}$$

If $h(t) = t^s$ for $0 \le s < \infty$, we obtain the s-dimensional Hausdorff measure which we denote by $\mathbf{H}^s$.

For more details on Hausdorff measures, one can consult [8, Chapter 2, Section 10].

A measure μ is regular with dimension $s > 0$, if there is $c \ge 1$ such that, for each $x \in X$ and $0 < r \le \operatorname{diam}(X)$,

$$c^{-1}r^s \le \mu(B(x,r)) \le cr^s. \tag{5.23}$$

Note that if μ is regular with dimension s, then μ is doubling. Moreover, X has Hausdorff dimension s and there is a constant $c > 0$ such that, for every $E \subset X$,

$$c^{-1}\mathbf{H}^s(E) \le \mu(E) \le c\mathbf{H}^s(E). \tag{5.24}$$

Theorem 5.4. *Let Φ be an N-function satisfying the Δ_2 condition and $\alpha = \alpha(\Phi) = \sup_{t>0}(t\varphi(t)/\Phi(t))$. Let μ be a regular measure with dimension s such that $s > \alpha$. Define $h : [0,+\infty[\to [0,+\infty[$ by $h(t) = t^{s/\alpha-1}$. Then for every $E \subset X$,*

$$C_{\Phi,\mu}(E) \le c\mathbf{H}^h(E), \tag{5.25}$$

where the constant c depends only on Φ and the doubling constant. In particular, if $\mathbf{H}^h(E) = 0$, then $C_{\Phi,\mu}(E) = 0$.

Proof. Let $B(x_i, r_i)$, $i = 1, 2, \ldots$, be any covering of E such that the radii r_i satisfy $r_i \le 1/2$, $i = 1, 2, \ldots$. By (5.5) we get, for all i, that

$$C_{\Phi,\mu}(B(x_i, r_i)) \le Cr^{-1} \frac{1}{\Phi^{-1}(1/\mu(B(x_i, r_i)))}. \tag{5.26}$$

From (5.23) we have $\mu(B(x_i, r_i)) \le cr_i^s$, for all i; and hence

$$\Phi^{-1}\left(\frac{1}{\mu(B(x_i, r_i))}\right) \ge \Phi^{-1}\left(\frac{1}{cr_i^s}\right) \ge c^{-1}\Phi^{-1}\left(\frac{1}{r_i^s}\right). \tag{5.27}$$

It remains to evaluate $\Phi^{-1}(1/r_i^s)$. For this goal we distinguish two cases

(i) if $\Phi(1) \le 1$, then $\Phi^{-1}(1/r_i^s) \ge \Phi^{-1}(\Phi(1)(1/r_i^s))$;

(ii) if $\Phi(1) \ge 1$, then

$$\Phi^{-1}\left(\frac{1}{r_i^s}\right) = \Phi^{-1}\left(\frac{\Phi(1)}{\Phi(1)r_i^s}\right) \ge \frac{1}{\Phi(1)}\Phi^{-1}\left(\Phi(1)\frac{1}{r_i^s}\right). \tag{5.28}$$

Hence in the two cases we have $\Phi^{-1}(1/r_i^s) \ge C'\Phi^{-1}(\Phi(1)(1/r_i^s))$. We get

$$\Phi^{-1}\left(\frac{1}{\mu(B(x_i, r_i))}\right) \ge C'\Phi^{-1}\left(\Phi(1)\frac{1}{r_i^s}\right). \tag{5.29}$$

On the other hand we know that $\Phi(t) \le \Phi(1)t^\alpha$, for all $t \ge 1$. This implies that $t^{1/\alpha} \le \Phi^{-1}(\Phi(1)t)$, for all $t \ge 1$. Whence, for all i

$$\Phi^{-1}\left(\frac{1}{\mu(B(x_i, r_i))}\right) \ge C' \frac{1}{r_i^{s/\alpha}}. \tag{5.30}$$

Thus, for all i

$$C_{\Phi,\mu}(B(x_i, r_i)) \le C'' r^{s/\alpha - 1}. \tag{5.31}$$

The subadditivity of $C_{\Phi,\mu}$ yields

$$C_{\Phi,\mu}(E) \le C'' \sum_{i=1}^{\infty} C_{\Phi,\mu}(B(x_i, r_i)) \le C'' \sum_{i=1}^{\infty} h(r_i). \tag{5.32}$$

By taking the infimum over all coverings by balls and letting the radii tend to zero, we finish the proof. □

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