ON THE EXISTENCE OF A POSITIVE SOLUTION FOR A SEMILINEAR ELLIPTIC EQUATION IN THE UPPER HALF SPACE

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Received 14 September 2001

We show the existence of a nontrivial solution to the semilinear elliptic equation $-\Delta u + u = b(x)|u|^{p-2}u$, $u > 0$, $u \in H_0^1(\mathbb{R}_+^N)$ under some suitable conditions.

1. Introduction

In this paper, we study the following problem:

$$-\Delta u + u = b(x)|u|^{p-2}u, \quad u > 0, \; u \in H_0^1(\mathbb{R}_+^N), \tag{1.1}$$

where $2 < p < 2^*$, $2^* = 2N/(N-2)$ when $N > 2$ and $2^* = +\infty$ when $N = 2$, $\mathbb{R}_+^N = \{(x_1, x_2, \dots, x_N) \mid x_N > 0\}$ is the upper half space of $\mathbb{R}^N$. We put the following conditions on b:

(B_1) $b \in C(\mathbb{R}_+^N)$, and

$$1 = \inf_{x \in \mathbb{R}_+^N} b(x) = \lim_{|x| \to +\infty, x \in \mathbb{R}_+^N} b(x); \tag{1.2}$$

(B_2) there exists $\delta \in (0, 1)$, such that

$$\lim_{|x| \to \infty} \big(b(x) - 1\big) e^{(1-\delta)|x|} = +\infty. \tag{1.3}$$

Our main result is the following result.

Theorem 1.1. *If b satisfies (B_1) and (B_2), then (1.1) has a nontrivial solution.*

Similar problems have been studied extensively. In [3], Esteban and Lions showed that $-\Delta u + u = |u|^{p-2}u$, $u \in H_0^1(\mathbb{R}_+^N)$ does not have nontrivial solution.

Abstract and Applied Analysis 7:1 (2002) 29–33
2000 Mathematics Subject Classification: 35J25
URL: http://dx.doi.org/10.1155/S1085337502000726

However, with the help of the potential $b(x)$ and the comparison technique derived in [2], we proved the existence of a nontrivial solution to problem (1.1).

2. Proof of Theorem 1.1

Let

$$\varphi(u)=\int_{\mathbb{R}_+^N}\left(\frac{1}{2}|\nabla u|^2+\frac{1}{2}|u|^2\right)dx-\frac{1}{p}\int_{\mathbb{R}_+^N}b(x)|u|^p\,dx. \tag{2.1}$$

Lemma 2.1. *Denote $S_p=\inf_{u\in H_0^1(\mathbb{R}^N),\,|u|_p=1}\|u\|^2$. Then*

$$\inf_{u\in H_0^1(\mathbb{R}_+^N),\,|u|_p=1}\|u\|^2=S_p. \tag{2.2}$$

Lemma 2.2. *If $0<c<c^*:=(1/2-1/p)S_p^{p/(p-2)}$, then φ satisfies $(PS)_c$ condition.*

One can see [2] or [5] for the proof of Lemmas 2.1 and 2.2.

Lemma 2.3. *The equation $-\Delta u+u=|u|^{p-2}u$ has a unique positive solution w in $\mathbb{R}^N$ and satisfies the following conditions:*

(i) $w>0$ *in* $\mathbb{R}^N$, $w\in C^\infty(\mathbb{R}^N)$.

(ii) $w(x)=w(|x|)$.

(iii) $w(r)r^{(N-1)/2}\exp(r)\longrightarrow C>0$ *and* $w'(r)r^{(N-1)/2}\exp(r)\longrightarrow -C$, *where* $r=|x|\longrightarrow+\infty$.

