A TURNPIKE THEOREM FOR CONTINUOUS-TIME CONTROL SYSTEMS WHEN THE OPTIMAL STATIONARY POINT IS NOT UNIQUE

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We study the turnpike property for the nonconvex optimal control problems described by the differential inclusion $\dot{x} \in a(x)$. We study the infinite horizon problem of maximizing the functional $\int_0^T u(x(t))dt$ as T grows to infinity. The turnpike theorem is proved for the case when a turnpike set consists of several optimal stationary points.

1. Introduction

Let $x \in \mathbb{R}^n$ and let $\Omega \subset \mathbb{R}^n$ be a given compact set. Denote by $\Pi_c(\mathbb{R}^n)$ the set of all compact subsets of $\mathbb{R}^n$. We consider the following problem:

$$\dot{x} \in a(x), \qquad x(0) = x^0, \tag{1.1}$$

$$J_T(x(\cdot)) = \int_0^T u(x(t))dt \longrightarrow \max. \tag{1.2}$$

Here, $x^0 \in \Omega$ is an assigned initial point. The multivalued mapping $a : \Omega \to \Pi_c(\mathbb{R}^n)$ has compact images and is continuous in the Hausdorff metric. We also assume that at every point $x \in \Omega$ the set $a(x)$ is uniformly locally connected (see [2]). The function $u : \Omega \to \mathbb{R}^1$ is a given continuous function.

In this paper, we study the turnpike property for problem (1.1) and (1.2). The term of turnpike property was first coined by Samuelson (see [17]) where it is shown that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path. This property was further investigated by Radner [14], McKenzie [12], Makarov and Rubinov [7], and others for optimal trajectories of a von Neuman-Gale model with discrete time. In all these studies, the turnpike property was established under some convexity assumptions.

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In [11, 13], the turnpike property was defined using the notion of statistical convergence (see [3]) and it was proved that all optimal trajectories have the same unique statistical cluster point (which is also a statistical limit point). In these works, the turnpike property is proved when the graph of the mapping a is not a convex set.

The turnpike property for continuous-time control systems was studied by Rockafellar [15, 16], Cass and Shell [1], Scheinkman [6, 18], and others where, besides convexity assumptions, some additional conditions are imposed on the Hamiltonian. To prove turnpike theorem without these kind of additional conditions became a very important problem. This problem was further investigated by Zaslavski [19, 21], Mamedov [8, 9, 10], and others.

In [10], problem (1.1) and (1.2) is considered without convexity assumptions and the turnpike property is established assuming that the optimal stationary point is unique. In this paper, we consider the case when a turnpike set consists of several optimal stationary points.

Definition 1.1. An absolutely continuous function $x(\cdot)$ is called a trajectory (solution) to system (1.1) on the interval $[0,T]$ if $x(0)=x^0$ and almost everywhere on the interval $[0,T]$ the inclusion $\dot{x}(t)\in a(x(t))$ is satisfied.

We denote the set of trajectories defined on the interval [0,T] by X_T and we let

$$J_T^* = \sup_{x(\cdot)\in X_T} J_T(x(\cdot)). \tag{1.3}$$

Since $x(t)\in\Omega$ and the set Ω is bounded, the trajectories of system (1.1) are uniformly bounded, that is, there exists a number $L<+\infty$ such that

$$\|x(t)\|\le L, \quad \forall t\in[0,T],\ x(\cdot)\in X_T,\ T>0. \tag{1.4}$$

On the other hand, since the mapping a is continuous, then there is a number $K<+\infty$ such that

$$\|\dot{x}(t)\|\le K \quad \text{for almost all } t\in[0,T],\ \forall x(\cdot)\in X_T,\ T>0. \tag{1.5}$$

Note that in this paper we focus our attention on the turnpike property of optimal trajectories. So we did not study the existence of bounded trajectories defined on $[0,\infty]$. This problem for different control problems has been studied by Leizarowitz [4, 5], Zaslavsky [19, 20], and others.

Definition 1.2. The trajectory $x(\cdot)$ is called optimal if $J(x(\cdot))=J_T^*$ and is called ξ-optimal ($\xi>0$) if

$$J(x(\cdot))\ge J_T^*-\xi. \tag{1.6}$$

Definition 1.3. The point x is called a stationary point if $0\in a(x)$.

Stationary points play an important role in the study of asymptotical behavior of optimal trajectories. We denote the set of stationary points by M:

$$M = \{x \in \Omega : 0 \in a(x)\}. \tag{1.7}$$

We assume that the set M is nonempty. Since the mapping $a(x)$ is continuous, then the set M is also closed. Therefore M is a compact set.

Definition 1.4. The point $x^* \in M$ is called an optimal stationary point if

$$u(x^*) = u^* \triangleq \max_{x \in M} u(x). \tag{1.8}$$

We denote the set of optimal stationary point by M_{op}. Since the function u is continuous, then this set is not empty. In Turnpike theory, it is usually assumed that the optimal stationary point x^* is unique. In this paper, we consider non-convex problem (1.1) and (1.2) (i.e., the function u is not strictly concave and the graph of the mapping a is not convex) and therefore the optimal stationary point may be not unique.

We assume that the set M_{op} consists of m different points $x_1^*, x_2^*, \dots, x_m^*$; that is,

$$x_i^* \in M, \quad u(x_i^*) = u^*, \quad \forall i; \qquad u(x) < u^* \quad \text{if } x \in M \setminus \{x_1^*, \dots, x_m^*\}. \tag{1.9}$$

Consider an example for which this assumption holds.

Example 1.5. Assume that the set M is convex and

$$u(x) = \max\{u_i(x) : i \in \{1, 2, \dots, l\}\}, \quad x \in \Omega, \tag{1.10}$$

where the functions u_i are continuous and strictly concave. For every i, there exists a unique point $x_i' \in M$ for which

$$u_i(x_i') = u_i^* \triangleq \max_{x \in M} u_i(x). \tag{1.11}$$

Clearly, the function u is continuous and $u^* = \max\{u_i^* : i \in \{1, 2, \dots, l\}\}$. We also note that the function u may be not concave. In this example the number m and the points $x_1^*, x_2^*, \dots, x_m^*$ in (1.9) can be chosen out of the points x_i' ($i \in \{1, 2, \dots, l\}$) for which $u(x_i') = u^*$.

2. Main conditions and Turnpike theorem

The turnpike theorem will be proved under two main conditions, Conditions 2.1 and 2.2. The first condition is about the existence of "good" trajectories starting from the initial state x^0. The second is the main condition which provides the turnpike property.

Condition 2.1. There exists $b < +\infty$ such that, for every $T > 0$, there is a trajectory $x(\cdot) \in X_T$ satisfying the inequality

$$J_T(x(\cdot)) \geq u^* T - b. \tag{2.1}$$

Note that the satisfaction of this condition depends in an essential way on the initial point x^0, and in a certain sense it can be considered as a condition for the existence of trajectories converging to some points x_i^*, $i = 1, 2, \dots, m$. Thus, for example, if there exists a trajectory that hits some optimal stationary point x_i^* in finite time, then Condition 2.1 is satisfied.

