AN ELLIPTIC PROBLEM WITH CRITICAL EXPONENT AND POSITIVE HARDY POTENTIAL

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We give the existence result and the vanishing order of the solution in 0 for the following equation: $-\triangle u(x) + (\mu/|x|^2)u(x) = \lambda u(x) + u^{2^*-1}(x)$, where $x \in B_1$, $\mu > 0$, and the potential $\mu/|x|^2 - \lambda$ is positive in B_1.

1. Introduction

In this paper, we consider the following problem:

$$\begin{gathered}
-\triangle u(x) + \frac{\mu}{|x|^2}u(x) = \lambda u(x) + u^{2^*-1}(x), \quad x \in B_1, \\
u(x) \geq 0, \quad x \in B_1, \\
u(x) = 0, \quad x \in \partial B_1,
\end{gathered} \tag{1.1}$$

where $B_1 = \{x \in \mathbb{R}^N \mid |x| < 1\}$ is the unit ball in $\mathbb{R}^N$ ($N \geq 3$), $\lambda, \mu > 0$, $2^* := 2N/(N-2)$. When $\mu < 0$, this problem has been considered by many authors recently (cf. [5, 6, 7, 8]). But when $\mu > 0$, this problem has not been considered as far as we know. In fact, the existence of nontrivial solution for (1.1) when $\mu > 0$ is an open problem which was imposed in [7]. In this paper, we get the following results.

Theorem 1.1. *If $N = 3$ and $3/4 < \lambda \leq \mu$ or if $N \geq 4$ and $0 < \lambda \leq \mu$, then for (1.1) there exists a nontrivial radially symmetric solution.*

Remark 1.2. Condition $0 < \lambda \leq \mu$ shows that the potential $\mu/|x|^2 - \lambda$ is positive in B_1. Thus the Brézis-Nirenberg method (cf. [1]) cannot be used.

Theorem 1.3. *If $\mu > 0$ and $u \in H_0^1(B_1)$ is a solution of (1.1), then there are $C_1, C_2 > 0$ and $\delta > 0$ such that $C_2|x|^\alpha \geq u(x) \geq C_1|x|^\alpha$, for $x \in B_\delta$, where $\alpha = (1/2)(\sqrt{(N-2)^2 + 4\mu^2} - (N-2)) > 0$.*

Remark 1.4. One can easily deduce that if $u \in H_0^1(B_1)$ is a solution of (1.1), then $u \in C^2(B_1 \setminus \{\theta\})$ and $u > 0$ in $B_1 \setminus \{\theta\}$. Theorem 1.3 shows that $u(\theta) = 0$. It is greatly different from the case of $\mu \leq 0$ (see [6]).

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2. Proof of Theorem 1.1

Lemma 2.1. *Every radially symmetric nonnegative solution u of the equation*

$$-\triangle u+\frac{\mu}{|x|^2}u(x)=u^{2^*-1}(x),\quad u\in\mathcal{D}^{1,2}(\mathbb{R}^N), \tag{2.1}$$

can be represented by $u(x)=\rho^{(N-2)/2}U(\rho x)$ *for some positive number* ρ, *where*

$$U(x)=\frac{C_0|x|^{\tau-(N-2)/2}}{\left(1+|x|^{4\tau/(N-2)}\right)^{(N-2)/2}}, \tag{2.2}$$

$\tau=\sqrt{((N-2)/2)^2+\mu}$, *and* C_0 *is a constant.*

Proof. Let $t=-\ln|x|$, $\theta=x/|x|$, and $v(t,\theta):=e^{-((N-2)/2)t}u(e^{-t}\theta)$. Then by [3], we know that v satisfies the equation

$$-v_{tt}-\triangle_\theta v+\tau^2v=v^{2^*-1}\quad\text{in }\mathbb{R}\times\mathbb{S}^{N-1}. \tag{2.3}$$

Since u is radially symmetric, v depends only on t and satisfies $-v_{tt}+\tau^2v=v^{2^*-1}$, $v>0$ in $\mathbb{R}$. By [3], we know that the only positive solutions of the equation are translation of

$$v(t)=\left(\frac{\tau^2 2^*}{2}\right)^{1/(2^*-1)}\left(\cosh\left(\frac{2^*-2}{2}\tau t\right)\right)^{-2/(2^*-2)}. \tag{2.4}$$

Thus, every radially symmetric nonnegative solution u of (2.1) can be represented by $u(x)=\rho^{(N-2)/2}U(\rho x)$ for some positive number ρ. □

Define $\mathcal{D}_r^{1,2}(\mathbb{R}^N):=\{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\mid u\text{ is radially symmetric}\}$ and $H_{0,r}^1(B_1):=\{u\in H_0^1(B_1)\mid u\text{ is radially symmetric}\}$. Let

$$S_\mu:=\inf_{u\in\mathcal{D}_r^{1,2}(\mathbb{R}^N),u\neq 0}\frac{\int_{\mathbb{R}^N}|\nabla u|^2+\mu\int_{\mathbb{R}^N}(u^2/|x|^2)}{\left(\int_{\mathbb{R}^N}|u|^{2^*}\right)^{2/2^*}}. \tag{2.5}$$