Proof. See [1, 4]. □

Proof of Theorem 1.1. By Lemma 2.2, we need only to prove that the Mountain Pass value of φ,

$$c=\inf\left\{\left(\frac{1}{2}-\frac{1}{p}\right)\left(\frac{\|v\|^2}{\left(\int_{\mathbb{R}_+^N}b(x)|v|^p\,dx\right)^{2/p}}\right)^{p/(p-2)};\ v\in H_0^1(\mathbb{R}_+^N),\ v\neq\theta\right\}, \tag{2.3}$$

satisfies that $0<c<c^*$.

Set

$$\psi_n(x)=\begin{cases}1, & |x|\le n-1,\\ n-|x|, & n-1<|x|\le n,\\ 0, & |x|>n.\end{cases} \tag{2.4}$$

Let $x_n=(0,0,\dots,0,n)$, $u_n=\psi_n(\cdot-x_n)w(\cdot-x_n)$. Then $u_n\in H_0^1(\mathbb{R}_+^N)$, $n=1,2,\dots$.

Notice that

$$\begin{aligned}
\|u_n\|^2 &= \int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \int_{\mathbb{R}^N} u_n^2\,dx \\
&= \int_{\mathbb{R}^N} |\nabla(\psi_n(\cdot - x_n)w(\cdot - x_n))|^2\,dx + \int_{\mathbb{R}^N} |\psi_n(\cdot - x_n)w(\cdot - x_n)|^2\,dx \\
&= \int_{B(\theta,n)} |\nabla(\psi_n\cdot w)|^2\,dx + \int_{B(\theta,n)} |\psi_n\cdot w|^2\,dx \\
&= \int_{B(\theta,n-1)} |\nabla\omega|^2\,dx + \int_{B(\theta,n-1)} \omega^2\,dx \\
&\quad + \int_{B_{n-1,n}} \psi_n^2\cdot|\nabla\omega|^2\,dx + \int_{B_{n-1,n}} |\nabla\psi_n|^2\cdot\omega^2\,dx \\
&\quad + 2\int_{B_{n-1,n}} \omega\cdot\psi_n\cdot\nabla\omega\cdot\nabla\psi_n\,dx + \int_{B_{n-1,n}} \psi_n^2\cdot\omega^2\,dx \\
&= \int_{\mathbb{R}^N} |\nabla\omega|^2\,dx + \int_{\mathbb{R}^N} \omega^2\,dx \\
&\quad + \int_{\mathbb{R}^N\setminus B(\theta,n-1)} |\nabla\omega|^2\,dx + \int_{\mathbb{R}^N\setminus B(\theta,n-1)} |\omega|^2\,dx \\
&\quad + \int_{B_{n-1,n}} \psi_n^2\cdot|\nabla\omega|^2\,dx + 2\int_{B_{n-1,n}} \omega\cdot\psi_n\cdot\nabla\omega\cdot\nabla\psi_n\,dx \\
&\quad + \int_{B_{n-1,n}} |\nabla\psi_n|^2\cdot\omega^2\,dx + \int_{B_{n-1,n}} \psi_n^2\cdot\omega^2\,dx,
\end{aligned} \tag{2.5}$$

where $B_{n-1,n} = B(\theta,n)\setminus B(\theta,n-1)$.

From Lemma 2.3 we know that, when n is big enough,

$$\begin{aligned}
\int_{\mathbb{R}^N\setminus B(\theta,n-1)} |\nabla\omega|^2\,dx &= C_1\int_{n-1}^{+\infty} |\omega'(r)|^2\cdot r^{N-1}\,dr \\
&\le C_1'\int_{n-1}^{+\infty} e^{-2r}\,dr = \frac{C_1'}{e^{2(n-1)}}.
\end{aligned} \tag{2.6}$$

Similarly, when n is big enough, we get

$$\int_{\mathbb{R}^N\setminus B(\theta,n-1)} \omega^2\,dx \le \frac{C_2'}{e^{2(n-1)}}. \tag{2.7}$$

