THREE PERIODIC SOLUTIONS TO AN EIGENVALUE PROBLEM FOR A CLASS OF SECOND-ORDER HAMILTONIAN SYSTEMS

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We establish a multiplicity result to an eigenvalue problem related to second-order Hamiltonian systems. Under new assumptions, we prove the existence of an open interval of positive eigenvalues in which the problem admits three distinct periodic solutions.

1. Introduction

In this paper, we consider the following eigenvalue problem:

$$\begin{gathered} \ddot{u} - A(t)u = \lambda \nabla F(t,u) \quad \text{a.e. in } [0,T], \\ u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0, \end{gathered} \tag{1.1}$$

with $T > 0$ and $\lambda \geq 0$.

Throughout the paper, we assume that $A : [0,T] \to \mathbb{R}^N \times \mathbb{R}^N$ is a mapping into the space of N-order symmetric matrices with $A \in L^\infty([0,T])$ and there exists $\mu > 0$ such that

$$(A(t)x, x) \geq \mu |x|^2 \tag{1.2}$$

for almost everywhere $t \in [0,T]$ and each $x \in \mathbb{R}^N$. The function $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is measurable in t, for each $x \in \mathbb{R}^N$, continuously differentiable in x, for almost every $t \in [0,T]$, and satisfying the following condition:

$$\max\{|F(t,x)|, |\nabla F(t,x)|\} \leq a(|x|)b(t) \quad (\text{a.e. } t \in [0,T],\ x \in \mathbb{R}^N) \tag{1.3}$$

for some $a \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $b \in L^1(0,T;\mathbb{R}_+)$. For convenience, we also assume that $F(t,0) \equiv 0$.

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Under such assumptions, the functionals

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\left(\int_0^T |\dot{u}(t)|^2 dt + \int_0^T (A(t)u(t), u(t))\,dt\right),\\ \Psi(u) &= \int_0^T F(t, u(t))\,dt\end{aligned} \tag{1.4}$$

are continuously differentiable and weakly lower semicontinuous on H_T^1 (see [11]), where

$$H_T^1 = \left\{ u : [0,T] \longrightarrow \mathbb{R}^N \,\middle|\, \begin{array}{l} u \text{ is absolutely continuous,} \\ u(0) = u(T),\ \dot{u} \in L^2(0,T;\mathbb{R}^N) \end{array} \right\} \tag{1.5}$$

is a Hilbert space with norm defined by

$$\|u\| = \left(\int_0^T |u(t)|^2 dt + \int_0^T |\dot{u}(t)|^2 dt\right)^{1/2} \tag{1.6}$$

for all $u \in H_T^1$. The critical points of the functional $\Phi + \lambda\Psi$ are solutions of problem (1.1).

Then we define

$$k(A) = \sup_{u \in H_T^1 \setminus \{0\}} \frac{\|u\|_\infty}{\left(\int_0^T (A(t)u(t), u(t))\,dt + \int_0^T |\dot{u}(t)|^2 dt\right)^{1/2}}, \tag{1.7}$$

where $\|\cdot\|_\infty$ denotes the sup-norm in $C^0(0,T;\mathbb{R}^N)$.

Our main result, Theorem 2.1, gives conditions that assure the existence of an open interval $\Lambda \subset [0,\infty[$ and real number $\rho > 0$ such that (1.1), for each $\lambda \in \Lambda$, admits at least three distinct solutions whose norms are less than ρ. Moreover, we are able to give information about the location of such interval Λ since it results that $\Lambda \subseteq [0,\overline{a}]$, where $\overline{a}$ is a positive real number whose dependence from data is given.

The proof of Theorem 2.1 is essentially based on a result due to Bonanno [3], which specifies the three critical points theorem obtained by Ricceri [12, Theorem 1], which has been widely applied to obtain multiplicity results for some Dirichlet and Neumann problems [2, 4, 5, 6, 7, 9, 10, 12, 13].

Theorem 1.1 [3, Theorem 2.1]. *Let X be a separable and reflexive real Banach space and let $\Phi, J : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals. Assume that there exists $x_0 \in X$ such that $\Phi(x_0) = J(x_0)$ and $\Phi(x) \geq 0$ for every $x \in X$, and that there exist $x_1 \in X$ and $r > 0$ such that*

(i) $r < \Phi(x_1)$;

(ii) $\sup_{x \in \Phi^{-1}(]-\infty,r])} J(x) < r(J(x_1)/\Phi(x_1))$.

