PERIODIC SOLUTIONS OF NONLINEAR VIBRATING BEAMS

J. BERKOVITS, H. LEINFELDER, AND V. MUSTONEN

Received 1 October 2002

The aim of this paper is to prove new existence and multiplicity results for periodic semilinear beam equation with a nonlinear time-independent perturbation in case the period is not prescribed. Since the spectrum of the linear part varies with the period, the solvability of the equation depends crucially on the period which can be chosen as a free parameter. Since the period of the external forcing is generally unknown a priori, we consider the following natural problem. For a given time-independent nonlinearity, find periods T for which the equation is solvable for any T-periodic forcing. We will also deal with the existence of multiple solutions when the nonlinearity interacts with the spectrum of the linear part. We show that under certain conditions multiple solutions do exist for any small forcing term with suitable period T. The results are obtained via generalized Leray-Schauder degree and reductions to invariant subspaces.

1. Introduction

We consider beam equation of the form

$$\begin{gathered} \partial_t^2 u + \alpha_0^2 \partial_x^4 u - g(x,u) = h(x,t), \\ u(0,t) = u(L,t) = \partial_x^2 u(0,t) = \partial_x^2 u(L,t) = 0 \quad \left(x \in]0,L[,\ t \in \mathbb{R}\right), \\ u(x,t) = u(x,t+T), \end{gathered} \tag{1.1}$$

where h is the forcing term being T-periodic in t, and $\alpha_0 > 0$ is a constant. The function $g(x,s)$ from $[0,L] \times \mathbb{R}$ to $\mathbb{R}$ is measurable in x for each $s \in \mathbb{R}$ and continuous in s for a.a. $x \in [0,L]$. Moreover, we assume that $g(x,\cdot)$ has utmost linear growth. Essential to our considerations is that g is time independent. Hence we will look for periodic solutions where the period T is determined by the forcing term h alone.

Abstract and Applied Analysis 2003:14 (2003) 823–841
2000 Mathematics Subject Classification: 35L20, 47H11, 35P99
URL: http://dx.doi.org/10.1155/S1085337503301022

With suitable nonlinearity g, (1.1) provides a reasonable model for the road bed of a suspension bridge, where the road bed is being treated as a vibrating beam.

After rescaling, denoting $\omega = 2\pi/T$ and renaming $\alpha_0^2(\pi^4/L^4)$ again by α_0^2, we obtain the equivalent equation

$$\begin{gathered} \omega^2\partial_t^2 u + \alpha_0^2\partial_x^4 u - g(x,u) = h_\omega(x,t), \\ u(0,t) = u(\pi,t) = \partial_x^2 u(0,t) = \partial_x^2 u(\pi,t) = 0 \quad (x \in]0,\pi[,\ t \in \mathbb{R}), \\ u(x,t) = u(x,t+2\pi), \end{gathered} \tag{1.2}$$

where $h_\omega(x,t) = h(x,\omega^{-1}t)$. Note that with these notations the case $\omega = \alpha_0 = 1$ corresponds to the standard situation with period $T = 2\pi$ being widely studied in the literature.

We will study the existence of *weak solutions* of (1.2), that is, solutions of the operator equation

$$L_\omega u - N(u) = h_\omega, \quad u \in D(L_\omega) \tag{1.3}$$

in $H = L_2(\Omega;\mathbb{R})$ for $\Omega =]0,\pi[\times]0,2\pi[$, where N is the Nemytskii operator generated by g, and $L_\omega : D(L_\omega) \subset H \to H$ is the abstract realization of the beam operator. We will apply the extension of the Leray-Schauder degree introduced by Berkovits and Mustonen [3] for a class of mappings related to our model problem. Basically, homotopy arguments are used to obtain existence results for (1.3). Moreover, for certain nonlinearities, we can apply the Banach fixed-point theorem to obtain unique solutions.

By using suitable reductions to invariant subspaces, we find solutions for (1.3) provided that N and h satisfy some auxiliary symmetry conditions. Indeed, if the beam operator L_ω is reduced by a closed linear subspace V and $N(V) \subset V$, any solution of the reduced equation

$$L_\omega\big|_V u - N\big|_V(u) = h_\omega, \quad u \in D(L_\omega) \cap V,\ h_\omega \in V, \tag{1.4}$$

is also a solution for the original operator equation (1.3). In case of the wave equation, the method of reduction to suitable subspaces was already used by Vejvoda et al. [12] and Coron [7] (see also [4, 5]). The same idea was employed earlier in the study of periodic solutions for ordinary differential equations (see, e.g., [9]). The reader may observe that the reductions yield restrictions for the nonlinear operator N, leaving the treatment of the problem with general N open. From a purely abstract point of view, we may replace L_ω by $-L_\omega$ in our considerations. Hence, our results have a counterpart now concerning the equation

$$L_\omega u + N(u) = h_\omega, \quad u \in D(L_\omega). \tag{1.5}$$

Note however that due to the asymmetry of the spectrum of the beam operator, the conditions are different for the solvability for (1.3) and (1.5), respectively.

Semilinear wave and beam equations with fixed period are widely studied in the literature. We would like to mention here [1, 6, 8, 10, 11] and the references therein.

The paper is organized as follows. In Section 2, we define the operator equation and show how the spectrum of the linear part depends on the period $T = 2\pi/\omega$ (see Lemma 2.1). Some appropriate subspaces for the reduction method are collected in Lemmas 2.2 and 2.3. Section 3 is devoted to nonresonance. We obtain sufficient conditions for the time-independent nonlinearity $g = g(x,u)$ and the period T of the forcing term such that (1.1) admits a weak solution for any T-periodic forcing term h. Finally, in Section 4, we are looking for the existence of multiple solutions. We show that under certain conditions on the interaction between the nonlinearity and the spectrum of the beam operator, multiple solutions do exist for any small forcing term h with suitable period T. The methods and the results are illuminated by several examples.

2. Prerequisites

We recall first the basic properties of the linear operators involved. Denote $H = L_2(\Omega;\mathbb{R})$, $H_{\mathbb{C}} = H + iH = L_2(\Omega;\mathbb{C})$, and $\phi_{j,k}(x,t) = (1/\pi)\sin(jx)\exp(ikt)$, where $(x,t) \in \Omega$, $j \in \mathbb{Z}_+$, and $k \in \mathbb{Z}$. The set $\{\phi_{j,k}\}$ forms an orthonormal basis in $H_{\mathbb{C}}$. We will use the notations $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ for the inner product and norm in any real Hilbert space and the subscript "$\mathbb{C}$" whenever the product is complex. The beam operator $\omega^2\partial_t^2 + \alpha_0^2\partial_x^4$ with periodic Dirichlet boundary conditions has in H the abstract realization

$$L_\omega u = \sum_{j,k} \lambda_{j,k}^{\omega} \langle u, \phi_{j,k}\rangle_{\mathbb{C}} \phi_{j,k}, \tag{2.1}$$

with $\lambda_{j,k}^{\omega} = \alpha_0^2 j^4 - \omega^2 k^2$, $j \in \mathbb{Z}_+$, $k \in \mathbb{Z}$ and

$$D(L_\omega) = \left\{ u \in L_2(\Omega) \mid \sum_{j,k} \left|\lambda_{j,k}^{\omega}\right|^2 \left|\langle u, \phi_{j,k}\rangle_{\mathbb{C}}\right|^2 < \infty \right\}. \tag{2.2}$$

Clearly, each $\lambda_{j,k}^{\omega}$ is an eigenvalue of L_ω with corresponding eigenvector $\phi_{j,k}$. We will always assume that $\omega \in \alpha_0\mathbb{Q}$, that is, α_0/ω is rational. Otherwise, we encounter the hard problem of "small divisors," see [5], for instance. Clearly, L_ω is selfadjoint and has a compact partial inverse on $\operatorname{Im} L_\omega$. For more details on abstract operators like beam operators, we refer to [2].