Set

$$\mathcal{B} = \{x \in \Omega : u(x) \geq u^*\}. \tag{2.2}$$

We fix $p \in \mathbb{R}^n$, $p \neq 0$, and define a support function

$$c(x) = \max_{y \in a(x)} py. \tag{2.3}$$

Here, the notation py means the scalar product of the vectors p and y. By $|c|$ we denote the absolute value of c.

We also define the function

$$\varphi(x, y) = \frac{u(x) - u^*}{|c(x)|} + \frac{u(y) - u^*}{c(y)}. \tag{2.4}$$

Condition 2.2. There exists a vector $p \in \mathbb{R}^n$ such that

(H1) $c(x) < 0$ for all $x \in \mathcal{B}$ and $x \neq x_i^*$, $i = 1, 2, \dots, m$;

(H2) there exist points $\tilde{x}_i \in \Omega$ such that

$$p\tilde{x}_i = px_i^*, \quad c(\tilde{x}_i) > 0, \quad \forall i = 1, 2, \dots, m; \tag{2.5}$$

(H3) for all points x, y, for which

$$px = py, \qquad c(x) < 0, \qquad c(y) > 0, \tag{2.6}$$

the inequality $\varphi(x, y) < 0$ is satisfied; and also if

$$\begin{gathered} x_k \longrightarrow x_i^* \quad \text{for some } i = 1, 2, \dots, m, \\ y_k \longrightarrow y', \quad y' \neq x_i^*,\ i = 1, 2, \dots, m, \\ px_k = py_k, \qquad c(x_k) < 0, \qquad c(y_k) > 0, \end{gathered} \tag{2.7}$$

then $\limsup_{k \to \infty} \varphi(x_k, y_k) < 0$.

Note that if Condition 2.2 is satisfied for any vector p, then it is also satisfied for all λp, $(\lambda > 0)$. That is why we assume that $\|p\| = 1$.

Condition (H1) means that derivatives of system (1.1) are directed to one side with respect to p; that is, if $x \in \mathcal{B}$ and $x \neq x_i^*$, $i = 1, 2, \ldots, m$, then $py < 0$ for all $y \in a(x)$. It is also clear that $py \leq 0$ for all $y \in a(x_i^*)$ and $c(x_i^*) = 0$, $i = 1, 2, \ldots, m$.

The main condition here is (H3). It can be considered as a relation between the mapping a and the function u which provides the turnpike property. In [8] it is shown that conditions (H1) and (H3) hold if the graph of the mapping a is a convex set (in $\mathbb{R}^n \times \mathbb{R}^n$) and the function u is strictly concave. On the other hand, an example given in [10] shows that Condition 2.2 may hold for mappings a having nonconvex graphs and for functions u that are not strictly concave (in this example the function u is convex).

The main sense of the turnpike property is that optimal trajectories can stay just during a restricted time interval on the outside of the ε-neighborhood of the turnpike set M_{op}. When the set M_{op} consists of several different points, it is interesting to study a state transition of the trajectories from one optimal stationary point to another. We introduce the following definition. Take any number $\delta > 0$ and let $S_\delta(x)$ stands for the closed δ-neighborhood of the point x.

Definition 2.3. Say that on the interval $[t_1, t_2]$ a trajectory $x(t)$ makes a state transition from x_i^* to x_j^* $(i \neq j)$ if $x(t_1) \in S_\delta(x_i^*)$, $x(t_2) \in S_\delta(x_j^*)$, and

$$x(t) \notin S_\delta(x_k^*), \quad \forall t \in (t_1, t_2),\ k = 1, \ldots, m. \tag{2.8}$$

For a given number $\delta > 0$ and a given ξ-optimal trajectory $x(\cdot) \in X_T$, we denote by $N_T(\delta, \xi, x(\cdot))$ the number of disjoint intervals $[t_1, t_2]$ on which the trajectory $x(\cdot)$ makes a state transition from x_i^* to x_j^* $(i \neq j,\ i, j = 1, 2, \ldots, m)$. We call $N_T(\delta, \xi, x(\cdot))$ a number of state transitions.

Clearly in Definition 2.3 a small number δ should be used. We take

$$\delta \leq \frac{1}{4} \min\left\{ \|x_i^* - x_j^*\| : i \neq j,\ i, j = 1, 2, \ldots, m \right\}. \tag{2.9}$$

Now we formulate the main result of the present paper.

Theorem 2.4. *Suppose that Conditions 2.1 and 2.2 are satisfied and there are m different optimal stationary points x_i^*. Then*

(1) *there exists $C < +\infty$ such that*

$$\int_0^T [u(x(t)) - u^*]\,dt \leq C \tag{2.10}$$

for every $T > 0$ and every trajectory $x(\cdot) \in X_T$;

(2) *for every $\varepsilon > 0$, there exists $K_{\varepsilon,\xi} < +\infty$ such that*

$$\operatorname{meas}\left\{ t \in [0, T] : \|x(t) - x_1^*\| \geq \varepsilon, \ldots, \|x(t) - x_m^*\| \geq \varepsilon \right\} \leq K_{\varepsilon,\xi} \tag{2.11}$$

for every $T > 0$ and every ξ-optimal trajectory $x(\cdot) \in X_T$;

(3) *for every* $\xi > 0$ *and* $\delta > 0$ *(satisfying (2.9)), there exists a number* $N_{\delta,\xi} < +\infty$ *such that*

$$N_T(\delta,\xi,x(\cdot)) \le N_{\delta,\xi} \tag{2.12}$$

for every $T > 0$ *and every* ξ*-optimal trajectory* $x(\cdot) \in X_T$;

(4) *if* $x(\cdot)$ *is an optimal trajectory and* $x(t_1) = x(t_2) = x_i^*$ *for some* $i = 1,2,\dots,m$*, then* $x(t) = x_i^*$ *for all* $t \in [t_1,t_2]$.

The proof of this theorem is given in Section 4. In Section 3, we present preliminary results.

3. Preliminary results

3.1. Let $x \in \mathcal{B}$ and $x \ne x_i^*$, $i = 1,2,\dots,m$, that is $x \in \mathcal{B} \setminus M_{\text{op}}$. By the condition (H2) we have $c(x) < 0$. Since the function $c(x)$ is continuous, there is a number $\varepsilon_x > 0$ such that $c(x') < 0$ for all $x' \in V_{\varepsilon_x}(x) \cap \Omega$. We define the set $\mathcal{D}$ as follows:

$$\mathcal{D} = \text{cl}\big[\cup_{x\in\mathcal{B}\setminus M_{\text{op}}} V_{\varepsilon_x}(x)\big] \cap \Omega. \tag{3.1}$$

It is not difficult to show that the following conditions hold:

(a) $x \in \text{int}\mathcal{D}$ for all $x \in \mathcal{B} \setminus M_{\text{op}}$;
(b) $c(x) < 0$ for all $x \in \mathcal{D} \setminus M_{\text{op}}$;
(c) $\mathcal{D} \cap \mathcal{M}^* = M_{\text{op}}$ and $\mathcal{B} \subset \mathcal{D}$.