Further, put

$$\overline{a} = \frac{hr}{r(J(x_1)/\Phi(x_1)) - \sup_{x\in\Phi^{-1}(]-\infty,r])} J(x)}, \quad h > 1, \tag{1.8}$$

and assume that the functional $\Phi - \lambda J$ is sequentially weakly lower semicontinuous and satisfies the Palais-Smale condition and

(iii) $\lim_{\|x\|\to+\infty}(\Phi(x) - \lambda J(x)) = +\infty$ *for every* $\lambda \in [0,\overline{a}]$.

Then there exist an open interval $\Lambda \subseteq [0,\overline{a}]$ and a positive real number ρ such that, for each $\lambda \in \Lambda$, the equation

$$\Phi'(x) - \lambda\Psi'(x) = 0 \tag{1.9}$$

admits at least three solutions in X whose norms are less than ρ.

The authors who gave the major contribution to the existence of three solutions for second-order Hamiltonian systems are Tang and Wu [14, 15, 16]. Another interesting result on this topic has been recently obtained by Faraci in [8].

The condition present in all the papers of Tang is as follows: there exist $r > 0$ and an integer $k \geq 0$ such that

$$-\frac{1}{2}(k+1)^2 w^2 |x|^2 \leq F(t,x) - F(t,0) \leq -\frac{1}{2}k^2 w^2 |x|^2 \tag{1.10}$$

for each $|x| \leq r$ and almost everywhere $t \in [0,T]$, where $w = 2\pi/T$.

Tang, using condition (1.10) together with coercive assumption on F (weakened in [16]) or in presence of a sublinear behavior of the nonlinearity, proved that the problem

$$\begin{gathered} \ddot{u} = \nabla F(t,u) \quad \text{a.e. in } [0,T], \\ u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0 \end{gathered} \tag{1.11}$$

admits three solutions.

Problem (1.1), with $\lambda = 1$, has been studied by Faraci [8] when $F(t,x) = b(t)V(x)$, with $b \in L^1(0,T;\mathbb{R}_+) \setminus \{0\}$ and $V \in C^1(\mathbb{R}^N)$.

We also cite the result recently obtained by Barletta and Livrea in [1], where the authors deal with problem (1.1) when the nonlinearity is of the type $b(t)\nabla G(x)$.

Section 3 is dedicated to a comparison with some results cited above in order to stress the novelty of our conditions. In particular, we give a simple example which fits all the hypotheses of Proposition 3.1 but does not satisfy condition (1.10) and conditions (2) and (3) of [8, Theorem 2.1].

2. Main result

In this section, we state and prove our main result.

Theorem 2.1. *Besides the hypotheses in the introduction, assume also that the following conditions hold:*

(i) *there exist $r > 0$ and $c \in \mathbb{R}^N$ such that*

$$\begin{gathered}\int_0^T (A(t)c,c)\,dt > r,\\ \frac{\int_0^T F(t,c)dt}{\int_0^T (A(t)c,c)\,dt} < \frac{1}{r}\int_0^T \inf_{|x|\le k(A)\sqrt{r}} F(t,x)dt;\end{gathered} \tag{2.1}$$

(ii) *put*

$$\overline{a} = \frac{hr}{2\left(\int_0^T \inf_{|x|\le k(A)\sqrt{r}} F(t,x)dt - r\left(\int_0^T F(t,c)dt/\int_0^T (A(t)c,c)\,dt\right)\right)}, \tag{2.2}$$

with $h > 1$, and assume that there exist $M > 0$ and $\alpha \in L^1(0,T;\mathbb{R}_+)$, with $\|\alpha\|_{L^1} < 1/2k(A)^2\overline{a}$, such that

$$F(t,x) \ge -\alpha(t)|x|^2 \tag{2.3}$$

for every $x \in \mathbb{R}^N$, with $|x| > M$, and almost everywhere $t \in [0,T]$.

Then there exist an open interval $\Lambda \subset [0,\overline{a}]$ and a real number $\rho > 0$ such that, for each $\lambda \in \Lambda$, problem (1.1) has at least three distinct solutions whose norms in H_T^1 are less than ρ.

Proof. Our end is to apply Theorem 1.1 with $X = H_T^1$ and $J = -\Psi$, where the functionals Φ and Ψ have been defined in the introduction.

