ATTRACTORS OF ITERATED FUNCTION SYSTEMS AND MARKOV OPERATORS

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This paper contains a review of results concerning "generalized" attractors for a large class of iterated function systems $\{w_i : i \in I\}$ acting on a complete separable metric space. This generalization, which originates in the Banach contraction principle, allows us to consider a new class of sets, which we call semi-attractors (or semifractals). These sets have many interesting properties. Moreover, we give some fixed-point results for Markov operators acting on the space of Borel measures, and we show some relations between semi-attractors and supports of invariant measures for such Markov operators. Finally, we also show some relations between multifunctions and transition functions appearing in the theory of Markov operators.

1. Introduction

The fixed-point theory is not only a fruitful tool in diverse branches of mathematics, but sometimes even a crucial part of the construction of a theory. This paper deals with two such cases, namely, the theory of fractals defined by the iterated functions systems (see [1, 2, 12]) and the theory of invariant measures for Markov operators. In the sequel, instead of the term fractal, we rather prefer to use the term attractor.

Attractors can be considered a generalization of fixed points of contracting transformations. To explain this fact more precisely, assume that (X, ρ) is a complete metric space and $w : X \to X$ is a strict contraction, that is, a Lipschitz function with a Lipschitz constant $L < 1$. According to the Banach principle, there is a unique fixed point x_0 of w such that $\rho(w^n(x), x_0) \to 0$ for every $x \in X$. Moreover, the set $\{x_0\}$ is the unique fixed point of the transformation $A \to w(A)$ which maps the family of nonempty compact subsets of X into itself. If $\{w_1, \dots, w_N\}$ is a finite family of strict contractions, we may consider the Barnsley-Hutchinson

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multifunction given by the formula

$$F(A) = \bigcup_{i=1}^{N} w_i(A). \tag{1.1}$$

Again it can be proved that there is a unique compact set A_0 such that $F(A_0) = A_0$ and that for every nonempty compact subset A of X the sequence of iterates $(F^n(A))$ converges in the Hausdorff distance to A_0. According to Barnsley and Hutchinson (see [1, 9]), the set A_0 is called the fractal generated by the iterated function system $\{w_1, \dots, w_N\}$.

Fractals are strongly related to Markov operators acting on the space of all Borel measures. If the functions $w_1, \dots, w_N$ are given and $p_1, \dots, p_N$ is a probability vector (i.e., $p_i \geq 0, \sum p_i = 1$), then we may define the Markov operator

$$P\mu = \sum_{i=1}^{N} p_i \left(\mu \circ w_i^{-1}\right) \quad \text{for } \mu \in \mathcal{M}_1, \tag{1.2}$$

where $\mathcal{M}_1$ denotes the family of all probability Borel measures on X. If all w_i are Lipschitzian (not necessarily contractions) with Lipschitz constants L_i, then the condition $\sum p_i L_i < 1$ implies the asymptotic stability of P. If all w_i are strictly contractive, then the support of the unique probability measure μ_* invariant with respect to P is equal to the fixed point of the Barnsley-Hutchinson multifunction F defined by (1.1).

It is interesting that the class of sets which can be obtained as a support of invariant measures of asymptotically stable Markov operators contains not only attractors (fractals). This leads to the notion of semi-attractors (semifractals). Roughly speaking, a semiattractor is the support of an invariant measure corresponding to an asymptotically stable Markov operator generated by an iterated function system with probabilities.

Semi-attractors, likewise attractors, can also be defined by topological methods without any use of probability theory. Namely, having the Barnsley-Hutchinson multifunction, we may introduce in a natural way semi-attractors which correspond in the classical setting to the notion of fractals. What is more, this approach can be applied to a large class of multifunctions. In order to define the convergence of a sequence of sets, we use the topological (Kuratowski) limits instead of the traditional Hausdorff distance in the fractal theory.

It is worth emphasizing that multifunctions with closed values are strongly related to transition functions appearing in the theory of Markov operators. Namely, the support of a transition function is a closed-valued measurable multifunction. Reciprocally, given a closed-valued measurable multifunction F, there exists a transition function such that its support is equal to F. Analogous relationship can be established between lower semicontinuous multifunctions and Fellerian transition functions. Moreover, the support of an invariant measure for asymptotically stable Markov operators is equal to the semiattractor of the corresponding multifunctions.

The paper is organized as follows. Section 2 contains some notions and definitions concerning measures on metric spaces and Markov operators acting on the space of measures. In Section 3 we recall Barnsley's and Hutchinson's approach to the fractal theory. In Section 4 we give some sufficient conditions for the existence of an invariant measure for Markov operators acting on a Polish space. Moreover, this section contains also some results concerning asymptotic stability of Markov operators. In Section 5 we extend Barnsley's and Hutchinson's approach to a more general class of iterated functions systems. This leads to the definition of a new class of sets called semi-attractors (or semifractals). In Section 6 we introduce the concept of semistability of multifunctions. In this way we obtain a further generalization of notion of semifractals. Section 7 is devoted to the relation between multifunctions and transition functions. Finally, in Section 8 we present some results concerning the numerical possibility of the construction of semi-attractors.

2. Preliminaries

Let (X,ρ) be a metric space. By $B(x,r)$ (resp., $B^o(x,r)$) we denote the closed (resp., open) ball with center at x and radius r. For a subset A of X, $\operatorname{cl}A$, $\operatorname{diam}A$, and 1_A stand for the closure of A, the diameter of A, and the characteristic function of A, respectively. By $\mathbb{R}$ we denote the set of all reals and by $\mathbb{N}$ the set of all positive integers.

By $\mathcal{B}$ we denote the σ-algebra of Borel subsets of X and by $\mathcal{M}$ the family of all finite Borel measures on X. By $\mathcal{M}_1$ we denote the space of all $\mu \in \mathcal{M}$ such that $\mu(X) = 1$ and by $\mathcal{M}_s$ the space of all finite signed Borel measures on X. The elements of $\mathcal{M}_1$ are called *distributions*.

Given $\mu \in \mathcal{M}$, we define the support of μ by the formula

$$\operatorname{supp}\mu = \{x \in X : \mu(B(x,r)) > 0 \text{ for every } r > 0\}. \tag{2.1}$$

As usual, by $B(X)$ we denote the space of all bounded Borel measurable functions $f : X \to \mathbb{R}$ and by $C(X)$ the subspace of all continuous functions. Both spaces are considered with the supremum norm.

For $f \in B(X)$ and $\mu \in \mathcal{M}_s$, we write

$$\langle f,\mu\rangle = \int_X f(x)\mu(dx). \tag{2.2}$$

We admit that $\mathcal{M}_s$ is endowed with the *Fortet-Mourier norm* given by

$$\|\mu\| = \sup\{\,|\langle f,\mu\rangle| : f \in \mathcal{L}\} \quad \text{for } \mu \in \mathcal{M}_s, \tag{2.3}$$

where $\mathcal{L}$ is the set of all $f \in C(X)$ such that $|f(x)| \le 1$ and $|f(x) - f(y)| \le \rho(x, y)$ for $x, y \in X$ (see [8]).

We say that a sequence $(\mu_n) \subset \mathcal{M}$ converges weakly to a measure $\mu \in \mathcal{M}$ if

$$\lim_{n\to\infty} \langle f, \mu_n \rangle = \langle f, \mu \rangle \quad \text{for every } f \in C(X). \tag{2.4}$$

It is well known (see [4]) that the convergence in the Fortet-Mourier norm is equivalent to the weak convergence.

An operator $P : \mathcal{M} \to \mathcal{M}$ is called a *Markov operator* if

$$\begin{gathered} P(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 P\mu_1 + \lambda_2 P\mu_2 \quad \text{for } \lambda_1, \lambda_2 \in \mathbb{R}_+,\ \mu_1, \mu_2 \in \mathcal{M}, \\ P\mu(X) = \mu(X) \quad \text{for } \mu \in \mathcal{M}. \end{gathered} \tag{2.5}$$

The theory of Markov operators started in 1906 when Markov showed that the asymptotic properties of some stochastic processes can be studied using stochastic matrices [17]. Such matrices define positive linear operators on $\mathbb{R}^n$. Markov's ideas were generalized in many directions. In particular, Feller developed the theory of the Markov operators acting on Borel measures defined on some topological spaces. Hopf proposed to study Markov operators on L^1 spaces. Some historical remarks and a vast literature can be found in the book of Nummelin [19] (see also [3, 6, 7, 12, 20]).

