PROPERNESS AND TOPOLOGICAL DEGREE FOR GENERAL ELLIPTIC OPERATORS

V. VOLPERT AND A. VOLPERT

Received 29 November 2001

The paper is devoted to general elliptic operators in Hölder spaces in bounded or unbounded domains. We discuss the Fredholm property of linear operators and properness of nonlinear operators. We construct a topological degree for Fredholm and proper operators of index zero.

1. Introduction

In this paper, we study elliptic operators in Hölder spaces: the Fredholm property for linear operators, properness and topological degree for nonlinear operators. The construction of the degree uses both the Fredholm property and the properness. In this section we briefly discuss main ideas underlying normal solvability, properness and topological degree for elliptic operators.

1.1. Normal solvability. Consider a linear operator L acting from a Banach space $E_0(\Omega)$ to another space $E(\Omega)$. Here Ω denotes a domain in $\mathbb{R}^n$, and the notation $E(\Omega)$ is used for a Banach space of functions defined in Ω. We are basically interested in the case where the domain Ω is unbounded though all results remain applicable and in many cases even simpler for bounded domains. Suppose that $E_0(G)$ is compactly embedded in a space $E'(G)$ for any bounded domain G and that the estimate

$$\|u\|_{E_0(\Omega)} \le K\left(\|Lu\|_{E(\Omega)} + \|u\|_{E'(\Omega)}\right) \tag{1.1}$$

holds with a constant K independent of u.

As an example, we can take the spaces

$$E_0(\Omega) = \{u \in C^{2+\alpha}(\Omega),\ u|_{\partial\Omega} = 0\}, \qquad E(\Omega) = C^{\alpha}(\Omega), \qquad E'(\Omega) = C^2(\Omega) \tag{1.2}$$

Abstract and Applied Analysis 2003:3 (2003) 129–181
2000 Mathematics Subject Classification: 35J30, 47H11
URL: http://dx.doi.org/10.1155/S1085337503204073

and the operator

$$-Lu = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} a_j(x)\frac{\partial u}{\partial x_j} + c(x)u. \tag{1.3}$$

Then estimate (1.1) follows from the Schauder estimate. It holds under certain conditions on the domain Ω and on the coefficients of the operator (see [1, 2]). If we use known estimate for Hölder spaces

$$\|u\|_{E'(\Omega)} \leq \epsilon \|u\|_{E_0(\Omega)} + c_\epsilon \|u\|_{C(\Omega)} \tag{1.4}$$

with a small ϵ and a constant c_ϵ depending on ϵ, then (1.1) becomes equivalent to the Schauder estimate.

Assume that the operator L satisfies the following condition: if $f_n \to f_0$ in $E(\Omega)$, $Lu_n = f_n$, $\|u_n\|_{E_0(\Omega)} \leq M$, and $u_n \to u_0$ in $E'(\Omega)$, then $u_0 \in E_0(\Omega)$ and $Lu_0 = f_0$. For the example considered above this property is satisfied.

It is known that if this condition is satisfied and (1.1) holds, then the operator is normally solvable, that is, its image is closed, and has a finite-dimensional kernel (see [19]). This is a simple though an important result valid in the case if the domain Ω is bounded. If it is unbounded, we should add one more condition. To formulate it we define limiting problems. In the simplest case where $\Omega = \mathbb{R}^1$,

$$-Lu = a(x)u'' + b(x)u' + c(x)u, \tag{1.5}$$

and the functions a, b, and c have limits at infinity

$$a_\pm = \lim_{x\to\pm\infty} a(x), \qquad b_\pm = \lim_{x\to\pm\infty} b(x), \qquad c_\pm = \lim_{x\to\pm\infty} c(x), \tag{1.6}$$

the limiting operators are

$$-L^\pm u = a_\pm u'' + b_\pm u' + c_\pm u. \tag{1.7}$$

If in addition to the previous two conditions we require that the limiting problems

$$L^\pm u = 0 \tag{1.8}$$

have only zero solutions in $E_0(\Omega)$, then the operator L is normally solvable with a finite-dimensional kernel. The values λ such that the equation

$$L^{\pm}u = \lambda u \tag{1.9}$$

has a nonzero solution, belongs to the essential spectrum of the operator L.

Limiting problems and the normal solvability for linear elliptic operators in unbounded domains were studied in a number of works for $\mathbb{R}^n$, and for domains with cylindrical and conical ends (see [3, 21, 25, 26, 33] and the references therein).

In this paper, we consider general elliptic operators in the Doulgis-Nirenberg sense [7]. Since the domain Ω is also generic, we define in Section 2 limiting problems, which includes limiting domains and limiting operators. To define limiting domains for an unbounded domain Ω, consider a sequence $x_m \in \Omega$, $|x_m| \to \infty$. Let $\chi(x)$ be the characteristic function of Ω. Consider the shifted functions $\chi(x+x_m)$ and the corresponding domains Ω_m. Thus we have a sequence of domains. If their boundaries $\partial\Omega_m$ are uniformly Hölder continuous, then from the sequence Ω_m we can choose a subsequence Ω_{mk} converging to some limiting function Ω_* in any ball $B \subset \mathbb{R}^n$, that is,

$$\Omega_{mk} \cap B \longrightarrow \Omega_* \cap B. \tag{1.10}$$

If we take an increasing sequence of balls, we can extend the limiting domain Ω_* to the whole $\mathbb{R}^n$. It can depend on the choice of the sequence x_m and of converging subsequences Ω_{mk}.

To define limiting operators, we consider the shifted coefficients and choose subsequences converging to a limiting function on any bounded set. Limiting operators are operators with the limiting coefficients. Thus we define limiting problems.

We prove in Section 2 that the operator is normally solvable with a finite-dimensional kernel if and only if the limiting problems do not have nonzero solutions (we will call it Condition NS). If we require that it is Fredholm, that is, the codimension of its image is also finite, then the limiting operators are invertible.

This result gives a useful property for the class of operators, which coincide with their limiting operators: their spectrum consists only of the essential spectrum, that is, there are no points of the spectrum where the operator is Fredholm. In particular, this property applies for operators with periodic or quasiperiodic coefficients in cylindrical domains.

1.2. Index. In what follows we are mostly interested in Fredholm operators of index zero for which the topological degree will be constructed. The index of elliptic operators in unbounded domains is computed only for some particular cases (see [5, 15, 22] and the references therein). However we do not need here an explicit computation of the index. We can use the stability of the index for semi-Fredholm operators. Consider the operator $L_\lambda = L + \lambda I$ and assume that Condition NS is satisfied for all $\lambda \geq 0$. Then it is semi-Fredholm. If it is invertible for large λ, which is the case for elliptic operators, then the operator L has a zero index.

The condition that the operator L_λ satisfies Condition NS is much more restrictive than the same condition just for the operator L. We will see however that it is exactly the same condition, which is used for the degree construction.

It is interesting to note that there are different homotopy classes of Fredholm operators of index zero. For example, if we consider the operator (1.5), then its essential spectrum is given by two parabolas

$$-\lambda = -a_\pm \xi^2 + b_\pm i\xi + c_\pm, \quad \xi \in \mathbb{R}^1. \tag{1.11}$$

In both cases, if c_+ and c_- are negative and if they are positive, the index equals 0 [5]. However, they are not homotope in this class of operators and they are different from the point of view of the degree construction (see Section 1.3 below). If they are both positive, Condition NS for the operator L_λ is not satisfied for some $\lambda > 0$.

1.3. Properness. Everywhere below we will say that an operator $A(u) : E_0 \to E$ is proper if the intersection of an inverse image of a compact set with any bounded closed ball $B \subset E_0$ is compact. We recall that a linear operator is proper if and only if it is normally solvable with a finite-dimensional kernel.

Assume that a nonlinear operator $A(u)$ is differentiable in the following sense: for each $u_0 \in E_0$ there exists a linear operator $A'(u_0)$ such that

$$A(u) - A(u_0) = A'(u_0)(u - u_0) + \phi(u, u_0), \tag{1.12}$$

where

$$||\phi(u, u_0)||_E \leq K(u, u_0)||u - u_0||_{E'} \tag{1.13}$$

and $K(u, u_0) \to 0$ as $u \to u_0$ in E'.

We note that this condition is more restrictive than the Fréchet differentiability because the norm $\|u - u_0\|_{E_0}$ in the right-hand side of (1.13) is replaced by $\|u - u_0\|_{E'}$.

Similar to linear operator discussed above, we assume that the operator $A(u)$ satisfies the following condition: if $f_n \to f_0$ in E,

$$A(u_n) = f_n, \tag{1.14}$$

$u_n \in B$, $u_n \to u_0$ in E', then $u_0 \in E_0$ and

$$A(u_0) = f_0. \tag{1.15}$$

If these conditions are satisfied and the domain Ω is bounded, then the operator $A(u)$ is proper. Indeed, from (1.14) and the compact embedding $E_0(\Omega)$ in $E'(\Omega)$ follows (1.15). From the differentiability

$$A'(u_0)(u_n - u_0) = f_n - f_0 - \phi(u_n, u_0), \tag{1.16}$$

and from (1.1), $u_n \to u_0$ in $E(\Omega)$.

We call a nonlinear operator $A(u)$ elliptic if a linearized operator $A'(u_0)$ is elliptic. A precise definition of ellipticity used in this paper is given in Section 3.

Nonlinear elliptic operators satisfy all the conditions above. Their properness is known for scalar equations in bounded domains [34].

If the domain Ω is unbounded, then elliptic operators are not generically proper. Consider the following example:

$$A(u) = u'' + F(u), \quad A : C^{2+\alpha}(\mathbb{R}) \longrightarrow C^{\alpha}(\mathbb{R}). \tag{1.17}$$

Here $F(u)$ is a sufficiently smooth function such that

$$\begin{gathered} F(0) = 0, \qquad F(u) < 0 \quad \text{for } 0 < u < u_0, \quad F(u) > 0 \quad \text{for } u > u_0, \\ \int_0^1 F(u)\,du = 0. \end{gathered} \tag{1.18}$$

Then there exists a solution $u(x)$ of the equation $A(u) = 0$ such that $u(x) \to 0$ as $x \to \pm\infty$. Obviously, all functions $u(x+h)$, $h \in \mathbb{R}$, are also solutions of this problem, and this family of solutions is not compact. Therefore, the inverse image of the set $\{0\}$ is not compact. The choice of Hölder spaces is not essential here. The same problem arises in Sobolev spaces: $u(x)$ decays exponentially at infinity if $F'(0) \neq 0$, and the solution is integrable with its derivatives.

To avoid this problem we introduce weighted spaces. In the example above, it is the weighted Hölder spaces $C_\mu^{2+\alpha}(\mathbb{R})$ and $C_\mu^{\alpha}(\mathbb{R})$ with the norms

$$\|u\|_{C_\mu^{2+\alpha}(\mathbb{R})} = \|u\mu\|_{C^{2+\alpha}(\mathbb{R})}, \qquad \|u\|_{C_\mu^{\alpha}(\mathbb{R})} = \|u\mu\|_{C^{\alpha}(\mathbb{R})}, \tag{1.19}$$

respectively. The weight function $\mu(x)$ can be taken for example $1 + x^2$. The precise conditions on it will be given in Section 2.1. Here we note that the weight should be weaker than the exponential one in order not to lose solutions decaying exponentially at infinity.

If we consider now the operator $A(u)$ in the weighted spaces, then the family of functions $u(x+h)$ is still a solution of the problem $A(u) = 0$. However the norm $\|u(x+h)\mu(x)\|$ tends to infinity as $h \to \pm\infty$. So in every bounded ball $B \subset C_\mu^{2+\alpha}(\mathbb{R})$, the inverse image of $\{0\}$ is compact.

This example explains the situation with the properness of elliptic operators in unbounded domains. We will see in Section 3 that weighted spaces allow the convergence $u_n \to u_0$ in $E'(\Omega)$. The strong convergence in $E(\Omega)$ will follow from Condition NS.

We note that various particular cases of properness for nonlinear elliptic operators in weighted Hölder spaces are considered in [3, 33]. Weighted Sobolev spaces are considered in [30, 31, 32] where some estimates of semilinear elliptic operators from below are obtained. Fredholm property and properness follow from these estimates, and the degree is constructed using the approach developed in [27]. It is also possible to consider spaces without weight. However, in this case it is necessary to impose some additional conditions on the operators [25].

In Section 3, we prove properness of general elliptic problems. We consider weighted Hölder spaces with an infinitely differentiable, positive weight function $\mu(x)$ such that $\mu(x) \to \infty$ and $|D^\beta \mu(x)/\mu(x)| \to 0$ as $|x| \to \infty$. Here β is a multi-index, $|\beta| > 0$. The conditions on the weight function mean in particular that its growth at infinity is slower than exponential. Therefore, it does not change the limiting operators and the location of the essential spectrum. Condition NS, which is a necessary and sufficient condition of properness of linear elliptic operators, is also sufficient for properness of nonlinear operators.

1.4. Topological degree. One of the approaches to define the topological degree is based on the theory of Fredholm operators [4, 8, 9, 10, 11, 23]. Consider an operator $A : E_0 \to E$ assuming that it is Fredholm, proper, and that it has a zero index. Let $\mathfrak{D} \subset E_0$ be a bounded domain, $A(u) \neq 0$ for $u \in \partial\mathfrak{D}$. Let $a \in E$ be a regular value with a sufficiently small norm. Its existence is known (see [24, 28]). Then there exists a finite number of solutions $u_1, \dots, u_N \in \mathfrak{D}$ of the equation

$$A(u) = a, \tag{1.20}$$

and there is no solutions of this equation at the boundary $\partial\mathfrak{D}$.

For each solution u_j, $j = 1, \dots, N$, we can associate a value

$$o(u_j) = 1 \qquad \text{or} \qquad o(u_j) = -1 \tag{1.21}$$

called orientation. The topological degree $\gamma(A, \mathfrak{D})$ can be defined as

$$\gamma(A, \mathfrak{D}) = \sum_{j=1}^{N} o(u_j). \tag{1.22}$$

It should be shown that it does not depend on the choice of a regular value a, and that it is a homotopy invariant.

This scheme has been realized in a number of works under different conditions on spaces, operators, and with different definitions of the orientation [4, 10, 11, 23].

Let $U(u_j)$ be a small neighbourhood of the point u_j in E_0, which does not contain other solutions of (1.20). Then by the definition of the index $\operatorname{ind}(A, u_j)$

of a stationary point u_j, we have

$$\gamma(A, U(u_j)) = \operatorname{ind}(A, u_j). \tag{1.23}$$

On the other hand, if we define the topological degree through the orientation, we obtain

$$\gamma(A, U(u_j)) = o(u_j). \tag{1.24}$$

For finite-dimensional mappings, for the Leray-Schauder degree, and for some of its generalizations

$$o(u_j) = (-1)^{\nu}, \tag{1.25}$$

where ν is a number of negative eigenvalues of the operator $A'(u_j)$ together with their multiplicities (see also [16, 27], where the index of stationary points is also calculated in terms of sum of multiplicities of eigenvalues of some operators). If this definition of orientation is applicable, any other definition should be equivalent if the degree is unique [33].

The relation between the orientation and the eigenvalues of the linearized operator imposes some condition on the spectrum. Indeed, the homotopy invariance of the degree implies that $\gamma(A_\tau, U_\tau(u_j^\tau))$ remains constant for the operator A_τ depending on parameter τ if the linearized operator $A'_\tau(u_j^\tau)$ does not have zero eigenvalues. Therefore, the number of negative eigenvalues modulus 2 should not change either. Generally speaking, this condition can be guaranteed only if the essential spectrum does not intersect the negative half-axis. Thus, not only the operator $A'(u_j)$ should be Fredholm but also $A'(u_j) + \sigma I$ for all $\sigma \geq 0$.

The topological degree for Fredholm, proper operators of index zero satisfying the condition above was constructed in [10] in the case of bounded operators acting in the same space. However, elliptic operators can be considered as bounded if acting in different spaces, or unbounded if acting in the same space.

Suppose that $A : E_0 \to E$ is a bounded operator, and $E_0 \neq E$. Then the construction of the degree should be modified. First of all, instead of the eigenvalues of the linearized operator we can consider negative solutions of the equation

$$A'(u_j)u - \lambda J u = 0, \tag{1.26}$$

where J is a normalization operator. If it is invertible, we can consider the operator

$$\tilde{A} = J^{-1}A : E_0 \longrightarrow E_0 \tag{1.27}$$

acting in the space E_0. The degree $\tilde{\gamma}$ for it can be defined through the degree for the operator A

$$\tilde{\gamma}(\tilde{A}, \mathcal{D}) = \gamma(A, \mathcal{D}). \tag{1.28}$$

If the operator $\tilde{A}' + \sigma I$ is Fredholm for all $\sigma \geq 0$, then the operator $A' + \sigma J$ is Fredholm for all $\sigma \geq 0$.

An important class of linear elliptic operators are operators whose essential spectrum does not intersect the real positive half-axis, that is, $A' - \lambda I$ is Fredholm for all $\lambda \leq 0$. For some particular cases these two classes of operators are equivalent [3, 6].

In the general case, a priori we cannot expect that these two classes of Fredholm operators coincide. Denote $J_k u = \Delta u - ku$ the normalization operator to show its dependence on k. Let Φ_k be the class of linear elliptic operators L such that $L + \sigma J_k$ is Fredholm for any $\sigma \geq 0$. Then

$$\cdots \subset \Phi_k \subset \Phi_{k+1} \subset \cdots . \tag{1.29}$$

Put

$$\Phi = \bigcup_{k=1}^{\infty} \Phi_k. \tag{1.30}$$

We show in [33] that for second order elliptic operators in unbounded cylinders under some additional conditions, the class Φ coincides with all operators L such that $L - \lambda I$ is Fredholm for all $\lambda \leq 0$. As above, it allows to consider the operators acting in the same space. For each Φ_k we define the degree γ_k and prove that the degree is unique. Therefore the degree is defined for the whole class Φ.

For general elliptic operators this construction becomes complicated, and it is not quite clear what class of operators it allows to consider. So in this paper we construct the degree for bounded operators acting in different spaces directly, without reduction to the same space (Section 4). The class of operators we consider here consists of all Fredhom and proper operators $A(u)$ such that the operator $A' - \lambda I$ is Fredholm for all $\lambda \leq 0$ and it is invertible for λ sufficiently large. In fact, we consider a pair $(A(u), B(u))$ of operators acting from a space $E_0(\Omega)$ to a product of spaces $E(\Omega) = E_1(\Omega) \times E_2(\partial\Omega)$. Here $B(u)$ corresponds to the boundary operator. The nonlinear boundary conditions change the degree construction. Indeed, after linearization we obtain the operator A', and include the set of functions u satisfying the condition $B'u = 0$ in the domain of the operator A'. Therefore, during homotopies not only the operators but also the spaces are changed. In this case the previous degree constructions cannot be used.