It follows from Lemma 2.1 that $S_\mu=(\int_{\mathbb{R}^N}|\nabla U|^2+\mu\int_{\mathbb{R}^N}(U^2/|x|^2))/(\int_{\mathbb{R}^N}U^{2^*})^{2/2^*}$. Let $\Sigma=\{u\in H_{0,r}^1(B_1)\mid \|u\|_{2^*}=1\}$. For $u\in\Sigma$, define

$$S_{\lambda,\mu}(u)=\int_{B_1}|\nabla u|^2+\mu\int_{B_1}\frac{u^2}{|x|^2}-\lambda\int_{B_1}u^2. \tag{2.6}$$

Lemma 2.2. *If* $N=3$ *and* $3/4<\lambda\le\mu$ *or if* $N\ge 4$ *and* $0<\lambda\le\mu$, *then* $S_{\lambda,\mu}:=\inf_{u\in\Sigma}S_{\lambda,\mu}(u)<S_\mu$.

Proof. Let $\eta \in C_0^\infty(\mathbb{R}^N)$ be a cut function which satisfies $0 \le \eta(x) \le 1$, $|\nabla \eta| \le 2$ in $\mathbb{R}^N$, $\eta(x) \equiv 1$ in $B_{1/2}$, and $\eta(x) \equiv 0$ in $\mathbb{R}^N \setminus B_1$. Let $U_\rho(x) := \rho^{(N-2)/2} U(\rho x)$ and $u_\rho(x) = \eta(x) U_\rho(x)$. By (2.2), we know that when $|x|$ is big enough, there are constants $C_1, C_2 > 0$ such that

$$|U(x)| \le \frac{C_1}{|x|^{\tau+N/2-1}}, \qquad |\nabla U(x)| \le \frac{C_2}{|x|^{\tau+N/2}}, \tag{2.7}$$

since

$$\begin{aligned}
\int_{B_1} |\nabla u_\rho|^2 &= \int_{B_1} \eta^2 |\nabla u_\rho|^2 + \int_{B_1} u_\rho^2 |\nabla \eta|^2 + 2\int_{B_1} u_\rho \cdot \eta \cdot \nabla u_\rho \cdot \nabla \eta \\
&\le \int_{B_1} |\nabla u_\rho|^2 + 4\int_{B_1 \setminus B_{1/2}} u_\rho^2 + 4\left(\int_{B_1 \setminus B_{1/2}} u_\rho^2\right)^{1/2} \left(\int_{B_1 \setminus B_{1/2}} |\nabla u_\rho|^2\right)^{1/2} \\
&= \int_{\mathbb{R}^N} |\nabla U|^2 + \int_{\mathbb{R}^N \setminus B_\rho} |\nabla U|^2 + \frac{4}{\rho^2} \int_{B_\rho \setminus B_{\rho/2}} U^2 \\
&\quad + \frac{4}{\rho}\left(\int_{B_\rho \setminus B_{\rho/2}} U^2\right)^{1/2} \left(\int_{B_\rho \setminus B_{\rho/2}} |\nabla U|^2\right)^{1/2}.
\end{aligned} \tag{2.8}$$

By (2.7), when $N = 3$ and $3/4 < \lambda \le \mu$ or when $N \ge 4$ and $0 < \lambda \le \mu$, for ρ big enough,

$$\begin{aligned}
\int_{B_\rho \setminus B_{\rho/2}} U^2 &\le \int_{B_\rho \setminus B_{\rho/2}} \frac{C_1}{|x|^{2\tau+N-2}} dx = \frac{C_3}{\rho^{2\tau-2}}, \\
\int_{\mathbb{R}^N \setminus B_\rho} |\nabla U|^2 &\le \int_{\mathbb{R}^N \setminus B_\rho} \frac{C_2}{|x|^{2\tau+N}} dx = \int_\rho^{+\infty} \frac{C_2}{r^{2\tau+1}} dr = \frac{C_4}{\rho^{2\tau}},
\end{aligned} \tag{2.9}$$

$$\int_{B_1} |\nabla u_\rho|^2 \le \int_{\mathbb{R}^N} |\nabla U|^2 + \frac{C_5}{\rho^{2\tau}}, \tag{2.10}$$

$$\begin{gathered}
\int_{B_1} \frac{u_\rho^2}{|x|^2} \le \int_{\mathbb{R}^N} \frac{U^2}{|x|^2} + \frac{C_6}{\rho^{2\tau}}, \qquad \int_{B_1} |u_\rho|^{2^*} \ge \int_{\mathbb{R}^N} U^{2^*} - \frac{C_7}{\rho^{2^*\tau}}, \\
\int_{B_1} u_\rho^2 \ge \frac{C_8}{\rho^2}.
\end{gathered} \tag{2.11}$$