A Markov operator P is called *nonexpansive* if

$$\|P\mu_1 - P\mu_2\| \le \|\mu_1 - \mu_2\| \quad \text{for } \mu_1, \mu_2 \in \mathcal{M}_1. \tag{2.6}$$

A Markov operator P is called a *Markov-Feller* operator if there is an operator $U : B(X) \to B(X)$ such that

(i) $\langle Uf, \mu \rangle = \langle f, P\mu \rangle$ for $f \in B(X)$ and $\mu \in \mathcal{M}$;
(ii) $Uf \in C(X)$ for $f \in C(X)$.

The operator U is called *dual* to P.

It can be proved that every nonexpansive Markov operator is a Markov-Feller operator.

A measure μ is called *invariant* (or *stationary*) with respect to P if $P\mu = \mu$. A Markov operator P is called *asymptotically stable* if there exists a stationary measure $\mu_* \in \mathcal{M}_1$ such that

$$\lim_{n\to\infty} P^n\mu = \mu_* \quad \text{for every } \mu \in \mathcal{M}_1. \tag{2.7}$$

Obviously, the measure μ_* satisfying the condition above is unique.

3. Barnsley's and Hutchinson's approach to the fractal theory

Let (X,ρ) be a complete separable metric space.

An iterated function system (shortly IFS) is given by a family of continuous functions

$$w_i : X \longrightarrow X, \quad i \in I. \tag{3.1}$$

If, in addition, there is given a family of continuous functions

$$p_i : X \longrightarrow [0,1], \quad i \in I \tag{3.2}$$

satisfying

$$\sum_{i\in I} p_i(x) = 1 \quad \text{for every } x \in X, \tag{3.3}$$

then the family $\{(w_i, p_i);\ i \in I\}$ is called an IFS with probabilities. Here, for simplicity we assume that the index set I is finite. Note that only some of these results remain true for the case of the countable index set as well.

Having an IFS $\{w_i : i \in I\}$, we define the corresponding Barnsley-Hutchinson multifunction F by

$$F(x) = \{w_i(x) : i \in I\} \quad \text{for } x \in X, \tag{3.4}$$

and having an IFS with probabilities $\{(w_i, p_i);\ i \in I\}$, we define the Markov operator P acting on measures by

$$P\mu(A) = \sum_{i\in I} \int_X 1_A(w_i(x))\, p_i(x)\mu(dx) \quad \text{for } \mu \in \mathcal{M},\ A \in \mathcal{B}. \tag{3.5}$$

It is easy to verify that P is a Markov-Feller operator and its dual U is given by

$$Uf(x) = \sum_{i\in I} p_i(x) f(w_i(x)). \tag{3.6}$$

A set A_0 such that $F(A_0) = A_0$ is called *invariant* with respect to the IFS $\{w_i : i \in I\}$. If, in addition, for every nonempty compact subset A of X, the sequence $(F^n(A))$ converges in the Hausdorff distance to A_0, the set A_0 is called an *attractor* (or *fractal*) corresponding to the IFS $\{w_i : i \in I\}$.

Assume that I is finite. Moreover, assume that for every $i \in I$, the function w_i is Lipschitzian with the Lipschitz constant L_i and the function p_i is constant. The following facts are well known (see [1, 12]).

Fact 3.1. If $L_i < 1$ for $i \in I$, then the IFS $\{w_i : i \in I\}$ is asymptotically stable (on sets), the operator P given by (3.5) is asymptotically stable (on measures), and

$$A_0 = \operatorname{supp}\mu_*, \tag{3.7}$$

where A_0 is the attractor corresponding to the IFS $\{w_i : i \in I\}$ and μ_* is the invariant measure with respect to P.

Fact 3.2. If

$$\sum_{i\in I} p_i L_i < 1, \tag{3.8}$$

then an IFS with probabilities $\{(w_i, p_i) : i \in I\}$ is asymptotically stable.

The following natural questions arise. What are the geometric properties of the set $\operatorname{supp}\mu_*$? Can we define this set using only the transformations w_i also in the case when some constants $L_i \geq 1$? Note that the assumption $L_i < 1$, $i \in I$, is quite restrictive and excludes such natural transformations as shifts and rotations. Can we extend these results to a more general family of maps w_i or, directly, to some classes of multifunctions. The positive answers to these questions are given in the next sections.

4. Existence of an invariant measure for Markov operators

In this section, we give some sufficient conditions for the existence of an invariant measure for Markov operators defined on a Polish space X. Similar results for the case of compact spaces and locally compact spaces can be found in the literature, starting from the classical book by Foguel [7]. In the case of a compact space, the proof of existence goes as follows: first, we construct a positive, invariant functional defined on the space of all continuous and bounded functions $f : X \to \mathbb{R}$, and then using the Riesz representation theorem, we define an invariant measure (see [3, 6, 19]). The case of locally compact spaces requires some caution. The first existence results were established by Lasota and Yorke (see [16]) by using the concept of nonexpansiveness and lower bound technique. When X is a Polish space, the ideas above break down, since a positive functional may not correspond to a measure. General existence theorems for Markov operators on Polish spaces have been established quite recently (see [21, 22, 23, 24]). The proofs of these results are based on the concept of tightness and suitable concentration properties of Markov operators. Here we recall some of them.

Given a set $A \subset X$ and a number $r > 0$, we denote by $\mathcal{N}^o(A,r)$ (resp., $\mathcal{N}(A,r)$) the open (resp., closed) r-neighbourhood of the set A, that is,

$$\mathcal{N}^o(A,r) = \{x \in X : \rho(x,A) < r\}, \qquad \mathcal{N}(A,r) = \{x \in X : \rho(x,A) \leq r\}, \tag{4.1}$$

where

$$\rho(x,A) = \inf\{\rho(x,y) : y \in A\}. \tag{4.2}$$

We denote by $\mathcal{C}_\varepsilon$, $\varepsilon > 0$, the family of all closed sets C for which there exists a finite set $\{x_1, x_2, \dots, x_n\} \subset X$ (ε-net) such that $C \subset \bigcup_{i=1}^n B(x_i, \varepsilon)$.

Let $A \in \mathcal{B}$. We say that a measure $\mu \in \mathcal{M}$ is *concentrated* on A if $\mu(X \setminus A) = 0$. By $\mathcal{M}_1^A$ we denote the set of all distributions concentrated on A.

A sequence of distributions (μ_n) $(\mu_n \in \mathcal{M}_1)$ is called *tight* if for every $\varepsilon > 0$ there exists a compact set $K \subset X$ such that $\mu_n(K) \geq 1 - \varepsilon$ for every $n \in \mathbb{N}$.

It is well known (see [4, 6]) that every tight sequence of distributions contains a weakly convergent subsequence.

We say that a Markov operator $P : \mathcal{M} \to \mathcal{M}$ is *tight* if for every $\mu \in \mathcal{M}_1$ the sequence of iterates $(P^n\mu)$ is tight.

An operator P is called *globally concentrating* if for every $\varepsilon > 0$ and every Borel bounded set A, there exist a Borel bounded set B and a number $n_0 \in \mathbb{N}$ such that

$$P^n\mu(B) \geq 1 - \varepsilon \quad \text{for } n \geq n_0,\ \mu \in \mathcal{M}_1^A. \tag{4.3}$$

An operator P is called *locally concentrating* if for every $\varepsilon > 0$, there is $\alpha > 0$ such that for every Borel bounded set A, there exist a set $B \in \mathcal{B}$ with $\operatorname{diam} B \leq \varepsilon$ and $n_0 \in \mathbb{N}$ satisfying

$$P^{n_0}\mu(B) > \alpha \quad \text{for } \mu \in \mathcal{M}_1^A. \tag{4.4}$$

Remark 4.1. There exists a Markov operator which is locally concentrating but not globally concentrating.

An operator P is called *concentrating* if for every $\varepsilon > 0$, there exist a set $B \in \mathcal{B}$ with $\operatorname{diam} B \leq \varepsilon$ and a number $\alpha > 0$ such that

$$\liminf_{n\to\infty} P^n\mu(B) > \alpha \quad \text{for } \mu \in \mathcal{M}_1. \tag{4.5}$$

An operator P is called *semiconcentrating* if for every $\varepsilon > 0$, there exist $B \in \mathcal{C}_\varepsilon$ and $\alpha > 0$ such that condition (4.5) holds.