Here,

$$\mathcal{M}^* = \{x \in \Omega : c(x) \ge 0\} \tag{3.2}$$

and we recall that $\mathcal{B} = \{x \in \Omega : u(x) \ge u^*\}$. Clearly $\mathcal{M} \subset \mathcal{M}^*$.

Lemma 3.1. *For every* $\varepsilon > 0$*, there exists* $\nu_\varepsilon > 0$ *such that*

$$u(x) \le u^* - \nu_\varepsilon \tag{3.3}$$

for every $x \in \Omega$, $x \notin \text{int}\mathcal{D}$*, and* $\|x - x_1^*\| \ge \varepsilon,\dots,\|x - x_m^*\| \ge \varepsilon$.

Proof. Assume on the contrary that for any $\varepsilon > 0$, there exists a sequence x_k such that $x_k \notin \text{int}\mathcal{D}$, $\|x_k - x_i^*\| \ge \varepsilon$ $(i = 1,\dots,m)$, and $u(x_k) \to u^*$ as $k \to \infty$. Since the sequence x_k is bounded, it has a limit point, say x'. Clearly $x' \ne x_i^*$ $(i = 1,\dots,m)$, $x' \notin \text{int}\mathcal{D}$, and also $u(x') = u^*$, which implies $x' \in \mathcal{B}$. This contradicts property (a) of the set $\mathcal{D}$. □

Lemma 3.2. *For every* $\varepsilon > 0$*, there exists* $\eta_\varepsilon > 0$ *such that*

$$c(x) < -\eta_\varepsilon, \quad \forall x \in \mathcal{D}, \qquad \|x - x_1^*\| \ge \varepsilon,\dots,\|x - x_m^*\| \ge \varepsilon. \tag{3.4}$$

Proof. Assume on the contrary that for any $\varepsilon > 0$, there exists a sequence x_k such that $x_k \in \mathcal{D}$, $\|x_k - x_i^*\| \geq \varepsilon$ $(i = 1,\dots,m)$, and $c(x_k) \to 0$. Let x' be a limit point of the sequence x_k. Then $x' \in \mathcal{D}$, $x' \neq x_i^*$ $(i = 1,\dots,m)$, and $c(x') = 0$. This contradicts property (b) of the set $\mathcal{D}$. □

3.2. Given the interval $[p_2, p_1] \subset (-\infty, +\infty)$, we define two classes of subsets of the time interval $[0, T]$. We denote these classes by $\mathbf{T}^1[p_2, p_1]$ and $\mathbf{T}^2[p_2, p_1]$.

Definition 3.3. The set $\pi \subset [0, T]$ belongs to the class $\mathbf{T}^1[p_2, p_1]$ if the following conditions hold:

(a) the set π can be presented as a union of two sets, $\pi = \pi_1 \cup \pi_2$, such that

$$x(t) \in \operatorname{int}\mathcal{D}, \quad \forall t \in \pi_1, \qquad x(t) \notin \operatorname{int}\mathcal{D}, \quad \forall t \in \pi_2; \tag{3.5}$$

(b) the set π_1 consists of at most countable number of intervals Δ_k, with endpoints $t_1^k < t_2^k$, such that
  (i) the intervals $(px(t_2^k), px(t_1^k))$, $k = 1, 2, \dots$, are disjoint (clearly in this case, the intervals $\Delta_k^0 = (t_1^k, t_2^k)$ are also disjoint);
  (ii) $[px(t_2^k), px(t_1^k)] \subset [p_2, p_1]$ for all $k = 1, 2, \dots$.

Definition 3.4. The set $\omega \subset [0, T]$ belongs to the class $\mathbf{T}^2[p_2, p_1]$ if the following conditions hold:

(a) $x(t) \notin \operatorname{int}\mathcal{D}$, for all $t \in \omega$;
(b) the set ω contains at most countable number of intervals $[s_2^k, s_1^k]$ such that the intervals $(px(s_2^k), px(s_1^k))$, $k = 1, 2, \dots$, are nonempty and disjoint, and

$$p_1 - p_2 = \sum_k [px(s_1^k) - px(s_2^k)]. \tag{3.6}$$

Note that the inclusion $x(t) \in \operatorname{int}\mathcal{D}$ means that $u(x(t)) > u^*$ whereas the condition $x(t) \notin \operatorname{int}\mathcal{D}$ implies $u(x(t)) \leq u^*$.

LEMMA 3.5. *Assume that $x(\cdot) \in X_T$ is a continuously differentiable function, π $(= \pi_1 \cup \pi_2) \in \mathbf{T}^1[p_2, p_1]$, and $\omega \in \mathbf{T}^2[p_2, p_1]$. Then,*

$$\int_{\pi \cup \omega} u(x(t))\,dt \leq u^* \cdot \operatorname{meas}(\pi \cup \omega) - \int_Q [u^* - u(x(t))]\,dt - \int_E \delta^2(x(t))\,dt, \tag{3.7}$$

where

(a) $Q \cup E = \omega \cup \pi_2 = \{t \in \pi \cup \omega : x(t) \notin \operatorname{int}\mathcal{D}\}$;
(b) *for every $\varepsilon > 0$, there exists a number $\delta_\varepsilon > 0$ such that*

$$\delta^2(x) \geq \delta_\varepsilon, \quad \forall x, \text{ for which } \|x - x_i^*\| \geq \varepsilon \ (i = 1,\dots,m); \tag{3.8}$$

(c) *for every* $\delta > 0$, *there exists a number* $K(\delta) < \infty$ *such that*

$$\operatorname{meas}[(\pi \cup \omega) \cap Z_\delta] \leq K(\delta) \cdot \operatorname{meas}[(Q \cup E) \cap Z_\delta], \tag{3.9}$$

here $Z_\delta = \{t \in [0,T] : |px(t) - p_i^*| \geq \delta,\ i = 1,\dots,m\}$ *and* $p_i^* = px_i^*$, $i = 1,\dots,m$.

The proof of this lemma is similar to the proof of [10, Lemma 5.4], so we do not give it. We also present the next two lemmas without proofs. Their proofs can be done in a similar way to the proofs of [10, Lemmas 6.6 and 6.7].