Another remark concerns the orientation of Fredholm operators. It is well known that topological degree cannot be constructed in the class of all proper C^1 Fredholm mappings of index zero. The class of mappings should be also orientable (see [4, 12]). Therefore, the problem is how to describe the class of orientable mappings. For example, consider the class of all elliptic operators which are Fredholm with the zero index, proper and sufficiently smooth. The question whether this class of operators is orientable remains open. In this paper, we introduce a class of nonlinear elliptic operators which is orientable. The orientation is defined through the number of negative eigenvalues under the assumption that the essential spectrum does not intersect the real positive half-axis (Section 4). The condition on the essential spectrum allows to prove that it is a homotopy invariant and to construct the topological degree.

2. Linear operators

2.1. Operators and spaces. Let $\beta = (\beta_1, \dots, \beta_n)$ be a multi-index, β_i nonnegative integers, $|\beta| = \beta_1 + \cdots + \beta_n$, $D^\beta = D_1^{\beta_1} \cdots D_n^{\beta_n}$, $D_i = \partial/\partial x_i$. We consider the following operators:

$$
\begin{aligned}
A_i u &= \sum_{k=1}^{p} \sum_{|\beta| \le \beta_{ik}} a_{ik}^{\beta}(x) D^\beta u_k \quad (i = 1, \dots, p),\ x \in \Omega, \\
B_i u &= \sum_{k=1}^{p} \sum_{|\beta| \le \gamma_i k} b_{ik}^{\beta}(x) D^\beta u_k \quad (i = 1, \dots, r),\ x \in \partial\Omega.
\end{aligned}
\tag{2.1}
$$

Following [2] we consider integers $s_1, \dots, s_p; t_1, \dots, t_p; \sigma_1, \dots, \sigma_r$ such that

$$
\beta_{ij} \le s_i + t_j, \quad i, j = i, \dots, p; \qquad \gamma_{ij} \le \sigma_i + t_j, \quad i = 1, \dots, r,\ j = 1, \dots, p; \qquad s_i \le 0.
\tag{2.2}
$$

We suppose that the number $m = \sum_{i=1}^{p}(s_i + t_i)$ is even and put $r = m/2$.

We assume that the problem is elliptic [1, 7, 29], that is, the ellipticity condition

$$
\det \left(\sum_{|\beta| = \beta_{ik}} a_{ik}^{\beta}(x) \xi^\beta \right)_{ik=1}^{p} \ne 0, \quad \beta_{ik} = s_i + t_k,
\tag{2.3}
$$

is satisfied for any $\xi \in \mathbb{R}^n$, $\xi \ne 0$, $x \in \bar{\Omega}$, as well as the condition of proper ellipticity and the Shapiro-Lopatinskii conditions (or the complementing boundary condition in [1]). Here $\xi = (\xi, \dots, \xi_n)$, $\xi^\beta = \xi_1^\beta \cdots \xi_n^\beta$. The system is uniformly elliptic if the last determinant is bounded from below by a positive constant for all $|\xi| = 1$ and $x \in \bar{\Omega}$.

Everywhere below $C^{k+\alpha}(\Omega)$ denotes the standard Hölder space of functions bounded in Ω together with their derivatives up to the order k, and the latter satisfies the Hölder condition uniformly in x.

Denote by E_0 a space of vector-valued functions $u(x) = (u_1(x), \dots, u_p(x))$, $u_j \in C^{l+t_j+\alpha}(\Omega)$, $j = 1, \dots, p$, where l and α are given numbers, $l \ge \max(0, \sigma_i)$, $0 < \alpha < 1$. Therefore,

$$
E_0 = C^{l+t_1+\alpha}(\Omega) \times \cdots \times C^{l+t_p+\alpha}(\Omega).
\tag{2.4}
$$

The domain Ω is supposed to be of the class $C^{l+\lambda+\alpha}$, where $\lambda = \max(-s_i, -\sigma_i, t_j)$, and the coefficients of the operator satisfy the following regularity conditions:

$$a_{ij}^{\beta} \in C^{l-s_i+\alpha}(\Omega), \qquad b_{ij}^{\beta} \in C^{l-\sigma_i+\alpha}(\partial\Omega). \tag{2.5}$$

The operator A_i acts from E_0 to $C^{l-s_i+\alpha}(\Omega)$, and B_i from E_0 to $C^{l-\sigma_i+\alpha}(\partial\Omega)$. Denote $A = (A_1,\dots,A_p)$, $B = (B_1,\dots,B_r)$. Then

$$A : E_0 \longrightarrow E_1, \qquad B : E_0 \longrightarrow E_2, \qquad (A,B) : E_0 \longrightarrow E, \tag{2.6}$$

where $E = E_1 \times E_2$,

$$\begin{aligned} E_1 &= C^{l-s_1+\alpha}(\Omega) \times \cdots \times C^{l-s_p+\alpha}(\Omega), \\ E_2 &= C^{l-\sigma_1+\alpha}(\partial\Omega) \times \cdots \times C^{l-\sigma_r+\alpha}(\partial\Omega). \end{aligned} \tag{2.7}$$

We will consider weighted Hölder spaces $E_{0,\mu}$ and E_μ with the norms

$$\|u\|_{E_{0,\mu}} = \|u\mu\|_{E_0}, \qquad \|u\|_{E_\mu} = \|u\mu\|_E. \tag{2.8}$$

We use also the notation $C_\mu^{k+\alpha}$ for a weighted Hölder space with the norm $\|u\|_{C_\mu^{k+\alpha}} = \|u\mu\|_{C^{k+\alpha}}$.

We suppose that the weight function μ is a positive infinitely differentiable function defined for all $x \in \mathbb{R}^n$, $\mu(x) \to \infty$ as $|x| \to \infty$, $x \in \Omega$, and

$$\left| \frac{1}{\mu(x)} D^\beta \mu(x) \right| \longrightarrow 0, \quad |x| \longrightarrow \infty,\ x \in \Omega \tag{2.9}$$

for any multi-index β, $|\beta| > 0$. In fact, we will use its derivative only up to a certain order (see Sections 2.6 and 3.2).

Operator (A,B) considered in weighted Hölder spaces acts from $E_{0,\mu}$ into E_μ.

2.2. Limiting domains. In this section, we define limiting domains for unbounded domains in $\mathbb{R}^n$, show their existence, and study some of their properties. We consider an unbounded domain $\Omega \subset \mathbb{R}^n$, which satisfies the following condition.

Condition D. For each $x_0 \in \partial\Omega$, there exists a neighbourhood $U(x_0)$ such that

(1) $U(x_0)$ contains a sphere with the radius δ and the center x_0, where δ is independent of x_0,
(2) there exists a homeomorphism $\psi(x;x_0)$ of the neighbourhood $U(x_0)$ on the unit sphere $B = \{y : |y| < 1\}$ in $\mathbb{R}^n$ such that the images of $\Omega \cap U(x_0)$ and $\partial\Omega \cap U(x_0)$ coincide with $B_+ = \{y : y_n > 0,\ |y| < 1\}$ and $B_0 = \{y : y_n = 0,\ |y| < 1\}$, respectively,

(3) the function $\psi(x;x_0)$ and its inverse belong to the Hölder space $C^{k+\alpha}$, with $k=\max(1,l+\lambda)$. Their $\|\cdot\|_{k+\alpha}$-norms are bounded uniformly in x_0.

For definiteness we suppose that $\delta<1$.

Remark 2.1. In what follows, we suppose that ψ is extended such that $\psi\in C^{k+\alpha}(\mathbb{R}^n)$ and $\|\psi\|_{C^{k+\alpha}(\mathbb{R}^n)}\le M$ with M independent of x_0.

It is easy to see that δ and ψ in Condition D can be chosen such that this requirement can be satisfied. Indeed, denote by V_δ the sphere with the center at x_0 and the radius δ and let $W_\delta=\psi(V_\delta)$.

Obviously, there exists a sphere Q_ε with the center at $y_0=\psi(x_0;x_0)$ and the radius ε such that $Q_\varepsilon\subset W_\delta$ and ε does not depend on x_0. Indeed, denote $\varphi=\psi^{-1}$ and let y_1 be an arbitrary point on the boundary of W_δ. We have $\delta=|\varphi(y_1)-\varphi(y_0)|\le K|y_1-y_0|$, where K is the Lipschitz constant which does not depend on x_0. Let $\varepsilon<\delta/K$. We have $|y_1-y_0|>\varepsilon$ which proves the existence of the desired sphere Q_ε.

Let $\widetilde{U}(x_0)=\varphi(Q_\varepsilon)$. There exists a sphere S with the center at x_0 and the radius $\widetilde{\delta}$ such that $S\subset\widetilde{U}(x_0)$ and $\widetilde{\delta}$ does not depend on x_0. Indeed, let x_1 be an arbitrary point of the boundary of $\widetilde{U}(x_0)$. Then we have $\varepsilon=|\psi(x_0;x_0)-\psi(x_1;x_0)|\le K_1|x_0-x_1|$, where K_1 is the Lipschitz constant of ψ, which does not depend on x_0. So for $\widetilde{\delta}<\varepsilon/K_1$ we have $|x_0-x_1|>\widetilde{\delta}$, which proves the existence of the mentioned sphere S.

We can take $\widetilde{U}(x_0)$ as a new neighborhood of x_0 and $\widetilde{\psi}(x;x_0)=(1/\varepsilon)(\psi(x;x_0)-\psi(x_0;x_0))$ as a new function ψ. Since $\widetilde{\psi}(x;x_0)$ is defined in the sphere V_δ it can be extended on $\mathbb{R}^n$.

To define convergence of domains we use the following Hausdorff metric space. Let M and N denote two nonempty closed sets in $\mathbb{R}^n$. Denote

$$\varsigma(M,N)=\sup_{a\in M}\rho(a,N),\qquad \varsigma(N,M)=\sup_{b\in N}\rho(b,M),\tag{2.10}$$

where $\rho(a,N)$ denotes the distance from a point a to a set N, and let

$$\varrho(M,N)=\max\big(\varsigma(M,N),\varsigma(N,M)\big).\tag{2.11}$$

We denote Ξ a metric space of bounded closed nonempty sets in $\mathbb{R}^n$ with the distance given by (2.11). We say that a sequence of domains Ω_m converges to a domain Ω in Ξ_{loc} if

$$\varrho(\bar{\Omega}_m\cap\bar{B}_R,\bar{\Omega}\cap\bar{B}_R)\longrightarrow 0,\quad m\longrightarrow\infty\tag{2.12}$$

for any $R>0$ and $B_R=\{x:|x|<R\}$. Here the bar denotes the closure of domains.

Definition 2.2. Let $\Omega\subset\mathbb{R}^n$ be an unbounded domain, $x_m\in\Omega$, $|x_m|\to\infty$ as $m\to\infty$, let $\chi(x)$ be a characteristic function of Ω, and let Ω_m be a shifted domain defined by the characteristic function $\chi_m(x)=\chi(x+x_m)$. We say that Ω_* is a *limiting domain* of the domain Ω if $\Omega_m\to\Omega_*$ in Ξ_{loc} as $m\to\infty$.

We denote $\Lambda(\Omega)$ the set of all limiting domains of the domain Ω (for all sequences x_m). We will show below that if Condition D is satisfied, then the limiting domains exist and also satisfy this condition.

Theorem 2.3. *If a domain Ω satisfies Condition D, then there exists a function $f(x)$ defined in $\mathbb{R}^n$ such that*

(1) $f(x) \in C^{k+\alpha}(\mathbb{R}^n)$,
(2) $f(x) > 0$ *if and only if* $x \in \Omega$,
(3) $|\nabla f(x)| \geq 1$ *for* $x \in \partial\Omega$,
(4) $\min(d(x), 1) \leq |f(x)|$, *where $d(x)$ is the distance from x to $\partial\Omega$.*

Proof. There exists a number N such that from the covering $U(x_0)$ of ∂D we can choose a countable subcovering U_i such that the following conditions are satisfied:

(i) $\bigcup_i U_i$ cover the $\delta/2$-neighborhood of $\partial\Omega$,
(ii) any N distinct sets U_i have an empty intersection.

Indeed, denote by V the $\delta/2$-neighborhood of $\partial\Omega$. Obviously for any point $x_0 \in V$ there exists a point $x_0' \in \partial\Omega$ such that $B_{\delta/2}(x_0) \subset B_\delta(x_0') \subset U(x_0')$. Here and in what follows $B_r(x)$ denotes a ball in $\mathbb{R}_n$ with the center at x and with radius r. So we have a covering $U'(x_0) = U(x_0')$ of V such that the centers of balls are at the boundary of the domain. Denote $\Gamma = \bigcup U'$.

Consider an ε-mesh in $\mathbb{R}^n$. We denote by K the union of all n-dimensional ε-intervals of this mesh which have a nonempty intersection with V. For any $Q_i \in K$, we take a point $x_i \in Q_i \cap V$ $(i = 1, 2, \ldots)$ and consider the neighbourhood $U_i \in \Gamma$, which contains the point x_i. We suppose that ε is taken such that the diameter of Q_i is less than $\delta/2$. Then $Q_i \subset U_i$ and

$$V \subset \Gamma_0 = \bigcup_i U_i. \tag{2.13}$$

Therefore the covering Γ_0 satisfies condition (i).

To each $Q_i \in K$ corresponds no more than one neighbourhood $U_i \in \Gamma_0$. From Condition D it follows that the diameter of U_i is less than a constant independent of i. Hence (ii) is also satisfied.

Let $\omega_i \in C^{k+\alpha}(\mathbb{R}^n)$ be a partition of unity subordinate to the covering Γ_0, that is, $\operatorname{supp}\omega_i \subset U_i$. Denote by ψ_i the vector-valued function $\psi(x, x_0)$ which corresponds to U_i in Condition D and

$$f_0(x) = c\Sigma_i \psi_{in}(x)\omega_i(x), \tag{2.14}$$

where $\psi_{in}(x)$ is the last component of $\psi_i(x)$ and the constant c will be chosen later. We note that this sum contains no more than N terms.

For any points $x \in U_i$ and $x^1 \in \partial\Omega \cap U_i$ we have $|x - x^1| \le M|y - y^1|$, where $y = \psi_i(x)$, $y^1 = \psi_i(x^1)$ and the constant M does not depend on i. So

$$d(x) \le M \inf_{y_n^1=0} |y - y^1| = M|\psi_{in}(x)|. \tag{2.15}$$

It follows that for all $x \in V$ we have $d(x) \le M|f_0(x)|/c$. We have used the fact that $\psi_{in}(x)$ have the same sign for all i. We take $c \ge M$ and then

$$\min(d(x), 1) \le |f_0(x)|. \tag{2.16}$$

Therefore (4) is proved for $f_0(x)$.

We prove (3). Denote $\varphi_i = \psi_i^{-1}$ and by φ_i' and ψ_i' the Jacobian matrices of φ_i and ψ_i, respectively. Then for any $x \in U_i$ we have $\psi_i'(x) \cdot \varphi_i'(\psi(x)) = I$ (identity matrix). Let a_i be the kth row of ψ_i' and b_i be the kth column of φ_i', then $|a_i||b_i| \ge 1$. From Condition D, $|b_i| \le M_1$, where M_1 is a constant independent of i. So $|a_i| \ge 1/M_1$. In particular,

$$|\nabla\psi_{in}(x)| \ge \frac{1}{M_1}. \tag{2.17}$$

From (2.14) for $x \in \partial D$ we have

$$\nabla f_0(x) = c\sum_i \nabla\psi_{in}(x)\omega_i(x). \tag{2.18}$$

Let ν be the unit inward normal to ∂D. Then

$$(\nabla f_0(x), \nu(x)) = (\nabla f_0(x), \nabla f_0(x)/|\nabla f_0(x)|) = |\nabla f_0(x)|. \tag{2.19}$$

For $x \in \partial\Omega \cap U_i$ we have similarly

$$(\nabla\psi_{in}(x), \nu(x)) = |\nabla\psi_{in}(x)|. \tag{2.20}$$

Multiplying (2.18) by $\nu(x)$ we get

$$|\nabla f_0(x)| = c\sum_i |\nabla\psi_{in}(x)|\omega_i(x). \tag{2.21}$$

From (2.17), taking $c \ge M_1$ we obtain $|\nabla f_0(x)| \ge 1$, $x \in \partial\Omega$ and (3) is proved for $f_0(x)$.

We have defined the function $f_0(x)$ in a neighbourhood of the boundary $\partial\Omega$. We can easily extend it on the whole $\mathbb{R}^n$ in such a way that its regularity is preserved, it is greater than a positive constant inside the domain Ω and less than a negative constant outside the domain. Multiplying it by a large positive number, we will have the last two assertions of the theorem also satisfied. The theorem is proved. □

Let Ω be an unbounded domain satisfying Condition D and $f(x)$ be the function satisfying conditions of Theorem 2.3. Consider a sequence $x_m \in \Omega$, $|x_m| \to \infty$. Denote

$$f_m(x) = f(x + x_m). \tag{2.22}$$

Theorem 2.4. *Let $f_m(x) \to f_*(x)$ in $C^k_{\text{loc}}(\mathbb{R}^n)$, where k is not greater than that in Theorem 2.3. Denote*

$$\Omega_* = \{x : x \in \mathbb{R}^n,\ f_*(x) > 0\}. \tag{2.23}$$

Then

(1) $f_*(x) \in C^{k+\alpha}(\mathbb{R}^n)$,
(2) Ω_* *is a nonempty open set.*

If $\Omega_* \neq \mathbb{R}^n$, then

(3) $|\nabla f_*(x)|_{\partial\Omega_*} \geq 1$,
(4) $\min(d_*(x), 1) \leq |f_*(x)|$, *where $d_*(x)$ is the distance from x to $\partial\Omega_*$.*

Proof. The first assertion of the theorem is obvious. To prove the second assertion, we note that the origin O belongs to all domains Ω_m. Denote by d_m the distance from O to the boundary $\partial\Omega_m$. If $d_m \to 0$, then from the properties of the functions $f_m(x)$ it follows that

$$f_*(O) = 0, \qquad |\nabla f_*(O)| \geq 1. \tag{2.24}$$

Hence there are points in a neighbourhood of the origin where the function $f_*(x)$ is positive. Consequently Ω_* is nonempty.

If d_m does not converge to zero, then $d_{m_i} \geq a > 0$ for some positive a. From Theorem 2.3 we conclude that

$$f_{m_i}(O) \geq \min(a, 1). \tag{2.25}$$

Therefore

$$f_*(O) \geq \min(a, 1) > 0, \tag{2.26}$$

and we obtain again that the set Ω_* is not empty. The fact that it is open is obvious.