By $\mathcal{C}_\varepsilon^+$ we denote the family of all $B \in \mathcal{C}_\varepsilon$ such that

$$\inf_{\mu\in\mathcal{M}_1} \liminf_{n\to\infty} P^n\mu(B) > 0. \tag{4.6}$$

Remark 4.2. It is obvious that a concentrating Markov operator is semiconcentrating.

Theorem 4.3. *Let P be a Markov operator. Assume that P is continuous in the weak topology. If P is tight, then P admits an invariant measure $\mu_* \in \mathcal{M}_1$.*

Proof. Fix $\mu \in \mathcal{M}_1$ and set

$$\bar{\mu}_n = \frac{\mu + P\mu + \cdots + P^{n-1}\mu}{n} \quad \text{for } n \in \mathbb{N}. \tag{4.7}$$

Since P is tight, the sequence of distributions $(\bar{\mu}_n)$ is tight. From the Prokhorov theorem (see [4, 6, 19]), it follows that there exists a subsequence $(\bar{\mu}_{n_k})$ of $(\bar{\mu}_n)$ which converges weakly to some distribution $\bar{\mu}$. Since P is continuous, we obtain $P\bar{\mu}_{n_k} \to P\bar{\mu}$ in the weak topology. From (4.7) it follows that $\|P\bar{\mu}_{n_k} - \bar{\mu}_{n_k}\| \to 0$ and consequently $P\bar{\mu} = \bar{\mu}$. The proof is completed. □

We also need the following known facts concerning tightness.

Lemma 4.4. *Let (μ_n) be a sequence of distributions such that for every $\varepsilon > 0$, there exists a set $C \in \mathscr{C}_\varepsilon$ satisfying $\mu_n(C) \geq 1 - \varepsilon$, $n \in \mathbb{N}$. Then (μ_n) is tight.*

The proof can be found in [22].

Lemma 4.5. *Every sequence of distributions (μ_n) satisfying the Cauchy condition is tight.*

Sketch of the proof. Since (μ_n) satisfies the Cauchy condition, for every $\varepsilon > 0$, we may choose $n_0 \in \mathbb{N}$ such that

$$||\mu_p - \mu_q|| \leq \frac{\varepsilon^2}{4} \quad \text{for } p, q \geq n_0. \tag{4.8}$$

Further, by the Ulam theorem there exists a compact set $K \subset X$ such that

$$\mu_n(K) \geq 1 - \frac{\varepsilon}{2} \quad \text{for } n = 1, \ldots, n_0. \tag{4.9}$$

By (4.8), [22, Lemma 3.1], and (4.9), we have

$$\mu_n\left(\mathcal{N}\left(K, \frac{\varepsilon}{2}\right)\right) \geq \mu_{n_0}(K) - \frac{\varepsilon}{2} \quad \text{for } n \geq n_0. \tag{4.10}$$

Observe that $\mathcal{N}(K, \varepsilon/2) \in \mathscr{C}_\varepsilon$. An application of Lemma 4.4 finishes the proof. □

Let a Markov operator P be fixed. For $A \in \mathscr{B}$ and $\eta \in [0,1]$ we set

$$\mathcal{M}_1^{A,\eta} = \{\mu \in \mathcal{M}_1 : P^n\mu(A) \geq 1 - \eta \text{ for } n \in \mathbb{N}\}. \tag{4.11}$$

Now we define a function $\varphi : \mathscr{B} \times [0,1] \to [0,2] \cup \{-\infty\}$ by the formula

$$\varphi(A, \eta) = \limsup_{n\to\infty} \sup\left\{||P^n\mu_1 - P^n\mu_2|| : \mu_1, \mu_2 \in \mathcal{M}_1^{A,\eta}\right\}. \tag{4.12}$$

As usual, we admit that the supremum of an empty set is equal to $-\infty$.

Lemma 4.6. *Let P be a nonexpansive and locally concentrating Markov operator and let $\eta \in (0, 1/2)$. Then, for every Borel bounded set A,*

$$\varphi\left(A, \eta\left(1 - \frac{\alpha}{2}\right)\right) \leq \left(1 - \frac{\alpha}{2}\right)\varphi(A, \eta) + \frac{\alpha\varepsilon}{2}, \tag{4.13}$$

where $\varepsilon > 0$ is arbitrary and $\alpha > 0$ corresponds to ε according to the locally concentrating property.

The proof is rather technical and long, and can be found in [22].

Theorem 4.7. *Let P be a nonexpansive and locally concentrating Markov operator. If for every $\mu \in \mathcal{M}_1$ and every $\varepsilon > 0$ there is a Borel bounded set $A \subset X$ such that*

$$\liminf_{n\to\infty} P^n\mu(A) \geq 1 - \varepsilon, \tag{4.14}$$

then P admits an invariant measure $\mu_* \in \mathcal{M}_1$.

Sketch of the proof. By virtue of Lemma 4.5 and Theorem 4.3, it is sufficient to show that for every $\mu \in \mathcal{M}_1$ the sequence $(P^n\mu)$ satisfies the Cauchy condition. Fix $\varepsilon > 0$ and $\mu \in \mathcal{M}_1$. Let $\alpha > 0$ correspond to $\varepsilon/2$ according to the locally concentrating property. Let $k \in \mathbb{N}$ be such that $4(1 - \alpha/2)^k < \varepsilon$. By hypothesis we can choose a Borel bounded set A and a number $\bar{n}$ such that

$$P^n\mu(A) \geq 1 - \frac{(1 - \alpha/2)^k}{3} \quad \text{for } n \geq \bar{n}. \tag{4.15}$$

Using Lemma 4.6 and an induction argument, we can prove that

$$\varphi\left(A, \frac{(1 - \alpha/2)^k}{3}\right) < \varepsilon. \tag{4.16}$$

Since $P^m\mu, P^n\mu \in \mathcal{M}_1^{A,(1-\alpha/2)^k/3}$ for $m, n \in \mathbb{N}$, from (4.16) it follows that there exists $n_0 \in \mathbb{N}$ $(n_0 \geq \bar{n})$ such that

$$\|P^{n_0}P^n\mu - P^{n_0}P^m\mu\| < \varepsilon. \tag{4.17}$$

Consequently,

$$\|P^p\mu - P^q\mu\| < \varepsilon \quad \text{for } p, q \geq n_0, \tag{4.18}$$

which completes the proof. □

We also need the following lemma, proved in [23].

Lemma 4.8. *Let P be a nonexpansive Markov operator and let $\varepsilon > 0$. Assume that $A \in \mathcal{B}$ is such that* $\operatorname{diam} A \leq \varepsilon^2/16$. *Moreover, assume that there exists $\mu \in \mathcal{M}_1$ such that*

$$\liminf_{n\to\infty} P^n\mu(A) > 0. \tag{4.19}$$

Then, there exists $C \in \mathcal{C}_\varepsilon$ such that

$$P^n\nu(C) \geq 1 - \frac{\varepsilon}{2} \quad \forall n \in \mathbb{N},\ \nu \in \mathcal{M}_1^A. \tag{4.20}$$

Theorem 4.9. *If P is a nonexpansive and concentrating Markov operator, then P admits an invariant measure $\mu_* \in \mathcal{M}_1$.*

Sketch of the proof. By virtue of Theorem 4.3 and Lemma 4.4, it is sufficient to show that for every $\varepsilon > 0$ and $\mu \in \mathcal{M}_1$, there exists $C \in \mathcal{C}_\varepsilon$ such that $P^n\mu(C) \geq 1 - \varepsilon$ for $n \in \mathbb{N}$. Fix $\varepsilon > 0$ and set $\tilde{\varepsilon} = \varepsilon^2/16$. Let $\alpha > 0$ and $A \in \mathcal{B}$ correspond to $\tilde{\varepsilon}$ according to the concentrating property. By Lemma 4.4 there exists $C \in \mathcal{C}_\varepsilon$ such that

$$P^n\nu(C) \geq 1 - \frac{\varepsilon}{2} \quad \forall n \in \mathbb{N},\ \nu \in \mathcal{M}_1^A. \tag{4.21}$$