Lemma 3.6. *Assume that* $x(\cdot) \in X_T$ *is a continuously differentiable function. Then, the interval* $[0,T]$ *can be divided into subintervals such that*

$$[0,T] = \cup_n (\pi_n \cup \omega_n) \cup (F_1 \cup F_2 \cup F_3) \cup E, \tag{3.10}$$

$$\int_0^T u(x(t))\,dt = \sum_n \int_{\pi_n \cup \omega_n} u(x(t))\,dt + \int_{F_1 \cup F_2 \cup F_3} u(x(t))\,dt + \int_E u(x(t))\,dt. \tag{3.11}$$

Here, we have

(1) $\pi_n \in \mathbf{T}^1[p_n^2, p_n^1]$ *and* $\omega_n \in \mathbf{T}^2[p_n^2, p_n^1]$, $n = 1,2,\dots$;
(2) *for each* $i \in \{1,2,3\}$, *the set* $F_i \in \mathbf{T}^1[p_i'', p_i']$ *for some interval* $[p_i'', p_i']$ *and*

$$x(t) \in \operatorname{int}\mathfrak{D}, \quad \forall t \in F_1 \cup F_2 \cup F_3, \tag{3.12}$$

$$p_i' - p_i'' \leq C < +\infty, \quad i = 1,2,3; \tag{3.13}$$

(3) *the set* E *such that*

$$x(t) \notin \operatorname{int}\mathfrak{D}, \quad \forall t \in E; \tag{3.14}$$

(4) *for every* $\delta > 0$, *there is a number* $C(\delta)$ *such that*

$$\operatorname{meas}[(F_1 \cup F_2 \cup F_3) \cap Z_\delta] \leq C(\delta), \tag{3.15}$$

where

$$Z_\delta = \{t \in [0,T] : |px(t) - p_i^*| \geq \delta,\ i = 1,\dots,m\} \tag{3.16}$$

and the number $C(\delta) < +\infty$ *does not depend on the trajectory* $x(\cdot)$, *on* T, *and on the intervals of (3.10).*

Lemma 3.7. *Assume that* $x(\cdot) \in X_T$ *is a continuously differentiable function and the sets* F_i $(i = 1,2,3)$ *are defined in Lemma 3.6. Then, there is a number* $L < +\infty$ *such that*

$$\int_{F_1 \cup F_2 \cup F_3} [u(x(t)) - u^*]\,dt < L, \tag{3.17}$$

where the number L *does not depend on the trajectory* $x(\cdot)$, *on* T, *and on the intervals in (3.10).*

4. Proof of Theorem 2.4

From Condition 2.1, it follows that, for every $T>0$, there is a trajectory $x_T(\cdot)\in X_T$, for which

$$\int_{[0,T]} u(x_T(t))\,dt \geq u^* T - b. \tag{4.1}$$

(1) First we consider the case when $x(t)$ is a continuously differentiable function. In this case we can use the results obtained in Section 3.

From Lemmas 3.6 and 3.7, we have

$$\begin{aligned}\int_{[0,T]} u(x(t))\,dt \leq \sum_n \int_{\pi_n\cup\omega_n} u(x(t))\,dt + \int_E u(x(t))\,dt \\ + L + u^* \cdot \operatorname{meas}(F_1\cup F_2\cup F_3).\end{aligned} \tag{4.2}$$

Then from Lemma 3.5, we obtain (see, also, (3.10))

$$\begin{aligned}\int_{[0,T]} u(x(t))\,dt &\leq \sum_n \bigg(u^* \operatorname{meas}(\pi_n\cup\omega_n) \\ &\qquad - \int_{Q_n} [u^* - u(x(t))]\,dt - \int_{E_n} \delta^2(x(t))\,dt \bigg) \\ &\qquad + \int_E u(x(t))\,dt + L + u^* \cdot \operatorname{meas}(F_1\cup F_2\cup F_3) \\ &= u^*\Big(\sum_n \operatorname{meas}(\pi_n\cup\omega_n) + \operatorname{meas}(F_1\cup F_2\cup F_3) + \operatorname{meas} E\Big) \\ &\qquad - \int_Q [u^* - u(x(t))]\,dt - \int_A \delta^2(x(t))\,dt + L \\ &= u^* \operatorname{meas}[0,T] - \int_Q [u^* - u(x(t))]\,dt - \int_A \delta^2(x(t))\,dt + L.\end{aligned} \tag{4.3}$$

Therefore,

$$\int_{[0,T]} [u(x(t)) - u^*]\,dt \leq -\int_Q [u^* - u(x(t))]\,dt - \int_A \delta^2(x(t))\,dt + L. \tag{4.4}$$

Here, $Q=(\cup_n Q_n)\cup E$ and $A=\cup_n E_n$. Taking into account (4.1), we have

$$\begin{aligned}\int_{[0,T]} u(x(t))\,dt - \int_{[0,T]} u(x_T(t))\,dt \leq -\int_Q [u^* - u(x(t))]\,dt \\ - \int_A \delta^2(x(t))\,dt + L + b,\end{aligned} \tag{4.5}$$

that is,

$$J_T(x(\cdot)) - J_T(x_T(\cdot)) \leq -\int_Q [u^* - u(x(t))]\,dt - \int_A \delta^2(x(t))\,dt + L + b. \tag{4.6}$$

Here,

$$Q = (\cup_n Q_n) \cup E, \qquad A = \cup_n E_n, \tag{4.7}$$

and the following conditions hold:

(a) (see Lemma 3.5(a), (3.12) and (3.14))

$$Q \cup A = \{t \in [0,T] : x(t) \notin \operatorname{int}\mathcal{D}\}; \tag{4.8}$$

(b) (see (3.10))

$$[0,T] = \cup_n (\pi_n \cup \omega_n) \cup (F_1 \cup F_2 \cup F_3) \cup E; \tag{4.9}$$

(c) for every $\delta > 0$, there exist $K(\delta) < +\infty$ and $C(\delta) < +\infty$ such that (see Lemma 3.5(c) and (3.15))

$$\begin{gathered} \operatorname{meas}[(\pi_n \cup \omega_n) \cap Z_\delta] \le K(\delta) \operatorname{meas}[(Q_n \cup E_n) \cap Z_\delta], \\ \operatorname{meas}[(F_1 \cup F_2 \cup F_3) \cap Z_\delta] \le C(\delta); \end{gathered} \tag{4.10}$$

we recall that $Z_\delta = \{t \in [0,T] : |px(t) - p_i^*| \ge \delta,\ i = 1,2,\dots,m\}$;

(d) for every $\varepsilon > 0$, there exist $\delta_\varepsilon > 0$ such that (see Lemma 3.5(b))

$$\delta^2(x) \ge \delta_\varepsilon, \quad \forall x, \qquad ||x - x_i^*|| \ge \varepsilon, \quad i = 1,2,\dots,m. \tag{4.11}$$

The first assertion of the theorem follows from (4.4), (4.8), and (4.11) for the case under consideration (i.e., $x(\cdot)$ is continuously differentiable). We show the second assertion.