We verify now the third assertion of the theorem. Let $f_*(x_0) = 0$ for some x_0. Then $f_m(x_0) \to 0$. From (4) of Theorem 2.3 it follows that $d_m(x_0) \to 0$, where $d_m(x_0)$ is the distance of x_0 to $\partial\Omega_m$. So there exists $z_m \in \partial D_m$ such that $|z_m - x_0| \to 0$. Since $|\nabla f_m(z_m)| \geq 1$ by (3) of Theorem 2.3, then passing to limit we get $|\nabla f_*(x_0)| \geq 1$.

We prove finally the last assertion of the theorem. For any $x_0 \in \mathbb{R}^n$ we have

$$\min\left(d_m(x_0), 1\right) \leq \left| f_m(x_0) \right|. \tag{2.27}$$

So we should verify that $d_m(x_0)$ converges to $d_*(x_0)$ as $m \to \infty$.

Suppose that x_0 belongs to a ball B_R. Denote

$$\Gamma_m = \{x : f_m(x) = 0,\ x \in B_R\}, \qquad \Gamma_* = \{x : f_*(x) = 0,\ x \in B_R\}. \tag{2.28}$$

It is sufficient to prove that

$$\varrho(\Gamma_m, \Gamma_*) \longrightarrow 0, \quad m \longrightarrow \infty. \tag{2.29}$$

Let Γ_m^ϵ, Γ_*^ϵ be ϵ-neighbourhoods of these sets, respectively. From convergence $f_m(x) \to f_*(x)$ in $C^k(B_R)$ it follows that $\Gamma_m \subset \Gamma_*^\epsilon$ for m sufficiently large. We show that $\Gamma_* \subset \Gamma_m^\epsilon$ for m large. Indeed,

$$\left| f_m(x) - f_*(x) \right| < \epsilon, \quad x \in B_R \tag{2.30}$$

for $m \geq m_\epsilon$ and some m_ϵ. If $x \in \Gamma_*$, then $f_*(x) = 0$ and $|f_m(x)| < \epsilon$. From the last assertion of Theorem 2.3 it follows that $d_m(x) < \epsilon$, and $x \in \Gamma_m^\epsilon$. Convergence (2.29) follows from this. The theorem is proved. □

Remark 2.5. The limiting set Ω_* is not necessarily connected even if the domain Ω is connected.

Theorem 2.6. *If $f_m(x) \to f_*(x)$ in C^k_{loc} as $m \to \infty$, then $\partial\Omega_m \to \partial\Omega_*$ in Ξ_{loc}.*

The proof of the theorem follows from convergence (2.29).

Theorem 2.7. *If $f_m(x) \to f_*(x)$ in C^k_{loc} as $m \to \infty$, then the limiting domain Ω_* satisfies Condition D, or $\Omega_* = \mathbb{R}^n$.*

Proof. Suppose that $\Omega_* \neq \mathbb{R}^n$ and $x_0 \in \partial\Omega_*$. Then there exists a sequence $\hat{x}_m$ such that

$$\hat{x}_m \longrightarrow x_0, \quad \hat{x}_m \in \partial\Omega_m, \tag{2.31}$$

where Ω_m are the domains where the functions $f_m(x)$ are positive. For each point $\hat{x}_m$ and domain Ω_m, there exists a neighbourhood $U(\hat{x}_m)$ and the function $\psi(x;\hat{x}_m)$ defined in Condition D.

Since the domain Ω satisfies Condition D, the functions $\psi(x;\hat{x}_m)$ are uniformly bounded in the $C^{k+\alpha}$-norm with $k \geq 1$. The domain of definition of each of these functions is an inverse image of the unit sphere in $\mathbb{R}^n$. Choosing a converging subsequence of the inverse images and of the functions $\psi(x;\hat{x}_m)$, we obtain a limiting neighbourhood $U(x_0)$ and a limiting function $\psi(x;x_0)$ which satisfy Condition D. The theorem is proved. □

From the previous theorems the main result of this section follows.

Theorem 2.8. *Let Ω be an unbounded domain satisfying Condition D, $x_m \in \Omega$, $|x_m| \to \infty$, and $f(x)$ be the function constructed in Theorem 2.3. Then there exists a subsequence x_{m_i} and a function $f_*(x)$ such that*

$$f_{m_i}(x) \equiv f(x + x_{m_i}) \longrightarrow f_*(x) \tag{2.32}$$

in $C^k_{\text{loc}}(\mathbb{R}^n)$, and the domain

$$\Omega_* = \{x : f_*(x) > 0\} \tag{2.33}$$

satisfies Condition D, or $\Omega_* = \mathbb{R}^n$.

Moreover

$$\bar{\Omega}_{m_i} \longrightarrow \bar{\Omega}_* \quad \text{in } \Xi_{\text{loc}}, \tag{2.34}$$

where

$$\Omega_{m_i} = \{x : f_{m_i}(x) > 0\}. \tag{2.35}$$

2.3. Limiting problems. In the previous section we introduced limiting domains. Here we will define the corresponding limiting problems.

Let Ω be a domain satisfying Condition D and $\chi(x)$ be its characteristic function. Consider a sequence $x_m \in \Omega$, $|x_m| \to \infty$ and the shifted domains Ω_m defined by the shifted characteristic functions $\chi_m(x) = \chi(x + x_m)$. We suppose that the sequence of domains Ω_m converge in Ξ_{loc} to some limiting domain Ω_*. In this section we suppose that $0 \le k \le l + \lambda$.

Definition 2.9. Let $u_m(x) \in C^k(\Omega_m)$, $m = 1, 2, \ldots$. We say that $u_m(x)$ converges to a limiting function $u_*(x) \in C^k(\Omega_*)$ in $C^k_{\text{loc}}(\Omega_m \to \Omega_*)$ if there exists an extension $v_m(x) \in C^k(\mathbb{R}^n)$ of $u_m(x)$, $m = 1, 2, \ldots$ and an extension $v_*(x) \in C^k(\mathbb{R}^n)$ of $u_*(x)$ such that

$$v_m \longrightarrow v_* \quad \text{in } C^k_{\text{loc}}(\mathbb{R}^n). \tag{2.36}$$

Definition 2.10. Let $u_m(x) \in C^k(\partial\Omega_m)$, $m = 1, 2, \ldots$. We say that $u_m(x)$ converges to a limiting function $u_*(x) \in C^k(\partial\Omega_*)$ in $C^k_{\text{loc}}(\partial\Omega_m \to \partial\Omega_*)$ if there exists an extension $v_m(x) \in C^k(\mathbb{R}^n)$ of $u_m(x)$, $m = 1, 2, \ldots$ and an extension $v_*(x) \in C^k(\mathbb{R}^n)$ of $u_*(x)$ such that

$$v_m \to v_* \quad \text{in } C^k_{\text{loc}}(\mathbb{R}^n). \tag{2.37}$$

Remark 2.11. In these definitions $u_*(x)$ does not depend on the choice of the extensions $v_m(x)$ and $v_*(x)$. Indeed, in Definition 2.9 for any point $x \in \bar{\Omega}_*$ there exists a sequence $\hat{x}_m \in \bar{\Omega}_m$ such that $\hat{x}_m \to x$. Therefore

$$u_*(x) = v_*(x) = \lim_{m\to\infty} v_m(\hat{x}_m) = \lim_{m\to\infty} u_m(\hat{x}_m). \tag{2.38}$$

Similarly it can be checked for Definition 2.10.

THEOREM 2.12. *Let* $u_m \in C^{k+\alpha}(\Omega_m)$, $\|u_m\|_{C^{k+\alpha}} \le M$, *where the constant* M *is independent of* m. *Then there exists a function* $u_* \in C^{k+\alpha}(\Omega_*)$ *and a subsequence* u_{m_k} *such that* $u_{m_k} \to u_*$ *in* $C^k_{\text{loc}}(\Omega_{m_k} \to \Omega_*)$.

Let $u_m \in C^{k+\alpha}(\partial\Omega_m)$, $\|u_m\|_{C^{k+\alpha}} \le M$. *Then there exists a function* $u_* \in C^{k+\alpha}(\partial\Omega_*)$ *and a subsequence* u_{m_k} *such that* $u_{m_k} \to u_*$ *in* $C^k_{\text{loc}}(\partial\Omega_{m_k} \to \partial\Omega_*)$.

Proof. Let $u_m \in C^{k+\alpha}(\Omega_m)$, $\|u_m\|_{C^{k+\alpha}} \le M$. It follows from Condition D that there exists an extension $v_m(x)$ of $u_m(x)$ on the whole space $\mathbb{R}^n$ such that

$$v_m \in C^{k+\alpha}(\mathbb{R}^n), \quad \|v_m\|_{C^{k+\alpha}(\mathbb{R}^n)} \le M_0, \qquad v_m(x) = u_m(x), \quad x \in \Omega_m, \tag{2.39}$$

where M_0 is independent of m.

Passing to a subsequence and retaining the same notation we can suppose that there exists a function $v_*(x) \in C^{k+\alpha}(\mathbb{R}^n)$ such that $\|v_*\|_{C^{k+\alpha}(\mathbb{R}^n)} \le M_0$ and

$$v_m \longrightarrow v_* \quad \text{in } C^k_{\text{loc}}(\mathbb{R}^n). \tag{2.40}$$

So

$$u_m \longrightarrow u_* \quad \text{in } C^k_{\text{loc}}(\Omega_m \to \Omega_*) \tag{2.41}$$

in the sense of Definition 2.9. Here $u_*(x)$ is the restriction of $v_*(x)$ on Ω_*.

The second part of the theorem for $u_m \in C^{k+\alpha}(\partial\Omega_m)$ is proved similarly. The theorem is proved. □

The operator L consists of a pair of operators, $L = (L_1, L_2)$ where the operator L_1 acts inside the domain and L_2 is a boundary operator. So we can represent the boundary problem as

$$L_1 u = f_1, \qquad L_2 u = f_2, \tag{2.42}$$

where $u \in E_0(\Omega)$, $f_1 \in E_1(\Omega)$, $f_2 \in E_2(\partial\Omega)$, $E = E_1 \times E_2$. The coefficients $a_{ij}(x)$ of the operator L_1 are defined in $\bar{\Omega}$ and the coefficients $b_{ij}(x)$ of L_2 in $\partial\Omega$. We recall that

$$a_{ij}(x) \in C^{l-s_i+\alpha}(\Omega), \qquad b_{ij}(x) \in C^{l-\sigma_i+\alpha}(\partial\Omega) \tag{2.43}$$

(see Section 2.1). Then obviously the shifted coefficients $a_{ij}(x+x_m)$ and $b_{ij}(x+x_m)$ satisfy conditions of Theorem 2.12. Therefore we can define the *limiting problem*

$$\hat{L}_1 u = f, \qquad \hat{L}_2 u = g, \tag{2.44}$$

where $u \in E_0(\Omega_*)$, $f_1 \in E_1(\Omega_*)$, $f_2 \in E_2(\partial\Omega_*)$, $\hat{L}_1$ and $\hat{L}_2$ are operators with limiting coefficients $a_{ij}^*(x) \in C^{l-s_i+\alpha}(\Omega_*)$, $b_{ij}^*(x) \in C^{l-\sigma_i+\alpha}(\partial\Omega_*)$.

We note that for a given problem (2.42) there can exist a set of limiting problems corresponding to different sequences x_m and to different converging subsequences of coefficients of the operators.

2.4. Normal solvability. We consider the operator $L : E_0(\Omega) \to E(\Omega)$ and introduce limiting domains and limiting operators defined above.

In what follows we will use also the spaces E_0' and E', which are obtained from E_0 and E, respectively, if we put $\alpha = 0$.

From Theorem 2.12 it follows that for any sequences $u_m \in E_0(\Omega_m)$, $f_m \in E(\Omega_m)$ with uniformly bounded norms there exist subsequences u_{m_k} and f_{m_k} converging to some limiting functions $u_* \in E_0(\Omega_*)$ and $f_* \in E(\Omega_*)$ in $E'_{0,\mathrm{loc}}(\Omega_{m_k} \to \Omega_*)$ and $E'_{\mathrm{loc}}(\Omega_{m_k} \to \Omega_*)$, respectively.

If L_m is a sequence of operators with shifted coefficients and $L_m u_m = f_m$, then there exists a limiting operator $\hat{L}$ such that $\hat{L}u_* = f_*$.

This is true in particular for the case where $\Omega_m = \Omega$ for all m and $L_m = L$.

It is known that for a domain Ω satisfying Condition D and an operator L the following estimate

$$\|u\|_{E_0} \le K\left(\|Lu\|_E + \|u\|_C\right) \tag{2.45}$$

holds, where the constant K is independent of the function $u \in E_0(\Omega)$ and $\|\cdot\|_C$ is the norm in $C(\Omega)$.

Condition NS. For any limiting domain Ω_* and any limiting operator $\hat{L}$ the problem

$$\hat{L}u = 0, \quad u \in E_0(\Omega_*) \tag{2.46}$$

has only zero solution.

Theorem 2.13. *Let Condition NS be satisfied. Then the operator L is normally solvable and its kernel is finite dimensional.*

Proof. Let the limiting problems have only zero solution. It is sufficient to prove that the operator L is proper. Consider the equation

$$Lu_n = f_n, \tag{2.47}$$

where $f_n \in E(\Omega)$ and $f_n \to f_0$. Suppose that $\|u_n\|_{E_0(\Omega)} \le M$. We will prove that there exists a function $u_0 \in E_0(\Omega)$ and a subsequence u_{n_k} such that

$$\left\|u_{n_k} - u_0\right\|_{E_0(\Omega)} \longrightarrow 0. \tag{2.48}$$

There exists a function $u_0 \in E_0(\Omega)$ such that $u_{n_k} \to u_0$ in $E'_{0,\mathrm{loc}}(\Omega)$ and $Lu_0 = f_0$. Without loss of generality we can assume, here as well as below, that it is the same sequence. We prove first that

$$||u_n - u_0||_{C(\Omega)} \longrightarrow 0. \tag{2.49}$$

Suppose that this convergence does not take place. Since $u_n \to u_0$ in $C_{\mathrm{loc}}(\Omega)$, we conclude that there exists a sequence x_m, $|x_m| \to \infty$ and a subsequence u_{nm} of u_n such that

$$||u_{nm}(x_m) - u_0(x_m)|| \geq \epsilon > 0. \tag{2.50}$$

Consider the shifted domains Ω_n with characteristic functions $\chi(x+x_m)$, the operators with shifted coefficients and the functions $v_{nm}(x) = u_{nm}(x+x_m) - u_0(x+x_m)$. Passing to a subsequence we conclude that there exists a limiting domain Ω_*, a limiting operator $\hat{L}$, and a nonzero limiting function $v_0 \in E_0(\Omega_*)$ such that

$$\hat{L}v_0 = 0. \tag{2.51}$$

This contradiction proves (2.49).

From this convergence, from the convergence $f_n \to f_0$ in $E(\Omega)$, and estimate (2.45) it follows that $u_n \to u_0$ in $E_0(\Omega)$. The theorem is proved. □

The next theorem will provide a necessary condition of normal solvability. In fact, it is the same Condition NS. However we need now more restrictive conditions on the coefficients of the operator and on the domain Ω.

We suppose here that

$$a_{ik}^{\beta} \in C^{l-s_i+\delta}(\Omega), \quad b_{ik}^{\beta} \in C^{l-\sigma_i+\delta}(\partial\Omega), \quad \text{the domain } \Omega \text{ is of class } C^{l+\lambda+\delta} \tag{2.52}$$

with $\alpha < \delta < 1$.

Lemma 2.14. *Let as above Ω_m and Ω_* be shifted and limiting domains, respectively. Then for any N there exists m_0 such that for $m > m_0$ there exists a diffeomorphism*

$$h_m(x) : \bar{\Omega}_m \cap B_N \longrightarrow \bar{\Omega}_* \cap B_N \tag{2.53}$$

satisfying the condition

$$||h_m(x) - x||_{C^{l+\lambda+\alpha}(\bar{\Omega}_m \cap B_N)} \longrightarrow 0 \tag{2.54}$$

as $m \to \infty$.

Proof. Consider a domain G such that $\bar{G} \subset \Omega_m \cap \Omega_*$ for all m sufficiently large. Let $x_0 \in \partial\Omega_*$. Denote by $n(x_0)$ the normal to $\partial\Omega_*$ at $x = x_0$. If m is sufficiently large, then in a neighbourhood of x_0, $n(x_0)$ intersects $\partial\Omega_m$ only at one point. The domain G can be chosen such that it satisfies the same property.

We put $h_m(x) = x$ for $x \in G$. We define then $h_m(x)$ along each normal $n(x_0)$ by mapping the interval, which belongs to Ω_m on the interval in Ω_*. It can be done in such a way that we have the required regularity. The lemma is proved. □

Theorem 2.15. *Suppose that problem (2.46) has a nonzero solution u_0 for some limiting operator $\hat{L}$ and limiting domain Ω_*. Then the operator L is not proper.*

Proof. Let $\varphi(x)$ be an infinitely differentiable function of $x \in \mathbb{R}^n$ such that $0 \le \varphi(x) \le 1$, $\varphi(x) = 1$ for $|x| < 1$, $\varphi(x) = 0$ for $|x| > 2$. If $\{x_m\}$ is the sequence for which the limiting operator $\hat{L}$ is defined, denote

$$\varphi_m(x) = \varphi\left(\frac{x}{r_m}\right), \tag{2.55}$$

where $r_m \to \infty$ and $r_m \le |x_m|/3$. Some other conditions on the sequence r_m will be formulated below.

Let $V_j = \{y : y \in \mathbb{R}^n,\ |y| < j\}$, $j = 1, 2, \ldots$. Denote by n_j a number such that for $m \ge n_j$ the diffeomorphism h_m defined in Lemma 2.14 can be constructed in $\Omega_m \cap V_{j+1}$ and

$$||h_m(y) - y||_{C^{l+\lambda+\alpha}(\Omega_m \cap V_{j+1})} < 1. \tag{2.56}$$

For arbitrary $m_j \ge n_j$ we take $r_{m_j} = \min(j/2, |x_{m_j}|/3)$.