Let $\mu \in \mathcal{M}_1$. Using an induction argument, we define a sequence of integers (n_k) and two sequences of distributions (μ_k), (ν_k) in the following way. If $k = 0$, we set $n_0 = 0$ and $\mu_0 = \nu_0 = \mu$. If $k \geq 1$ and n_{k-1}, μ_{k-1}, ν_{k-1} are given, we choose, according to the concentrating property, an integer n_k such that

$$P^{n_k}\mu_{k-1}(A) \geq \frac{\alpha}{2}. \tag{4.22}$$

Now we define

$$\nu_k(B) = \frac{P^{n_k}\mu_{k-1}(B \cap A)}{P^{n_k}\mu_{k-1}(A)} \quad \text{for } B \in \mathcal{B}, \tag{4.23}$$

$$\mu_k(B) = \frac{1}{1-\alpha/2}\left(P^{n_k}\mu_{k-1}(B) - \frac{\alpha}{2}\nu_k(B)\right) \quad \text{for } B \in \mathcal{B}. \tag{4.24}$$

Observe that $\nu_k \in \mathcal{M}_1^A$. Using (4.24), it is easy to verify by an induction argument that

$$\begin{aligned} P^{n_1+\cdots+n_k}\mu = {} & \frac{\alpha}{2}P^{n_2+\cdots+n_k}\nu_1 + \frac{\alpha}{2}\left(1 - \frac{\alpha}{2}\right)P^{n_3+\cdots+n_k}\nu_2 \\ & + \cdots + \frac{\alpha}{2}\left(1 - \frac{\alpha}{2}\right)^{k-1}\nu_k + \left(1 - \frac{\alpha}{2}\right)^{k}\mu_k. \end{aligned} \tag{4.25}$$

Let $k \in \mathbb{N}$ be such that

$$\left(1-\left(1-\frac{\alpha}{2}\right)^k\right)\left(1-\frac{\varepsilon}{2}\right) \geq 1-\varepsilon. \tag{4.26}$$

Since $\nu_i \in \mathcal{M}_1^A$ for $i = 1, 2, \ldots, k$, from (4.25) and (4.21) it follows that

$$\begin{aligned} P^n\mu(C) &\geq \frac{\alpha}{2}P^{n-n_1}\nu_1(C) + \frac{\alpha}{2}\left(1-\frac{\alpha}{2}\right)P^{n-n_1-n_2}\nu_2(C) \\ &\quad + \cdots + \frac{\alpha}{2}\left(1-\frac{\alpha}{2}\right)^{k-1}P^{n-n_1-\cdots-n_k}\nu_k(C) \\ &\geq \left(1-\left(1-\frac{\alpha}{2}\right)^k\right)\left(1-\frac{\varepsilon}{2}\right) \geq 1-\varepsilon \end{aligned} \tag{4.27}$$

for $n \geq n_1 + \cdots + n_k$. By the Ulam theorem, we can find a compact set $K \subset X$ such that

$$P^n\mu(K \cup C) \geq 1-\varepsilon \quad \text{for } n \in \mathbb{N}. \tag{4.28}$$

Since $K \cup C \in \mathcal{C}_\varepsilon$, Lemma 4.4 and Theorem 4.3 show that P admits an invariant measure. This completes the proof. □

The following technical lemma is crucial for further consideration.

Lemma 4.10. *Let P be a nonexpansive and semiconcentrating Markov operator. Then for every $\varepsilon > 0$, there exist a finite sequence of Borel sets $A_1, \ldots, A_k$ with* $\operatorname{diam} A_i \leq \varepsilon$, *$i = 1, \ldots, k$ and a measure $\mu_0 \in \mathcal{M}_1$ such that*

$$\bigcup_{i=1}^k A_i \in \mathcal{C}_\varepsilon^+, \quad \liminf_{n\to\infty} P^n\mu_0(A_i) > 0 \quad \textit{for } i = 1, \ldots, k. \tag{4.29}$$

The proof can be found in [23].

Theorem 4.11. *Every nonexpansive and semiconcentrating Markov operator P admits an invariant measure $\mu_* \in \mathcal{M}_1$.*

Sketch of the proof. Fix $\varepsilon > 0$ and set $\tilde{\varepsilon} = \varepsilon^2/16$. By virtue of Lemma 4.10, there exist a sequence of Borel sets $(A_1, \ldots, A_k)$ with $\operatorname{diam} A_i \leq \tilde{\varepsilon}$ for $i = 1, \ldots, k$, and a measure $\mu_0 \in \mathcal{M}_1$ such that the set $\bigcup_{i=1}^k A_i \in \mathcal{C}_{\tilde{\varepsilon}}^+$ and

$$\liminf_{n\to\infty} P^n\mu_0(A_i) > 0 \quad \text{for } i = 1, \ldots, k. \tag{4.30}$$

By Lemma 4.8, for every $i \in \{1,\dots,k\}$ there exists a set $C_i \in \mathscr{C}_\varepsilon$ such that

$$P^n\nu(C_i) \geq 1 - \frac{\varepsilon}{2} \quad \text{for } n \in \mathbb{N},\ \nu \in \mathcal{M}_1^{A_i},\ i = 1,\dots,k. \tag{4.31}$$

Set $C = \bigcup_{i=1}^k C_i$ and observe that $C \in \mathscr{C}_\varepsilon$. Moreover, we have

$$P^n\nu(C) \geq 1 - \frac{\varepsilon}{2} \quad \text{for } n \in \mathbb{N},\ \nu \in \bigcup_{i=1}^k \mathcal{M}_1^{A_i}. \tag{4.32}$$

Since $\bigcup_{i=1}^k A_i \in \mathscr{C}_{\tilde{\varepsilon}}^+$, it follows that there exists $\tilde{\alpha} > 0$ such that

$$\liminf_{n\to\infty} P^n\mu\left(\bigcup_{i=1}^k A_i\right) > \tilde{\alpha} \quad \text{for } \mu \in \mathcal{M}_1. \tag{4.33}$$

Define

$$\eta = \sup\left\{\gamma \geq 0 : \liminf_{n\to\infty} P^n\mu(C) \geq \gamma\ \forall \mu \in \mathcal{M}_1\right\}. \tag{4.34}$$

We claim that $\eta \geq 1 - \varepsilon/2$. To see this, suppose for a contradiction that $\eta < 1 - \varepsilon/2$. Set $\alpha = \tilde{\alpha}/k$. Clearly,

$$\eta > \frac{\eta}{1-\alpha} - \frac{\alpha}{1-\alpha}\left(1 - \frac{\varepsilon}{2}\right). \tag{4.35}$$

Choose $\gamma > 0$ such that

$$\eta > \gamma > \frac{\eta}{1-\alpha} - \frac{\alpha}{1-\alpha}\left(1 - \frac{\varepsilon}{2}\right). \tag{4.36}$$

By the definition of η we have

$$\liminf_{n\to\infty} P^n\mu(C) \geq \gamma \quad \text{for } \mu \in \mathcal{M}_1. \tag{4.37}$$

Fix $\mu \in \mathcal{M}_1$. Analysis similar to that in the proof of Theorem 4.9 (see (4.25)) shows that there exist $n_0 \in \mathbb{N}$, $\tilde{\mu} \in \mathcal{M}_1$, and $\nu \in \bigcup_{i=1}^k \mathcal{M}_1^{A_i}$ such that

$$P^{n_0}\mu = (1-\alpha)\tilde{\mu} + \alpha\nu. \tag{4.38}$$

By (4.32), (4.37), the linearity of P, and the choice of γ we have

$$\begin{aligned}\liminf_{n\to\infty} P^{n_0+n}\mu(C) &\geq (1-\alpha)\liminf_{n\to\infty} P^n\tilde{\mu}(C) + \alpha \liminf_{n\to\infty} P^n\nu(C) \\ &\geq (1-\alpha)\gamma + \alpha\left(1 - \frac{\varepsilon}{2}\right) > \eta.\end{aligned} \tag{4.39}$$

Since $\mu \in \mathcal{M}_1$ is arbitrary, we have

$$\liminf_{n\to\infty} P^n\mu(C) \geq (1-\alpha)\gamma + \alpha\left(1-\frac{\varepsilon}{2}\right) > \eta \quad \text{for } \mu \in \mathcal{M}_1, \tag{4.40}$$

which contradicts the definition of η. Finally, we can easily check that for every $\mu \in \mathcal{M}_1$ the sequence $(P^n\mu)$ is tight. An application of Theorem 4.3 completes the proof. □