We consider more closely the spectrum $\sigma(L_\omega)$ of the operator L_ω. For any $\omega \in \alpha_0\mathbb{Q}_+$, it is easy to see that $\operatorname{Ker} L_\omega$ is infinite dimensional and L_ω has pure point spectrum $\sigma(L_\omega) = \{\lambda_{j,k}^{\omega} \mid j \in \mathbb{Z}_+,\ k \in \mathbb{Z}\}$. The eigenvalues are isolated and all nonzero eigenvalues have finite geometric multiplicity. Note that the spectrum is unbounded from below and from above.

For the readers convenience, we recall here some basic facts about the topological degree theory introduced in [3]. Let H be a real separable Hilbert space and $L : D(L) \subset H \to H$ a linear densely defined normal operator with $\operatorname{Im} L = (\operatorname{Ker} L)^{\perp}$. The inverse K of the restriction of L to $\operatorname{Im} L \cap D(L)$ is a bounded linear operator on $\operatorname{Im} L$. We will further assume that the inverse K is compact. Denote by P and $Q = I - P$ the orthogonal projections to $\operatorname{Ker} L$ and $\operatorname{Im} L = (\operatorname{Ker} L)^{\perp}$, respectively. For any map $N : H \to H$, the equation

$$Lu \pm N(u) = 0, \quad u \in D(L), \tag{2.3}$$

can be written equivalently as

$$Q(u \pm KQN(u)) + PN(u) = 0, \quad u \in H. \tag{2.4}$$

Above we have used the fact that $KQ \pm P$ is the right inverse of $L \pm P$. If N is bounded, demicontinuous, and of class (S_+), then there exists a topological degree for mappings of the form $F = Q(I + C) + PN$, where C is compact (see [3]). We recall that N is *of class* (S_+) if for any sequence with $u_j \rightharpoonup u$, $\limsup \langle N(u_j), u_j - u \rangle \le 0$, it follows that $u_j \to u$.

The degree theory in [3] is a unique extension of the classical Leray-Schauder degree. It is single valued and has the usual properties of degree, such as additivity of domains and invariance under homotopies. We denote the corresponding degree function by d_H. In order to simplify our notations, we define a further degree function "deg" by setting

$$\deg(L \pm N, G, 0) \equiv d_H(Q(I \pm KQN) + PN, G, 0) \tag{2.5}$$

for any open set $G \subset H$ such that $0 \notin (L \pm N)(\partial G \cap D(L))$. In the sequel, the term "admissible map" refers to any map for which the degree is well defined. Similarly, we use the term "admissible homotopy." We will employ the fact that

$$\deg(L - N, G, 0) \neq 0 \tag{2.6}$$

for any linear admissible injection such that $0 \in (L - N)(G)$ (see [3]). Note that if N is strongly monotone, then it is of class (S_+). If N is only monotone, then it is pseudomonotone and we can replace N by $N + \epsilon I$, $\epsilon > 0$, which is of class (S_+), and then let $\epsilon \to 0$.

We start with some results on the distribution of the spectrum of L_ω in compact intervals depending on the parameter ω.

Lemma 2.1. *Let $[a, b]$ be a given compact interval in $\mathbb{R}$. Then*

(i) *there exist arbitrarily small and arbitrarily large values of $\omega \in \alpha_0\mathbb{Q}_+$ such that $[a, b] \cap \sigma(L_\omega) \neq \emptyset$;*

(ii) *there exists a limit value $\tilde{\omega} > 0$ such that $[a, b] \cap \sigma(L_\omega) \neq \emptyset$ for all $\omega \le \tilde{\omega}$ satisfying $\omega \in \alpha_0\mathbb{Q}_+$;*

(iii) *if $\omega \in \alpha_0\mathbb{Z}_+$ satisfies $\alpha_0\omega \ge \max\{|a|, |b|\}$, then*

$$[a, b] \cap (\sigma(L_\omega) \setminus \{0\}) = \{\alpha_0^2 j^4 \mid a \le \alpha_0^2 j^4 \le b,\ j \in \mathbb{Z}_+\}. \tag{2.7}$$

Proof. Assume that $k > 0$. Denote $c_j^2 = j^4 - b/\alpha_0^2$ and $d_j^2 = j^4 - a/\alpha_0^2$, where $j \geq j_0$ and $j_0 = \min\{j \in \mathbb{Z}_+ \mid j^4 > b/\alpha_0^2\}$. Then $a \leq \lambda_{j,k}^{\omega} \leq b$ if and only if $k\omega/\alpha_0 \in [c_j, d_j]$ for any $j \geq j_0$. Take some $r_j \in [c_j, d_j] \cap \mathbb{Q}$ and denote $\omega_{j,k} = \alpha_0 r_j/k$. Then $a \leq \lambda_{j,k}^{\omega_{j,k}} \leq b$ for all $j \geq j_0$. By taking $k = 1$ and $k = j^3$, respectively, one can see that there are arbitrarily small and arbitrarily large values of $\omega \in \alpha_0\mathbb{Q}_+$ such that $[a,b] \cap \sigma(L_\omega) \neq \varnothing$. If $\tilde{\omega} \leq \alpha_0(d_{j_0} - c_{j_0})$, then, it is easy to see that for any $\omega \leq \tilde{\omega}$, $k\omega/\alpha_0 \in [c_{j_0}, d_{j_0}]$ for some $k \in \mathbb{Z}_+$. Due to the special status of the eigenvalues $\lambda = 0$ and $\lambda_{j,0}^{\omega}$, $j \in \mathbb{Z}_+$, it is sufficient to consider eigenvalues $\lambda_{j,k}^{\omega}$, $j \in \mathbb{Z}_+$, $k \in \mathbb{Z}_+$, to prove (iii). Indeed, if $\omega_q = \alpha_0 q$, where $q \in \mathbb{Z}_+$, then

$$|\lambda_{j,k}^{\omega_q}| = \alpha_0^2 |j^2 - qk| (j^2 + qk) \geq |j^2 - qk| \alpha_0^2 (1+q). \tag{2.8}$$

Hence, if $\lambda_{j,k}^{\omega_q} \neq 0$, then $|\lambda_{j,k}^{\omega_q}| > \max\{|a|, |b|\}$ and the conclusion follows. □

