

ON THE MODULUS OF u -CONVEXITY OF JI GAO

EVA MARÍA MAZCUÑÁN-NAVARRO

Received 19 November 2001

We consider the modulus of u -convexity of a Banach space introduced by Ji Gao (1996) and we improve a sufficient condition for the fixed-point property (FPP) given by this author. We also give a sufficient condition for normal structure in terms of the modulus of u -convexity.

Let X be a Banach space and let C be a nonempty subset of X . A mapping $T : C \rightarrow C$ is said to be *nonexpansive* whenever

$$\|Tx - Ty\| \leq \|x - y\| \quad (1)$$

for all $x, y \in C$. A Banach space X has the *weak fixed-point property* (WFPP) (resp., *fixed-point property* (FPP)) if for each nonempty weakly compact convex (resp., bounded, closed, and convex) set $C \subset X$ and each nonexpansive mapping $T : C \rightarrow C$, there is an element $x \in C$ such that $T(x) = x$.

It is well known that the WFPP holds for Banach spaces with certain geometrical properties. Among such properties, weak normal structure is, maybe, the most widely studied (see [5, Chapter 3.2]). In order to give sufficient conditions for the WFPP or weak normal structure, different moduli of convexity of Banach spaces have been introduced by several authors (see [5, Chapter 4.5]).

At the origin of these moduli is the classical modulus of convexity introduced by J. A. Clarkson in 1936 to define uniformly convex spaces. It is the function $\delta : [0, 2] \rightarrow [0, 1]$ given by

$$\delta(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in B_X, \|x - y\| \geq \varepsilon \right\}. \quad (2)$$

The related characteristic of convexity is the number

$$\varepsilon_0(X) = \sup \{ \varepsilon > 0 : \delta(\varepsilon) = 0 \}. \quad (3)$$

It is well known (see [5, Theorem 5.12, page 122]) that Banach spaces, with $\varepsilon_0(X) < 1$, are superreflexive and enjoy a uniform normal structure and hence the FPP.

On the other hand, uniformly nonsquare Banach spaces (i.e., Banach spaces with $\varepsilon_0(X) < 2$) are superreflexive [4], but it remains unknown if the FPP holds for these spaces. Nevertheless, there are several partial results which guarantee the FPP for uniformly nonsquare Banach spaces with some additional properties such as the property WORTH, introduced in [6] by Sims as follows: a Banach space has the property WORTH provided that for every weakly null sequence (x_n) in X and any $x \in X$,

$$\liminf_{n \rightarrow \infty} \| \|x_n + x\| - \|x_n - x\| \| = 0. \quad (4)$$

In [7] the same author proved the following theorem.

THEOREM 1 (Sims [7]). *If X is a uniformly nonsquare Banach space with the property WORTH, then X has normal structure.*

In this paper, we concentrate on the modulus of u -convexity introduced in [1] by Gao.

The modulus of u -convexity of a Banach space X is defined by

$$u(\varepsilon) := \inf \left\{ 1 - \frac{1}{2} \|x + y\| : x, y \in S_X, f(x - y) \geq \varepsilon \text{ for some } f \in \nabla_x \right\}, \quad (5)$$

where

$$\nabla_x := \{ f \in X^* : \|f\| = 1, f(x) = \|x\| \}. \quad (6)$$

Gao proved in [1] that $u(\varepsilon) \geq \delta(\varepsilon)$ for any $\varepsilon \in [0, 2]$ and he gave the following results.

THEOREM 2 (Gao [1]). *Let X be a Banach space. If $u(1) > 0$, then X is uniformly nonsquare (hence X is superreflexive).*

THEOREM 3 (Gao [1]). *For any Banach space X , if there exists a $\delta > 0$ such that $u(1/2 - \delta) > 0$, then X has a uniform normal structure, and therefore X has the fixed-point property.*

In order to give a sufficient condition for the FPP, more general than the one required in the above theorem, we need some definitions and lemmas.

Recall that a Banach space X is weakly nearly uniform smooth (WNUS) provided that there exist $\varepsilon \in (0, 1)$ and $\delta > 0$ such that if $0 < t < \delta$ and (x_n) is a basic sequence in B_X , then there exists $k \geq 1$ such that $\|x_1 + tx_k\| \leq 1 + t\varepsilon$.

In [2] García-Falset defined the coefficient

$$R(X) = \sup \left\{ \liminf_{n \rightarrow \infty} \|x_n + x\| \right\}, \quad (7)$$

where the supremum is taken over all weak null sequences (x_n) in B_X and any $x \in B_X$. He gave the following characterization of WNUS in terms of this coefficient: a Banach space X is WNUS if and only if it is reflexive and $R(X) < 2$.