So

$$\|u_n\|^2 = \|\omega\|^2 + O\left(\frac{1}{e^{2(n-1)}}\right). \tag{2.8}$$

Let $a(x) = b(x) - 1$. Then

$$\begin{aligned}\int_{\mathbb{R}^N_+} b(x)\left|u_n\right|^p dx &= \int_{\mathbb{R}^N_+} \left|u_n\right|^p dx + \int_{\mathbb{R}^N_+} a(x)\left|u_n\right|^p dx \\ &= \int_{B(\theta,n-1)} |\omega|^p dx + \int_{B(\theta,n-1)} a(x+x_n)|\omega|^p dx \\ &\quad + \int_{B_{n-1,n}} a(x+x_n)\left|\psi_n(x)\cdot\omega(x)\right|^p dx \\ &= \int_{\mathbb{R}^N} |\omega|^p dx + \int_{B(\theta,n-1)} a(x+x_n)|\omega|^p dx \\ &\quad + \int_{B_{n-1,n}} a(x+x_n)\left|\psi_n(x)\cdot\omega(x)\right|^p dx + \int_{\mathbb{R}^N\setminus B(\theta,n-1)} |\omega|^p dx.\end{aligned} \tag{2.9}$$

Thus

$$\begin{gathered}\int_{\mathbb{R}^N\setminus B(\theta,n-1)} |\omega|^p dx = o\left(\frac{1}{e^{2(n-1)}}\right), \\ \int_{B_{n-1,n}} a(x+x_n)\left|\psi_n(x)\cdot\omega(x)\right|^p dx = o\left(\frac{1}{e^{2(n-1)}}\right).\end{gathered} \tag{2.10}$$

From (B_2), we see that there exists a positive constant λ, such that

$$a(x) \geq \frac{\lambda}{e^{(1-\delta)|x|}}. \tag{2.11}$$

So there exists a positive constant C_3, such that

$$\int_{\mathbb{R}^N_+} b(x)\left|u_n\right|^p dx \geq \int_{\mathbb{R}^N} |\omega|^p dx + \frac{C_3}{e^{2(1-\delta)(n-1)}}. \tag{2.12}$$

Then

$$\begin{aligned}&\left(\int_{\mathbb{R}^N_+} b(x)\left|u_n\right|^p dx\right)^{2/p} \\ &\quad \geq \left(\int_{\mathbb{R}^N} |\omega|^p dx + \frac{C_3}{e^{2(1-\delta)(n-1)}}\right)^{2/p} \\ &\quad = \left(\int_{\mathbb{R}^N} |\omega|^p dx\right)^{2/p}\left(1 + \frac{C_4}{e^{2(1-\delta)(n-1)}}\right)^{2/p} \\ &\quad = \left(\int_{\mathbb{R}^N} |\omega|^p dx\right)^{2/p}\left(1 + \frac{C_5}{e^{2(1-\delta)(n-1)}} + o\left(\frac{1}{e^{2(1-\delta)(n-1)}}\right)\right).\end{aligned} \tag{2.13}$$

Thus

$$\frac{\|u_n\|^2}{\left(\int_{\mathbb{R}^N_+} b(x)|u_n|^p\,dx\right)^{2/p}} = \frac{\|\omega\|^2}{\left(\int_{\mathbb{R}^N} \omega^p\,dx\right)^{2/p}} - \frac{C_6}{e^{2(1-\delta)(n-1)}} + o\left(\frac{1}{e^{2(1-\delta)(n-1)}}\right). \quad (2.14)$$

So

$$\frac{\|u_n\|^2}{\left(\int_{\mathbb{R}^N_+} b(x)|u_n|^p\,dx\right)^{2/p}} < \frac{\|\omega\|^2}{\left(\int_{\mathbb{R}^N} \omega^p\,dx\right)^{2/p}}, \quad (2.15)$$

when n is big enough.

Thus, we come to the conclusion that

$$c < \left(\frac{1}{2} - \frac{1}{p}\right) S_p^{p/(p-2)}. \quad (2.16)$$

□

Acknowledgement

This work was supported in part by the National Natural Science Foundation and the Fujian Province Natural Science Foundation of China.

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