By Lemma 2.1, one can see that the eigenvalues $\lambda = 0$ and $\lambda = \lambda_{j,0}^{\omega}$, $j \in \mathbb{Z}_+$, play a special role. In order to deal with these exceptional eigenvalues, we will look for solutions of (1.3) in suitable invariant subspaces in the sequel. If the operator L_ω is reduced by a closed subspace $V \subset H$, then $\sigma(L_\omega) = \sigma(L_\omega|_V) \cup \sigma(L_\omega|_{V^\perp})$, implying that the spectrum of L_ω in V is thinner than the spectrum of L_ω in H. The main problem is to find natural conditions ensuring $N(V) \subset V$. For a given continuous function $g : \mathbb{R} \to \mathbb{R}$, appropriate subspaces are provided by the following lemmas. Therein, the function g has to satisfy the condition

$$|g(s)| \leq c_1 + c_2|s| \quad (s \in \mathbb{R}) \tag{2.9}$$

with suitable nonnegative constants c_1 and c_2. The corresponding nonlinear operator N is then given by

$$N(u)(x,t) = g(u(x,t)) \quad ((x,t) \in \Omega,\ u \in H). \tag{2.10}$$

Lemma 2.2. *Assume that g satisfying (2.9) is odd, and that for $r \in \mathbb{Z}_+$, $r \geq 2$, the subspace V is given by*

$$\begin{aligned} V = V_r := \Big\{ & u \in H \mid u\Big(x + \frac{2\pi}{r}, t\Big) = u(x,t) \text{ for a.a. } t \in]0, 2\pi[,\ x \in \Big]0, \pi - \frac{2\pi}{r}\Big[, \\ & u\Big(\frac{2\pi}{r} - x, t\Big) = -u(x,t) \text{ for a.a. } t \in]0, 2\pi[,\ x \in \Big]0, \frac{2\pi}{r}\Big[\Big\} \\ = {} & \overline{\mathrm{sp}_{\mathbb{C}}}\{\phi_{j,k} \mid j \in r\mathbb{Z}_+,\ k \in \mathbb{Z}\} \cap H. \end{aligned} \tag{2.11}$$

Then $N(V) \subset V$ and the operator L_ω is reduced by V.

LEMMA 2.3. *Assume that g satisfies (2.9), $k_0 \in \mathbb{Z}_+$, and the subspace V is defined by one of the following cases:*

$$\begin{aligned}
V = W &:= \{u \in H \mid u(\pi - x, t) = u(x,t) \text{ for a.a. } x \in]0,\pi[,\ t \in]0,2\pi[\} \\
&= \overline{\mathrm{sp}_{\mathbb{C}}}\{\phi_{j,k} \mid j \in \mathbb{Z}_+,\ j \text{ is odd},\ k \in \mathbb{Z}\} \cap H, \\
V = Z_{k_0} &:= \left\{u \in H \mid u\left(\pi - x, t + \frac{\pi}{k_0}\right) = u(x,t) \text{ for a.a. } x \in]0,\pi[,\ t \in \left]0, 2\pi - \frac{\pi}{k_0}\right[\right\} \\
&= \overline{\mathrm{sp}_{\mathbb{C}}}\{\phi_{j,k} \mid j \in \mathbb{Z}_+,\ k = k_0 l,\ l \in \mathbb{Z},\ j + l \text{ is odd}\} \cap H, \quad k_0 \text{ is odd}, \\
V = E_{k_0} &:= \left\{u \in H \mid u\left(x, t + \frac{2\pi}{k_0}\right) = u(x,t) \text{ for a.a. } x \in]0,\pi[, t \in \left]0, 2\pi - \frac{2\pi}{k_0}\right[\right\} \\
&= \overline{\mathrm{sp}_{\mathbb{C}}}\{\phi_{j,k} \mid j \in \mathbb{Z}_+,\ k \in k_0\mathbb{Z}\} \cap H.
\end{aligned} \tag{2.12}$$

Then $N(V) \subset V$ and the operator L_ω is reduced by V.

Proof. To prove the lemmas above, one only has to show the validity of the claimed characterizations of the subspaces V_r, W, Z_{k_0}, and E_{k_0}. We start with the subspace $V = V_r$. Obviously it suffices to show that $\langle u, \phi_{j,k}\rangle = 0$ if $u \in V_r$ and $j \notin r\mathbb{Z}_+$. But this is clear since for fixed $t \in [0, 2\pi]$, the Fourier coefficients $1/\sqrt{\pi}\langle \hat{u}(\cdot, t), \sin j(\cdot)\rangle$ vanish if $j \notin r\mathbb{Z}_+$ because the odd 2π-periodic extension $\hat{u}(\cdot, t)$ of $u(\cdot, t)$ is actually $2\pi/r$-periodic. By similar arguments, one can show the characterizations of the spaces W and E_{k_0} above. In case $V = W$, one has to remember that, for $u \in W$ and for fixed $t \in [0, 2\pi]$, the function $\hat{u}(\cdot, t)$ is even with respect to $\pi/2$ and in case $V = E_{k_0}$, one argues that, for $u \in E_{k_0}$ and for fixed $x \in [0, \pi]$, the 2π-periodic extension $\hat{u}(\cdot, t)$ of $u(\cdot, t)$ is $2\pi/k_0$-periodic in x. The characterization of the subspace Z_{k_0} cannot be obtained in a similar way since now the underlying symmetries concern both variables x and t. Since only the case $k_0 = 1$ is actually used in this paper (see Theorem 3.6), we will give a proof just for this case (being really the crucial one). To do so, let $Q_1 = (0, \pi)^2$ and $Q_2 = (0, \pi) \times (\pi, 2\pi)$ so that $Q = Q_1 \cup Q_2$ and let $S(x, t) = (\pi - x, t + \pi)$ if $(x, t) \in Q_1$. We like to show that $\langle u, \phi_{j,k}\rangle = 0$ if $u \in Z_1$ and $j + k$ even. Clearly,

$$\langle u, \phi_{j,k}\rangle = \langle u, \phi_{j,k}\rangle_{L^2(Q_1)} + \langle u, \phi_{j,k}\rangle_{L^2(Q_2)}. \tag{2.13}$$

Since S, and thus also S^{-1}, is measure preserving and $u \circ S = u|_{Q_1}$ for $u \in Z_1$ as well as $\phi_{j,k} \circ S = (-1)^{j+k+1}\phi_{j,k}|_{Q_1}$, we get

$$\langle u, \phi_{j,k}\rangle_{L^2(Q_2)} = \langle u \circ S, \phi_{j,k} \circ S\rangle_{L^2(Q_1)} = (-1)^{j+k+1}\langle u, \phi_{j,k}\rangle_{L^2(Q_1)}. \tag{2.14}$$

This yields $\langle u, \phi_{j,k}\rangle = (1 + (-1)^{j+k+1})\langle u, \phi_{j,k}\rangle_{L^2(Q_1)} = 0$ if $j + k$ is even. □

We have formulated the definitions of the subspaces in Lemmas 2.2 and 2.3 in such a way that the necessity of the oddness assumption on g is clear whenever needed. This fact might appear not so obvious if the definitions were made by using odd 2π-periodic extensions in x, which is the easiest way. For instance, the space V_r in Lemma 2.2 consists of all functions u such that its odd 2π-periodic extension in x is also $2\pi/r$-periodic. Note also that in case $V = E_{k_0}$, we can allow g to be x-dependent.

3. Nonresonance: existence and surjectivity results

By nonresonance we generally refer to the surjectivity of the corresponding mapping. However, if we restrict the operator to some invariant subspace, the nonresonance in that space implies that the subspace is included in the range of the mapping. Hence, there is a minor ambiguity in the use of the term "nonresonance." As before, we assume that the function $g(x,s)$ from $[0,\pi]\times\mathbb{R}$ to $\mathbb{R}$ is measurable in x for each $s\in\mathbb{R}$ and continuous in s for a.a. $x\in[0,\pi]$. Moreover, we assume that $g(x,\cdot)$ has utmost linear growth, that is,

$$\left|g(x,s)\right| \leq c_0|s| + k_0(x) \tag{3.1}$$

for all $s\in\mathbb{R}$, a.a. $(x,t)\in\Omega$, and with some constant $c_0\geq 0$ and $k_0\in L_2(]0,\pi[)$. We denote by N the Nemytskii operator given by

$$N(u)(x,t) = g\big(x,u(x,t)\big) \quad \big((x,t)\in\Omega,\ u\in H\big). \tag{3.2}$$

Our first result is based on the standard use of the Banach fixed-point theorem.