Let

$$\begin{aligned} &v_{m_j}(y) = \varphi_{m_j}(y) u_0(h_{m_j}(y)) \quad \text{for } y \in \Omega_{m_j} \cap V_{j+1}, \\ &v_{m_j}(y) = 0 \quad \text{for } y \in \Omega_{m_j},\ |y| \ge j+1. \end{aligned} \tag{2.57}$$

Denote

$$u_{m_j}(x) = v_{m_j}(x - x_{m_j}), \quad x \in \Omega. \tag{2.58}$$

It is easy to see that $u_{m_j} \in E_0(\Omega)$ and

$$||u_{m_j}||_{E_0(\Omega)} \le M, \tag{2.59}$$

where M does not depend on m_j. Indeed, obviously

$$\varphi_{m_j}(y) = 0 \tag{2.60}$$

for y outside V_j. So to prove (2.59) it is sufficient to show that

$$||v_{m_j}||_{E_0(\Omega_{m_j}\cap V_{j+1})} \leq M_1, \tag{2.61}$$

or

$$||u_0(h_{m_j}(y))||_{E_0(\Omega_{m_j}\cap V_{j+1})} \leq M_2, \tag{2.62}$$

where M_1 and M_2 do not depend on m_j. This follows from (2.56) and the fact that $u_0 \in E_0(\Omega_*)$.

We will prove that choice of m_j in (2.58) can be specified so that

(i) $Lu_{m_j} \to 0$ in $E(\Omega)$ as $m_j \to \infty$,

(ii) the sequence $\{u_{m_j}\}$ is not compact in $E_0(\Omega)$.

The assertion of the theorem will follow from this.

(i) We consider operator A_i. The operator B_i is treated similarly. For any j and $m \geq n_j$ we have

$$A_i u_m = A_i^1 u_m + A_i^2 u_m, \tag{2.63}$$

where

$$A_i^1 u_m(x) = \varphi_m(x-x_m)\sum_{k=1}^{p}\sum_{|\beta|\leq\beta_{ik}} a_{ik}^{\beta}(x)D^{\beta}u_{0k}(h_m(x-x_m)), \quad x\in\Omega, \tag{2.64}$$

$u_0 = (u_{01},\dots,u_{0p})$ and A_i^2 contains derivatives of φ_m. Obviously

$$||A_i^2 u_m||_{C^{l-s_i+\alpha}(\Omega)} \longrightarrow 0 \tag{2.65}$$

as $m \to \infty$.

Denote $y = x - x_m$. From (2.64) we obtain

$$A_i^1 u_m(y+x_m) = \varphi_m(y)T_{im}(y), \quad y\in\Omega_m, \tag{2.66}$$

where

$$T_{im}(y) = \sum_{k=1}^{p}\sum_{|\beta|\leq\beta_{ik}} a_{ik,m}^{\beta}(y)D^{\beta}u_{0k}(h_m(y)), \quad y\in\Omega_m, \tag{2.67}$$

$a_{ik,m}^{\beta}(y) = a_{ik}^{\beta}(y+x_m)$.

We will prove that for any j fixed

$$||T_{im}||_{C^{l-s_i+\alpha}(\Omega_m\cap V_{j+1})} \longrightarrow 0 \tag{2.68}$$

as $m \to \infty$.

By definition of u_0 the following equality holds:

$$\sum_{k=1}^{p} \sum_{|\beta| \le \beta_{ik}} \hat{a}_{ik}^{\beta}(x) D_x^{\beta} u_{0k}(x) = 0, \quad x \in \Omega_*. \tag{2.69}$$

Here $\hat{a}_{ik}^{\beta}(x)$ are the limiting coefficients. So

$$T_{im}(y) = \sum_{k=1}^{p} \sum_{|\beta| \le \beta_{ik}} [S_{ik,m}^{\beta}(y) + P_{ik,m}^{\beta}(y)], \tag{2.70}$$

where

$$S_{ik,m}^{\beta}(y) = a_{ik,m}^{\beta}(y)[D_y^{\beta} u_{0k}(h_m(y)) - D_x^{\beta} u_{0k}(h_m(y))], \tag{2.71}$$

$$P_{ik,m}^{\beta}(y) = [a_{ik,m}^{\beta}(y) - \hat{a}_{ik}^{\beta}(h_m(y))] D_x^{\beta} u_{0k}(h_m(y)). \tag{2.72}$$

The first factor on the right in (2.71) is bounded since

$$||a_{ik,m}^{\beta}||_{C^{l-s_i+\alpha}(\Omega_m)} = ||a_{ik}^{\beta}||_{C^{l-s_i+\alpha}(\Omega)}. \tag{2.73}$$

From Lemma 2.14 it follows that the second factor tends to 0 in the norm $C^{l-s_i+\alpha}(\Omega_m \cap V_{j+1})$ as $m \to \infty$. So

$$||S_{ik,m}^{\beta}||_{C^{l-s_i+\alpha}(\Omega_m \cap V_{j+1})} \longrightarrow 0 \tag{2.74}$$

as $m \to \infty$.

Consider (2.72). Using Lemma 2.14 we easily prove that

$$||D_x^{\beta} u_{0k}(h_m(y))||_{C^{l-s_i+\alpha}(\Omega_m \cap V_{j+1})} \le M_3 \tag{2.75}$$

with M_3 independent of m.

To prove (2.68) it remains to show that, for any subsequence of m, T_{im} has a convergent to zero subsequence. If m_ν is a subsequence of m, then assumption (2.52) and Lemma 2.14 imply that

$$||a_{ik,m}^{\beta}(\cdot) - \hat{a}_{ik}^{\beta}(h_m(\cdot))||_{C^{l-s_i+\alpha}(\Omega_m \cap V_{j+1})} \longrightarrow 0 \tag{2.76}$$

as $m \to \infty$ by some subsequence of m_ν. So (2.68) is proved.

Now we specify the choice of m_j in (2.58). According to (2.68) for any j we can take p_j such that

$$||T_{im}||_{C^{l-s_i+\alpha}(\Omega_m \cap V_{j+1})} < \frac{1}{j} \tag{2.77}$$

for $m \geq p_j$. We take $m_j = \max(n_j, p_j)$. Then obviously

$$||\varphi_{m_j} T_{im_j}||_{C^{l-s_i+\alpha}(\Omega_{m_j})} \longrightarrow 0 \tag{2.78}$$

as $m_j \to \infty$.

It is easy to see that m_j can be chosen by the same manner in such a way that (2.78) is true for all $i = 1, \dots, p$ and also for operators B_i.

Thus the assertion (i) is proved.

(ii) We will prove that sequence (2.58) does not have convergent subsequence. Obviously $u_{m_j}(x) = 0$ for $|x| < r_{m_j}$ and so

$$u_{m_j}(x) \longrightarrow 0 \tag{2.79}$$

as $m_j \to \infty$ for any $x \in \Omega$ fixed.

For any subsequence s_i of m_j, there exists N such that

$$\sup_{x \in \Omega} \left| u_{s_i}(x) \right| > 0 \quad \text{for } s_i > N. \tag{2.80}$$

Indeed, denote $y = x - x_{s_i}$. Then

$$\sup_{x \in \Omega} \left| u_{s_i}(x) \right| \geq \sup_{y \in \Omega_{s_i} \bigcap V_{j+1}} \left| \varphi_{s_i}(y) u_0\big(h_{s_i}(y)\big) \right|. \tag{2.81}$$

Let $x_0 \in \Omega_*$ be a point such that $|u_0(x_0)| > 0$. Denote $y_{s_i} = h_{s_i}^{-1}(x_0)$, $y_{s_i} \in \Omega_{s_i}$. From Lemma 2.14 it follows that $|y_{s_i}|$ is bounded. So there exists N such that $|y_{s_i}| < r_{s_i}$, $|y_{s_i}| < j+1$ for $s_i > N$.

From (2.81), $\sup_{x \in \Omega} |u_{s_i}(x)| \geq |u_0(x_0)|$ and (2.80) follows.

We have obtained $\|u_{s_i}\|_{C(\Omega)} > 0$. This and (2.79) imply that u_{m_j} is not compact in $E_0(\Omega)$. The theorem is proved. □

2.5. Dual spaces: invertibility of limiting operators. We consider now the space $E = E(\Omega)$ defined in Section 2.1 and the space E^0, which consists of functions $u \in E$ converging to 0 at infinity in the norm E, that is,

$$\|u\|_{E(\Omega \cap \{|x| \geq N\})} \longrightarrow 0 \tag{2.82}$$

as $N \to \infty$. We say that $u_n \to u_0$ in $E_{\text{loc}}(\Omega)$ if this convergence holds in $\Omega \cap \{|x| \leq N\}$ for any N.

Lemma 2.16. *Let ϕ be a functional in the dual space $(E^0)^*$, $u \in E$ and $u \notin E^0$, $u_n \in E^0$, $\|u_n\|_E \leq 1$, and $u_n \to u$ in E_{loc}. Then there exists a limit*

$$\hat{\phi} = \lim_{n \to \infty} \phi(u_n). \tag{2.83}$$

Proof. Since ϕ is a bounded functional, then

$$|\phi(u_n)| \leq K\|u_n\|_E \leq K \tag{2.84}$$

with some positive constant K. Suppose that the limit (2.83) does not exist. We will construct a sequence $z_n \in E^0$ uniformly bounded in the norm E such that $\phi(z_n) \to \infty$.

We can choose two subsequences $\hat{u}_n$ and $\bar{u}_n$ such that

$$\phi(\hat{u}_n) \longrightarrow K_1, \quad \phi(\bar{u}_n) \longrightarrow K_2, \quad K_1 \neq K_2. \tag{2.85}$$

Without loss of generality, we can assume that $K_1 > K_2$ and that for all $n \geq 1$

$$\phi(\hat{u}_n) \geq K_1 - \delta > K_2 + \delta \geq \phi(\bar{u}_n) \tag{2.86}$$

for some positive δ. We put $v_1 = \hat{u}_{n_1} - \bar{u}_{n_1}$. Then

$$\phi(v_1) \geq K_1 - K_2 - 2\delta > 0. \tag{2.87}$$

For any given ball and any $\epsilon > 0$ we can choose n_1 sufficiently large such that the E-norm of v_1 in this ball is less than $\epsilon/2$. On the other hand, v_1 converges to 0 at infinity in the sense of definition of the space E^0. Therefore there exists a function $\omega_1 \in E^0$, $\|\omega_1\|_E \leq \epsilon$ such that $w_1 = v_1 + \omega_1$ has a finite support.

We choose ϵ such that

$$|\phi(\omega_1)| \leq K\|\omega_1\|_E \leq K\epsilon < K_1 - K_2 - 2\delta. \tag{2.88}$$

Then

$$\phi(w_1) > K_1 - K_2 - 2\delta - K\epsilon > 0. \tag{2.89}$$

We choose the functions $\hat{u}_{n_2}$, $\bar{u}_{n_2}$ such that $v_2 = \hat{u}_{n_2} - \bar{u}_{n_2}$ is sufficiently small in the support of w_1. Then there exists ω_2 such that $\|\omega_2\|_E \leq \epsilon$ and

$$\begin{gathered} \operatorname{supp} w_1 \cap \operatorname{supp} w_2 = \varnothing, \\ \phi(w_2) > K_1 - K_2 - 2\delta - K\epsilon > 0. \end{gathered} \tag{2.90}$$

In the same manner we construct other functions of the sequence w_n. We put

$$z_n = \sum_{i=1}^{n} w_i. \tag{2.91}$$

Then the functions z_n are uniformly bounded in the E norm and $\phi(z_n) \to \infty$. The lemma is proved. □

LEMMA 2.17. *The limit (2.83) does not depend on the sequence u_n.*

Proof. Suppose that there are two sequences $\hat{u}_n$ and $\bar{u}_n$ such that

$$\hat{u}_n \longrightarrow u, \qquad \bar{u}_n \to u \tag{2.92}$$

in E_{loc} and

$$\lim_{n\to\infty} \phi(\hat{u}_n) \neq \lim_{n\to\infty} \phi(\bar{u}_n). \tag{2.93}$$

Then we proceed as in the proof of the previous lemma. The lemma is proved. □

COROLLARY 2.18. *If $u_n \to 0$ in E_{loc}, then $\phi(u_n) \to 0$.*

We can extend now the functional ϕ to the space $E(\Omega)$. For any $u \in E(\Omega)$ we put $\hat{\phi}(u) = \phi(u)$ if $u \in E^0(\Omega)$ and

$$\hat{\phi}(u) = \lim_{n\to\infty} \phi(u_n), \tag{2.94}$$

where $u_n \in E^0(\Omega)$ is an arbitrary sequence converging to u in E_{loc}. This is a linear bounded functional on $E(\Omega)$.

Denote all such functionals $\hat{E}$. It is a linear subspace in E^*. Suppose that $\hat{E} \neq E^*$. We take a functional $\psi \in E^*$, which does not belong to $\hat{E}$. Let ψ_0 be a restriction of ψ on E^0. Then $\psi_0 \in (E^0)^*$. As above we can define the functional $\hat{\psi}_0 \in (E)^*$. By assumption $\psi \neq \hat{\psi}_0$. Denote $\tilde{\psi} = \psi - \hat{\psi}_0$. Then

$$\tilde{\psi} = 0, \quad \forall u \in E^0. \tag{2.95}$$

Thus we have proved the following theorem.

THEOREM 2.19. *The dual space E^* is a direct sum of the extension $\hat{E}$ of $(E^0)^*$ on E and of the subspace $\tilde{E}$ consisting of all functionals satisfying (2.95).*

Remark 2.20. For any function $v \in L^1(\Omega)$, we can define the functional $\phi \in \hat{E}$ as

$$\phi(u) = \int_\Omega v(x)u(x)\,dx. \tag{2.96}$$

We do not know whether $\hat{E} = (C^\alpha(\Omega))^*$. However, if instead of the space $C^\alpha(\Omega)$ we take, for example, the space of functions from $C^\alpha(\Omega)$ having limits at infinity, then all constructions above remain applicable and $\hat{E} \neq (C^\alpha(\Omega))^*$. Indeed, the functional

$$\psi(u) = \lim_{|x|\to\infty} u(x) \tag{2.97}$$

does not belong to $\hat{E}$. However the following lemma shows that the normal solvability is determined completely by the subspace $\hat{E}$.

LEMMA 2.21. *Suppose that the operator $L : E_0 \to E$ is normally solvable with a finite-dimensional kernel, and the problem*

$$Lu = f, \quad f \in E \tag{2.98}$$

is solvable if and only if

$$\psi_i(f) = 0, \quad i = 1, \dots, N, \tag{2.99}$$

where ψ_i are linearly independent functionals in E^*. Then $\psi_i \in \hat{E}$.

Proof. Suppose that the assertion of the lemma does not hold and

$$\psi_1 \notin \hat{E}, \qquad \psi_2, \dots, \psi_N \in \hat{E}. \tag{2.100}$$

We suppose first that $\psi_1 \in \tilde{E}$. We consider the functionals $\psi_i, i = 2, \dots, N$, as functionals on E^0. They are linearly independent. There exist functions $f_j \in E^0$, $j = 2, \dots, N$, such that

$$\psi_i(f_j) = \delta_{ij}, \quad i, j = 2, \dots, N, \tag{2.101}$$

where δ_{ij} is the Kronecker symbol.

Let $f^{(n)} \in E^0$, the norms $\|f^{(n)}\|_E$ be uniformly bounded and $f^{(n)} \to f$ in E_{loc}. Then the problem

$$Lu = g^{(n)}, \tag{2.102}$$

where

$$g^{(n)} = f^{(n)} - \sum_{i=2}^{N} \psi_i(f^{(n)}) f_i, \tag{2.103}$$

is solvable in E_0 since

$$\psi_1(f^{(n)}) = 0, \quad \psi_1(f_i) = 0, \quad \psi_i(g^{(n)}) = 0, \quad i = 2, \dots, N. \tag{2.104}$$

Denote by $u^{(n)}$ its solution and put $u^{(n)} = v^{(n)} + w^{(n)}$, where

$$v^{(n)} \in \operatorname{Ker} L, \qquad w^{(n)} \in (\operatorname{Ker} L)^{\perp}, \tag{2.105}$$

and $(\operatorname{Ker} L)^{\perp}$ denotes the supplement to the kernel of the operator L in the space E_0. Then

$$Lw^{(n)} = g^{(n)} \tag{2.106}$$

and the E_0 norms of the functions $w^{(n)}$ are uniformly bounded. Indeed, if $\|w^{(n)}\|_{E_0} \to \infty$, then for the functions

$$\tilde{w}^{(n)} = \frac{w^{(n)}}{\|w^{(n)}\|_{E_0}}, \qquad \tilde{g}^{(n)} = \frac{g^{(n)}}{\|w^{(n)}\|_{E_0}} \tag{2.107}$$

we have

$$L\tilde{w}^{(n)} = \tilde{g}^{(n)}, \quad \|\tilde{g}^{(n)}\|_E \longrightarrow 0. \tag{2.108}$$

Since the operator L is proper, then there exists a function w_0 such that $\tilde{w}^{(n_k)} \to w_0$. Hence $w_0 \in (\operatorname{Ker} L)^\perp$. On the other hand, $Lw_0 = 0$. This contradiction proves the boundedness of the sequence $w^{(n)}$.

Therefore there exists a subsequence $w^{(n_k)}$ converging in $E'_{0,\mathrm{loc}}$ (see Section 2.4) to a limiting function $\hat{w} \in E_0$. Passing to the limit in (2.106), we have

$$L\hat{w} = f - \sum_{i=2}^{N} \psi_i(f) f_i. \tag{2.109}$$

Since this problem is solvable for any f, then

$$0 = \psi_1\left(f - \sum_{i=2}^{N} \psi_i(f) f_i\right) = \psi_1(f). \tag{2.110}$$

This means that for any function $f \in E$, the value of the functional ψ_1 equals zero. This contradiction proves the lemma.

If $\psi_1 \notin \hat{E}$ and $\psi_1 \notin \tilde{E}$, then by virtue of Theorem 2.19, $\psi_1 = \tilde{\psi}_1 + \hat{\psi}_1$, where $\tilde{\psi}_1 \in \tilde{E}$, $\hat{\psi}_1 \in \hat{E}$. If the functionals $\hat{\psi}_1, \psi_2, \dots, \psi_N$ are linearly dependent, we can take their linear combination and reduce this case to the case considered above. If they are linearly independent, we repeat the same construction with all N functionals, that is, the sum in the expression for $g^{(n)}$ contains the term $\hat{\psi}_1(f^{(n)}) f_1$. The solvability condition

$$0 = \psi_1\left(f - \sum_{i=2}^{N} \psi_i(f) f_i - \hat{\psi}_1(f) f_1\right) = \tilde{\psi}_1(f) \tag{2.111}$$

gives $\psi_1 \in \hat{E}$.

The proof remains the same, as we suppose that more than one functional does not belong to $\hat{E}$. The theorem is proved. □

In the following theorem it is supposed that conditions (2.52) are satisfied.

Theorem 2.22. *If the operator L is Fredholm, then any of its limiting operator is invertible.*

Proof. It is sufficient to prove that the problem

$$\hat{L}u = f^* \tag{2.112}$$

is solvable for any $f^* \in E(\Omega^*)$ where Ω^* is the limiting domain.