The same author proved in [3] that Banach spaces with $R(X) < 2$ have the WFPP, obtaining as a corollary that WNUS Banach spaces have the FPP.

We obtain an equivalent expression for the coefficient $R(X)$ which will be useful later.

LEMMA 4. *Let X be a Banach space which is not Schur, then*

$$R(X) = \sup \left\{ \liminf_{n \rightarrow \infty} \|x_n + x\| \right\}, \quad (8)$$

where the supremum is taken over all $x \in S_X$ and weak null sequences (x_n) with $\liminf_{n \rightarrow \infty} \|x_n\| = 1$.

Proof. Let

$$\lambda = \sup \left\{ \liminf_{n \rightarrow \infty} \|x_n + x\| : (x_n) \text{ is weakly null, } \liminf_{n \rightarrow \infty} \|x_n\| = 1, x \in S_X \right\}. \quad (9)$$

If $R(X) = 1$, it is evident from the lower semi-continuity of the norm that $R(X) \leq \lambda$. Suppose that $R(X) > 1$. Let ε be an arbitrary scalar in $(0, R(X) - 1)$. By the definition of $R(X)$, we find a weakly null sequence (x_n) in B_X and $x \in B_X$ such that

$$\liminf_{n \rightarrow \infty} \|x_n + x\| > R(X) - \varepsilon. \quad (10)$$

If $x = 0$, then

$$\liminf_{n \rightarrow \infty} \|x_n\| = \liminf_{n \rightarrow \infty} \|x + x_n\| > R(X) - \varepsilon > 1, \quad (11)$$

which is a contradiction, so $x \neq 0$.

We have

$$\begin{aligned}
\|x_n + x\| &= \left\| (1 - \|x\|)x_n + \|x\| \left(x_n + \frac{x}{\|x\|} \right) \right\| \\
&\leq (1 - \|x\|)\|x_n\| + \|x\| \left\| x_n + \frac{x}{\|x\|} \right\| \\
&\leq (1 - \|x\|)(R(X) - \varepsilon) + \|x\| \left\| x_n + \frac{x}{\|x\|} \right\| \\
&< (1 - \|x\|) \liminf_{n \rightarrow \infty} \|x_n + x\| + \|x\| \left\| x_n + \frac{x}{\|x\|} \right\|
\end{aligned} \tag{12}$$

which shows that

$$\liminf_{n \rightarrow \infty} \|x_n + x\| \leq \liminf_{n \rightarrow \infty} \left\| x_n + \frac{x}{\|x\|} \right\|. \tag{13}$$

On the other hand, if $\liminf \|x_n\| = 0$, then

$$1 \leq \|x\| = \liminf_{n \rightarrow \infty} (\|x\| + \|x_n\|) \geq \liminf_{n \rightarrow \infty} \|x + x_n\| > 1, \tag{14}$$

so $\liminf \|x_n\| \neq 0$. Hence, we can write

$$\begin{aligned}
\left\| x_n + \frac{x}{\|x\|} \right\| &= \left\| \left(1 - \liminf_{n \rightarrow \infty} \|x_n\| \right) \frac{x}{\|x\|} + \liminf_{n \rightarrow \infty} \|x_n\| \left(\frac{x_n}{\liminf_{n \rightarrow \infty} \|x_n\|} + \frac{x}{\|x\|} \right) \right\| \\
&\leq \left(1 - \liminf_{n \rightarrow \infty} \|x_n\| \right) + \liminf_{n \rightarrow \infty} \|x_n\| \left\| \frac{x_n}{\liminf_{n \rightarrow \infty} \|x_n\|} + \frac{x}{\|x\|} \right\| \\
&\leq (1 - \liminf_{n \rightarrow \infty} \|x_n\|) \liminf_{n \rightarrow \infty} \left\| x_n + \frac{x}{\|x\|} \right\| \\
&\quad + \liminf_{n \rightarrow \infty} \|x_n\| \left\| \frac{x_n}{\liminf_{n \rightarrow \infty} \|x_n\|} + \frac{x}{\|x\|} \right\|
\end{aligned} \tag{15}$$

and conclude that

$$\liminf_{n \rightarrow \infty} \left\| x_n + \frac{x}{\|x\|} \right\| \leq \liminf_{n \rightarrow \infty} \left\| \frac{x_n}{\liminf_{n \rightarrow \infty} \|x_n\|} + \frac{x}{\|x\|} \right\|. \tag{16}$$

We, finally, obtain from (13) and (16) that

$$R(X) - \varepsilon < \liminf_{n \rightarrow \infty} \|x_n + x\| \leq \liminf_{n \rightarrow \infty} \left\| \frac{x_n}{\liminf_{n \rightarrow \infty} \|x_n\|} + \frac{x}{\|x\|} \right\| \leq \lambda. \quad (17)$$

Since ε was arbitrarily small, we conclude that $R(X) \leq \lambda$.