Taking into account (ii), for each $\lambda \in [0,\overline{a}]$ and $u \in H_T^1$, one has

$$\begin{aligned}\Phi(u)+\lambda\Psi(u) &= \Phi(u)+\lambda\int_{\{t\,|\,|u(t)|\le M\}} F(t,u(t))\,dt + \lambda\int_{\{t\,|\,|u(t)|>M\}} F(t,u(t))\,dt\\ &\ge \Phi(u) - \overline{a}\sup_{|x|\le M} a(|x|)\int_0^T b(t)dt - \overline{a}\int_0^T \alpha(t)\,|u(t)|^2 dt\\ &\ge \Phi(u) - \overline{a}\sup_{|x|\le M} a(|x|)\int_0^T b(t)dt - \overline{a}\|\alpha\|_{L^1}\|u\|_\infty^2\\ &\le \left(1-\overline{a}k(A)^2\|\alpha\|_{L^1}\right)\Phi(u) - \overline{a}\sup_{|x|\le M} a(|x|)\int_0^T b(t)dt.\end{aligned} \tag{2.4}$$

Hence, for every $\lambda \in [0,\overline{a}]$,

$$\lim_{\|u\|\to+\infty} \Phi(u)+\lambda\Psi(u) = +\infty. \tag{2.5}$$

Put $u_1(t) = c$ for every $t \in [0,T]$. We have $\Phi(0) = \Psi(0) = 0$ and $\Phi(u_1) > r/2$. Moreover, since

$$\left\{u \in H_T^1 \mid \Phi(u) \le \frac{r}{2}\right\} \subseteq \{u \in H_T^1 \mid \|u\|_\infty \le k(A)\sqrt{r}\} \tag{2.6}$$

and by (2.3), it results that

$$\begin{aligned} \inf_{\Phi^{-1}(]0,r/2])} \Psi &\ge \inf_{\|u\|_\infty \le k(A)\sqrt{r}} \Psi(u) \ge \int_0^T \inf_{|x| \le k(A)\sqrt{r}} F(t,x)dt \\ &> r \frac{\int_0^T F(t,c)dt}{\int_0^T (A(t)c,c)dt} = \frac{r}{2}\frac{\Psi(u_1)}{\Phi(u_1)}. \end{aligned} \tag{2.7}$$

Finally, we observe that the Gâteaux derivative of Ψ is compact due to the compact embedding of H_T^1 into $C^0(0,T;\mathbb{R}^N)$. The Gâteaux derivative of Φ admits a continuous inverse by [17, Theorem 26 A]. All the hypotheses of Theorem 1.1 hold, hence the thesis is its consequence. □

3. Some consequences

A consequence of Theorem 2.1 is the following proposition.

Proposition 3.1. *Let $a \in L^1(0,T;\mathbb{R}_+) \setminus \{0\}$ and $G, H \in C^1(\mathbb{R}^N)$ with $G(0) = H(0) = 0$. Assume that*

$$\liminf_{|x|\to+\infty} \frac{G(x)}{|x|^2} \ge 0, \qquad \liminf_{|x|\to+\infty} \frac{H(x)}{|x|^2} \ge 0, \tag{3.1}$$

and there exists $c \in \mathbb{R}^N \setminus \{0\}$ such that

$$\frac{G(c)}{k(A)^2 \int_0^T (A(t)c,c)dt} < \liminf_{|x|\to 0^+} \frac{G(x)}{|x|^2}. \tag{3.2}$$

Then, there exists $\delta > 0$ such that, for every $b \in L^1(0,T;\mathbb{R}_+)$ with $\int_0^T b(t)dt < \delta$, there exist an open interval $\Lambda \subset [0,+\infty[$ and a real number $\rho > 0$ so that, for every $\lambda \in \Lambda$, the problem

$$\begin{aligned} \ddot{u} - A(t)u &= \lambda\big(a(t)\nabla G(u) + b(t)\nabla H(u)\big) \quad \text{a.e. in } [0,T], \\ u(T) - u(0) &= \dot{u}(T) - \dot{u}(0) = 0, \end{aligned} \tag{3.3}$$

admits at least three distinct solutions whose norms are less than ρ.