Theorem 3.1. *Assume that $0\notin[a,b]$, $\{j\in\mathbb{Z}_+ \mid a\leq\alpha_0^2j^4\leq b\}=\emptyset$, and $g = g(x,s)$ satisfies*

$$a \leq \frac{g(x,s)-g(x,\hat{s})}{s-\hat{s}} \leq b \quad \forall s\neq\hat{s}. \tag{3.3}$$

Then, for all $\omega\in\alpha_0\mathbb{Z}_+$ such that $\alpha_0\omega\geq\max\{|a|,|b|\}$, the equation

$$L_\omega u - N(u) = h_\omega, \quad u\in D(L_\omega), \tag{3.4}$$

admits a unique solution for any $2\pi/\omega$-periodic forcing h term such that $h_\omega\in H$.

Proof. Let $\omega\in\alpha_0\mathbb{Z}_+$ such that $\alpha_0\omega\geq\max\{|a|,|b|\}$. Then, in view of Lemma 2.1(iii), we know that $[a,b]\cap\sigma(L_\omega)=\emptyset$. First we notice that u is a solution of (3.4) if and only if

$$u = G(u) := \left(L_\omega - c_0I\right)^{-1}\left(N(u)-c_0u\right), \tag{3.5}$$

where c_0 is the midpoint of the interval $[a,b]$. Due to our assumptions on $g(x,s)$,

$$\left|g(x,s)-g(x,\hat{s})-c_0(s-\hat{s})\right| \leq \frac{b-a}{2}|s-\hat{s}| \quad (x\in\,]0,\pi[,\ s,\hat{s}\in\mathbb{R}), \tag{3.6}$$

which yields

$$||N(u)-N(v)-c_0(u-v)|| \leq \frac{b-a}{2}\|u-v\| \quad (u,v \in H), \tag{3.7}$$

and thus

$$||G(u)-G(v)|| \leq \left|\left|(L_\omega - c_0 I)^{-1}\right|\right| \frac{(b-a)}{2}\|u-v\| \quad (u,v \in H). \tag{3.8}$$

Since $\|(L_\omega - c_0 I)^{-1}\| = \operatorname{dist}(c_0, \sigma(L_\omega))^{-1}$ and $\operatorname{dist}(c_0, \sigma(L_\omega)) > (b-a)/2$ in view of $[a,b] \cap \sigma(L_\omega) = \emptyset$, the operator G turns out to be a contraction. Hence, the proof is complete due to the Banach fixed-point theorem. □

We prove the following more general result using the degree theory constructed in [3].

Theorem 3.2. *Assume that* $0 \notin [a,b]$ *and* $\{j \in \mathbb{Z}_+ \mid a \leq \alpha_0^2 j^4 \leq b\} = \emptyset$. *Suppose* $g(x,\cdot)$ *is nondecreasing and satisfies*

$$a \leq \frac{g(x,s)}{s} \leq b \quad \forall s \neq 0. \tag{3.9}$$

Then, for all $\omega \in \alpha_0 \mathbb{Z}_+$ *such that* $\alpha_0 \omega > \max\{|a|,|b|\}$, *(3.4) admits at least one solution for any* $2\pi/\omega$*-periodic forcing* h *term such that* $h_\omega \in H$. *In the special case* $h = 0$, *(3.4) has the unique trivial solution* $u = 0$.

Proof. Note that $b > a > 0$ by the assumptions of the theorem. Assume first that N is strongly monotone. By Lemma 2.1(iii) and due to our assumptions, we can take $\omega \in \alpha_0 \mathbb{Q}_+$ such that $[a,b] \cap \sigma(L_\omega) = \emptyset$. Let h be a $2\pi/\omega$-periodic forcing term such that $h_\omega \in H$ and consider the admissible homotopy equation

$$L_\omega u - c_0 u = \mu(N(u) - c_0 u - h_\omega), \quad u \in D(L_\omega),\ 0 \leq \mu \leq 1, \tag{3.10}$$

where c_0 is the midpoint of the interval $[a,b]$. Then it is easy to see that

$$||N(u) - c_0 u|| \leq \frac{b-a}{2}\|u\|, \quad u \in H, \tag{3.11}$$

and consequently, for any solution u of the homotopy equation

$$\|u\| \leq \left|\left|(L_\omega - c_0 I)^{-1}\right|\right| \frac{(b-a)}{2}(\|u\| + ||h_\omega||). \tag{3.12}$$

Since $\|(L_\omega - c_0 I)^{-1}\| = \operatorname{dist}(c_0, \sigma(L_\omega))^{-1}$ and $\operatorname{dist}(c_0, \sigma(L_\omega)) > (b-a)/2$, the solution set of the homotopy equation remains bounded (in case $h = 0$ we have 0).

Hence, there exists $R > 0$ such that

$$\deg\left(L_\omega - N, B_R(0), h_\omega\right) = \deg\left(L_\omega - c_0 I, B_R(0), 0\right) \neq 0. \tag{3.13}$$

Consequently, a solution exists and the first part of the proof is complete.

If g is nondecreasing and hence N not necessarily strongly monotone, we follow the standard procedure, that is, replace N by $N + \epsilon I$ in the above proof and then let $\epsilon \to 0$. Indeed, it is easy to see that there exists $\epsilon_0 > 0$ such that the assumptions of the theorem remain valid for all $0 < \epsilon < \epsilon_0$ if we replace a and b by $a + \epsilon$ and $b + \epsilon$, respectively. Consequently,

$$L_\omega u - \left(N(u) + \epsilon u\right) = h_\omega, \quad u \in D(L_\omega), \tag{3.14}$$

admits a solution u_ϵ and it is not hard to see that $\|u_\epsilon\| < R$ for all $0 < \epsilon < \epsilon_0$ with R independent of ϵ. Take any sequence (ϵ_j) such that $\epsilon_j \to 0+$ and denote $v_j = u_{\epsilon_j}$. At least for a subsequence, we can assume that $v_j \rightharpoonup v \in \overline{B_R(0)}$. By the compactness of the partial inverse of L_ω, we get

$$\lim \langle N(v_j), v_j - v\rangle = 0, \tag{3.15}$$

and since N is monotone, hence pseudomonotone, we have $N(v_j) \rightharpoonup N(v)$. Thus $L_\omega v - N(v) = h_\omega$ completing the proof. □

Remark 3.3. Due to the asymmetry of the spectrum, the equation

$$L_\omega u + N(u) = h_\omega, \quad u \in D(L_\omega), \tag{3.16}$$

where $g(x,\cdot)$ is nondecreasing and satisfies (3.9) with $0 \notin [a,b]$ admitting at least one solution for any $2\pi/\omega$-periodic forcing h term such that $h_\omega \in H$, provided $\omega \in \alpha_0\mathbb{Z}_+$ is such that $\alpha_0\omega \geq \max\{|a|,|b|\}$. Indeed, if we replace L_ω by $-L_\omega$, the condition $\{j \in \mathbb{Z}_+ \mid a \leq -\alpha_0^2 j^4 \leq b\} = \varnothing$ is trivially satisfied.