Let $\varepsilon > 0$ and $\delta > 0$ be given numbers and let $x(\cdot)$ be a continuously differentiable ξ-optimal trajectory. We denote

$$\mathcal{X}_\varepsilon = \{t \in [0,T] : ||x(t) - x_i^*|| \ge \varepsilon,\ i = 1,2,\dots,m\}. \tag{4.12}$$

First we show that there is a number $\tilde{K}_{\varepsilon,\xi} < +\infty$ (which does not depend on $T > 0$) such that the following inequality holds

$$\operatorname{meas}[(Q \cup A) \cap \mathcal{X}_\varepsilon] \le \tilde{K}_{\varepsilon,\xi}. \tag{4.13}$$

Assume that (4.13) is not true. In this case, there exist sequences $T_k \to \infty$ and $K_{\varepsilon,\xi}^k \to \infty$, and sequences of trajectories $\{x^k(\cdot)\}$ (every $x^k(\cdot)$ is a ξ-optimal trajectory in the interval $[0,T_k]$) and $\{x_{T_k}(\cdot)\}$ (satisfying (4.1) for every $T = T_k$) such that

$$\operatorname{meas}[(Q^k \cup A^k) \cap \mathcal{X}_\varepsilon^k] \ge K_{\varepsilon,\xi}^k \quad \text{as } k \longrightarrow \infty. \tag{4.14}$$

From Lemma 3.1 and (4.11), we have

$$\begin{aligned} u^* - u(x^k(t)) &\geq \nu_\varepsilon \quad \text{if } t \in Q^k \cup \mathscr{X}_\varepsilon^k, \\ \delta^2(x^k(t)) &\geq \delta_\varepsilon^2 \quad \text{if } t \in A^k \cap \mathscr{X}_\varepsilon^k. \end{aligned} \tag{4.15}$$

Denote $\nu = \min\{\nu_\varepsilon, \delta_\varepsilon^2\} > 0$. From (4.6), it follows that

$$J_{T_k}(x^k(\cdot)) - J_{T_k}(x_{T_k}(\cdot)) \leq L + b - \nu \operatorname{meas}[(Q^k \cup A^k) \cap \mathscr{X}_\varepsilon^k]. \tag{4.16}$$

Therefore, for sufficient large numbers k, we have

$$J_{T_k}(x^k(\cdot)) \leq J_{T_k}(x_{T_k}(\cdot)) - 2\xi \leq J_{T_k}^* - 2\xi, \tag{4.17}$$

which means that $x^k(t)$ is not a ξ-optimal trajectory. This is a contradiction. Thus (4.13) is true.

Now, we show that, for every $\delta > 0$, there is a number $K_{\delta,\xi}^1 < +\infty$ such that

$$\operatorname{meas} Z_\delta \leq K_{\delta,\xi}^1. \tag{4.18}$$

From (4.9) and (4.10), we have

$$\begin{aligned} \operatorname{meas} Z_\delta &= \sum_n \operatorname{meas}[(\pi_n \cup \omega_n) \cap Z_\delta] \\ &\quad + \operatorname{meas}[(F_1 \cup F_2 \cup F_3) \cap Z_\delta] + \operatorname{meas}(E \cap Z_\delta) \\ &\leq \sum_n K(\delta) \operatorname{meas}[(Q_n \cup E_n) \cap Z_\delta] + C(\delta) + \operatorname{meas}(E \cap Z_\delta) \\ &\leq \tilde{K}(\delta) \operatorname{meas}[([\cup_n (Q_n \cup E_n)] \cap Z_\delta) \cup (E \cap Z_\delta)] + C(\delta) \\ &= \tilde{K}(\delta) \operatorname{meas}[(Q \cup A) \cap Z_\delta] + C(\delta). \end{aligned} \tag{4.19}$$

Here $\tilde{K}(\delta) = \max\{1, K(\delta)\}$.

Since $Z_\delta \subset \mathscr{X}_\delta$, then taking into account (4.13) we obtain (4.18), where

$$K_{\delta,\xi}^1 = \tilde{K}(\delta)\tilde{K}_{\delta,\xi} + C(\delta). \tag{4.20}$$

We denote

$$\mathscr{X}_{\varepsilon/2}^0 = \left\{ t \in [0,T] : \|x(t) - x_i^*\| > \frac{\varepsilon}{2},\ i = 1, 2, \ldots, m \right\}. \tag{4.21}$$

Clearly, $\mathscr{X}_{\varepsilon/2}^0$ is an open set and therefore it can be presented as a union of at most countable number of open intervals $\tilde{\tau}_k$. Out of these intervals, we chose the intervals τ_k, $k = 1, 2, \ldots$, which have nonempty intersections with $\mathscr{X}_\varepsilon$. Then we have

$$\mathscr{X}_\varepsilon \subset \cup_k \tau_k \subset \mathscr{X}^0_{\varepsilon/2}. \tag{4.22}$$

Since a derivative of the function $x(t)$ is bounded, it is not difficult to see that there is a number $\sigma_\varepsilon > 0$ such that

$$\operatorname{meas} \tau_k \geq \sigma_\varepsilon, \quad \forall k. \tag{4.23}$$

But the interval $[0, T]$ is bounded and therefore the number of intervals τ_k is finite too. Let $k = 1, 2, 3, \ldots, N_T(\varepsilon)$. We divide every interval τ_k into two parts:

$$\tau_k^1 = \{t \in \tau_k : x(t) \in \operatorname{int}\mathscr{D}\}, \qquad \tau_k^2 = \{t \in \tau_k : x(t) \notin \operatorname{int}\mathscr{D}\}. \tag{4.24}$$

From (4.8) and (4.22), we obtain

$$\cup_k \tau_k^2 \subset (Q \cup A) \cap \mathscr{X}^0_{\varepsilon/2}, \tag{4.25}$$

and therefore from (4.13) it follows that

$$\operatorname{meas}\left(\cup_k \tau_k^2\right) \leq \tilde{K}_{\varepsilon/2,\xi}. \tag{4.26}$$

Now we apply Lemma 3.2. We have

$$p\dot{x}(t) \leq -\eta_{\varepsilon/2}, \quad t \in \cup_k \tau_k^1. \tag{4.27}$$

Denote $p_k^1 = \sup_{t\in\tau_k} px(t)$ and $p_k^2 = \inf_{t\in\tau_k} px(t)$. It is clear that

$$p_k^1 - p_k^2 \leq \tilde{C}, \quad k = 1, 2, 3, \ldots, N_T(\varepsilon), \tag{4.28}$$

$$\left| p\dot{x}(t) \right| \leq K, \quad \forall t. \tag{4.29}$$

Here, the numbers $\tilde{C}$ and K do not depend on $T > 0$, $x(\cdot)$, ε, and ξ. We divide the interval τ_k into three parts:

$$\begin{gathered} \tau_k^- = \{t \in \tau_k : p\dot{x}(t) < 0\}, \qquad \tau_k^0 = \{t \in \tau_k : p\dot{x}(t) = 0\}, \\ \tau_k^+ = \{t \in \tau_k : p\dot{x}(t) > 0\}. \end{gathered} \tag{4.30}$$