Proof. We have

$$F(t,x) = a(t)G(x) + b(t)H(x). \tag{3.4}$$

Fix $\epsilon > 0$; there exists $M > 0$ such that, for each $x \in \mathbb{R}^N$ with $|x| > M$,

$$G(x) \geq -\epsilon |x|^2, \qquad H(x) \geq -\epsilon |x|^2. \tag{3.5}$$

Hence, one has

$$F(t,x) \geq -\epsilon \big(a(t) + b(t)\big) |x|^2 \tag{3.6}$$

for $|x| > M$ and almost everywhere $t \in [0,T]$. Then, owing to the arbitrariness of ϵ, condition (ii) is satisfied.

One has

$$\frac{\int_0^T F(t,c)dt}{\int_0^T (A(t)c,c)dt} = \frac{\int_0^T a(t)dt}{\int_0^T (A(t)c,c)dt} G(c) + \frac{\int_0^T b(t)dt}{\int_0^T (A(t)c,c)dt} H(c). \tag{3.7}$$

Moreover, we have

$$\begin{aligned} \frac{1}{r}\int_0^T \inf_{|x|\leq k(A)\sqrt{r}} F(t,x)dt \geq{}& \frac{1}{r}\left(\int_0^T a(t)dt\right) \inf_{|x|\leq k(A)\sqrt{r}} G(x) \\ &+ \frac{1}{r}\left(\int_0^T b(t)dt\right) \inf_{|x|\leq k(A)\sqrt{r}} H(x) \end{aligned} \tag{3.8}$$

for every $r > 0$. It is easily seen that there exists a sequence of positive real numbers $\{r_n\}_{n\in\mathbb{N}}$ decreasingly convergent to zero and such that

$$\lim_{n\to\infty} \frac{\inf_{|x|\leq k(A)\sqrt{r_n}} G(x)}{k(A)^2 r_n} \geq \liminf_{|x|\to 0^+} \frac{G(x)}{|x|^2}. \tag{3.9}$$

Hence, by hypothesis, it follows that

$$\lim_{n\to\infty} \frac{\inf_{|x|\leq k(A)\sqrt{r_n}} G(x)}{r_n} > \frac{G(c)}{\int_0^T (A(t)c,c)dt}. \tag{3.10}$$

So fix $r_{\overline{n}} < \int_0^T (A(t)c,c)dt$ for which one has

$$\frac{\inf_{|x|\leq k(A)\sqrt{r_{\overline{n}}}} G(x)}{r_{\overline{n}}} > \frac{G(c)}{\int_0^T (A(t)c,c)dt}, \tag{3.11}$$

condition (ii) is satisfied when $\int_0^T b(t)dt$ is small enough. □

Remark 3.2. Although the hypotheses of Proposition 3.1 do not allow us to give a bound for $\overline{a}$, it can be calculated in concrete applications as the proof implicitly shows.

Now we want to make a comparison with some results cited in the introduction. For the reader's convenience, we cite the Faraci's main theorem and, subsequently, the one of Barletta and Livrea.

Theorem 3.3 [8, Theorem 2.1]. *Assume that*

(1) *there exist* $\sigma > 0$ *and* $u_0 \in \mathbb{R}^N$ *with*

$$|u_0| < \sqrt{\frac{\sigma}{\sum_{i,j} \|a_{i,j}\|_{L^\infty(0,T)}} T} \tag{3.12}$$

such that

$$V(u_0) = \inf_{|x| \le c_1 \sqrt{\sigma}} V(x), \tag{3.13}$$

where c_1 *is the constant of the embedding of* H_T^1 *into* $C^0(0,T;\mathbb{R}^N)$*;*

(2) *one has*

$$\liminf_{|x| \to +\infty} \frac{V(x)}{|x|^2} > -\frac{1}{2c_1^2 \|b\|_{L^1(0,T)}}; \tag{3.14}$$

(3) *there exists* $u_1 \in \mathbb{R}^N$ *such that*

$$V(u_0) - V(u_1) > \frac{\sum_{i,j} \|a_{i,j}\|_{L^\infty(0,T)} T}{2\|b\|_{L^1(0,T)}} |u_1|^2. \tag{3.15}$$

Then the problem

$$\begin{gathered} \ddot{u} - A(t)u = b(t)\nabla V(u) \quad a.e.\ [0,T], \\ u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0, \end{gathered} \tag{3.16}$$

admits at least three solutions in H_T^1.