Example 3.4. Assume that $\alpha_0 = 1$ and define

$$g(x,s) = a(x)s + d(x)\left((s+1)^{1/3} - 1\right), \quad s \in \mathbb{R}, \tag{3.17}$$

where $a, d \in L_\infty(]0,\pi[)$, $a(x) \geq a > 1$, $d(x) \geq 0$, and $\|a\|_\infty + (4/3)\|d\|_\infty < 16$. It is easy to see that N is monotone and

$$a \leq \frac{g(x,s)}{s} \leq b \quad \forall s \neq 0, \tag{3.18}$$

with $b = \|a\|_\infty + (4/3)\|d\|_\infty$. By Theorem 3.2, (3.4) admits a solution for any $2\pi/\omega$-periodic forcing h term such that $h_\omega \in H$, provided that $\omega \in \mathbb{Z}_+$ and

$\omega \geq b = \max\{|a|, |b|\}$. Note that

$$\sup_{s \neq \hat{s}} \frac{g(x,s) - g(x,\hat{s})}{s - \hat{s}} = \infty \tag{3.19}$$

and thus Theorem 3.1 does not apply.

We would like to mention a slightly more abstract version of Theorems 3.1 and 3.2 which will be used later on.

Proposition 3.5. *Suppose A is a selfadjoint operator in a real separable Hilbert space V. Assume that the spectrum of A consists only of isolated eigenvalues λ and that their multiplicities are finite for all $\lambda \neq 0$. Suppose $\sigma(A) \cap [a,b] = \emptyset$ and the nonlinear operator $N : V \to V$ is demicontinuous.*

(1) If N is monotone and satisfies condition (3.11), then the equation

$$Au - N(u) = h \tag{3.20}$$

admits at least one solution $u \in D(A)$ for any right-hand side $h \in V$. Moreover, if the multiplicity of $\lambda = 0$ is finite, then no monotonicity of N is needed.

(2) If N satisfies condition (3.7), then (3.20) admits a unique solution u for any right-hand side $h \in V$.

(3) If N satisfies condition (3.11), then the homogeneous equation

$$Au - N(u) = 0 \tag{3.21}$$

has only the trivial solution $u = 0$.

Proof. Proposition 3.5 follows immediately by inspecting and adapting the proofs of Theorems 3.1 and 3.2. □

We note that if N is given by $g = g(x,s)$, condition (3.9) implies (3.11), and (3.7) follows directly from condition (3.3).

The existence results in Theorems 3.1 and 3.2 were obtained in the case where $\sigma(L_\omega) \cap [a,b] = \emptyset$. If, on the other hand, $\sigma(L_\omega) \cap [a,b] \neq \emptyset$, then in view of Proposition 3.5 and Lemmas 2.2 and 2.3, we can still look for possible invariant subspaces V of H to eliminate that part of the spectrum of L_ω belonging to $[a,b]$. Hence, in some cases, we can find an invariant subspace such that $\sigma(L_\omega|_V) \cap [a,b] = \emptyset$. In the following two theorems, we will deal separately with the cases where $0 \in [a,b]$ or $\{j \in \mathbb{Z}_+ \mid a \leq \alpha_0^2 j^4 \leq b\} \neq \emptyset$.

Theorem 3.6. *Assume $0 \in [a,b]$ and $\{j \in \mathbb{Z}_+ \mid a \leq \alpha_0^2 j^4 \leq b\} = \emptyset$. Choose $\omega \in \alpha_0 \mathbb{Q}_+$ such that $\sigma(L_\omega) \cap [a,b] = \{0\}$ and write $\alpha_0 \omega^{-1} = p/q$ with p and q relatively prime. Suppose g satisfies*

$$a \leq \frac{g(s)}{s} \leq b \quad \forall s \neq 0. \tag{3.22}$$

Then, for any $2\pi/\omega$-periodic forcing term h such that $h_\omega \in W \cap E_{2^{n+1}}$ (see Lemma 2.3), the equation

$$\begin{gathered} \partial_t^2 u + \alpha_0^2 \partial_x^4 u - g(u) = h(x,t), \\ u(0,t) = u(\pi,t) = \partial_x^2 u(0,t) = \partial_x^2 u(\pi,t) = 0 \quad (x \in]0,\pi[,\ t \in \mathbb{R}), \\ u(x,t) = u(x,t+T) \quad \text{with } T = 2\pi\omega^{-1}, \end{gathered} \tag{3.23}$$

admits a weak solution u with $u_\omega \in W \cap E_{2^{n+1}}$. Here n is the power of the integer 2 in the decomposition of p into prime numbers.

Moreover, if both p and q are odd and if h is such that $h_\omega \in Z_1$, then (3.23) admits a weak solution u with $u_\omega \in Z_1$.

Proof. According to Proposition 3.5 and Lemmas 2.2 and 2.3, it suffices to find a reducing subspace V such that $0 \notin \sigma(L_\omega|_V)$. We will take $V = W \cap E_{2^{n+1}}$ and show that the eigenvalue $\lambda = 0$ of L_ω is "dropped" by reducing L_ω to the subspace V. Indeed, using $\alpha_0\omega^{-1} = p/q$, the identity

$$0 = \lambda = \alpha_0^2 j^4 - \omega^2 k^2 = \frac{\alpha_0^2}{p^2}(pj^2 - qk)(pj^2 + qk) \quad (j \in \mathbb{Z}_+,\ k \in \mathbb{Z}) \tag{3.24}$$

holds true if and only if

$$pj^2 = qk \quad \text{with } j,k \in \mathbb{Z}_+. \tag{3.25}$$

If we assume that $\lambda = 0$ is an eigenvalue of $L_\omega|_V$, then the integer j would be odd and $k \in 2^{n+1}\mathbb{Z}_+$ in identity (3.25). Moreover, in view of (3.25), the integer 2^{n+1} would divide pj^2. Hence, due to the definition of n, the prime number 2 would divide j contradicting the oddness of j. If, moreover, both p and q are odd, then we conclude from identity (3.25) that j and k are either both odd or both even. In any case, $j+k$ turns out to be even. It is now clear that one only needs to take $V = Z_1$ from Lemma 2.3 to see that 0 cannot be an eigenvalue of $L_\omega|_V$. □

THEOREM 3.7. *Assume that $0 \notin [a,b]$ and $J = \{j \in \mathbb{Z}_+ \mid a \le \alpha_0^2 j^4 \le b\} \neq \emptyset$ and denote $j_m = \max\{j \mid j \in J\}$. Suppose g is odd, nondecreasing, and satisfying*

$$a \le \frac{g(s)}{s} \le b \quad \forall s \neq 0. \tag{3.26}$$

Let $\omega \in \alpha_0\mathbb{Q}_+$ be such that $\sigma(L_\omega) \cap [a,b] = \{\alpha_0^2 j^4 \mid j \in J\}$.

Then, given $r > j_m$, (3.23) admits a weak solution u satisfying $u_\omega \in V_r$ for any $2\pi/\omega$-periodic forcing term h such that $h_\omega \in V_r$.

Proof. To "drop" the eigenvalues $\lambda_{j,0}^\omega = \alpha_0^2 j^4$ from the spectrum of L_ω, we choose, according to Lemma 2.2, the reducing subspace $V = V_r$ with $r > j_m$. Then $\sigma(L_\omega|_{V_r}) \cap [a,b] = \emptyset$ and Proposition 3.5 applies. □

We close this section by an example related to the Fučík spectrum of the beam operator.