Then we have

$$p_k^1 - p_k^2 \geq \left| \int_{\tau_k} p\dot{x}(t)dt \right| = \left| \int_{\tau_k^-} p\dot{x}(t)dt + \int_{\tau_k^+} p\dot{x}(t)dt \right|. \tag{4.31}$$

We denote $\alpha = -\int_{\tau_k^-} p\dot{x}(t)dt$ and $\beta = \int_{\tau_k^+} p\dot{x}(t)dt$. Clearly $\alpha > 0$, $\beta > 0$, and

$$p_k^1 - p_k^2 \geq \begin{cases} -\alpha + \beta & \text{if } \alpha < \beta, \\ \alpha - \beta & \text{if } \alpha \geq \beta. \end{cases} \tag{4.32}$$

From (4.29), we obtain

$$0 < \beta \leq K \operatorname{meas} \tau_k^+. \tag{4.33}$$

On the other hand, $\tau_k^1 \subset \tau_k^-$ and therefore from (4.27) we have

$$\alpha \geq \eta_{\varepsilon/2} \operatorname{meas} \tau_k^- \geq \eta_{\varepsilon/2} \operatorname{meas} \tau_k^1. \tag{4.34}$$

Consider the following two cases.

(1) If $\alpha \geq \beta$, then from (4.32), (4.33), and (4.34) we obtain

$$\tilde{C} \geq p_k^1 - p_k^2 \geq \alpha - \beta \geq \eta_{\varepsilon/2} \operatorname{meas} \tau_k^1 - K \operatorname{meas} \tau_k^+. \tag{4.35}$$

Since $\tau_k^+ \subset \tau_k^2$, then from (4.26) it follows that $\operatorname{meas} \tau_k^+ \leq \tilde{K}_{\varepsilon/2,\xi}$. Therefore, from (4.35), we have

$$\operatorname{meas} \tau_k^1 \leq C'_{\varepsilon,\xi}, \tag{4.36}$$

where $C'_{\varepsilon,\xi} = (C + K \cdot \tilde{K}_{\varepsilon/2,\xi})/\eta_{\varepsilon/2}$.

(2) If $\alpha < \beta$, then from (4.33) and (4.34) we obtain

$$\eta_{\varepsilon/2} \operatorname{meas} \tau_k^1 < K \operatorname{meas} \tau_k^+ \leq K \cdot \tilde{K}_{\varepsilon/2,\xi}, \tag{4.37}$$

or

$$\operatorname{meas} \tau_k^1 < C''_{\varepsilon,\xi}, \tag{4.38}$$

where $C''_{\varepsilon,\xi} = K \cdot \tilde{K}_{\varepsilon/2,\xi}/\eta_{\varepsilon/2}$.

Thus from (4.36) and (4.38) we obtain

$$\operatorname{meas} \tau_k^1 \leq C_{\varepsilon,\xi} = \max \{C'_{\varepsilon,\xi}, C''_{\varepsilon,\xi}\}, \quad k = 1, 2, \ldots, N_T(\varepsilon), \tag{4.39}$$

and then

$$\operatorname{meas} \left(\cup_k \tau_k^1 \right) \leq N_T(\varepsilon) C_{\varepsilon,\xi}. \tag{4.40}$$

Now we show that, for every $\varepsilon > 0$ and $\xi > 0$, there is a number $K'_{\varepsilon,\xi} < +\infty$ such that

$$\operatorname{meas} \left(\cup_k \tau_k^1 \right) \leq K'_{\varepsilon,\xi}. \tag{4.41}$$

Assume that (4.41) is not true. Then from (4.40), it follows that $N_T(\varepsilon) \to \infty$ as $T \to \infty$. Consider the intervals τ_k for which the following conditions hold:

$$\operatorname{meas} \tau_k^1 \geq \frac{1}{2} \sigma_\varepsilon, \qquad \operatorname{meas} \tau_k^2 \leq \lambda \operatorname{meas} \tau_k^1, \tag{4.42}$$

where λ is any fixed number. Since $N_T(\varepsilon) \to \infty$, then from (4.23) and (4.26) it follows that the number of intervals τ_k satisfying (4.42) infinitely increases as $T \to \infty$.

On the other hand, the number of intervals τ_k, for which the conditions $\alpha < \beta$ and

$$\operatorname{meas} \tau_k^2 > \lambda \operatorname{meas} \tau_k^1, \quad \lambda = \frac{\eta_{\varepsilon/2}}{K}, \tag{4.43}$$

hold, is finite. Therefore, the number of intervals τ_k, for which the conditions $\alpha \leq \beta$ and (4.42) hold, infinitely increases as $T \to \infty$. We denote the number of such intervals by N_T and for the sake of definiteness assume that these are intervals τ_k, $k = 1, 2, \ldots, N_T$.

We set $\lambda = \eta_{\varepsilon/2}/2K$ for every τ_k. Then from (4.35) and (4.42), we have

$$p_k^1 - p_k^2 \geq \eta_{\varepsilon/2} \operatorname{meas} \tau_k^1 - K \cdot \frac{\eta_{\varepsilon/2}}{2K} \operatorname{meas} \tau_k^1 = \frac{1}{2} \eta_{\varepsilon/2} \operatorname{meas} \tau_k^1. \tag{4.44}$$

Taking into account (4.23), we obtain

$$p_k^1 - p_k^2 \geq e_\varepsilon, \quad k = 1, 2, \ldots, N_T, \tag{4.45}$$

where

$$e_\varepsilon = \frac{1}{2} \eta_{\varepsilon/2} \sigma_\varepsilon > 0, \qquad N_T \longrightarrow \infty \text{ as } T \longrightarrow \infty. \tag{4.46}$$

Let $\delta = (1/8) e_\varepsilon$. From (4.45), it follows that, for every τ_k, there exists an interval $\Delta_k \triangleq [s_k^1, s_k^2] \subset \tau_k$ such that

$$\begin{gathered} \left| px(t) - p_i^* \right| \geq \delta, \quad \forall i = 1, 2, \ldots, m, \ t \in \Delta_k, \\ px(s_k^1) = \sup_{t \in \Delta_k} px(t), \qquad px(s_k^2) = \inf_{t \in \Delta_k} px(t), \qquad px(s_k^1) - px(s_k^2) = \delta. \end{gathered} \tag{4.47}$$

From (4.29), we have

$$\delta = \left| \int_{[s_k^1, s_k^2]} p\dot{x}(t) dt \right| \leq \int_{[s_k^1, s_k^2]} \left| p\dot{x}(t) \right| dt \leq \int_{\Delta_k} \left| p\dot{x}(t) \right| dt \leq K \cdot \operatorname{meas} \Delta_k. \tag{4.48}$$

Then $\operatorname{meas} \Delta_k \geq \delta/K > 0$. Clearly, $\Delta_k \subset Z_\delta$ and therefore

$$\operatorname{meas} Z_\delta \geq \operatorname{meas} \cup_{k=1}^{N_T} \Delta_k = \sum_{k=1}^{N_T} \operatorname{meas} \Delta_k \geq N_T \frac{\delta}{K}. \tag{4.49}$$

This means that $\operatorname{meas} Z_\delta \to \infty$ as $T \to \infty$, which contradicts (4.18).