The problem

$$Lu = f, \quad u \in E_0(\Omega), \ f \in E(\Omega), \tag{2.113}$$

is solvable if and only if

$$\psi_i(f) = 0, \quad i = 1, \dots, N, \tag{2.114}$$

where ψ_i are linearly independent functionals in $\hat{E}$ (see Lemma 2.21). Let $f_j \in E^0(\Omega)$, $j = 1, \dots, N$, be functions which form the biorthogonal system to these functionals. For any $f \in E(\Omega)$ the problem

$$Lu = f - \sum_{i=1}^{N} \psi_i(f) f_i \tag{2.115}$$

has a solution $u \in E_0(\Omega)$.

Let $\{x_m\}$ be the sequence for which the limiting operator $\hat{L}$ is defined. Denote $T_m f(x) = f(x + x_m)$ and consider the shifted problem. Then from (2.115)

$$L_m T_m u = T_m f - \sum_{i=1}^{N} \psi_i(f) T_m f_i, \tag{2.116}$$

where L_m is the operator with shifted coefficients.

So for any $f \in E(\Omega_m)$ the equation

$$L_m u = f - \sum_{i=1}^{N} \psi_i(T_m^{-1} f) T_m f_i \tag{2.117}$$

has a solution $u \in E_0(\Omega_m)$.

To prove the existence of solutions of (2.112), we use the construction given in the proof of Theorem 2.15. Let φ_m, V_j, n_j, m_j be the same as in Theorem 2.15. Suppose that (2.52) is satisfied. Denote $g_{m_j}(y) = \varphi_{m_j}(y) f^*(h_{m_j}(y))$ for $y \in \bar{\Omega}_{m_j} \cap V_{j+1}$ and suppose $g_{m_j}(y) = 0$ for $y \in \bar{\Omega}_{m_j}$, $|y| > j + 1$.

Consider the equation

$$L_{m_j} u_{m_j} = g_{m_j} - \sum_{i=1}^{N} \psi_i(T_{m_j}^{-1} g_{m_j}) T_{m_j} f_i, \tag{2.118}$$

which has the type (2.117), and so it has a solution $u_{m_j} \in E_0(\Omega_{m_j})$.

Since $\|g_{m_j}\|_{E(\Omega_{m_j})}$ is bounded, we obtain from (2.118) that $\|u_{m_j}\|_{E_0(\Omega_{m_j})}$ is bounded. By Theorem 2.12 there exists a function $u \in E_0(\Omega^*)$ and a subsequence $u_{m_{j_k}} \to u$ in $E'_{0,\mathrm{loc}}(\Omega_{m_{j_k}} \to \Omega^*)$. Moreover, the subsequence can be taken so that $g_{m_{j_k}}$ is convergent in $E'_{\mathrm{loc}}(\Omega_{m_{j_k}} \to \Omega^*, \partial\Omega_{m_{j_k}} \to \partial\Omega^*)$. (The notation corresponds to that in Definitions 2.9 and 2.10 and to the fact that $E = E_1 \times E_2$.) Obviously the limit of $g_{m_{j_k}}$ is f^*.

Passing to the limit in (2.118) by this subsequence and taking into account that $T_{m_j} f_i \to 0$, we obtain solvability of problem (2.112). The theorem is proved. □

Corollary 2.23. *If an operator L coincides with its limiting operator, and it is Fredholm, then it is invertible.*

The last result shows, in particular, that the spectrum of operators with constant, periodic or quasiperiodic coefficients in unbounded cylinders does not contain eigenvalues and consists only of points of the essential spectrum. We understand here essential spectrum as points of the complex plane where the operator $L - \lambda$ is not Fredholm. By eigenvalues, the points where it is Fredholm but its kernel is nonempty.

2.6. Weighted spaces. In this section, we discuss the Fredholm property in weighted spaces. Consider the problem

$$Lu = f, \tag{2.119}$$

where $u \in E_{0,\mu}$, $f \in E_\mu$ (see Section 2.1). Denote $v = u\mu$, $g = f\mu$. We have

$$Lv + Ku = g, \tag{2.120}$$

where

$$Ku = \mu Lu - L(\mu u). \tag{2.121}$$

Lemma 2.24. *Suppose that the operator $L : E_0 \to E$ is normally solvable and has a finite-dimensional kernel, the operator*

$$Ku \equiv \mu Lu - L(\mu u) : E_{0,\mu} \longrightarrow E \tag{2.122}$$

is compact. Then the operator $L : E_{0,\mu} \to E_\mu$ is normally solvable and has a finite-dimensional kernel.

Proof. Let f_k be a convergent sequence in E_μ, $Lu_k = f_k$,

$$\|u_k\|_{E_{0,\mu}} \le 1. \tag{2.123}$$

We will show that the sequence u_k is compact, and by this the operator $L : E_{0,\mu} \to E_\mu$ is proper. We have

$$Lv_k + Ku_k = g_k, \tag{2.124}$$

where

$$v_k = \mu u_k, \qquad g_k = \mu f_k. \tag{2.125}$$

Let $w_k = Ku_k$ and let w_{k_l} be a subsequence converging in E. Then

$$Lv_{k_l} = g_{k_l} - w_{k_l}, \tag{2.126}$$

and the sequence v_{k_l} is compact in E_0 since the operator $L : E_0 \to E$ is proper. Therefore the sequence u_{k_l} is compact in $E_{0,\mu}$. The lemma is proved. □

THEOREM 2.25. *Suppose that the conditions of Theorem 2.13 are satisfied. Then the operator $L : E_{0,\mu} \to E_\mu$ is normally solvable and has a finite-dimensional kernel.*

Proof. We consider the operators A_i defined in Section 2.1. The boundary operators B_i are treated similarly. Denote

$$K_i u = \mu A_i u - A_i(\mu u). \tag{2.127}$$

According to Lemma 2.24, it is sufficient to prove that the operator $K_i : E_{0,\mu} \to C^{l-s_i+\alpha}(\Omega)$ is compact. Obviously

$$K_i u = \sum_{k=1}^{p} \sum_{0<|\sigma|\le\beta ik,\, |\tau|<\beta_{ik}} c_{\sigma\tau}(x) D^\sigma \mu D^\tau u_k, \tag{2.128}$$

where $c_{\sigma\tau}$ is a linear combination of the coefficients $a_{ik}^{\beta}(x)$ of the operator A_i. So

$$c_{\sigma\tau}(x) \in C^{l-s_i+\alpha}(\Omega). \tag{2.129}$$

Suppose we have a sequence $\{u^\nu\}$, $\nu = 1, 2, \ldots$,

$$||u^\nu||_{E_{0,\mu}} = ||u^\nu \mu||_{E_0} \le M \tag{2.130}$$

with M independent of ν. We will prove that from the sequence $K_i u^\nu$ we can find a convergent subsequence in $C^{l-s_i+\alpha}(\Omega)$.

Indeed, denote $v^\nu = \mu u^\nu$. Then $\|v^\nu\|_{E_0} \le M$. So we can find a subsequence $w_j = v^{\nu_j}$ convergent in $\hat{E} \equiv C^{l+t_1}(\Omega) \times \cdots \times C^{l+t_p}(\Omega)$ locally to some limiting function $w_0 \in E_0$. Denote $u_0 = w_0/\mu$. Then we have

$$||K_i u^{\nu_j} - K_i u_0||_{C^{l-s_i+\alpha}(\Omega)} = \left|\left|K_i \frac{z_j}{\mu}\right|\right|_{C^{l-s_i+\alpha}(\Omega)}, \tag{2.131}$$

where

$$z_j = w_j - w_0, \quad ||z_j||_{E_0} \le M_1, \; z_j \longrightarrow 0 \tag{2.132}$$

in $\hat{E}$ locally, M_1 does not depend on j. Denote $y_j = K_i(z_j/\mu)$. We have to prove that

$$||y_j||_{C^{l-s_i+\alpha}(\Omega)} \longrightarrow 0 \tag{2.133}$$

as $j \to \infty$. It follows from (2.128) that

$$y_j = \sum_{k=1}^{p} \sum_{|\gamma|<\beta_{ik}} T_{k\gamma}(x) D^\gamma z_{jk}, \tag{2.134}$$

where $z_j = (z_{j1}, \ldots, z_{jp})$,

$$T_{k\gamma}(x) = \sum_{0<|\sigma|\le\beta ik, |\tau|<\beta ik, |\lambda|<\beta_{ik}} c_{\sigma\tau}(x) b_{\lambda\gamma} D^\sigma \mu D^\lambda \frac{1}{\mu}, \tag{2.135}$$

$b_{\lambda\gamma}$ are constants. From (2.134) we get

$$||y_j||_{C^{l-s_i+\alpha}(G)} \le M_2 \sum_{k=1}^{p} \sum_{|\gamma|<\beta_{ik}} ||T_{k\gamma}||_{C^{l-s_i+\alpha}(G)} ||D^\gamma z_{jk}||_{C^{l-s_i+\alpha}(G)}, \tag{2.136}$$

where $G = \Omega_{N+1}$ or $G = \hat{\Omega}_N$, $\Omega_{N+1} = \Omega \cap \{|x| < N+1\}$, $\hat{\Omega}_N = \Omega \cap \{|x| > N\}$. For any $\varepsilon > 0$ we can find N_0 such that for $N > N_0$ we have

$$||y_j||_{C^{l-s_i+\alpha}(\hat{\Omega}_N)} < \varepsilon \tag{2.137}$$

for all j. This follows from the fact that

$$D^\beta\left(D^\sigma \mu(x) D^\lambda \frac{1}{\mu(x)}\right) \longrightarrow 0 \tag{2.138}$$

as $|x| \to \infty$, $x \in \Omega$ for any $|\sigma| > 0$, λ and β. Boundedness of the last norm in the right-hand side of (2.136) follows from (2.132).

From (2.132), (2.136), and (2.137) with $G = \Omega_{N+1}$ we get (2.133). The theorem is proved. □

3. Properness of nonlinear operators

We consider general nonlinear elliptic operators

$$F_i(x, D^{\beta_{i1}} u_1, \ldots, D^{\beta_{ip}} u_p) = 0, \quad i = 1, \ldots, p, \ x \in \Omega, \tag{3.1}$$

with nonlinear boundary operators

$$G_j(x, D^{\gamma_{j1}} u_1, \ldots, D^{\gamma_{jp}} u_p) = 0, \quad j = 1, \ldots, r, \ x \in \partial\Omega, \tag{3.2}$$

in an unbounded domain $\Omega \in \mathbb{R}^n$. Here $D^{\beta_{ik}} u_k$ is a vector with the components $D^\alpha u_k = \partial^{|\alpha|} u_k / \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$ where the multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ takes all values such that $0 \le |\alpha| = \alpha_1 + \cdots + \alpha_n \le \beta_{ik}$, β_{ik} are given integers. The vectors $D^{\gamma_{jk}} u_k$ are defined similarly. The regularity of the functions F_i, G_i, $u = (u_1, \ldots, u_p)$, and of the domain Ω is determined by β_{ik}, γ_{jk}, $i, k = 1, \ldots, p$, $j = 1, \ldots, r$ (see below).

In what follows we will use also the notations

$$\begin{aligned} F_i(x, \mathcal{D}_i u) &= F_i(x, D^{\beta_{i1}} u_1, \ldots, D^{\beta_{ip}} u_p), \\ G_j(x, \mathcal{D}_j^b u) &= G_j(x, D^{\gamma_{j1}} u_1, \ldots, D^{\gamma_{jp}} u_p). \end{aligned} \tag{3.3}$$

The corresponding linear operators are

$$A_i(v, \eta_i) = \sum_{k=1}^{p} \sum_{|\alpha| \le \beta_{ik}} a_{ik}^\alpha(x, \eta_i) D^\alpha v_k, \quad i = 1, \ldots, p, \ x \in \Omega, \tag{3.4}$$

$$B_j(v, \zeta_i) = \sum_{k=1}^{p} \sum_{|\alpha| \le \gamma_{jk}} b_{jk}^\alpha(x, \zeta_i) D^\alpha v_k, \quad j = 1, \ldots, r, \ x \in \partial\Omega, \tag{3.5}$$

where

$$a_{ik}^\alpha(x, \eta_i) = \frac{\partial F_i(x, \eta_i)}{\partial \eta_{ik}^\alpha}, \qquad b_{jk}^\alpha(x, \zeta_i) = \frac{\partial G_j(x, \zeta_j)}{\partial \zeta_{jk}^\alpha}, \tag{3.6}$$

$\eta_i \in \mathbb{R}^{n_i}$ and $\zeta_j \in \mathbb{R}^{m_j}$ are the vectors with the components η_{ik}^α and ζ_{jk}^α, respectively, ordered in the same way as the derivatives in (3.1) and (3.2).

The system (3.1) and (3.2) is called *elliptic* if the corresponding system (3.4) and (3.5) is elliptic in the sense of [1] for all values of parameters η_i, ζ_j. When we mention the *Shapiro-Lopatinskii* condition for operators (3.1) and (3.2) we mean the corresponding condition for operators (3.4) and (3.5) for any $\eta_i \in \mathbb{R}^{n_i}$ and $\zeta_j \in \mathbb{R}^{m_j}$.

We suppose that F_i (G_i) satisfies the following conditions: for any positive number M and for all multi-indices β and γ: $|\beta+\gamma| \le l - s_i + 2$ ($|\beta+\gamma| \le l - \sigma_i + 2$), $|\beta| \le l - s_i$ ($|\beta| \le l - \sigma_i$) the derivatives $D_x^\beta D_\eta^\gamma F_i(x,\eta)$ ($D_x^\beta D_\zeta^\gamma G_i(x,\zeta)$) as functions of $x \in \Omega$, $\eta \in \mathbb{R}^{n_i}$, $|\eta| \le M$ ($x \in V$, $\zeta \in \mathbb{R}^{m_i}$, $|\zeta| \le M$) satisfy Hölder condition in x uniformly in η (ζ) and Lipschitz condition in η (ζ) uniformly in x (with constants possibly depending on M). We use the notations introduced in Section 2.1.

The domain Ω is supposed to be of class $C^{l+\lambda+\alpha}$, where $\lambda = \max(-s_i, -\sigma_i, t_j)$ and to satisfy the conditions of Section 2.

Denote $F = (F_1, \dots, F_p)$, $G = (G_1, \dots, G_r)$. Then (F, G) acts from $E_{0,\mu}$ into E_μ.

In Section 3.2 we study properness of the operator (F, G). We preface the study with a result on properness of operators in Banach spaces (Section 3.1).

3.1. Lemma on properness of operators in Banach spaces. Let E_0 and E be two Banach spaces. Suppose that a topology is introduced in E_0 such that the convergence in this topology, which we denote $\rightharpoonup$, has the following property: for any sequence $\{u_n\}$, $u_n \in E_0$, bounded in E_0-norm, there is a subsequence $\{u_{n_k}\} : u_{n_k} \rightharpoonup u_0 \in E_0$.

We consider the operator $T(u) : D \to E$, where $D \subset E_0$. Suppose that this operator is *closed* with respect to the convergence $\rightharpoonup$ in the following sense: if $T(u_k) = f_k$, $u_k \in D$, $f_k \in E$ and $u_k \rightharpoonup u_0 \in E_0$, $f_k \to f_0$ in E, then $u_0 \in D$ and $T(u_0) = f_0$.

Lemma 3.1. *Suppose that D is a bounded closed set in E_0, the operator $T(u)$ is closed with respect to the convergence $\rightharpoonup$ and for any $u_0 \in D$ there exists a linear bounded operator $S(u_0) : E_0 \to E$, which has a closed range and finite-dimensional kernel, such that for any sequence $\{v_k\}$, $v_k \in D$, $v_k \rightharpoonup u_0 \in D$, we have*

$$||T(u_0) - T(v_k) - S(u_0)(u_0 - v_k)||_E \longrightarrow 0. \tag{3.7}$$

Then $T(u)$ is a proper operator.

Proof. Consider a sequence $\{u_n\}$ in D such that $f_n = T(u_n) \to f_0$ in E. We have to prove that there exists a subsequence of $\{u_n\}$ which is convergent in E_0. Consider a subsequence $\{u_{n_i}\}$ such that $u_{n_i} \rightharpoonup u_0 \in E_0$. Then since $T(u)$ is closed, we have $u_0 \in D$ and $T(u_0) = f_0$. Denote $v_i = u_{n_i} - u_0$ and $h_i = S(u_0)v_i$. Then $h_i = [S(u_0)(u_{n_i} - u_0) - (T(u_{n_i}) - T(u_0))] + (T(u_{n_i}) - T(u_0)) \to 0$ in E. Suppose that $w_1, \dots, w_k$ is a basis of $\operatorname{Ker} S(u_0)$ and $\{\varphi_i\}$ is a biorthogonal sequence of functionals in the dual to E_0 space. Denote $E_1 = \{u \in E_0, \langle \varphi_i, u \rangle = 0,\ i = 1, \dots, k\}$. Then we have

$$v_i = \sum_{j=1}^{k} \langle \varphi_j, v_i \rangle w_j + v_i^1, \quad v_i^1 \in E_1. \tag{3.8}$$

Denote by S_1 the restriction of $S(u_0)$ on E_1. Then $S_1 v_i^1 = h_i$. By Banach theorem, S_1 has bounded inverse. So v_i^1 is a convergent in E_0 sequence. Since $u_{n_i} \in D$ and so v_i is a bounded sequence in E_0, it follows from (3.8) that we can find a convergent subsequence of v_i. The lemma is proved. □

3.2. Properness of elliptic operators. In this section, we prove that the operator $T = (F, G) : E_{0,\mu} \to E_\mu$ defined above satisfies the conditions of Lemma 3.1 under the assumptions formulated below. The convergence $\rightharpoonup$ is convergent in the space $E_{0,\mu}(\Omega_R)$ for $\alpha = 0$ and any $R > 0$. Here Ω_R is the intersection of Ω with a ball B_R in $\mathbb{R}^n$ with a radius R and the center at 0. It is clear that any bounded sequence in $E_{0,\mu}$ has a $\rightharpoonup$ convergent subsequence.

As a domain D we take a closed ball in $E_{0,\mu}$ with the center at zero. Obviously the operator $T = (F, G)$ is closed with respect to the convergence $\rightharpoonup$.