Theorem 3.4 [1, Theorem 1]. *Assume that* $G(0) \ge 0$ *and that there exist a positive constant* d *and* $c \in \mathbb{R}^N$ *such that*

(a) $|c| > d/k(A)\sqrt{\mu T}$;

(b) $(\max_{|x| \le d} G(x))/d^2 < (1/k(A)^2 T \sum_{i,j} \|a_{ij}\|_\infty)(G(c)/|c|^2)$.

Put

$$\lambda^* = \frac{pd^2}{(d^2/k(A)^2 T \sum_{i,j} \|a_{ij}\|_\infty)(G(c)/|c|^2) - k(A)^2 \max_{|x| \le d} G(x)} \tag{3.17}$$

with $p > 1$ *and suppose that*

(c) $\limsup_{|x| \to +\infty} (G(x)/|x|^2) < 1/2k(A)^2 \lambda^*$.

Then, for every function $b \in L^1(0,T;\mathbb{R}_+) \setminus \{0\}$*, there exist an open interval* $\Lambda \subseteq [0, \lambda^*/\|b\|_{L^1}]$ *and a positive real number* ρ *such that for every* $\lambda \in \Lambda$*, the problem*

$$\begin{gathered} \ddot{u} = A(t)u - \lambda b(t)\nabla G(u) \quad a.e.\ in\ [0,T], \\ u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0, \end{gathered} \tag{3.18}$$

admits at least three solutions in H_T^1 *whose norms are less than* ρ.

The last theorem may be obtained as a consequence of Theorem 2.1. In fact, in our settings, it is $F(t,x) = -b(t)(G(x) - G(0))$. Then, taking into account that

$$\mu T|c|^2 \le \int_0^T (A(t)c,c)\,dt \le T\sum_{i,j} \|a_{i,j}\|_\infty |c|^2, \tag{3.19}$$

it is easily seen that (a) and (b) imply condition (i) and (c) is equivalent to (ii).

Finally, we give the following simple application of Proposition 3.1.

Example 3.5. Consider the following eigenvalue problem:

$$\begin{gathered} \ddot{u}_i - \sum_j a_{i,j}(t)u_j = \lambda\left[a(t)\left(u_i^3 + 3u_i^2\right) + b(t)\right]e^{u_i}, \quad i = 1,2,\dots,N \text{ a.e. in } [0,T], \\ u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0, \end{gathered} \tag{3.20}$$

with $\lambda \ge 0$, $a,b \in L^1(0,T;\mathbb{R}_+) \setminus \{0\}$, and $a_{i,j}(t)$ denoting the (i,j)-entry of an N-order matrix $A(t)$ satisfying (1.2).

The potential F is defined by

$$F(t,x) = a(t)\sum_{i=1}^N x_i^3 e^{x_i} + b(t)\left(\sum_i e^{x_i} - N\right), \tag{3.21}$$

then $G(x) = \sum_i x_i^3 e^{x_i}$ and $H(x) = \sum_i e^{x_i} - N$.

It is easily seen that

$$\liminf_{|x|\to+\infty} \frac{G(x)}{|x|^2} \ge 0, \qquad \liminf_{|x|\to+\infty} \frac{H(x)}{|x|^2} \ge 0, \tag{3.22}$$

$$\lim_{|x|\to 0^+} \frac{G(x)}{|x|^2} = 0. \tag{3.23}$$

Moreover $G(c) < 0$ for some $c \in \mathbb{R}^N \setminus \{0\}$. So, all the hypotheses of Proposition 3.1 are satisfied without any other conditions on $A(t)$ and $a(t)$.

Condition (1.10) does not hold for this problem. In fact, in our case, the function F of condition (1.10) is given by

$$F(t,x) = \frac{1}{2}(A(t)x,x) + \lambda a(t)\sum_{i=1}^N x_i^3 e^{x_i} + \lambda b(t)\left(\sum_i e^{x_i} - N\right). \tag{3.24}$$

Hence (1.10) is not satisfied because $F(t,x)/|x|^2$ is not bounded in any neighborhood of the origin when $\lambda b(t) \ne 0$.

Now suppose that $b(t) = ba(t)$ with $b \in \mathbb{R}_+$. Theorem 3.3 cannot be applied. In fact

$$V(x) = \sum_{i=1}^{N} (x_i^3 + b)e^{x_i} - Nb, \tag{3.25}$$

for every $b \in \mathbb{R}_+$, has only one local minimum which is global.

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