Thus (4.41) is true. Then taking into account (4.26), we obtain

$$\operatorname{meas} \cup_k \tau_k = \sum_k \left(\operatorname{meas} \tau_k^1 + \operatorname{meas} \tau_k^2 \right) \leq \tilde{K}_{\varepsilon/2, \xi} + K'_{\varepsilon, \xi}. \tag{4.50}$$

Therefore, from (4.22), it follows that

$$\operatorname{meas}\mathscr{X}_{\varepsilon} = \operatorname{meas}\cup_k \tau_k \le K_{\varepsilon,\xi}, \tag{4.51}$$

where $K_{\varepsilon,\xi} = \tilde{K}_{\varepsilon/2,\xi} + K'_{\varepsilon,\xi}$.

Thus we have proved that the second assertion of the theorem is true for the case when $x(\cdot)$ is a continuously differentiable function.

(2) Now we take any trajectory $x(\cdot)$ to system (1.1). It is known that (see, for example, [2]) for a given number $\delta > 0$ (we take $\delta < \varepsilon/2$), there exists a continuously differentiable trajectory $\tilde{x}(\cdot)$, to system (1.1), such that

$$\|x(t) - \tilde{x}(t)\| \le \delta, \quad \forall t \in [0, T]. \tag{4.52}$$

Since the function u is continuous, then there is $\eta(\delta) > 0$ such that

$$u(\tilde{x}(t)) \ge u(x(t)) - \eta(\delta), \quad \forall t \in [0, T]. \tag{4.53}$$

Therefore,

$$\int_{[0,T]} u(\tilde{x}(t))\,dt \ge \int_{[0,T]} u(x(t))\,dt - T\eta(\delta). \tag{4.54}$$

Let $\xi > 0$ be a given number. For every $T > 0$, we choose a number δ such that $T\eta(\delta) \le \xi$. Then,

$$\int_{[0,T]} u(x(t))\,dt \le \int_{[0,T]} u(\tilde{x}(t))\,dt + T\eta(\delta) \le \int_{[0,T]} u(\tilde{x}(t))\,dt + \xi, \tag{4.55}$$

that is,

$$\int_{[0,T]} [u(x(t)) - u^*]\,dt \le \int_{[0,T]} [u(\tilde{x}(t)) - u^*]\,dt + \xi. \tag{4.56}$$

Since the function $\tilde{x}(\cdot)$ is continuously differentiable, then the second integral in this inequality is bounded (see the first part of the proof), and therefore the first assertion of Theorem 2.4 is proved.

Now, we prove the second assertion of Theorem 2.4. We will use (4.55). Take a number $\varepsilon > 0$ and assume that $x(\cdot)$ is a ξ-optimal trajectory; that is,

$$J_T(x(\cdot)) \ge J_T^* - \xi. \tag{4.57}$$

From (4.55), we have

$$J_T(\tilde{x}(\cdot)) \ge J_T(x(\cdot)) - \xi \ge J_T^* - 2\xi. \tag{4.58}$$

Thus $\tilde{x}(\cdot)$ is a continuously differentiable 2ξ-optimal trajectory. That is why (see the first part of the proof) for the numbers $\varepsilon/2 > 0$ and $2\xi > 0$, there is $K_{\varepsilon,\xi} < +\infty$

such that

$$\operatorname{meas}\left\{t \in [0,T] : \|\tilde{x}(t) - x_i^*\| \geq \frac{\varepsilon}{2},\ i = 1,2,\dots,m\right\} \leq K_{\varepsilon,\xi}. \tag{4.59}$$

If $\|x(t') - x_i^*\| \geq \varepsilon$ for any t', then

$$\|\tilde{x}(t') - x_i^*\| \geq \|x(t') - x_i^*\| - \|x(t') - \tilde{x}(t')\| \geq \varepsilon - \delta \geq \frac{\varepsilon}{2}. \tag{4.60}$$

Therefore,

$$\begin{aligned} &\{t \in [0,T] : \|x(t) - x_i^*\| \geq \varepsilon,\ i = 1,2,\dots,m\} \\ &\qquad \subset \left\{t \in [0,T] : \|\tilde{x}(t) - x_i^*\| \geq \frac{\varepsilon}{2},\ i = 1,2,\dots,m\right\}, \end{aligned} \tag{4.61}$$

which implies that the proof of the second assertion of the theorem is completed; that is,

$$\operatorname{meas}\{t \in [0,T] : \|x(t) - x^*\| \geq \varepsilon,\ i = 1,2,\dots,m\} \leq K_{\varepsilon,\xi}. \tag{4.62}$$

(3) Now, we prove the third assertion of Theorem 2.4.

We take any numbers $\varepsilon > 0$ and $\delta > 0$ (satisfying (2.9)). Consider a ξ-optimal trajectory $x(\cdot) \in \mathscr{X}_T$, $T > 0$, and let $N = N_T(\delta, \xi, x(\cdot))$ be a number of state transitions. By Definition 2.3, there are intervals $[t_1^n, t_2^n]$, $n = \overline{1,N}$, for which

$$\begin{aligned} x(t_j^n) \in S_\delta(x_{n_j}^*) \quad &\text{for some } n_j \in \{1,2,\dots,m\},\ j = 1,2, \\ x(t) \notin S_\delta(x_i^*), \quad &\forall t \in (t_1^n, t_2^n),\ i = 1,\dots,m. \end{aligned} \tag{4.63}$$

Then there exist intervals $\Delta_n \subset [t_1^n, t_2^n]$, $n = 1,2,\dots,N$, such that

$$\|x(t) - x_i^*\| \geq \delta, \quad \forall t \in \Delta_n,\ n = 1,2,\dots,N,\ i = 1,2,\dots,m. \tag{4.64}$$

Since $\|\dot{x}(t)\| \leq K < \infty$ (see (1.5)), there is a number $\eta > 0$ such that $\operatorname{meas}\Delta_n \geq \eta$ for all $n = 1,2,\dots,N$. Therefore,

$$\begin{aligned} N\eta &\leq \sum_{n=1}^{N} \operatorname{meas}\Delta_n = \operatorname{meas}\cup_n \Delta_n \\ &\leq \operatorname{meas}\{t \in [0,T] : \|x(t) - x_i^*\| \geq \delta,\ i = 1,\dots,m\} \\ &\leq K_{\delta,\xi}. \end{aligned} \tag{4.65}$$

The third assertion of the theorem is proved if we take $N_{\delta,\xi} = K_{\delta,\xi}/\eta < \infty$.