Theorem 4.12. *Every nonexpansive, locally and globally concentrating Markov operator P is asymptotically stable.*

Sketch of the proof. First, we check that the assumptions of Theorem 4.7 are satisfied. Indeed, let $\mu \in \mathcal{M}_1$ and $\varepsilon > 0$. Let A be a Borel bounded set such that $\mu(A) > 1 - \varepsilon/2$. Define

$$\tilde{\mu}(B) = \frac{\mu(B\cap A)}{\mu(A)} \quad \text{for } B \in \mathcal{B}. \tag{4.41}$$

Clearly, $\tilde{\mu} \in \mathcal{M}_1^A$ and $\mu \geq (1-\varepsilon/2)\tilde{\mu}$. Since P satisfies the globally concentrating property, there exists a Borel bounded set B such that

$$\liminf_{n\to\infty} P^n\tilde{\mu}(B) \geq 1 - \frac{\varepsilon}{2}. \tag{4.42}$$

Consequently,

$$\liminf_{n\to\infty} P^n\mu(B) \geq \left(1-\frac{\varepsilon}{2}\right)\left(1-\frac{\varepsilon}{2}\right) > 1 - \varepsilon, \tag{4.43}$$

which proves that the hypothesis of Theorem 4.7 holds.

Now we claim that

$$\liminf_{n\to\infty} \|P^n\mu_1 - P^n\mu_2\| = 0 \quad \forall \mu_1,\mu_2 \in \mathcal{M}_1. \tag{4.44}$$

Indeed, fix $\varepsilon > 0$ and $\mu_1,\mu_2 \in \mathcal{M}_1$. Let $\alpha > 0$ correspond to $\varepsilon/2$ according to the locally concentrating property. Let $k \in \mathbb{N}$ be such that $4(1-\alpha/2)^k < \varepsilon$ and let A be a Borel bounded set such that

$$P^n\mu_i(A) \geq 1 - \frac{(1-\alpha/2)^k}{3} \quad \text{for } n \in \mathbb{N},\ i = 1,2. \tag{4.45}$$

Similarly as in the proof of Theorem 4.7, we obtain

$$\varphi\left(A, \frac{(1-\alpha/2)^k}{3}\right) < \varepsilon. \tag{4.46}$$

From this and the definition of $\varphi(A,\eta)$, it follows that there exists $n_0 \in \mathbb{N}$ such that

$$\|P^{n_0}\mu_1 - P^{n_0}\mu_2\| < \varepsilon. \tag{4.47}$$

Since P is nonexpansive and $\varepsilon > 0$ is arbitrary, the last condition implies (4.44). The proof is completed. □

The details of the proofs of Theorems 4.9, 4.11, and 4.12 can be found in [24]. In the same spirit, we can prove the following theorem.

Theorem 4.13. *Every nonexpansive and concentrating Markov operator P is asymptotically stable.*

5. Semi-attractors given by IFSs

In this section, we develop the Barnsley-Hutchinson approach described in Section 3. In this purpose, we need the concept of Kuratowski topological limits.

Let (A_n) be a sequence of subsets of a metric space X. The *lower bound* $\mathrm{Li}\,A_n$ and the *upper bound* $\mathrm{Ls}\,A_n$ are defined by the following conditions. A point x belongs to $\mathrm{Li}\,A_n$ if for every $\varepsilon > 0$ there is an integer n_0 such that $A_n \cap B(x,\varepsilon) \neq \emptyset$ for $n \geq n_0$. A point x belongs to $\mathrm{Ls}\,A_n$ if for every $\varepsilon > 0$ the condition $A_n \cap B(x,\varepsilon) \neq \emptyset$ is satisfied for infinitely many n. If $\mathrm{Li}\,A_n = \mathrm{Ls}\,A_n$, we say that the sequence (A_n) is topologically convergent and we denote this common limit by $\mathrm{Lt}\,A_n$. It is called the *topological* (or *Kuratowski*) *limit* of the sequence (A_n).

Observe that $\mathrm{Li}\,A_n$ and $\mathrm{Ls}\,A_n$ are always closed sets. The basic properties of topological limits can be found in [10]. Here we recall that $\mathrm{Li}\,A_n = \mathrm{Li}(\mathrm{cl}\,A_n)$, $\mathrm{Ls}\,A_n = \mathrm{Ls}(\mathrm{cl}\,A_n)$, and $\mathrm{Li}\,A_n \subset B$ provided $A_n \subset B$ for sufficiently large n and B is closed. Moreover, every increasing sequence of sets (A_n) is topologically convergent and $\mathrm{Lt}\,A_n = \mathrm{cl}\bigcup_{n=1}^{\infty} A_n$. In the case when X is a compact space, $\mathrm{Lt}\,A_n = A$ if and only if the sequence (A_n) converges to A in the sense of the Hausdorff distance.

We say that an IFS $\{w_i : i \in I\}$ is *regular* if there is a nonempty subset I_0 of I such that an IFS $\{w_i : i \in I_0\}$ is asymptotically stable. The attractor of the subsystem $\{w_i : i \in I_0\}$ is called a *nucleus* of the system $\{w_i : i \in I\}$.

Proposition 5.1. *Let $\{w_i : i \in I\}$ be a regular IFS and let A_0 be a nucleus of this system. Let F be the multifunction given by (3.4). Then, the set*

$$A_*(A_0) = \mathrm{cl}\bigcup_{n=1}^{\infty} F^n(A_0) \tag{5.1}$$

has the following properties:

(i) $A_*(A_0) = \mathrm{Lt}\,F^n(A_0)$;
(ii) $F(A_*(A_0)) = A_*(A_0)$;
(iii) $A_*(A_0) \subset A$ *for every nonempty closed subset A of X such that $F(A) \subset A$.*

Proof. Since $A_0 \subset F(A_0)$, the sequence $(F^n(A_0))$ is increasing, whence (i) follows. Using the relation $\mathrm{cl}(w_i(\mathrm{cl}A)) = \mathrm{cl}\,w_i(A)$, we can easily verify (ii). Finally, let B be a bounded nonempty subset of A and F_0 the Barnsley-Hutchinson multifunction corresponding to $\{w_i : i \in I_0\}$. We have $F_0^n(B) \subset F^n(B) \subset F^n(A) \subset A$. It follows that $A_0 = \mathrm{Lt}\,F_0^n(B) \subset A$. Consequently, $F^n(A_0) \subset A$, $n \in \mathbb{N}$, whence (iii) follows. □

From Proposition 5.1, it follows that for a given regular IFS $\{w_i : i \in I\}$, the set $A_*(A_0)$ does not depend on the choice of the nucleus A_0.

The set

$$A_* = A_*(A_0) = \mathrm{Lt}\,F^n(A_0), \tag{5.2}$$

where A_0 is an arbitrary nucleus of an IFS $\{w_i : i \in I\}$, is called a *semiattractor* (or *semifractal*) corresponding to the regular IFS $\{w_i : i \in I\}$.

Using Proposition 5.1, we can prove the following theorem.

Theorem 5.2. *Let $\{w_i : i \in I\}$ be a regular IFS and let A_* be the corresponding semifractal given by (5.2). Then,*

(i) *A_* is the smallest nonempty closed set such that $\mathrm{cl}\,F(A_*) = A_*$;*
(ii) *$\mathrm{Lt}\,F^n(A) = A_*$ for every $A \subset A_*$, $A \neq \varnothing$.*

To establish the relation between semi-attractors and the support of invariant measures, we need some properties of a weakly convergent sequence of measures.

Let a sequence $(\mu_n) \subset \mathcal{M}$ and a measure $\mu \in \mathcal{M}$ be given. Then, the following conditions are equivalent:

(1) (μ_n) weakly converges to μ;
(2) $\limsup_{n\to\infty} \mu_n(A) \leq \mu(A)$ for every closed subset A of X;
(3) $\liminf_{n\to\infty} \mu_n(A) \geq \mu(A)$ for every open subset A of X.

The equivalence above is known as the Alexandrov theorem. This theorem allows us to prove the following property of supports.