We construct below the operator S introduced in Lemma 3.1. Let $F = (F_1, \dots, F_p)$, where F_i is the operator (3.1), and let $\eta_i = (\eta_{i1}, \dots, \eta_{in_i})$ and $\eta_i^0 = (\eta_{i1}^0, \dots, \eta_{in_i}^0)$ be two vectors in $\mathbb{R}^{n_i}$. Then by Taylor's formula we can write

$$F_i(x, \eta_i) = F_i(x, \eta_i^0) + \sum_{j=1}^{n_i} F'_{i\eta_{ij}}(x, \eta_i^0)(\eta_{ij} - \eta_{ij}^0) + \int_0^1 (1-s) \sum_{j,k=1}^{n_i} F''_{i\eta_{ij}\eta_{ik}}(x, \eta_i^0 + s(\eta_i - \eta_i^0))\, ds (\eta_{ij} - \eta_{ij}^0)(\eta_{ik} - \eta_{ik}^0). \tag{3.9}$$

Therefore, for any $u, u^0 \in E_{0,\mu}$ we have

$$F_i(x, \mathcal{D}_i u) - F_i(x, \mathcal{D}_i u^0) = A_i(u - u^0, \mathcal{D}_i u^0) + \Phi_i(u, u^0), \tag{3.10}$$

where A_i is given by (3.4) and

$$\Phi_i(u, u^0) = \int_0^1 (1-s) \sum_{j,k=1}^{n_i} F''_{iv_jv_k}(x, v^0 + s(v - v^0))\, ds (v_j - v_j^0)(v_k - v_k^0), \tag{3.11}$$

$v(x) = \mathcal{D}_i u(x)$, $v^0(x) = \mathcal{D}_i u^0(x)$.

Lemma 3.2. *The convergence $\|\Phi_i(u^m, u^0)\|_{C_\mu^{l-s_i+\alpha}(\Omega)} \to 0$ takes place if $u^m \rightharpoonup u^0$ and $\|u^m\|_{E_{0,\mu}}$ is bounded.*

Proof. It is sufficient to prove that

$$||D^\beta (F''_{iv_jv_k}(x, v^0 + s(v^m - v^0))(v_j^m - v_j^0)(v_k^m - v_k^0)\mu)||_{C^\alpha(\Omega)} \longrightarrow 0 \tag{3.12}$$

for $|\beta| \le l - s_i$. Here $v^m(x) = \mathfrak{D}_i u^m(x)$. We will prove that

$$||D^\beta F''_{iv_jv_k}(x, v^0 + s(v^m - v^0))||_{C^\alpha(\Omega)} \le M, \quad |\beta| \le l - s_i, \tag{3.13}$$

where M is a constant and

$$||D^\beta((v_j^m - v_j^0)(v_k^m - v_k^0)\mu)||_{C^\alpha(\Omega)} \longrightarrow 0, \quad |\beta| \le l - s_i \text{ as } m \longrightarrow \infty. \tag{3.14}$$

We begin with (3.13). Let $u^m = (u_1^m, \dots, u_p^m)$. By assumption $\|u_k^m\|_{C_\mu^{l+t_k+\alpha}(\Omega)} \le M_1$ $(k = 1, \dots, p)$. (Here and below M with subscripts denotes constants independent of u and v.) It follows that

$$||u_k^m||_{C^{l+t_k+\alpha}(\Omega)} \le M_2. \tag{3.15}$$

Indeed, denote $w = \mu u_k^m$. Then $\|w\|_{C^{l+t_k+\alpha}(\Omega)} \le M_1$, $u_k^m = (1/\mu)w$, and (3.15) follows easily from the properties of the function $\mu(x)$ since by (2.9), $D^\beta(1/\mu)$ is bounded for any multi-index β.

Obviously (3.15) implies

$$||v^m||_{C^{l-s_i+\alpha}(\Omega)} \le M_3. \tag{3.16}$$

Inequality (3.13) follows from this inequality and from the conditions of smoothness of the functions F_i.

Now we prove (3.14). Denote $w_j^m = v_j^m - v_j^0$. Obviously $D^\beta(w_j^m w_k^m \mu)$ is a sum of expressions of the form

$$D^\gamma w_j^m D^\tau w_k^m D^\sigma \mu = [\mu D^\gamma w_j^m][\mu D^\tau w_k^m]\frac{1}{\mu}\left(\frac{1}{\mu}D^\sigma \mu\right) \tag{3.17}$$

with constant coefficients, where γ, τ, σ are multi-indices, $\gamma + \tau + \sigma \le \beta$. The last factor in (3.17) is bounded by virtue of (2.9). From the properties of the function μ we conclude that $1/\mu$ and $D_i(1/\mu)$ $(i = 1, \dots, n)$ tend to 0 as $|x| \to \infty$, $x \in \Omega$. So

$$\left|\left|\frac{1}{\mu}\right|\right|_{C^\alpha(\Omega_R^-)} \longrightarrow 0 \quad \text{as } R \longrightarrow \infty. \tag{3.18}$$

Here Ω_R^- is the intersection of Ω with the ball $|x| > R$.

We prove next that

$$||\mu D^\gamma w_j^m||_{C^\alpha(\Omega)} \le M_4, \tag{3.19}$$

where $|\gamma| \le l - s_i$. Denote $y_k^m = u_k^m - u_k^0$. Then w_j^m has the form $D^\sigma y_k^m$ with $|\sigma| \le s_i + t_k$. So $D^\gamma w_j^m$ has the form $D^{\sigma+\gamma} y_k^m$ with $|\sigma+\gamma| \le l + t_k$. By the conditions of the lemma $\|y_k^m\|_{C_\mu^{l+t_k+\alpha}(\Omega)} \le M_5$, and (3.19) follows.

From (3.18) and (3.19) we obtain the convergence $\|D^\beta(w_j^m w_k^m \mu)\|_{C^\alpha(\Omega_R^-)} \to 0$ as $R \to \infty$. So to prove (3.14) it is sufficient to verify that $\|D^\beta(w_j^m w_k^m \mu)\|_{C^\alpha(\Omega_R)} \to 0$ for any R as $m \to \infty$. This follows from (3.17) and the fact that $\|\mu D^\beta w_j^m\|_{C^\alpha(\Omega_R)}$ is bounded and $\|\mu D^\beta w_j^m\|_{C(\Omega_R)} \to 0$ as $m \to \infty$ for $|\beta| \le l - s_i$ since $u^m \rightharpoonup u^0$. Therefore, the Hölder norm of the product of the first two factors in the right-hand side of (3.17) converges to zero. The lemma is proved. □

Lemma 3.2 implies the convergence

$$||F_i(x, \mathfrak{D}_i u^m) - F_i(x, \mathfrak{D}_i u^0) - A_i(u^m - u^0, \mathfrak{D}_i u^0)||_{C_\mu^{l-s_i+\alpha}(\Omega)} \longrightarrow 0 \tag{3.20}$$

if $u^m \rightharpoonup u^0$ and $\|u^m\|_{E_{0,\mu}}$ is bounded.

Similarly, we have for the operators $G_j(x, \mathfrak{D}_j^b u(x))$ $(j = 1, \dots, r)$:

$$||G_j(x, \mathfrak{D}_j^b u^m) - G_j(x, \mathfrak{D}_j^b u^0) - B_j(u^m - u^0, \mathfrak{D}_j^b u^0)||_{C_\mu^{l-\sigma_j+\alpha}(\partial\Omega)} \longrightarrow 0 \tag{3.21}$$

if $u^m \rightharpoonup u^0$ and $\|u^m\|_{E_{0,\mu}}$ is bounded.

Consider the operator

$$\begin{aligned} S(u_0)u = \big(&A_1(u, \mathfrak{D}_1 u_0), \dots, A_p(u, \mathfrak{D}_p u_0), \\ &B_1(u, \mathfrak{D}_1^b u_0), \dots, B_r(u, \mathfrak{D}_r^b u_0)\big) : E_{0,\mu} \longrightarrow E_\mu. \end{aligned} \tag{3.22}$$

We are interested in limiting operators for $S(u_0)$ in the sense of the previous section. We consider also the operator

$$S_0 u = \big(A_1(u,0), \dots, A_p(u,0), B_1(u,0), \dots, B_r(u,0)\big) : E_{0,\mu} \longrightarrow E_\mu, \tag{3.23}$$

which does not depend on u_0.

LEMMA 3.3. *For any $u_0 \in E_{0,\mu}$ the limiting operators for $S(u_0)$ and S_0 coincide.*

Proof. Consider first the operator $A_i(u; \eta_i)$ defined by (3.4). Since $u_0 \in E_{0,\mu}$, then $\mu D^\beta u_{0k}(x) \in C^\alpha(\Omega)$ for $|\beta| \le l + t_k$. So $\mu \mathfrak{D}_i u_0 \in C^\alpha(\Omega)$ and therefore

$$\left|\mathfrak{D}_i u_0(x)\right| \le \frac{M}{\mu(x)} \longrightarrow 0. \tag{3.24}$$

Let $|x^m| \to \infty$, $x^m \in \Omega$. Then $|x+x^m| \to \infty$ for all $x \in B_R$. So there exists m_0 such that for all $m > m_0$ and all $x \in \Omega_* \cap B_R$, the inequality $|\mathcal{D}_i u_0(x+x^m)| \leq 1$ holds. Here Ω_* is a limiting domain which corresponds to the sequence x^m.

Denote $f_{ik}^{\beta}(x,\eta_i) = \partial F_i(x,\eta_i)/\partial\eta_k^{\beta}$. It follows from the properties of the function F_i that for $m > m_0$ we have

$$\begin{aligned} &\left| f_{ik}^{\beta}(x+x^m,0) - f_{ik}^{\beta}\left(x+x^m,\mathcal{D}_i u_0(x+x^m)\right)\right| \\ &\qquad \leq K\left|\mathcal{D}_i u_0(x+x^m)\right| \leq \frac{KM}{\mu(x+x^m)} \longrightarrow 0 \end{aligned} \tag{3.25}$$

as $|x^m| \to \infty$, $x \in B_R$. Therefore, if one of the functions

$$f_{ik}^{\beta}(x+x^m,0), \qquad f_{ik}^{\beta}\left(x+x^m,\mathcal{D}_i u_0(x+x^m)\right) \tag{3.26}$$

has a limit as $|x^m| \to \infty$, then the same is true for another one and the limits coincide. Thus the lemma is proved for the operator (3.4). The proof is similar for the operator (3.5). The lemma is proved. □

Theorem 3.4. *Suppose that the system of operators (3.1) is uniformly elliptic and for the system of operators (3.1) and (3.2) Shapiro-Lopatinskii conditions are satisfied. Assume further that all the limiting operators for the operator S_0 satisfy Condition NS. Then the operator $(F,G) : E_{0,\mu} \to E_\mu$ is proper.*

Proof. We use Lemma 3.1 for the operator $T = (F,G)$. For any $u_0 \in E_{0,\mu}$ we take

$$\begin{aligned} S(u_0) = &\left(A_1\left(u_0,\mathcal{D}_1 u_0\right),\dots,A_p\left(u_0,\mathcal{D}_p u_0\right),\right. \\ &\left. B_1\left(u_0,\mathcal{D}_1^b u_0\right),\dots,B_r\left(u_0,\mathcal{D}_r^b u_0\right)\right) : E_{0,\mu} \longrightarrow E_\mu. \end{aligned} \tag{3.27}$$

From (3.20) and (3.21) we obtain

$$\left|\left|T(u_0) - T(u^m) - S(u_0)(u_0 - u^m)\right|\right|_{E_\mu} \longrightarrow 0 \tag{3.28}$$

if $u^m \rightharpoonup u_0$ and $\|u^m\|_{E_{0,\mu}}$ is bounded. If all limiting operators for S_0 satisfy Condition NS, then according to Lemma 3.3 the same is true for all limiting operators for $S(u_0)$ for any $u_0 \in E_{0,\mu}$. The results of the previous section imply that $S(u_0)$ has a closed range and a finite-dimensional kernel. The theorem is proved. □

Remark 3.5. Functions from the weighted space $E_{0,\mu}$ tend to zero at infinity. If we look for solutions, which are not zero at infinity, we can represent them in the form $u+\psi$, where ψ is a given function with a needed behavior at infinity, and u belongs to $E_{0,\mu}$. (See, for example, [30] where such reduction is done for travelling waves.)

3.3. Operators depending on parameter. Consider an operator $T(u,t) : D \times [0,1] \to E$, $D \subset E_0$ depending on parameter $t \in [0,1]$. We suppose here as in Section 3.1 that E_0 and E are arbitrary Banach spaces. We will obtain conditions of its properness with respect to both variables u and t.

First of all, we modify the definition of closed operators given in Section 3.1.

Let $T(u_k, t_k) = f_k$, $t_k \to t_0$, $u_k \in D$, $f_k \in E$, $u_k \rightharpoonup u_0 \in E_0$, $f_k \to f_0$ in E, then $u_0 \in D$ and $T(u_0, t_0) = f_0$.

Lemma 3.6. *Suppose that D is a bounded set in E_0, the operator $T(u,t)$ is closed, and for any $u_0 \in D$ there exists a linear bounded operator $S(u_0) : E_0 \to E$, which has a closed range and a finite-dimensional kernel, such that for any sequence $\{v_k\}$, $v_k \in D$, $v_k \rightharpoonup u_0 \in D$ and $t_k \to t_0$, we have*

$$\left\|T(u_0,t_0) - T(v_k,t_k) - S(u_0)(u_0 - v_k)\right\|_E \longrightarrow 0. \tag{3.29}$$

Then $T(u,t)$ is a proper operator.

The proof of the lemma remains the same as above.

Suppose now that the operator $T(u,t)$ satisfies the conditions of Lemma 3.1 for any $t \in [0,1]$ fixed, and it depends on t continuously in the operator norm, that is,

$$\left\|T(u,t) - T(u,t_0)\right\|_E \le c(t,t_0), \quad \forall u \in D, \tag{3.30}$$

where $c(t,t_0) \to 0$ as $t \to t_0$. Then

$$\begin{aligned} &\left\|T(u_0,t_0) - T(v_k,t_k) - S(u_0)(u_0 - v_k)\right\|_E \\ &\quad \le \left\|T(u_0,t_0) - T(v_k,t_0) - S(u_0)(u_0 - v_k)\right\|_E \\ &\qquad + \left\|T(v_k,t_0) - T(v_k,t_k)\right\|_E. \end{aligned} \tag{3.31}$$

Therefore, if the conditions of Lemma 3.1 are satisfied for each t fixed and the operator depends continuously on parameter, then Lemma 3.6 holds.

On the other hand, if the operator $T(u,t)$ is closed in the sense of Section 3.1 for each t fixed, and if it depends continuously on parameter, then it is also closed in the sense of the definition given in this section.

Thus, under the conditions of Section 3.2, elliptic operators depending continuously on parameter are proper with respect to two variables.

4. Topological degree

In this section, we construct a topological degree for a class of operators in Banach spaces. We recall the definition of a topological degree (see, e.g., [18, 20]). Let E_0 and E be two Banach spaces. Suppose we are given a class F of operators acting from E_0 to E and a class H of homotopies, that is, the mappings

$$A_\tau(u) : E_0 \times [0,1] \longrightarrow E, \quad \tau \in [0,1],\ u \in E_0 \tag{4.1}$$

such that $A_\tau(u) \in F$ for any $\tau \in [0,1]$. Assume, moreover, that for any bounded open set $D \subset E_0$ and any operator $A \in F$ such that

$$A(u) \neq 0, \quad u \in \partial D \tag{4.2}$$

(∂D denotes the boundary of D), there is an integer $\gamma(A,D)$ satisfying the following conditions.

(i) *Homotopy invariance*. Let $A_\tau(u) \in H$ and

$$A_\tau(u) \neq 0, \quad u \in \partial D,\ \tau \in [0,1]. \tag{4.3}$$

Then

$$\gamma(A_0, D) = \gamma(A_1, D). \tag{4.4}$$

(ii) *Additivity*. Let $D \subset E_0$ be an arbitrary bounded open set in E_0, and let $D_1, D_2 \subset D$ be open sets such that $D_1 \cap D_2 = \emptyset$. Suppose that $A \in F$ and

$$A(u) \neq 0, \quad u \in \bar{D} \backslash (D_1 \cup D_2). \tag{4.5}$$

Then

$$\gamma(A,D) = \gamma(A,D_1) + \gamma(A,D_2). \tag{4.6}$$

(iii) *Normalization*. There exists a bounded linear operator $J : E_0 \to E$ with a bounded inverse defined on all of E such that for any bounded open set $D \subset E_0$ with $0 \in D$,

$$\gamma(J,D) = 1. \tag{4.7}$$

The integer $\gamma(A,D)$ is called *topological degree*.

In Section 4.1, we study orientation of linear operators used for construction of the topological degree. In Section 4.2, topological degree is constructed for a class of operators. It contains in particular elliptic operators, which are Fredholm, proper, and for which the Fréchet differentials satisfy some spectral properties (Section 4.3). Fredholm property and properness of elliptic operators are discussed in the previous sections. The needed spectral properties follow in particular from the sectoriality of elliptic operators (see [13, 17] and the references therein).

4.1. Orientation of operators. Let E_0, E_1, and E_2 be Banach spaces. We suppose that $E_0 \subset E_1$. This means that if $u \in E_0$, then $u \in E_1$ and $\|u\|_{E_1} \le K\|u\|_{E_0}$, where K does not depend on u. Denote $E = E_1 \times E_2$. We consider linear operators

$A_1 : E_0 \to E_1$, $A_2 : E_0 \to E_2$, $A = (A_1, A_2) : E_0 \longrightarrow E$, and the following class of operators: class O is a class of bounded operators $A : E_0 \to E$ satisfying the following conditions:

(i) operator $(A_1 + \lambda I, A_2) : E_0 \to E$ is Fredholm with index zero for all $\lambda \geq 0$,
(ii) equation $A_1 u = 0$, $A_2 u = 0$ $(u \in E_0)$ has only zero solution,
(iii) there exists $\lambda_0 = \lambda_0(A)$ such that the equation

$$(A_1 + \lambda I)u = 0, \quad A_2 u = 0 \quad (u \in E_0) \tag{4.8}$$

has only zero solution for all $\lambda > \lambda_0$. Here I is the identity operator in E_0.

Proposition 4.1. *Let the operator $A = (A_1, A_2)$ belong to class O. Then the eigenvalue problem*

$$A_1 u + \lambda u = 0, \quad A_2 u = 0 \quad (u \in E_0) \tag{4.9}$$

has only finite number of positive eigenvalues λ. Each of them has a finite multiplicity.

Remark 4.2. Instead of the eigenvalue problem (4.9) we can consider the eigenvalue problem

$$A_{1,2} u + \lambda u = 0, \quad u \in E_{0,2}, \tag{4.10}$$

where $E_{0,2}$ is the space of all $u \in E_0$ such that $A_2 u = 0$, and $A_{1,2}$ is the restriction of A_1 on the space $E_{0,2}$. By multiplicity of λ in (4.9) we mean the multiplicity of λ in (4.10).