Example 3.8. Consider the equation

$$L_\omega u = \nu u^+ - \mu u^- + h_\omega, \quad u \in D(L_\omega). \tag{3.27}$$

It is well known that the complete structure of the Fučík spectrum of the beam operator is still unknown. We will deal only with some special choices of the constants ν and μ in (3.27). For the following, it is convenient to introduce the nonlinear function

$$g(s) = \nu s^+ - \mu s^-, \tag{3.28}$$

which satisfies the estimate

$$\min(\nu,\mu) \le \frac{g(s) - g(\hat{s})}{s - \hat{s}} \le \max(\nu,\mu) \quad \forall s \ne \hat{s}. \tag{3.29}$$

(a) Assume first that $\mu > \nu > 0$. If $[\nu,\mu] \cap \{\alpha_0^2 j^4 \mid j \in \mathbb{Z}_+\} = \emptyset$. Then, for any $\omega \in \alpha_0\mathbb{Z}_+$ such that $\alpha_0\omega \ge \mu = \max(|\nu|,|\mu|)$, we have $\sigma(L_\omega) \cap [\nu,\mu] = \emptyset$. Consequently, (ν,μ) does not belong to the Fučík spectrum of L_ω and, by Theorem 3.1, (3.27) admits a unique solution for any $2\pi/\omega$-periodic forcing term h such that $h_\omega \in H$. Similar result holds if $\nu > \mu > 0$.

(b) Assume that $\mu < \nu < 0$. Then, for all $\omega \in \alpha_0\mathbb{Z}_+$ such that $\alpha_0\omega \ge |\mu|$, we have $\sigma(-L_\omega) \cap [\mu,\nu] = \emptyset$. Hence, (ν,μ) is not in the Fučík spectrum of L_ω and (3.27) admits a unique solution for any $2\pi/\omega$-periodic function h such that $h_\omega \in H$. Similar result holds if $\nu < \mu < 0$.

(c) Assume that $\mu = 0$ and $-\nu = b > 0$. Then (3.27) reduces to

$$L_\omega u + b u^+ = h_\omega, \quad u \in D(L_\omega), \tag{3.30}$$

with $b > 0$, being a good one-dimensional model for the main span of a suspension bridge, where the supporting cable stays are assumed not to exert restoring forces whenever compressed. The constant b is determined by Hooke's Law and it represents the stiffness of the cables. We consider the problem in space Z_1. It is easy to see that there exists $\omega \in \alpha_0\mathbb{Z}_+$ such that $\operatorname{Ker}(L_\omega|_{Z_1}) = \{0\}$ and $\sigma(-L_\omega|_{Z_1}) \cap [0,b] = \emptyset$ (any $\omega = \alpha_0 q, \alpha_0\omega \ge b, q$ odd will do). Hence, (3.30) admits a unique solution for any $2\pi/\omega$-periodic forcing term h such that $h_\omega \in Z_1$.

4. Interaction with the spectrum: multiple solutions

We will now study the existence of multiple weak solutions for the semilinear beam equation

$$\begin{gathered}
\partial_t^2 u + \alpha_0^2 \partial_x^4 u - g(x,u) = h(x,t), \\
u(0,t) = u(\pi,t) = \partial_x^2 u(0,t) = \partial_x^2 u(\pi,t) = 0 \quad (x \in]0,\pi[,\ t \in \mathbb{R}), \\
u(x,t) = u(x,t+T) \quad \text{with } T = 2\pi\omega^{-1},
\end{gathered} \tag{4.1}$$

whenever h is "small" and the nonlinearity interacts with the spectrum of the linear part. Note however that

$$||h_\omega||_H = \sqrt{\omega}\|h\|_{L_2((0,\pi)\times(0,T))}. \tag{4.2}$$

Let $g(x,s)$ be a Carathéodory function from $]0,\pi[\times\mathbb{R}$ to $\mathbb{R}$ such that $g(x,\cdot)$ is strictly increasing and, for a.a. $x\in[0,\pi]$,

$$a \le \frac{g(x,s)}{s} \le b, \quad s \ne 0, \tag{4.3}$$

where $b>a>0$ are constants. Then the Nemytskii operator N generated by g from $H=L_2(\Omega)$ to H is bounded continuous and $N(0)=0$. Hence, we are interested in the case where N interacts with the spectrum of L_ω in the sense that a finite number of eigenvalues with finite multiplicity are crossed by the function $g(x,s)/s$ when $|s|$ runs from 0 to ∞. We refer to the set

$$\sigma(L_\omega)\cap[a,b] \tag{4.4}$$

as the set of interacting or crossed eigenvalues. Put

$$r_0(x)=\liminf_{s\to 0}\frac{g(x,s)}{s}, \qquad r_\infty(x)=\limsup_{|s|\to\infty}\frac{g(x,s)}{s}, \tag{4.5}$$

and denote

$$\Lambda=[a,b]\cap\{\alpha_0^2 j^4 \mid j\in\mathbb{Z}_+\}. \tag{4.6}$$

Assuming that $\Lambda\ne\varnothing$, we can write $\Lambda=\{\alpha_0^2 j^4 \mid j_L\le j\le j_U\}$. We suppose that there exist constants $\bar a$ and $\bar b$ such that for a.a. $x\in[0,\pi]$,

$$a\le r_\infty(x)\le \bar a<\lambda<\bar b\le r_0(x)\le b \quad (\lambda\in\Lambda). \tag{4.7}$$

Moreover, assume that there exist further constants $d>c>0$ such that for a.a. $x\in[0,\pi]$,

$$\begin{aligned} a \le \frac{g(x,s)}{s} &\le \bar a \quad \forall |s|>d,\\ \bar b\le \frac{g(x,s)}{s} &\le b \quad \forall 0<|s|\le c. \end{aligned} \tag{4.8}$$

Then, clearly, $\Lambda\subset\sigma(L_\omega)\cap[a,b]$ for any $\omega\in\alpha_0\mathbb{Q}_+$. Denote by $m(\lambda)$ the geometric multiplicity of any eigenvalue λ. Note that an eigenvalue λ has odd multiplicity if and only if $\lambda\in\{\alpha_0^2 j^4 \mid j\in\mathbb{Z}_+\}$. By Lemma 2.1(iii), we can take $\omega\in\alpha_0\mathbb{Q}_+$

such that

$$\sigma(L_\omega) \cap [a,b] = \Lambda, \qquad m(\lambda) = 1, \quad \lambda \in \Lambda. \tag{4.9}$$

We will study two different cases.

(A) First assume that $\Lambda = \{\lambda_0\}$ with $m(\lambda_0) = 1$, that is, only one simple eigenvalue is crossed. Then we have the following result.