Theorem 5.3. *Assume that a sequence $(\mu_n) \subset \mathcal{M}$ weakly converges to μ. Then,*

$$\mathrm{supp}\,\mu \subset \mathrm{Li}\,\mathrm{supp}\,\mu_n. \tag{5.3}$$

A sequence of measures $(\mu_n) \subset \mathcal{M}$ is called *condensed* at a point $x \in X$ if for every $\varepsilon > 0$ there is $\eta > 0$ such that

$$\inf\{\mu_n(B^o(x,\varepsilon)) : n \in \mathbb{N}_\eta\} > 0, \tag{5.4}$$

where

$$\mathbb{N}_\eta = \{n \in \mathbb{N} : B^o(x,\eta) \cap \mathrm{supp}\,\mu_n \neq \varnothing\}. \tag{5.5}$$

We say that a sequence (μ_n) is condensed on X if it is condensed at every point $x \in X$.

THEOREM 5.4. *Assume that a sequence of measures $(\mu_n) \subset \mathcal{M}$ weakly converges to a measure $\mu \in \mathcal{M}$. Then, the following conditions are equivalent:*

(i) *(μ_n) is condensed on X;*
(ii) *$\operatorname{Lt} \operatorname{supp} \mu_n = \operatorname{supp} \mu$.*

Sketch of the proof. (i)⇒(ii). In the presence of Theorem 5.3, it is sufficient to verify that $\operatorname{Ls} \operatorname{supp} \mu_n \subset \operatorname{supp} \mu$. For a contradiction, suppose that $x \in \operatorname{Ls} \operatorname{supp} \mu_n \setminus \operatorname{supp} \mu$ and let $\varepsilon > 0$ be such that $\mu(B(x, \varepsilon)) = 0$. Since the sequence (μ_n) is condensed on X, there exist $\eta > 0$ and $\alpha > 0$ such that $\operatorname{supp} \mu_n \cap B^o(x, \eta) \neq \emptyset$ and $\mu_n(B(x, \varepsilon)) \geq \alpha$. The Alexandrov theorem furnishes a contradiction.

To prove (ii)⇒(i), fix $x \in X$ and $\varepsilon > 0$. If $x \in \operatorname{supp} \mu$, then $\mu(B^o(x, \varepsilon)) > 0$ and by the Alexandrov theorem $\mu_n(B^o(x, \varepsilon)) > \mu(B^o(x, \varepsilon))/2$ for n sufficiently large. This implies that (μ_n) is condensed at x. If $x \notin \operatorname{supp} \mu$, hence $x \notin \operatorname{Ls} \operatorname{supp} \mu$. Thus there exist $\eta > 0$ and $n_0 \in \mathbb{N}$ such that $\operatorname{supp} \mu_n \cap B^o(x, \eta) = \emptyset$ for $n \geq n_0$, which again implies that (μ_n) is condensed at x. □

THEOREM 5.5. *Let X be a Polish space. Assume that an IFS with probabilities $\{(w_i, p_i) : i \in I\}$ is asymptotically stable and that the IFS $\{w_i : i \in I\}$ is regular. Then,*

$$A_* = \operatorname{supp} \mu_*, \tag{5.6}$$

where A_* is the semiattractor of $\{w_i : i \in I\}$ and μ_* is the invariant measure with respect to the IFS $\{(w_i, p_i) : i \in I\}$.

Proof. Let $u \in A_*$ and let δ_u be a δ-Dirac measure supported at u. Simple calculation shows that $\operatorname{supp} P^n \delta_u = F^n(u)$, $n \in \mathbb{N}$. Since $(P^n \delta_u)$ weakly converges to μ_*, by Theorems 5.3 and 5.2 we have

$$\operatorname{supp} \mu_* \subset \operatorname{Li} F^n(u) = \operatorname{Lt} F^n(u) = A_*. \tag{5.7}$$

Now let $u \in \operatorname{supp} \mu_*$. Clearly, $u \in A_*$ and by Theorem 5.2 we have

$$\operatorname{Lt} F^n(u) = A_*. \tag{5.8}$$

On the other hand, since $F(\operatorname{supp} \mu_*) \subset \operatorname{supp} \mu_*$, hence $F(u) \subset \operatorname{supp} \mu_*$. Consequently, $\operatorname{Ls} F^n(u) \subset \operatorname{supp} \mu_*$. From this and the first inclusion in (5.7), we have

$$\operatorname{Lt} F^n(u) = \operatorname{supp} \mu_*. \tag{5.9}$$

From (5.8) and (5.9), the statement of Theorem 5.5 follows. □

COROLLARY 5.6. *Let an IFS $\{(w_i, p_i) : i \in I\}$ satisfy condition (3.8). Clearly, the set I_0 of all $i \in I$ such that $L_i < 1$ is nonempty and the IFS $\{w_i : i \in I_0\}$ is asymptotically stable. Consequently, the IFS $\{w_i : i \in I\}$ is regular. According to Theorem 5.5, the support of the invariant measure with respect to the IFS $\{(w_i, p_i) : i \in I\}$ is equal to the semiattractor of $\{w_i : i \in I\}$.*

Theorem 5.5 makes in evidence the importance of the results concerning asymptotic stability of IFSs with probabilities. Now we give an example of such results.

THEOREM 5.7. *Let (X, ρ) be a Polish space. Assume that the transformations $w_i : X \to X$, $i \in I$, are Lipschitzian on every bounded subset of X. Moreover, assume that there is $i \in I$ such that the function w_i is a strict contraction. Then, there exist continuous functions $p_i : X \to (0,1)$, $i \in I$, satisfying $\sum_{i \in I} p_i(x) = 1$ for $x \in X$ and such that the IFS with probabilities $\{(w_i, p_i) : i \in I\}$ is asymptotically stable.*

Proof. The proof, based on Theorem 4.12, is rather technical and so it is omitted here. □

For further results concerning the semi-attractors given by IFSs and detailed proofs of the theorem above, see [13, 14, 15].

6. Semi-attractors of multifunctions

Let X be a metric space. A multifunction $F : X \to X$ is a subset of $X \times X$ such that for every $x \in X$ the set $F(x) = \{y : (x, y) \in F\}$ is nonempty. For $A \subset X$ we define $F^-(A) = \{x \in X : F(x) \cap A \neq \emptyset\}$.

A multifunction F is called *Borel measurable* or simply *measurable* (resp., *lower semicontinuous* or shortly *l.s.c.*) if the set $F^-(A)$ is Borel (resp., open) for every open subset A of X.

For the convenience of the reader, we recall some well-known properties of lower semicontinuous multifunctions.

PROPOSITION 6.1. *The following conditions are equivalent:*

(i) *F is l.s.c.;*
(ii) *$F(\mathrm{cl}\, A) \subset \mathrm{cl}\, F(A)$ for every $A \subset X$;*
(iii) *for every sequence $(x_n) \subset X$,*

$$\lim x_n = x \Longrightarrow F(x) \subset \mathrm{Li}\, F(x_n); \tag{6.1}$$

(iv) *for every sequence $(x_n) \subset X$,*

$$\lim x_n = x \Longrightarrow F(x) \subset \mathrm{Ls}\, F(x_n). \tag{6.2}$$

A set $A \subset X$ is called *subinvariant* (resp., *invariant*) with respect to a multifunction F if $F(A) \subset A$ (resp., $F(A) = A$).

We say that a multifunction F is *asymptotically stable* if there is a closed subset A_0 of X such that

(i) $\mathrm{cl}\, F(A_0) = A_0$;
(ii) $\mathrm{Lt}\, F^n(A) = A_0$ for every nonempty bounded subset A of X.

Given a multifunction $F : X \to X$, consider the set

$$C = \bigcap_{x \in X} \mathrm{Li}\, F^n(x). \tag{6.3}$$

If the set C is nonempty, then the multifunction F is called *asymptotically semistable* and the set C is called the *semiattractor* of F.

Theorem 6.2. *Assume that F is an asymptotically stable l.s.c. multifunction with the semiattractor C. Then, the following conditions hold:*

(i) *$C \subset \mathrm{Li}\, F^n(A)$ for every $A \subset X$, $A \neq \emptyset$;*
(ii) *$\mathrm{cl}\, F(C) = C$;*
(iii) *$\mathrm{Lt}\, F^n(A) = C$ for every $A \subset C$, $A \neq \emptyset$;*
(iv) *$C \subset A$ for every nonempty closed subset A of X such that $F(A) \subset A$.*

Proof. Condition (i) is obvious. From (6.3) it follows that

$$F(C) \subset \bigcap_{x \in X} F\big(\mathrm{Li}\, F^n(x)\big). \tag{6.4}$$

Using Proposition 6.1 and the semicontinuity of F, it is easy to verify that

$$F\big(\mathrm{Li}\, F^n(x)\big) \subset \mathrm{Li}\, F^n(x). \tag{6.5}$$

From the last inclusion it follows that

$$F(C) \subset C. \tag{6.6}$$

Since C is a closed set, we also have $\mathrm{cl}\, F(C) \subset C$. To prove the opposite inclusion, observe that $F^n(C) \subset F(C)$ for $n \geq 1$, which, in turn, implies that $\mathrm{Li}\, F^n(C) \subset \mathrm{cl}\, F(C)$. Since $C \subset \mathrm{Li}\, F^n(C)$, this completes the proof of (ii).