Proof of Proposition 4.1. Since $A \in O$, the operator $A_{1,2} + \lambda I$ is Fredholm with index zero for all $\lambda \geq 0$ and invertible for $\lambda = 0$ and $\lambda > \lambda_0$. The proposition follows from known properties of Fredholm operators (see [14]). □

Definition 4.3. The number

$$o(A) = (-1)^{\nu}, \tag{4.11}$$

where ν is the sum of multiplicities of all positive eigenvalues of problem (4.9), is called *orientation* of the operator A. Operators A belonging to class O are called *orientable.*

Definition 4.4. Operators $A^0 \in O$ and $A^1 \in O$ are said to be *homotopic* if there exists an operator $A(\tau) : E_0 \times [0,1] \to E$ such that $A(\tau) \in O$ for all $\tau \in [0,1]$, $A(\tau)$ is continuous in the operator norm with respect to τ, $\lambda_0(A(\tau))$ is bounded, and

$$A(0) = A^0, \qquad A(1) = A^1. \tag{4.12}$$

THEOREM 4.5. *If A^0 and A^1 are homotopic, then*

$$o(A^0) = o(A^1). \tag{4.13}$$

Proof. Let $\tau_0 \in [0,1]$. It is sufficient to prove that

$$o(A(\tau)) = o(A(\tau_0)) \tag{4.14}$$

for τ in some neighborhood of τ_0. Indeed, covering the interval $[0,1]$ by such neighborhoods and taking a finite subcovering we get (4.13).

To prove (4.14) consider the eigenvalue problems

$$A_1(\tau_0)u+\lambda u = 0, \quad A_2(\tau_0)u = 0, \quad u \in E_0, \tag{4.15}$$

$$A_1(\tau)u+\lambda u = 0, \quad A_2(\tau)u = 0, \quad u \in E_0. \tag{4.16}$$

We should prove that for τ close to τ_0 the sum of multiplicities of positive eigenvalues λ of problems (4.15) and (4.16) coincide modulo 2. It is convenient to consider the problem

$$A_1(\tau_0)u+\lambda u = 0, \quad A_2(\tau)u = 0, \quad u \in E_0 \tag{4.17}$$

and to compare (4.15) and (4.16) with (4.17).

Consider first problems (4.15) and (4.17). Consider also the operators $A_{1,2}(\tau_0)$ and $A_{1,2}(\tau)$, the restrictions of $A_1(\tau_0)$ on the spaces

$$E_{0,2}(\tau_0) = \{u : u \in E_0,\ A_2(\tau_0)u = 0\}, \tag{4.18}$$

$$E_{0,2}(\tau) = \{u : u \in E_0,\ A_2(\tau)u = 0\}, \tag{4.19}$$

respectively. By (i) and (ii) of the definition of class O, $A_{1,2}(\tau_0)$ is invertible.

It is easy to see that for τ sufficiently close to τ_0, the operator $A_{1,2}(\tau)$ is also invertible and has uniformly bounded inverse. Indeed, denote $K(\tau) = (A_1(\tau_0), A_2(\tau)) : E_0 \to E$. Obviously $\|K(\tau) - K(\tau_0)\| \le \|A_2(\tau) - A_2(\tau_0)\|$. Since $K(\tau_0)$ is invertible, we conclude that if τ is sufficiently close to τ_0, then $K(\tau)$ has uniformly bounded inverse. Consider the equation $A_{1,2}(\tau)u = f$, $u \in E_{0,2}(\tau)$ or $A_1(\tau_0)u = f$, $A_2(\tau)u = 0$, $u \in E_0$, $f \in E_1$. Since $K(\tau)$ is invertible, this equation has a unique solution for any $f \in E_1$. So $A_{1,2}(\tau)$ is invertible and $\|A_{1,2}^{-1}(\tau)\| \le \|K^{-1}(\tau)\|$.

Denote

$$J = A_{1,2}^{-1}(\tau)A_{1,2}(\tau_0) : E_{0,2}(\tau_0) \longrightarrow E_{0,2}(\tau). \tag{4.20}$$

Problems (4.15) and (4.17) can be written as

$$A_{1,2}(\tau_0)v + \lambda v = 0, \quad v \in E_{0,2}(\tau_0), \tag{4.21}$$

$$A_{1,2}(\tau)u + \lambda u = 0, \quad u \in E_{0,2}(\tau). \tag{4.22}$$

Let $u = Jv$, $v \in E_{0,2}(\tau_0)$, $u \in E_{0,2}(\tau)$. Then from (4.22)

$$\frac{1}{\lambda}v + S_1 v = 0, \quad v \in E_{0,2}(\tau_0), \tag{4.23}$$

where $S_1 = J^{-1}A_{1,2}^{-1}(\tau)J = A_{1,2}^{-1}(\tau_0)A_{1,2}^{-1}(\tau)A_{1,2}(\tau_0)$. We have from (4.21)

$$\frac{1}{\lambda}v + S_0 v = 0, \quad v \in E_{0,2}(\tau_0), \tag{4.24}$$

where $S_0 = A_{1,2}^{-1}(\tau_0)$.

We will prove that for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$||S_1 - S_0|| < \epsilon \quad \text{if } |\tau - \tau_0| < \delta. \tag{4.25}$$

Consider the problems

$$A_1(\tau_0)u = f, \quad A_2(\tau_0)u = 0, \quad u \in E_0, \ f \in E_1, \tag{4.26}$$

$$A_1(\tau_0)u_1 = f, \quad A_2(\tau)u_1 = 0, \quad u_1 \in E_0, \ f \in E_1 \tag{4.27}$$

or

$$A_{1,2}(\tau_0)u = f, \quad A_{1,2}(\tau)u_1 = f, \quad u \in E_{0,2}(\tau_0), \ u_1 \in E_{0,2}(\tau). \tag{4.28}$$

Let $B = A_2(\tau_0) - A_2(\tau)$. Denote $w = u - u_1$. Then from (4.26) and (4.27)

$$A_1(\tau_0)w = 0, \qquad A_2(\tau_0)w = -Bu_1. \tag{4.29}$$

We have from (4.28)

$$A_1(\tau_0)w = 0, \qquad A_2(\tau_0)w = -BA_{1,2}^{-1}(\tau)f. \tag{4.30}$$

Denote

$$L = (A_1(\tau_0), A_2(\tau_0)) : E_0 \longrightarrow E. \tag{4.31}$$

Then (4.30) implies

$$\|w\|_{E_0} \le ||L^{-1}|| \|B\| ||A_{1,2}^{-1}|| \|f\|_{E_1}. \tag{4.32}$$

By (4.28) we have

$$||A_{1,2}^{-1}(\tau_0)f - A_{1,2}^{-1}(\tau)f||_{E_0} \le ||L^{-1}|| \|B\| ||A_{1,2}^{-1}|| \|f\|_{E_1}. \tag{4.33}$$

Therefore,

$$||A_{1,2}^{-1}(\tau_0) - A_{1,2}^{-1}(\tau)|| \le ||L^{-1}|| \|B\| ||A_{1,2}^{-1}(\tau)||. \tag{4.34}$$

Since $A_2(\tau) \to A_2(\tau_0)$ as $\tau \to \tau_0$ in the operator norm, we get $\|B\| \to 0$ as $\tau \to \tau_0$, and from (4.34) we obtain (4.25).

Using (4.25) we prove that if τ is sufficiently close to τ_0, then the sum of multiplicities of the negative eigenvalues of the operator S_1 coincides modulo 2 with the sum of multiplicities of the negative eigenvalues of the operator S_0. Indeed, taking into account that $\|S_1\|$ and $\lambda_0(A(\tau))$ are uniformly bounded, we conclude that there exists an interval $[\alpha,\beta]$, $\alpha<\beta<0$, such that all negative eigenvalues of the operators S_1 and S_0 lie in this interval. Let Γ be a rectifiable contour in the λ-plane which contains the interval $[\alpha,\beta]$ and such that all points inside this contour, except for negative eigenvalues of the operator S_0, are regular points of this operator. From the known results on root spaces (see [14]), it follows that the sum of multiplicities of all eigenvalues of S_1 lying inside Γ coincides with the sum of the multiplicities of the negative eigenvalues of S_0 if δ in (4.25) is sufficiently small. Therefore, the sum of multiplicities of negative eigenvalues of S_0 and S_1 coincide modulo 2. It follows that the sum of the multiplicities of positive eigenvalues of problems (4.21) and (4.22), and consequently of problems (4.15) and (4.17) coincide modulo 2.

We obtain now the same results for problems (4.16) and (4.17). Denote by $B(\tau_0)$ and $B(\tau)$ the restrictions of $A_1(\tau_0)$ and $A_1(\tau)$ on the space $E_{0,2}(\tau)$ (see (4.19)), respectively. Then obviously

$$||B(\tau) - B(\tau_0)|| \le ||A(\tau) - A(\tau_0)|| \longrightarrow 0 \tag{4.35}$$

as $\tau \to \tau_0$. By the same arguments that we used for the operators S_0 and S_1 above we prove that the sum of multiplicities of the negative eigenvalues of the operators $B(\tau)$ and $B(\tau_0)$ coincide modulo 2. The theorem is proved. □

Remark 4.6. The requirement $\lambda_0(A(\tau))$ is bounded in Definition 4.4 can be omitted if we replace (iii) in class O by the following:

(iii∗) *There exists $\lambda_0 = \lambda_0(A)$ such that the operator $(A_1 + I\lambda, A_2) : E_0 \to E$ has inverse for $\lambda > \lambda_0$ which is uniformly bounded.*

Indeed, denote $A(\tau,\lambda) = (A_1(\tau) + I\lambda, A_2(\tau))$. Let $\tau_0 \in [0,1]$. Then $A(\tau,\lambda) = A(\tau_0,\lambda) + B(\tau)$, where $B(\tau) = A(\tau) - A(\tau_0)$. For $\lambda > \lambda_0(A(\tau_0))$ we have

$$A(\tau,\lambda) = A(\tau_0,\lambda)\left[I + A^{-1}(\tau_0,\lambda)B(\tau)\right]. \tag{4.36}$$

Since $\|B(\tau)\| \to 0$ as $\tau \to \tau_0$, we can take $\delta(\tau_0) > 0$ such that $\|A^{-1}(\tau_0,\lambda)B(\tau)\| \le 1/2$ for all $\lambda > \lambda_0(A(\tau_0))$, $|\tau - \tau_0| < \delta(\tau_0)$. So for these values of τ and λ the operator $A(\tau,\lambda)$ has a uniformly bounded inverse. Taking the corresponding covering of the interval $[0,1]$ and choosing a finite subcovering, we obtain that $\lambda_0(A(\tau))$ is bounded for $\tau \in [0,1]$.

Class O with the property (iii$*$) instead of (iii) will be used in the construction of the topological degree.

4.2. Topological degree for Fredholm operators. Let E_0, E_1, E_2, and $E = E_1 \times E_2$ be the same spaces as in Section 4.1, and let $G \subset E_0$ be an open bounded set. We consider the following classes of linear (Φ) and nonlinear (F) operators.

Class Φ is a class of bounded linear operators $A = (A_1, A_2) : E_0 \to E$ satisfying the following conditions:

(i) the operator $(A_1 + I\lambda, A_2) : E_0 \to E$ is Fredholm for all $\lambda \ge 0$,
(ii) there exists $\lambda_0 = \lambda_0(A)$ such that operators $(A_1 + I\lambda, A_2) : E_0 \to E$ have inverse which are uniformly bounded for all $\lambda > \lambda_0$.

Class F is a class of proper operators $f \in C^1(G, E)$ such that for any $x \in G$ the Fréchet derivative $f'(x)$ belongs to Φ.

We introduce also the following class of homotopies.

Class H is a class of proper operators $f(x,t) \in C^1(G \times [0,1], E)$ which for any $t \in [0,1]$ belong to class F.

Two operators $f_0(x) : G \to E$ and $f_1(x) : G \to E$ are said to be *homotopic* if there exists $f(x,t) \in H$ such that

$$f_0(x) = f(x,0), \qquad f_1(x) = f(x,1). \tag{4.37}$$

In this section, we construct a topological degree for the classes F and H. In what follows D denote an open set such that $\bar{D} \subset G$.

Let $a \in E$, $f \in C^1(G,E)$,

$$f(x) \ne a \quad (x \in \partial D), \tag{4.38}$$

where ∂D is the boundary of D. Suppose that the equation

$$f(x) = a \quad (x \in D) \tag{4.39}$$

has finite number of solutions $x_1, \dots, x_m$ and $f'(x_k)$ $(k = 1, \dots, m)$ are invertible operators belonging to the class Φ. Then the orientation of these operators is defined. We will use the following notation:

$$\gamma(f, D; a) = \sum_{k=1}^{m} o(f'(x_k)). \tag{4.40}$$

If (4.39) does not have solutions, it is supposed that $\gamma(f, D; a) = 0$.

LEMMA 4.7. *Let $f(x,t) \in H$, $a \in E$ be a regular value of $f(\cdot, 0)$ and $f(\cdot, 1)$. Suppose that*

$$f(x,t) \neq a \quad (x \in \partial D,\ t \in [0,1]). \tag{4.41}$$

Then

$$\gamma(f(\cdot,0), D; a) = \gamma(f(\cdot,1), D; a). \tag{4.42}$$

Proof. The main part of the proof of the lemma is done under the assumption that a is a regular value of the homotopy under consideration. Since this is not supposed in the formulation of the lemma, we replace $f(x,t)$ by a close function $g(x,t)$ for which a is a regular value and

$$\gamma(g(\cdot,0), D; a) = \gamma(f(\cdot,0), D; a), \tag{4.43}$$

$$\gamma(g(\cdot,1), D; a) = \gamma(f(\cdot,1), D; a) \tag{4.44}$$

(see [23]). Then we prove that

$$\gamma(g(\cdot,0), D; a) = \gamma(g(\cdot,1), D; a). \tag{4.45}$$

To construct the function $g(x,t)$, we use the following result (see [23]). For any $\eta > 0$ an operator $h \in C^1(G \times [0,1] \times [0,1], E)$ with the following properties can be constructed:

(i) $\|h(\cdot,\tau) - f\|_{1,G\times[0,1]} < \eta$ for any $\tau \in [0,1]$,
(ii) h is proper,
(iii) for $\tau \in [0,1]$, $h(\cdot,\tau)$ is Fredholm of index 1,
(iv) $h(\cdot,0) = f$ and a is a regular value of $h(\cdot,1)$.

Here we use the notation $\|f\|_{1,G\times[0,1]} = \sup\|f(x,t)\| + \sup\|f'(x,t)\|$ for $f \in C^1(G \times [0,1], E)$ (the supremum is taken over $(x,t) \in G \times [0,1]$ and f' is the Fréchet derivative of f).

We can put now $g(x,t) = h(x,t,1)$, $x \in G$, $t \in [0,1]$. From (4.41) it follows that $\eta > 0$ can be taken such that

$$g(x,t) \neq a \quad (x \in \partial D,\ t \in [0,1]). \tag{4.46}$$

We will prove that for a proper choice of $\eta > 0$ equality (4.43) holds. Since a is a regular value of $f(x,0)$, $f(x,0) \neq a$, $x \in \partial D$ and $f(x,0)$ is a proper operator, it follows that the equation

$$f(x,0) = a, \quad x \in D, \tag{4.47}$$

has finite number of solutions.

If (4.47) does not have solutions, then taking η sufficiently small we conclude that the equation

$$g(x,0) = a, \quad x \in D, \tag{4.48}$$

does not have solutions either. In this case both parts of equality (4.43) are equal 0.

Suppose that (4.47) has solutions. We denote them by $x_1, \dots, x_m$. Let B_k ($k = 1, \dots, m$) be open balls with centers at x_k and radius r. We suppose that r is taken such that the closures of the balls are disjoint and belong to D. If $\eta > 0$ is taken sufficiently small, then (4.48) has exactly m solutions and moreover the equation $g(x,0) = a$, $x \in B_k$ has one and only one solution ($k = 1, \dots, m$) (see [23]). Denote this solution by ξ_k.

Taking into account that $f'_x(x_k, 0)$ belongs to Φ and that it is invertible, it is easy to prove, for a proper choice of r and η, that $g'_x(\xi_k, 0)$ also belongs to Φ and is invertible. So the orientation of this operator is defined. Moreover using Theorem 4.5 to the homotopy $(1-\tau)f'_x(x_k,0) + \tau g'_x(\xi_k,0)$, $\tau \in [0,1]$ we obtain

$$o(g'_x(\xi_k,0)) = o(f'_x(x_k,0)) \tag{4.49}$$

and (4.43) follows from this. Decreasing η, if necessary, we obtain (4.44) in the same way.

We prove now (4.45). If both of the equations

$$g(x,0) = a, \quad g(x,1) = a \quad (x \in D) \tag{4.50}$$

have no solutions, then (4.45) is true, both parts of the equality are equal 0.

Suppose that at least one of (4.50) has a solution. Then the set $S = g^{-1}(a) \cap \bar{D} \times [0,1]$ is not empty. Since a is a regular value of g, $g^{-1}(a)$ is a one-dimensional submanifold of $\bar{D} \times [0,1]$. The set S is compact since the map is proper. Because of (4.46) the set S cannot have joint points with the set $\partial D \times [0,1]$. Suppose that the equation $g(x,0) = a$ has m solutions $(m > 0) : \xi_1, \dots, \xi_m$,

$$g(\xi_k, 0) = a \quad (k = 1, \dots, m). \tag{4.51}$$

We denote by l_k the connected component of S which contains the point $(\xi_k, 0)$. The set l_k is homeomorphic to a closed interval $\Delta = [0,1]$. We denote the end-points of l_k by $P_0 = (\xi_k, 0)$ and P_1 and suppose that P_0 corresponds to the point 0 in Δ and P_1 to 1.

Denote $y = (x,t)$ $(x \in G,\ t \in [0,1])$. We introduce local coordinates on l_k by finite number of sets $\{U_i\}$ such that each of them is homeomorphic to an open or half-open interval Δ_i. Moreover, we can suppose that U_i is given by the equation

$$y = y(s) \quad (s \in \Delta_i) \tag{4.52}$$

and that there exists a derivative in the norm $\|y\| = \|x\| + |t|$. We have $g(y(s)) = a$ and therefore

$$g'(y(s))y'(s) = 0. \tag{4.53}$$

Since a is a regular value, then the range of the operator $g'(y(s))$ coincides with E. Moreover, the index of $g'(y(s))$ is 1. So $y'(s)$ is the only (up to a real factor) solution of (4.53). We have $y(s) = (x(s), t(s))$, where $x(s) \in E_0$, $t(s)$ is a real valued function. It is easy to see that we can construct a functional $\phi(s) \in E_0^*$ which is continuous with respect to $s \in \Delta_i$ and

$$\langle \phi(s), x'(s) \rangle > 0 \quad \text{if } ||x'(s)|| > 0, \tag{4.54}$$

where $\langle \cdot, \cdot \rangle$ denotes the action of a functional.