Theorem 4.1. *Assume that (4.3), (4.7), and (4.8) hold and* $\Lambda = [a,b] \cap \{\alpha_0^2 j^4 \mid j \in \mathbb{Z}_+\} = \{\lambda_0\}$. *Let* $\omega \in \alpha_0\mathbb{Q}_+$ *be such that (4.9) holds with* $\Lambda = \{\lambda_0\}$. *Then there exists an* $\epsilon > 0$ *such that (4.1) admits at least two solutions for any* $2\pi/\omega$*-periodic forcing term* h *such that* $h_\omega \in H$, *provided that* $\|h_\omega\| \le \epsilon$. *If* $h = 0$, *then there exists at least two nontrivial solutions.*

Proof. First assume that $h = 0$. Then the proof is a variant of that given in [4]. Indeed, by [4, Theorem 2], there exist disjoint open bounded sets G_1 and G_2 such that $0 \notin G_1 \cup G_2$ and

$$\deg(L_\omega - N, G_i, 0) \neq 0, \quad i = 1,2. \tag{4.10}$$

Since 0 belongs to an open component of the open set $H \setminus (L-N)(\partial G_i \cap D(L_\omega))$, there exists $\epsilon > 0$ such that

$$\deg(L_\omega - N, G_i, h_\omega) = \deg(L_\omega - N, G_i, 0) \neq 0 \quad \forall \|h_\omega\| < \epsilon,\ i = 1,2. \tag{4.11}$$

Hence, the conclusion follows by the basic properties of topological degree. □

(B) If the spectrum interacts with more than one eigenvalue, we can use suitable invariant subspaces to obtain the following variant of Theorem 4.1.

Theorem 4.2. *Assume that* $g = g(s)$ *is odd and (4.3), (4.7), and (4.8) are satisfied. Let* $\omega \in \alpha_0\mathbb{Q}_+$ *be such that (4.9) holds with* $\Lambda = [a,b] \cap \{\alpha_0^2 j^4 \mid j_L \le j \le j_U\}$. *Then there exists an* $\epsilon > 0$ *such that (4.1) admits at least two solutions for each* h *such that* $h_\omega \in V_r$, $r = j_U = \max\Lambda$, *and* $\|h_\omega\| \le \epsilon$. *If* $h = 0$, *then there exist at least two nontrivial solutions.*

Note that if g is odd and $h = 0$, the pairs of solutions may be of the form $(u, -u)$ and be time independent. In a more specific situation, the results can usually be improved by using other invariant subspaces. In order to illuminate this, we close this section by two examples.

Example 4.3. Assume that $\alpha_0 = 1$ and $g(s) = as + d\arctan(s)$, with $a = 2$ and $d = 10$. Then g is odd and strongly monotone, and

$$2 \le \frac{g(x,s)}{s} \le 12 \quad \forall s \neq 0. \tag{4.12}$$

We consider now the equation

$$\begin{gathered}\partial_t^2 u+\partial_x^4 u+au+d\arctan(u)=0,\\ u(0,t)=u(\pi,t)=\partial_x^2 u(0,t)=\partial_x^2 u(\pi,t)=0 \quad (x\in\,]0,\pi[,\ t\in\mathbb{R}),\\ u(x,t)=u(x,t+T) \quad \text{with } T=2\pi\omega^{-1}.\end{gathered}\tag{4.13}$$

Because of the sign of the nonlinearity, the set of interacting eigenvalues is now $\sigma(-L_\omega)\cap[2,12]$. Theorems 4.1 and 4.2 cannot be applied directly. We take $\omega=q\in\mathbb{Z}_+$. It is not hard to see that $\sigma(-L_1)\cap[2,12]=\{3,8,9\}$, where $\lambda^1_{1,\pm2}=-3$, $\lambda^1_{1,\pm3}=-8$, and $\lambda^1_{2,\pm5}=-9$. Similarly $\sigma(-L_2)\cap[2,12]=\{3\}$, $\sigma(-L_3)\cap[2,12]=\{8\}$, and $\sigma(-L_5)\cap[2,12]=\{9\}$. Hence, in each case the multiplicity of the crossed eigenvalue is 2. Moreover, $\sigma(-L_\omega)\cap[2,12]=\varnothing$ for any $\omega=q\in\mathbb{Z}_+$, $q\neq 1,2,3,5$. As a consequence, we have the following result: *equation (4.13) admits at least six mutually different pairs of nontrivial 2π-periodic solutions.*

To be more precise, the periods of the pairs of solutions are $T=2\pi q^{-1}$, $q=2,3,5$, respectively.

To prove our assertion, we consider the problem in the subspace

$$\begin{aligned}V_{\text{even}}&=\{u\in H\mid u(x,2\pi-t)=u(x,t)\ \text{ for a.a. } x\in\,]0,\pi[,\ t\in\,]0,2\pi[\}\\ &=\overline{\operatorname{sp}}\{\phi_{j,k}+\phi_{j,-k}\mid j\in\mathbb{Z}_+,\ k\in\mathbb{Z}_+^0\}.\end{aligned}\tag{4.14}$$

In each of the cases $q=2,3,5$, the multiplicity of the crossed eigenvalue in V is 1, and thus, an obvious modification of Theorem 4.1 implies that the equation admits a pair of nontrivial solutions with corresponding period $T=2\pi q^{-1}$, $q=2,3,5$. To show that the solutions do not coincide, we use Proposition 3.5(iii). Indeed, there are two possibilities. First, the solutions may be time independent. But this is not possible, since the reduced equation in subspace

$$V_{\text{const}}=\{u\in H\mid u \text{ is constant in } t\}=\overline{\operatorname{sp}}\{\phi_{j,0}\mid j\in\mathbb{Z}_+\}\tag{4.15}$$

has only the trivial solution by Proposition 3.5(iii). The second possibility is that two solutions with different periods, say T_1 and T_2, coincide. Then, necessarily, T_1 and T_2 are multiples of some T_0 and the solution is actually T_0-periodic. In our case, this is excluded by Proposition 3.5(iii) and the fact that $\sigma(-L_\omega)\cap[2,12]=\varnothing$ for any $\omega=q\in\mathbb{Z}_+$, $q\neq 1,2,3,5$.

By a similar argument, we find three additional pairs of nontrivial solutions by reducing the problem to the subspace

$$\begin{aligned}V_{\text{odd}}&=\{u\in H\mid u(x,2\pi-t)=-u(x,t)\ \text{for a.a. } x\in\,]0,\pi[,\ t\in\,]0,2\pi[\}\\ &=\overline{\operatorname{sp}}\{i(\phi_{j,k}-\phi_{j,-k})\mid j,k\in\mathbb{Z}_+\}.\end{aligned}\tag{4.16}$$

Since $V_{\text{even}}\cap V_{\text{odd}}=\{0\}$, all the solutions are mutually disjoint.

Example 4.4. Assume that $\alpha_0 = 1$ and $g(s) = as + d\arctan(s)$ with positive constants a and d. Then g is odd, strongly monotone and

$$a \leq \frac{g(x,s)}{s} \leq a+d \quad \forall s \neq 0. \tag{4.17}$$

Assume that $a = 2$ and $d = 3^4$. We consider the equation

$$\begin{gathered}\partial_t^2 u + \partial_x^4 u - au - d\arctan(u) = 0, \\ u(0,t) = u(\pi,t) = \partial_x^2 u(0,t) = \partial_x^2 u(\pi,t) = 0 \quad (x \in]0,\pi[,\ t \in \mathbb{R}), \\ u(x,t) = u(x,t+T) \quad \text{with } T = 2\pi\omega^{-1}.\end{gathered} \tag{4.18}$$

We will consider the existence of 2π-periodic solutions. By Theorem 4.2, the existence of a pair of nontrivial solutions is achieved. However, a much better result can be proved using appropriate invariant subspaces and Lemma 2.1(iii). Indeed, we will show the following result.