To verify (iii), observe that (6.6) implies that $\mathrm{Ls}\, F^n(C) \subset C$. Thus, for an arbitrary nonempty set $A \subset C$ we have

$$C \subset \mathrm{Li}\, F^n(A) \subset \mathrm{Ls}\, F^n(A) \subset \mathrm{Ls}\, F^n(C) \subset C. \tag{6.7}$$

Condition (iv) can be verified as follows. The inclusion $F(A) \subset A$ implies that $F^n(A) \subset A$ for $n \in \mathbb{N}$. Consequently,

$$C \subset \mathrm{Li}\, F^n(A) \subset A. \tag{6.8}$$

□

Theorem 6.3. *Let $F : X \to X$ be a l.s.c. multifunction. Assume that there exists a l.s.c. and asymptotically semistable multifunction $F_0 : X \to Y$ such that $F_0(x) \subset F(x)$, $x \in X$. Then, F is asymptotically semistable and its semiattractor C is given by the formula*

$$C = \operatorname{Lt} F^n(C_0) = \operatorname{cl} \bigcup_{n=1}^{\infty} F^n(C_0), \tag{6.9}$$

where C_0 is the semiattractor of F_0.

Proof. Since $C_0 \subset C$, the multifunction F is asymptotically semistable. The first equality in (6.9) follows from condition (iii) of Theorem 6.2 with $A = C_0$. Now observe that $F^n(C_0) \subset C$ for $n \in \mathbb{N}$. Hence

$$\operatorname{cl} \bigcup_{n=1}^{\infty} F^n(C_0) \subset C. \tag{6.10}$$

Using this inclusion and the first equality in (6.9), we obtain the second equality of (6.9). The proof is completed. □

7. Markov multifunctions

A mapping $\pi : X \times \mathcal{B} \to [0,1]$ is called a *transition function* if $\pi(x,\cdot)$ is a probability measure for every $x \in X$ and $\pi(\cdot,A)$ is a measurable function for every $A \in \mathcal{B}$.

We say that a transition function $\pi : X \times \mathcal{B} \to [0,1]$ is *Fellerian* if the function $x \to \pi(x,\cdot)$ from X into $\mathcal{M}_1$ (endowed with the Fortet-Mourier norm) is continuous.

Given a transition function $\pi : X \times \mathcal{B} \to [0,1]$, the corresponding Markov operator P, its dual U, and the *Markov set function* Γ are given by

$$\begin{gathered} P\mu(A) = \int_X \pi(x,A)\mu(dx) \quad \text{for } A \in \mathcal{B}, \\ Uf(x) = \int_X f(y)\pi(x,dy) \quad \text{for } f \in B(X), \\ \Gamma(x) = \operatorname{supp} \pi(x,\cdot). \end{gathered} \tag{7.1}$$

The function Γ is also called the *Markov multifunction generated* by π, or shortly the *support* of π (see [14]). It is easy to see that Γ is closed valued and measurable. Vice versa we have the following theorem.

Theorem 7.1. *Let $F : X \to X$ be a measurable, closed-valued multifunction. Then, there exists a transition function $\pi : X \times \mathcal{B} \to [0,1]$ such that F is the support of π.*

Proof. According to the Kuratowski-Ryll Nardzewski theorem (see [11]), there exists a sequence (f_n) of measurable functions $f_n : X \to X$ such that

$$F(x) = \operatorname{cl}\{f_n(x) : n \in \mathbb{N}\} \quad \text{for } x \in X. \tag{7.2}$$

We define the function $\pi : X \times \mathcal{B} \to [0,1]$ by

$$\pi(x, A) = \sum_{n=1}^{\infty} p_n \delta_{f_n(x)}(A), \tag{7.3}$$

where (p_n) is a sequence of positive numbers such that $\sum_{n=1}^{\infty} p_n = 1$ and δ_u stands for the δ-Dirac measure supported at u. A simple calculation shows that π is a transition function and that F is the support of π. □

Theorem 7.2. *Assume that $\pi : X \times \mathcal{B} \to [0,1]$ is a Fellerian transition function. Then, the corresponding Markov multifunction Γ is l.s.c.*

Proof. Fix an $x \in X$ and consider a sequence $(x_n) \subset X$ converging to x. Since π is Fellerian, the corresponding sequence of measures $(\pi(x_n, \cdot))$ converges weakly to the measure $\pi(x, \cdot)$. By virtue of Theorem 5.3, we have $\Gamma(x) \subset \operatorname{Li}\Gamma(x_n)$. Thus the statement of Theorem 7.2 follows from Proposition 6.1. □

Theorem 7.3. *Assume that $F : X \to X$ is a l.s.c. multifunction with closed values. Then, there exists a Fellerian transition function $\pi : X \times \mathcal{B} \to [0,1]$ such that F is the support of π.*

Proof. Consider a multifunction $\Phi : X \to \mathcal{M}_1$ given by the formula

$$\Phi(x) = \{\mu \in \mathcal{M}_1 : \operatorname{supp}\mu \subset F(x)\}. \tag{7.4}$$

Clearly, Φ is convex and closed valued. It is easy to verify that Φ is l.s.c. Observe that $\mathcal{M}_1$ is a convex subset of the linear space $\mathcal{M}_s$ and $\mathcal{M}_1$ is complete with respect to the Fortet-Mourier norm (see [4, 6]). Thus the conditions of the Michael selection theorem (see [18]) are satisfied, and so there exists a sequence (φ_n) of continuous functions $\varphi_n : X \to \mathcal{M}_1$ such that

$$\Phi(x) = \operatorname{cl}\{\varphi_n(x) : n \in \mathbb{N}\}. \tag{7.5}$$

Let (p_n) be a sequence of positive numbers such that $\sum p_n = 1$. Define $\pi : X \times \mathcal{B} \to [0,1]$ by

$$\pi(x, A) = \sum_{n=1}^{\infty} p_n \varphi_n(x)(A). \tag{7.6}$$

Obviously, π is a transition function. To complete the proof, it suffices to verify that F is equal to the support of π. □

In order to prove the next result, we need two simple lemmas concerning the support of the measure $P\mu$ (see [14]).

LEMMA 7.4. *Let $P : \mathcal{M} \to \mathcal{M}$ be a Fellerian operator. If $\mu_1, \mu_2 \in \mathcal{M}$ and* $\operatorname{supp}\mu_1 \subset \operatorname{supp}\mu_2$, *then* $\operatorname{supp}P\mu_1 \subset \operatorname{supp}P\mu_2$.

LEMMA 7.5. *Let $P : \mathcal{M} \to \mathcal{M}$ be a Markov operator corresponding to a Fellerian transition function $\pi : X \times \mathcal{B} \to [0,1]$. Further, let Γ denote the support of π. Then, for every $\mu \in \mathcal{M}$ and $n \in \mathbb{N}$,*

$$\operatorname{supp}P^n\mu = \operatorname{cl}\Gamma^n(\operatorname{supp}\mu). \tag{7.7}$$

THEOREM 7.6. *If a Fellerian Markov operator P is asymptotically stable, then the corresponding Markov multifunction Γ is asymptotically semistable and*

$$C = \operatorname{supp}\mu_*, \tag{7.8}$$

where C is the semiattractor of Γ and μ_* is the measure invariant with respect to P.

Proof. Fix an arbitrary $x \in X$ and let $\mu = \delta_x$. Since P is asymptotically stable, the sequence $(P^n\mu)$ converges weakly to μ_*. By Theorem 5.3 and Lemma 7.5, we have

$$\operatorname{supp}\mu_* \subset \operatorname{Li}\operatorname{supp}P^n\mu = \operatorname{Li}\Gamma^n(x). \tag{7.9}$$

This implies that $\operatorname{supp}\mu_* \subset C$.