(4) Now, we prove the fourth assertion of Theorem 2.4.

Let $x(\cdot)$ be an optimal trajectory and $x(t_1) = x(t_2) = x^* \triangleq x_i^*$ for some $i \in \{1,2,\dots,m\}$. Consider a trajectory $x^*(\cdot)$ defined by the formula

$$x^*(t) = \begin{cases} x(t) & \text{if } t \in [0,t_1] \cup [t_2,T], \\ x^* & \text{if } t \in [t_1,t_2]. \end{cases} \tag{4.66}$$

Assume that the third assertion of Theorem 2.4 is not true; that is, there is a point $t' \in (t_1, t_2)$ such that $\|x(t') - x^*\| = c > 0$.

Consider the function $x(\cdot)$. In [2], it is proved that there is a sequence of continuously differentiable trajectories $x_n(\cdot)$, $t \in [t_1, T]$, which is uniformly convergent to $x(\cdot)$, on $[t_1, T]$, and $x_n(t_1) = x(t_1) = x^*$. That is, for every $\delta > 0$, there is a number N_δ such that

$$\max_{t \in [t_1, T]} \|x_n(t) - x(t)\| \leq \delta, \quad \forall n \geq N_\delta. \tag{4.67}$$

On the other hand, for every $\delta > 0$, there is a number $\eta(\delta) > 0$ such that $\eta(\delta) \to 0$ as $\delta \to 0$ and

$$|u(x(t)) - u(x_n(t))| \leq \eta(\delta) \quad \forall t \in [t_1, T]. \tag{4.68}$$

Then we have

$$\int_{[t_1, T]} u(x(t))\,dt \leq \int_{[t_1, T]} u(x_n(t))\,dt + T\eta(\delta). \tag{4.69}$$

Take a sequence of points $t^n \in (t', t_2)$ such that $t^n \to t_2$ as $n \to \infty$. Clearly, in this case $x_n(t^n) \to x^*$. We apply Lemma 3.6 for the interval $[t_1, t^n]$ and obtain

$$\int_{[t_1, t^n]} u(x_n(t))\,dt = \sum_k \int_{\pi_k^n \cup \omega_k^n} u(x_n(t))\,dt + \int_{F^n} u(x_n(t))\,dt + \int_{E^n} u(x_n(t))\,dt. \tag{4.70}$$

Here, $x(t) \in \operatorname{int}\mathcal{D}$ for all $t \in F^n$, and $F^n \in \mathbf{T}^1[px_n(t^n), p^*]$ if $px_n(t^n) < p^*$ ($p^* = px_i^*$).

Since $x_n(t^n) \to x^*$ and $px_n(t^n) \to p^*$, then for every $t \in F^n$ we have $u(x_n(t)) \to u^*$ as $n \to \infty$. Therefore,

$$\alpha_n = \int_{F^n} [u(x_n(t)) - u^*]\,dt \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{4.71}$$

We also note that from $x_n(t) \notin \operatorname{int}\mathcal{D}$, $t \in E^n$, it follows that

$$\int_{E^n} u(x_n(t))\,dt \leq u^* \operatorname{meas} E^n. \tag{4.72}$$

Now, we use Lemma 3.5 and obtain

$$\begin{aligned}\sum_k \int_{\pi_k^n \cup \omega_k^n} u(x_n(t))\,dt = {} & u^* \operatorname{meas}\left[\cup_k (\pi_k^n \cup \omega_k^n)\right] \\ & - \int_{\cup_k Q_k^n} [u^* - u(x_n(t))]\,dt - \int_{\cup_k E_k^n} \delta^2(x_n(t))\,dt.\end{aligned} \tag{4.73}$$

We take a number $\delta < c/2$. Then there is a number $\tilde{\beta} > 0$ such that

$$\operatorname{meas}\left[\cup_k \left(Q_k^n \cup E_k^n\right)\right] \geq \tilde{\beta}. \tag{4.74}$$

Then, there is a number $\beta > 0$, for which

$$\sum_k \int_{\pi_k^n \cup \omega_k^n} u(x_n(t))\,dt \leq u^* \operatorname{meas}\left[\cup_k \left(\pi_k^n \cup \omega_k^n\right)\right] - \beta. \tag{4.75}$$

Therefore, from (4.70), we have

$$\int_{[t_1,t^n]} u(x_n(t))\,dt \leq u^* \left\{\operatorname{meas}\left[\cup_k \left(\pi_k^n \cup \omega_k^n\right)\right] + \operatorname{meas} F^n + \operatorname{meas} E^n\right\} + \alpha_n - \beta \tag{4.76}$$

or

$$\int_{[t_1,t^n]} u(x_n(t))\,dt \leq u^* \left(t^n - t_1\right) + \alpha_n - \beta. \tag{4.77}$$

From (4.68), we obtain

$$\int_{[t_2,T]} u(x_n(t))\,dt \leq \int_{[t_2,T]} u(x(t))\,dt + T\eta(\delta) = \int_{[t_2,T]} u(x^*(t))\,dt + T\eta(\delta). \tag{4.78}$$

Thus, from (4.69), (4.77), and (4.78), we have

$$\begin{aligned}
\int_{[t_1,T]} u(x(t))\,dt &\leq \int_{[t_1,T]} u(x_n(t))\,dt + T\eta(\delta) \\
&= \int_{[t_1,t^n]} u(x_n(t))\,dt + \int_{[t^n,t_2]} u(x_n(t))\,dt \\
&\quad + \int_{[t_2,T]} u(x_n(t))\,dt + T\eta(\delta) \\
&\leq u^* \left(t^n - t_1\right) + u^* \left(t_2 - t^n\right) \\
&\quad + \int_{[t_2,T]} u(x^*(t))\,dt + \alpha_n - \beta + \lambda_n + 2T\eta(\delta) \\
&= \int_{[t_1,T]} u(x^*(t))\,dt + \alpha_n - \beta + \lambda_n + 2T\eta(\delta).
\end{aligned} \tag{4.79}$$

Here,

$$\lambda_n = \int_{[t^n,t_2]} \left[u(x_n(t)) - u^*\right] dt \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{4.80}$$

because of $t^n \to t_2$. We choose the numbers $\delta > 0$ and n such that the following inequality holds:

$$\alpha_n + \lambda_n + 2T\eta(\delta) < \beta. \tag{4.81}$$

In this case, we have

$$\int_{[t_1,T]} u(x(t))\,dt < \int_{[t_1,T]} u(x^*(t))\,dt \tag{4.82}$$

and therefore

$$\int_{[0,T]} u(x(t))\,dt < \int_{[0,T]} u(x^*(t))\,dt, \tag{4.83}$$

which means that $x(\cdot)$ is not optimal. This is a contradiction.

Then Theorem 2.4 is proved.

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