The inequality $\lambda \leq R(X)$ is clear from the definitions, so the proof is complete. \square

Now we are in a position to prove the following theorem.

THEOREM 5. *Let X be a Banach space. If there exists $\delta > 0$ such that $u(1 - \delta) > 0$, then $R(X) < 2$.*

Proof. Since the function u is clearly increasing, from our hypothesis, we can find $\eta > 0$ such that $u(1 - \eta) > \eta$.

Assume that $R(X) = 2$. From Lemma 4, we can find a weakly null sequence (x_n) such that $\liminf_{n \rightarrow \infty} \|x_n\| = 1$ and $x \in S_X$ satisfying the inequality

$$\liminf_{n \rightarrow \infty} \|x_n + x\| > 2(1 - \eta). \quad (18)$$

Consider $f \in \nabla_x$. We have

$$1 - \eta < 1 = f(x) = \lim_{n \rightarrow \infty} f\left(x - \frac{x_n}{\|x_n\|}\right), \quad (19)$$

so there exists $n_0 \geq 1$ such that for any $n \geq n_0$,

$$f\left(x - \frac{x_n}{\|x_n\|}\right) > 1 - \eta. \quad (20)$$

In consequence, by the definition of the modulus u , we must have

$$\left\| x + \frac{x_n}{\|x_n\|} \right\| \leq 2(1 - u(1 - \eta)) \quad (21)$$

for all $n \geq n_0$, and in consequence,

$$\liminf_{n \rightarrow \infty} \|x + x_n\| \leq 2(1 - u(1 - \eta)) < 2(1 - \eta), \quad (22)$$

which contradicts (18).

Therefore, our assumption is false, that is, $R(X) < 2$, as desired. \square

From Theorems 2, 5, and the characterization of WNUS spaces given in [2], we obtain the following corollary.

COROLLARY 6. *For any Banach space X , if there exists $\delta > 0$ such that $u(1 - \delta) > 0$, then X is WNUS, and in consequence, it has the FPP.*

This corollary provides a sufficient condition for the FPP of a Banach space in terms of its modulus of u -convexity which generalizes the one given in [Theorem 3](#), and—according to [Theorem 2](#)—presents a family of uniformly nonsquare Banach spaces for which the FPP holds.

We do not know if the hypothesis in [Corollary 6](#) implies normal structure, but from [Theorems 1](#) and [2](#) the following corollary is immediate.

COROLLARY 7. *If X is a Banach space with the property WORTH such that $u(1) > 0$, then X has normal structure.*

Acknowledgment

The author was partially supported by Programa Sectorial de Formación de Profesorado Universitario PSFPU AP98, Ministerio Educación y Cultura, Spain.

References

- [1] J. Gao, *Normal structure and modulus of u -convexity in Banach spaces*, Function Spaces, Differential Operators and Nonlinear Analysis (Paseky nad Jizerou, 1995), Prometheus, Prague, 1996, pp. 195–199.
- [2] J. García-Falset, *Stability and fixed points for nonexpansive mappings*, Houston J. Math. **20** (1994), no. 3, 495–506.
- [3] ———, *The fixed point property in Banach spaces with the NUS-property*, J. Math. Anal. Appl. **215** (1997), no. 2, 532–542.
- [4] R. C. James, *Uniformly non-square Banach spaces*, Ann. of Math. (2) **80** (1964), 542–550.
- [5] W. A. Kirk and B. Sims (eds.), *Handbook of Metric Fixed Point Theory*, Kluwer Academic Publishers, Dordrecht, 2001.
- [6] B. Sims, *Orthogonality and fixed points of nonexpansive maps*, Workshop/Miniconference on Functional Analysis and Optimization (Canberra, 1988), Proc. Centre Math. Anal. Austral. Nat. Univ., vol. 20, Austral. Nat. Univ., Canberra, 1988, pp. 178–186.
- [7] ———, *A class of spaces with weak normal structure*, Bull. Austral. Math. Soc. **49** (1994), no. 3, 523–528.

Eva María Mazcuñán-Navarro: Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Valencia, Doctor Moliner 50, 46100 Burjassot, Valencia, Spain
E-mail address: eva.m.mazcunan@uv.es