To prove the opposite inclusion, fix a point $z \notin \operatorname{supp}\mu_*$ and choose $\varepsilon > 0$ such that

$$B(z,\varepsilon) \cap \operatorname{supp}\mu_* = \varnothing. \tag{7.10}$$

Let $x \in \operatorname{supp}\mu_*$ and $\mu = \delta_x$. By Lemmas 7.4 and 7.5, we have

$$\Gamma^n(x) \subset \operatorname{supp}P^n\mu \subset \operatorname{supp}P^n\mu_* = \operatorname{supp}\mu_* \quad \text{for } n \in \mathbb{N}. \tag{7.11}$$

Thus

$$\Gamma^n(x) \cap B(z,\varepsilon) = \varnothing. \tag{7.12}$$

It follows that $z \notin \operatorname{Li}\Gamma^n(x)$ and consequently $z \notin C$. The proof is complete. □

8. Random iteration algorithms

In this section, we show an effective way for the construction of semifractals, using the well-known fact that an IFS with probabilities determines in a natural way a dynamical system.

Assume that there is given a family of Lipschitz functions $w_i : X \to X$, $i = 1,\dots,N$ and a probability vector $(p_1,\dots,p_N)$. Moreover, assume that we have given a probability space $(\Omega,\Sigma,\text{prob})$ and a sequence of random elements $\xi_n : \Omega \to I$, $n \in \mathbb{N}$, equally distributed, namely,

$$\text{prob}\,(\xi_n = i) = p_i \quad \text{for } i = 1,\dots,N,\ n \in \mathbb{N}. \tag{8.1}$$

The dynamical system corresponding to the IFS $\{(w_i, p_i) : i \in I\}$ is described by the formula

$$x_{n+1} = w_{\xi_n}(x_n) \quad \text{for } n \in \mathbb{N}, \tag{8.2}$$

where x_0 is a given initial point independent of the sequence (ξ_n).

Setting

$$\mu_n(A) = \text{prob}\,(x_n \in A) \quad \text{for } A \in \mathcal{B}, \tag{8.3}$$

we obtain a sequence of distribution (μ_n). It is well known that the operator P given by (1.2) is the transition function for this sequence, that is,

$$\mu_{n+1} = P\mu_n. \tag{8.4}$$

The following fact is basic for the computational construction of fractals. If the IFS under consideration satisfies condition (3.8), then for every $\epsilon > 0$ there exist n_0 and k_0 such that $\text{dist}(\{x_n,\dots,x_{n+k}\}, A_*) < \epsilon$ for every $n > n_0$ and $k > k_0$ (here, dist stands for the Hausdorff distance).

Using the Elton ergodic theorem (see [5]), we can obtain the following result.

Theorem 8.1. *Let $\{(w_i, p_i) : i \in I\}$ be an IFS satisfying (3.8) and let A_* be the semifractal corresponding to $\{w_i : i \in I\}$. Then, for every $x_0 \in A_*$,*

$$\text{Lt}\,\{x_0,\dots,x_n\} = A_* \quad a.s., \tag{8.5}$$

where the sequence (x_n) is defined by (8.2).

A disadvantage of Theorem 8.1 is that we must start from a point x_0 belonging to the set A_*. However, if all the maps w_i are nonexpansive, then the initial point x_0 can be arbitrarily chosen in X. In fact, we have the following theorem.

Theorem 8.2. *Let* $\{(w_i, p_i) : i \in I\}$ *be an IFS satisfying (3.8) and let* $L_i \leq 1$ *for* $i \in I$. *Then, for every* $x_0 \in X$ *and* $\varepsilon > 0$, *there exist* n_0 *and* k_0 *such that*

$$\text{prob}\left(\text{dist}\left(\{x_n, \dots, x_{n+k}\}, A_*\right) \leq \varepsilon\right) \geq 1 - \varepsilon \tag{8.6}$$

for every $n \geq n_0$ *and* $k \geq k_0$, *where* (x_n) *denotes the sequence defined by (8.2).*

Theorems 8.1 and 8.2 suggest a natural numerical algorithm to construct semifractals. Namely, given a family of transformations $\{w_i : i \in I\}$, we look for probabilities $\{p_i : i \in I\}$, for which the iterated function system with probabilities $\{(w_i, p_i) : i \in I\}$ is asymptotically stable (for one can use Theorem 5.7). Then, it is sufficient to find a point x_0 which belongs to a nucleus of the IFS $\{(w_i, p_i) : i \in I\}$ and construct a sequence (x_n) by the formula

$$x_{n+1} = w_{i_n}(x_n), \tag{8.7}$$

where i_n are randomly chosen step by step in such a way that the probability of choosing $i_n = k$ is equal to $p_k(x_n)$.

If all maps w_i are nonexpansive, by virtue of Theorem 8.2 the started point x_0 can be arbitrarily chosen in X.

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References

[1] M. F. Barnsley, *Fractals Everywhere*, Academic Press, Massachusetts, 1993.

[2] M. F. Barnsley, S. G. Demko, J. H. Elton, and J. S. Geronimo, *Invariant measures for Markov processes arising from iterated function systems with place-dependent probabilities*, Ann. Inst. H. Poincaré Probab. Statist. **24** (1988), no. 3, 367–394.

[3] E. Çinlar, *Introduction to Stochastic Processes*, Prentice-Hall, New Jersey, 1975.

[4] R. M. Dudley, *Probabilities and Metrics*, Lecture Notes Series, vol. 45, Aarhus Universitet, Denmark, 1978.

[5] J. H. Elton, *An ergodic theorem for iterated maps*, Ergodic Theory Dynam. Systems **7** (1987), no. 4, 481–488.

[6] S. N. Ethier and T. G. Kurtz, *Markov Processes*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons, New York, 1986.

[7] S. R. Foguel, *The Ergodic Theory of Markov Processes*, Van Nostrand Mathematical Studies, no. 21, Van Nostrand Reinhold, New York, 1969.

[8] R. Fortet and E. Mourier, *Convergence de la répartition empirique vers la répartition théorique*, Ann. Sci. Ecole Norm. Sup. (3) **70** (1953), 267–285 (French).

[9] J. E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), no. 5, 713–747.
[10] K. Kuratowski, *Topology. Vol. I*, Academic Press, New York, 1966.
[11] K. Kuratowski and C. Ryll-Nardzewski, *A general theorem on selectors*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **13** (1965), 397–403.
[12] A. Lasota and M. C. Mackey, *Chaos, Fractals and Noise, Stochastic Aspects of Dynamics*, Applied Mathematical Sciences, Springer-Verlag, New York, 1994.
[13] A. Lasota and J. Myjak, *Semifractals*, Bull. Polish Acad. Sci. Math. **44** (1996), no. 1, 5–21.
[14] ———, *Markov operators and fractals*, Bull. Polish Acad. Sci. Math. **45** (1997), no. 2, 197–210.
[15] ———, *Semifractals on Polish spaces*, Bull. Polish Acad. Sci. Math. **46** (1998), no. 2, 179–196.
[16] A. Lasota and J. A. Yorke, *Lower bound technique for Markov operators and iterated function systems*, Random Comput. Dynam. **2** (1994), no. 1, 41–77.
[17] A. A. Markov, *Extensions of the law of large numbers for dependent variables*, Izv. Fiz.-Mat. Obshch. Kazansk. Univ. **15** (1906), 135–156 (Russian).
[18] E. Michael, *Continuous selections. I*, Ann. of Math. (2) **63** (1956), 361–382.
[19] E. Nummelin, *General Irreducible Markov Chains and Nonnegative Operators*, Cambridge Tracts in Mathematics, vol. 83, Cambridge University Press, Cambridge, 1984.
[20] D. Revuz, *Markov Chains*, North-Holland Publishing, Amsterdam, 1975.
[21] T. Szarek, *Invariant measures for iterated function systems*, Ann. Polon. Math. **75** (2000), no. 1, 87–98.
[22] ———, *The stability of Markov operators on Polish spaces*, Studia Math. **143** (2000), no. 2, 145–152.
[23] ———, *Invariant measures for Markov operators with applications to function systems*, Studia Math. **154** (2003), 207–222.
[24] ———, *Invariant measures for nonexpansive Markov operators on Polish spaces*, to appear in Dissertationes Math., 2003.

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