Theorem 4.5. *Equation (4.18) admits at least two disjoint pairs of nontrivial time-independent solutions and at least* 14 *pairs of mutually disjoint pairs of time-dependent* 2π*-periodic solutions.*

We assume again that $\omega = q \in \mathbb{Z}_+$ making any ω-periodic solution also 2π periodic. For $\omega = 1$, the set $\sigma(L_\omega) \cap [2,83]$ of interacting eigenvalues is

$$\begin{aligned}&\{\lambda_{2,0}^1 = 16,\ \lambda_{2,1}^1 = 15,\ \lambda_{2,2}^1 = 12,\ \lambda_{2,3}^1 = 7,\ \lambda_{3,0}^1 = 80,\ \lambda_{3,1}^1 = 80, \\ &\lambda_{3,2}^1 = 77,\ \lambda_{3,3}^1 = 72,\ \lambda_{3,4}^1 = 65,\ \lambda_{3,5}^1 = 56,\ \lambda_{3,6}^1 = 45,\ \lambda_{3,7}^1 = 32, \\ &\lambda_{3,8}^1 = 17,\ \lambda_{4,14}^1 = 60,\ \lambda_{4,15}^1 = 31,\ \lambda_{5,24}^1 = 49,\ \lambda_{6,35}^1 = 71\}.\end{aligned} \tag{4.19}$$

The multiplicity of $\lambda_{2,0}^1$ and $\lambda_{3,0}^1$ is 1, all other crossed eigenvalues have multiplicity 2. Hence, the sum of multiplicities, which is the dimension of the space spanned by the corresponding eigenvectors, is 32.

(1) First consider the existence of stationary solutions. Indeed, in the space V_{const} (u constant in t), the crossed eigenvalues are 2^4 and 3^4, both having multiplicity 1. The further restriction into the space V_2 removes the eigenvalue 3^4. Hence, there exists at least one pair of nontrivial solutions in $V_{\text{const}} \cap V_2$. Similarly, the reduction to the space $V_{\text{const}} \cap V_3$ yields the existence of at least one pair of nontrivial solutions. Note however that the solutions may coincide and belong to $V_2 \cap V_3$. To avoid this, we take the subspace $V_{\text{const}} \cap W$ (see Lemma 2.3

for the definition of W). Since $W \cap V_2 = \{0\}$, we find another (different) pair of nontrivial solutions in $V_{\text{const}} \cap W$.

(2) In order to find time-dependent solutions, we first reduce the equation to the space V_{odd}. In V_{odd}, all the crossed eigenvalues have multiplicity 1, the eigenvalues $2^4, 3^4$ are removed, and $V_{\text{odd}} \cap V_{\text{const}} = \emptyset$. From the list of interacting eigenvalues, we conclude two lemmas giving necessary conditions for any time-dependent nontrivial solution in V_{odd}.

Lemma 4.6. *Assume that $\omega = q \in \mathbb{Z}_+$. Then (4.18) has only trivial solution in V_{odd} if $q \notin \{1,2,3,4,5,6,7,8,12,14,15,24,35\}$.*

Lemma 4.7. *Assume that $\omega = q \in \mathbb{Z}_+$, $q \in \{1,2,3,4,5,6,7,8,12,14,15,24,35\}$, and $r \in \mathbb{Z}_+$. Then (4.18) has only trivial solution in $V_{\text{odd}} \cap V_r$ if $r \geq 7$.*

Considering different values of q and r, we can find a large number of nontrivial solutions but most of them may coincide. One should carefully check which of the solutions are mutually disjoint. Using Lemma 4.6, we obtain four pairs of nontrivial solutions with periods $2\pi/35$, $2\pi/24$, $2\pi/15$, and $2\pi/14$, respectively.

Next, assume that $q = 8$ and reduce the problem into $V_{\text{odd}} \cap V_3$. It is easy to see that the only interacting eigenvalue left is $\lambda^8_{3,1} = 17$. Hence, there exists a pair of nontrivial solutions with period $2\pi/8$. Since the restriction of L_{24} into $V_{\text{odd}} \cap V_3$ has no interacting eigenvalues, these solutions are not $2\pi/24$ periodic.

By using the subspace $V_{\text{odd}} \cap V_3 \cap W$ and a similar reasoning, we find a pair of nontrivial solutions with period $2\pi/7$ which are neither $2\pi/14$- nor $2\pi/35$-periodic.

For $q = 6$, we find a pair of nontrivial solutions by reducing the problem into $V_{\text{odd}} \cap V_3$ (these solutions are not $2\pi/24$-periodic).

If $q = 5$, a pair of nontrivial solutions exists in $V_{\text{odd}} \cap V_3 \cap W$. It is easy to see that these solutions are neither $2\pi/15$- nor $2\pi/35$-periodic.

If $q = 3$, we will use the reducing subspace

$$\begin{aligned} W_r := \Big\{ & u \in H \mid u\left(x + \frac{\pi}{r}, t\right) = -u(x,t) \text{ for a.a. } t \in \left]0, 2\pi\right[,\ x \in \left]0, \pi - \frac{\pi}{r}\right[, \\ & u\left(\frac{\pi}{r} - x, t\right) = u(x,t) \text{ for a.a. } t \in \left]0, 2\pi\right[,\ x \in \left]0, \frac{\pi}{r}\right[\Big\} \\ = {} & \overline{\operatorname{sp}_{\mathbb{C}}}\{\phi_{j,k} \mid j \in r(2\mathbb{Z}_+ - 1),\ k \in \mathbb{Z}\} \cap H, \end{aligned} \tag{4.20}$$

where $r \in \mathbb{Z}_+$, $r \geq 2$. A further pair of nontrivial solutions with period $2\pi/3$ is obtained by reduction to the space $V_{\text{odd}} \cap V_2$.

If $q = 2$, the reduction to $V_{\text{odd}} \cap V_2$ gives a pair of nontrivial solutions, which do not have period $T = 2\pi/2l$ for any $l \geq 2$, $l \in \mathbb{Z}_+$.

To improve the result, we reduce the problem to the subspace V_{even} introduced in the previous example. Note that $V_{\text{even}} \cap V_{\text{odd}} = \{0\}$ but the reduction into V_{even} does not remove the eigenvalues 2^4 and 3^4. A further reduction to the space V_r will help. Using the subspaces $V_{\text{even}} \cap V_6$, $V_{\text{even}} \cap V_5$, and $V_{\text{even}} \cap V_4$, we obtain four further solutions with periods $2\pi/35$, $2\pi/24$, $2\pi/15$, and $2\pi/14$, respectively. Since the equation in $V_{\text{even}} \cap V_r \cap V_{\text{const}}$ has only the trivial solution for $r = 4, 5, 6,$ these solutions are time dependent.

Remark 4.8. From the physical point of view, part of the solutions in the previous examples may be equivalent in the sense that they are related by a time shift. Since the spaces V_{odd} and V_{even} are not invariant under the time shift $t \to t + \tau$, the number of pairs of nontrivial solutions, which are not physically equivalent, is 3 in Example 4.3 and 12 in Example 4.4.

Acknowledgments

H. Leinfelder thanks the Department of Mathematical Sciences, University of Oulu for the hospitality and the financial support during summer term 2001.

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J. Berkovits: Department of Mathematical Sciences, University of Oulu, P.O. Box 3000, FIN-90014 Oulu, Finland

E-mail address: juha.berkovits@oulu.fi

H. Leinfelder: Laboratory of Applied Mathematics, Ohm Polytechnic Nuremberg, P.O. Box 210320, D-90121 Nuremberg, Germany

E-mail address: herbert.leinfelder@fh-nuernberg.de

V. Mustonen: Department of Mathematical Sciences, University of Oulu, P.O. Box 3000, FIN-90014 Oulu, Finland

E-mail address: vesa.mustonen@oulu.fi