We can find η in (i) such that for all y satisfying the equation $g(y) = a$, the operators $g'_x(y)$ belong to Φ, and $\lambda_0(g'_x(y))$ are uniformly bounded. Indeed, denote by T the set of all solutions of the equation $f(y) = a$. From (i) and properness of f it follows that for any $\varepsilon > 0$ we can find $\eta > 0$ such that all solutions of the equation $g(y) = a$ belong to ε-neighborhood of T. Since T is compact, ε and η can be found such that $g'_x(y)$ has the mentioned property.

We represent g in the form $g = (g_1, g_2)$, where $g_1 : G \times [0,1] \to E_1$, $g_2 : G \times [0,1] \to E_2$. Denote by $g'_{ix}(x,t)$ and $g'_{it}(x,t)$ $(i = 1,2)$ the partial derivatives in x and t, respectively.

Consider the operators

$$A_1(s) = \begin{bmatrix} g'_{1x}(y(s)) & g'_{1t}(y(s)) \\ \phi(s) & t'(s) \end{bmatrix}, \qquad A_2(s) = \left(g'_{2x}(y(s)), g'_{2t}(y(s))\right), \tag{4.55}$$

where $A_1(s) : E_0 \times \mathbb{R} \to E_1 \times \mathbb{R}$, $A_2(s) : E_0 \times \mathbb{R} \to E_2$, $\mathbb{R}$ is the space of real numbers.

Denote $A(s) = (A_1(s), A_2(s)) : E_0 \times \mathbb{R} \to (E_1 \times \mathbb{R}) \times E_2$. It is easy to see that A is a Fredholm operator of index zero.

The equation $A(s)w = 0$, $w \in E_0 \times \mathbb{R}$ has only zero solution. Indeed, let $w = (u, v)$, $u \in E_0$, $v \in \mathbb{R}$. Then

$$g'\big(y(s)\big)w = 0, \qquad \langle \phi(s), u\rangle + t'(s)v = 0. \tag{4.56}$$

It follows that

$$w = \alpha(s)y'(s), \text{ that is, } u = \alpha(s)x'(s), \; v = \alpha(s)t'(s). \tag{4.57}$$

So

$$\langle \phi(s), u\rangle + t'(s)v = \alpha(s)\big(\langle \phi(s), x'(s)\rangle + t'^2(s)\big). \tag{4.58}$$

Since $y'(s) \neq 0$, then $\langle \phi(s), x'(s)\rangle + t'^2(s) \neq 0$, and therefore $\alpha(s) = 0$.

Let J be identity operator in $E_1 \times \mathbb{R}$. Then the operator

$$\big(A_1(s) + \lambda J, A_2(s)\big) : E_0 \times \mathbb{R} \longrightarrow (E_1 \times \mathbb{R}) \times E_2 \tag{4.59}$$

is Fredholm operator of index 0 for $\lambda \geq 0$.

Let $s \in \Delta_i$. We will prove that there exists $\lambda_0 > 0$ such that for $\lambda > \lambda_0$ the operator $(A_1(s) + \lambda J, A_2(s))$ has uniformly bounded inverse in λ. Indeed, consider the equation

$$\big(A_1(s) + \lambda J, A_2(s)\big)w = \psi, \quad w \in E_0 \times \mathbb{R}, \; \psi \in (E_1 \times \mathbb{R}) \times E_2. \tag{4.60}$$

Let $w = (w_1, w_2)$, $\psi = (\psi_1, \psi_2, \psi_3)$, $w_1 \in E_0$, $w_2 \in \mathbb{R}$, $\psi_1 \in E_1$, $\psi_2 \in \mathbb{R}$, $\psi_3 \in E_2$. We have

$$(g'_{1x} + \lambda I)w_1 + g'_{1t}w_2 = \psi_1 \tag{4.61}$$

$$(\phi, w_1) + (t' + \lambda)w_2 = \psi_2 \tag{4.62}$$

$$g'_{2x}w_1 + g'_{2t}w_2 = \psi_3. \tag{4.63}$$

We can find w_1 from (4.61) and (4.63) for $\lambda > \lambda_0$ since $(g'_{1x} + \lambda I, g'_{2x})$ has uniformly bounded inverse, and substitute in (4.62). Obviously the equation so obtained for w_2 can be solved for $\lambda > \lambda_0$ if λ_0 is sufficiently large. It is clear that the solution w_1, w_2 of (4.61), (4.62), and (4.63) is unique and can be estimated by a constant independent of λ. So we have proved that $(A_1(s) + \lambda J, A_2(s))$ has uniformly bounded inverse for $\lambda > \lambda_0$.

Operator $A(s)$ satisfies conditions formulated in the previous subsection. So the orientation $o(A(s))$ of operator $A(s)$ can be constructed, and it does not depend on s. By standard arguments we can prove that the orientation does not depend on the choice of covering of l_k.

Suppose now that for some s the operator $g'_x(y(s)) : E_0 \to E$ is invertible and $t'(s) \neq 0$. We will prove the following formula:

$$o(A(s)) = o(g'_x(y(s)))\,\mathrm{sgn}\,t'(s). \tag{4.64}$$

Consider the operator $A(s;\tau) = (A_1(s;\tau), A_2(s;\tau))$, $0 \leq \tau \leq 1$,

$$A_1(s;\tau) = \begin{bmatrix} g'_{1x}(y(s)) & \tau g'_{1t}(y(s)) \\ \tau\phi(s) & t'(s) \end{bmatrix}, \qquad A_2(s;\tau) = \big(g'_{2x}(y(s)), \tau g'_{2t}(y(s))\big). \tag{4.65}$$

As before we prove that this operator satisfies conditions of the previous subsection and, consequently,

$$o(A(s)) = o(A(s;0)). \tag{4.66}$$

Equality (4.64) easily follows from the definition of the orientation.

Consider now the operator $A(s)$ at the endpoints of the line $l_k : P_0 = (\xi_k, 0)$ and P_1. We begin with the point P_0. The operator $g'_x(\xi_k, 0)$ is invertible. For small t we can take $s = t$. Then $t'(s) = 1$.

There are two possibilities for the point P_1:

$$P_1 = (\xi_l, 0) \quad (l \neq k), \tag{4.67}$$

$$P_1 = (\bar{x}, 1), \tag{4.68}$$

where $(\bar{x}, 1)$ is a solution of the equation

$$g(\bar{x}, 1) = a. \tag{4.69}$$

Consider first the case (4.67). We can take $s = 1 - t$ in the neighborhood of the point P_1 (this corresponds to the positive orientation), and so $t'(s) = -1$. From (4.64) it follows that

$$o(g'_x(P_0)) = -o(g'_x(P_1)). \tag{4.70}$$

In the case (4.68) by the same reason we have

$$o(g_x'(P_0)) = o(g_x'(P_1)). \tag{4.71}$$

The proof of (4.45) follows directly from these equalities. The lemma is proved. □

THEOREM 4.8. *Let $f \in F$ and B be a ball $\|a\| < r$, $a \in E$ such that $f(x) \neq a$ ($x \in \partial D$) for all $a \in B$. Then for all regular values $a \in B$, $\gamma(f, D; a)$ does not depend on a.*

Proof. Let a_0 and a_1 be two regular values belonging to B. Denote $a_t = a_0(1-t) + a_1 t$, $t \in [0,1]$ and consider the operator $f(x,t) = f(x) - a_t$. It is easy to see that all conditions of Lemma 4.7 are satisfied for this operator if we set $a = 0$ in this lemma. So equality (4.42) is valid. From (4.40) we get $\gamma(f, D; a_0) = \gamma(f, D; a_1)$. The theorem is proved. □

Using this theorem we can give the following definition of the topological degree $\gamma(f, D)$.

Definition 4.9. Let $f \in F$ and $f(x) \neq 0$ ($x \in \partial D$). Let B be a ball $\|a\| < r$ in E such that $f(x) \neq a$ ($x \in \partial D$) for all $a \in B$. Then

$$\gamma(f, D) = \gamma(f, D; a) \tag{4.72}$$

for any regular value $a \in B$.

Existence of regular values $a \in B$ of f follows from the Sard-Smale theorem (see [24, 28]).

THEOREM 4.10 (homotopy invariance). *Let $f(x,t) \in H$ and (4.37) take place. Suppose that*

$$f(x,t) \neq 0 \quad (x \in \partial D,\ t \in [0,1]) \tag{4.73}$$

for an open set D, $\bar{D} \subset G$. Then

$$\gamma(f_0, D) = \gamma(f_1, D). \tag{4.74}$$

Proof. We take a number $\varepsilon > 0$ so small that

$$f(x,t) \neq a \quad (x \in \partial D,\ t \in [0,1]) \tag{4.75}$$

for all a such that $\|a\| < \varepsilon$. Let a be a regular value for both $f_0(x)$ and $f_1(x)$. Consider the function $\tilde{f}(x,t) = f(x,t) - a$. This function satisfies the conditions of Lemma 4.7 if we set $a = 0$ in this lemma. So

$$\gamma(\tilde{f}(\cdot,0), D; 0) = \gamma(\tilde{f}(\cdot,1), D; 0) \tag{4.76}$$

and therefore

$$\gamma(f_0, D; a) = \gamma(f_1, D; a). \tag{4.77}$$

This implies (4.74). The theorem is proved. □

Additivity of the topological degree follows from (4.40). We suppose that the class F is not empty. Let $f \in F$, $x \in G$, $f'(x) = (A_1, A_2)$, where $A_1 : E_0 \to E_1$, $A_2 : E_0 \to E_2$. Suppose that $\lambda > 0$ is so large that operator $J = (A_1 + \lambda I, A_2) : E_0 \to E$ is invertible. Then the operator J can be taken as a *normalization* operator.

Thus the topological degree for the class F of operators and class H of homotopies is constructed.

4.3. Application to elliptic problems. In this section, we briefly discuss application of the topological degree constructed above to elliptic problems. We recall that we consider the class F of nonlinear operators, class Φ of linearized operators, and class H of homotopies (Section 4.2).

Properness of nonlinear elliptic problems follows from Condition NS for the linearized problems (Theorem 3.4). This condition means that all limiting problems have only zero solution. It is a necessary and sufficient condition.

Condition (i) of the definition of the class Φ is satisfied in particular if the essential spectrum of the linearized operator is in the right-half plane. Condition (ii) would follow from sectoriality (see [17]).

Sectoriality of elliptic operators is well known (see [13]). However, to our knowledge it is not yet done for general elliptic problems in the sense of [2].

References

[1] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I*, Comm. Pure Appl. Math. **12** (1959), 623–727.

[2] ———, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II*, Comm. Pure Appl. Math. **17** (1964), 35–92.

[3] C. Barillon and V. A. Volpert, *Topological degree for a class of elliptic operators in* $\mathbf{R}^n$, Topol. Methods Nonlinear Anal. **14** (1999), no. 2, 275–293.

[4] P. Benevieri and M. Furi, *A simple notion of orientability for Fredholm maps of index zero between Banach manifolds and degree theory*, Ann. Sci. Math. Québec **22** (1998), no. 2, 131–148.

[5] J. F. Collet and V. A. Volpert, *Computation of the index of linear elliptic operators in unbounded cylinders*, J. Funct. Anal. **164** (1999), no. 1, 34–59.

[6] E. N. Dancer, *Boundary-value problems for ordinary differential equations on infinite intervals*, Proc. London Math. Soc. (3) **30** (1975), no. 3, 76–94.

[7] A. Douglis and L. Nirenberg, *Interior estimates for elliptic systems of partial differential equations*, Comm. Pure Appl. Math. **8** (1955), 503–538.

[8] K. D. Elworthy and A. J. Tromba, *Degree theory on Banach manifolds*, Nonlinear Functional Analysis (Proc. Sympos. Pure Math., Vol. XVIII, Part 1, Chicago, Ill, 1968), American Mathematical Society, Rhode Island, 1970, pp. 86–94.

[9] ———, *Differential structures and Fredholm maps on Banach manifolds*, Global Analysis (Proc. Sympos. Pure Math., Vol. XV, Berkeley, Calif, 1968), American Mathematical Society, Rhode Island, 1970, pp. 45–94.

[10] C. C. Fenske, *Extensio gradus ad quasdam applicationes Fredholmii*, Mitt. Math. Sem. Giessen (1976), no. Heft 121, 65–70 (Polish).

[11] P. Fitzpatrick and J. Pejsachowicz, *Orientation and the Leray-Schauder theory for fully nonlinear elliptic boundary value problems*, Mem. Amer. Math. Soc. **101** (1993), no. 483, vi+131.

[12] P. Fitzpatrick, J. Pejsachowicz, and P. J. Rabier, *Orientability of Fredholm families and topological degree for orientable nonlinear Fredholm mappings*, J. Funct. Anal. **124** (1994), no. 1, 1–39.

[13] G. Geymonat and P. Grisvard, *Alcuni risultati di teoria spettrale per i problemi ai limiti lineari ellittici*, Rend. Sem. Mat. Univ. Padova **38** (1967), 121–173 (Italian).

[14] I. C. Gohberg and M. G. Kreĭn, *The basic propositions on defect numbers, root numbers and indices of linear operators*, Amer. Math. Soc. Transl. (2) **13** (1960), 185–264.

[15] L. Hörmander, *The Analysis of Linear Partial Differential Operators. II. Differential Operators with Constant Coefficients*, Grundlehren der Mathematischen Wissenschaften, vol. 257, Springer-Verlag, Berlin, 1983.

[16] A. G. Kartsatos and I. V. Skrypnik, *The index of a critical point for densely defined operators of type $(S_+)_L$ in Banach spaces*, Trans. Amer. Math. Soc. **354** (2002), no. 4, 1601–1630.

[17] T. Kato, *Perturbation Theory for Linear Operators*, Die Grundlehren der Mathematischen Wissenschaften, vol. 132, Springer-Verlag, New York, 1966.

[18] M. A. Krasnosel'skiĭ and P. P. Zabreĭko, *Geometric Methods of Nonlinear Analysis*, Izdat. "Nauka", Moscow, 1975 (Russian).

[19] J.-L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications. Vol. I*, Die Grundlehren der mathematischen Wissenschaften, vol. 181, Springer-Verlag, New York, 1972.

[20] J. Mawhin, *Topological Degree Methods in Nonlinear Boundary Value Problems*, CBMS Regional Conference Series in Mathematics, no. 40, American Mathematical Society, Rhode Island, 1979.

[21] È. M. Mukhamadiev, *Normal solvability and Noethericity of elliptic operators in spaces of functions on $\mathbf{R}^n$. I*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **110** (1981), 120–140 (Russian), translated in J. Soviet Math. **25** (1984), no. 1, 884–901.

[22] S. A. Nazarov and K. Pileckas, *On the Fredholm property of the Stokes operator in a layer-like domain*, Z. Anal. Anwendungen **20** (2001), no. 1, 155–182.

[23] J. Pejsachowicz and P. J. Rabier, *Degree theory for C^1 Fredholm mappings of index* 0, J. Anal. Math. **76** (1998), 289–319.

[24] F. Quinn and A. Sard, *Hausdorff conullity of critical images of Fredholm maps*, Amer. J. Math. **94** (1972), 1101–1110.

[25] P. J. Rabier and C. A. Stuart, *C^1-Fredholm maps and bifurcation for quasilinear elliptic equations on $\mathbf{R}^N$*, Recent Trends in Nonlinear Analysis, Progress in Nonlinear Differential Equations and Their Applications, vol. 40, Birkhäuser, Basel, 2000, pp. 249–264.

[26] V. S. Rabinovich, *The Fredholm property of general boundary value problems on noncompact manifolds, and limit operators*, Dokl. Akad. Nauk **325** (1992), no. 2, 237–241 (Russian), translated in Russian Acad. Sci. Dokl. Math. **46** (1993), no. 1, 53–58.

[27] I. V. Skrypnik, *Nonlinear Elliptic Boundary Value Problems*, Teubner-Texte zur Mathematik, vol. 91, BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1986.

[28] S. Smale, *An infinite dimensional version of Sard's theorem*, Amer. J. Math. **87** (1965), 861–866.

[29] L. R. Volevič, *Solubility of boundary value problems for general elliptic systems*, Mat. Sb. (N.S.) **68(110)** (1965), 373–416 (Russian), translated in Amer. Math. Soc. Transl. (2) **67** (1968), 182–225.

[30] A. I. Volpert and V. A. Volpert, *Application of the theory of the rotation of vector fields to the investigation of wave solutions of parabolic equations*, Trans. Moscow Math. Soc. **1990** (1990), 59–108.

[31] ———, *Construction of the Leray-Schauder degree for elliptic operators in unbounded domains*, Ann. Inst. H. Poincaré Anal. Non Linéaire **11** (1994), no. 3, 245–273.

[32] A. I. Volpert, V. A. Volpert, and V. A. Volpert, *Traveling Wave Solutions of Parabolic Systems*, Translations of Mathematical Monographs, vol. 140, American Mathematical Society, Rhode Island, 1994.

[33] V. A. Volpert, A. I. Volpert, and J. F. Collet, *Topological degree for elliptic operators in unbounded cylinders*, Adv. Differential Equations **4** (1999), no. 6, 777–812.

[34] V. G. Zvyagin and V. T. Dmitrienko, *Properness of nonlinear elliptic differential operators in Hölder spaces*, Global Analysis—Studies and Applications, V, Lecture Notes in Mathematics, vol. 1520, Springer, Berlin, 1992, pp. 261–284.

V. Volpert: Mathématiques Appliquées, UMR 5585 CNRS, Université Lyon 1, 69622 Villeurbanne, France

E-mail address: volpert@maply.univ-lyon1.fr

A. Volpert: Department of Mathematics, Technion - Israel Institute of Technology, Haifa 32000, Israel

E-mail address: volp@techunix.technion.ac.il


Advances in
Operations Research
Advances in
Decision Sciences
Journal of
Applied Mathematics
Algebra
Journal of
Probability and Statistics
The Scientific
World Journal
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of
Differential Equations
International Journal of
Combinatorics
Advances in
Mathematical Physics
Journal of
Complex Analysis
Journal of
Mathematics
Mathematical Problems
in Engineering
Abstract and
Applied Analysis
Discrete Dynamics in
Nature and Society
International
Journal of
Mathematics and
Mathematical
Sciences
Journal of
Discrete Mathematics
Journal of
Function Spaces
International Journal of
Stochastic Analysis
Journal of
Optimization