When $N = 3$ and $3/4 < \lambda \le \mu$ or when $N \ge 4$ and $0 < \lambda \le \mu$, we have $2\tau > 2$. Thus by (2.10) and (2.11), we get

$$S_{\lambda,\mu} \frac{u_\rho}{||u_\rho||_{2^*}} \le S_\mu - \frac{C_9}{\rho^2} + o\left(\frac{1}{\rho^2}\right), \quad \text{as } \rho \longrightarrow \infty. \tag{2.12}$$

It proves the lemma. □

Proof of Theorem 1.1. By Lemma 2.2 and [10, Theorem 8.8], we deduce that $S_{\lambda,\mu}$ can be achieved by some $0 \le u \in H_{0,r}^1(B_1)$, then $S_{\lambda,\mu}^{-1/(2^*-2)} u$ is a nontrivial radially symmetric solution of (1.1). □

3. Proof of Theorem 1.3

Let E be the space which is the completion of $C_0^\infty(B_1)$ under the norm $\|u\|_E = (\int_{B_1} |x|^{2\alpha}|\nabla u|^2 dx)^{1/2}$.

Lemma 3.1 (see [2]). *For all $u \in C_0^\infty(\mathbb{R}^N)$ $(N \geq 3)$,*

$$\left(\int_{\mathbb{R}^N} |x|^{-bp}|u|^p dx\right)^{2/p} \leq C_{a,b}\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2 dx, \tag{3.1}$$

where $-\infty < a < (N-2)/2$, $a \leq b \leq a+1$, and $p = 2N/(N-2+2(b-a))$.

Choosing $a = -\alpha$, $p = 2$ and 2^*, respectively, in (3.1), we get the following lemma.

Lemma 3.2. *There is a constant $C > 0$ such that, for any $u \in C_0^\infty(\mathbb{R}^N)$,*

$$\begin{gathered}\left(\int_{\mathbb{R}^N} |x|^{2^*\alpha}|u|^{2^*} dx\right)^{2/2^*} \leq C\int_{\mathbb{R}^N} |x|^{2\alpha}|\nabla u|^2 dx,\\ \int_{\mathbb{R}^N} |x|^{2\alpha-2}|u|^2 dx \leq C\int_{\mathbb{R}^N} |x|^{2\alpha}|\nabla u|^2 dx.\end{gathered} \tag{3.2}$$

Proof of Theorem 1.3. If $v \in H_0^1(B_1)$ is a solution of (1.1), then by the standard regularity theory, one can easily deduce that $v \in C^2(B_1 \setminus \{\theta\})$. Let $u(x) = |x|^{-\alpha}v(x)$ (this kind of transform has been used in [9]). Direct calculation shows that, for any $x \in B_1 \setminus \{\theta\}$,

$$-\operatorname{div}\left(|x|^{2\alpha}\nabla u\right) = |x|^{2^*\alpha}u^{2^*-1} + \lambda|x|^{2\alpha}u. \tag{3.3}$$

Since $v \in E$, then by Lemma 3.1 we know that v is a weak solution of (3.3), that is, for any $\zeta \in C_0^\infty(B_1)$,

$$\int_{B_1} |x|^{2\alpha}\nabla u\nabla\zeta = \int_{B_1} |x|^{2^*\alpha}u^{2^*-1}\zeta + \int_{B_1} |x|^{2\alpha}u\zeta. \tag{3.4}$$

For $t > 2$, $k > 0$, define

$$h(r) = \begin{cases} r^{t/2}, & 0 \leq r \leq k,\\ \dfrac{t}{2}k^{t/2-1}r + \left(1 - \dfrac{t}{2}\right)k^{t/2}, & r \geq k,\end{cases} \tag{3.5}$$

and $\phi(r) = \int_0^r |h'(s)|^2 ds$. It is easy to verify that there exists a constant $C > 0$ independent of k such that

$$|r\phi(r)| \leq \frac{t^2}{4(t-1)}|h(r)|^2, \tag{3.6}$$

$$|\phi(r) - h(r)h'(r)| \leq C_t|h(r)h'(r)|, \tag{3.7}$$

where $C_t = (t-2)/2(t-1) < 1$.

Let $0 < r_2 < r_1 < 1$ and $\eta \in C_0^\infty(B(\theta, r_1))$ satisfying $0 \le \eta \le 1$, $\eta \equiv 1$ in $B(\theta, r_2)$, $\eta \equiv 0$ in $\mathbb{R}^N \setminus B(\theta, r_1)$, and $|\nabla \eta| \le 2/(r_1 - r_2)$. Notice that $\eta^2\phi(u) \in E$, then

$$\begin{aligned}\int_{B_1} |x|^{2\alpha} \nabla u \nabla (\eta^2 \phi(u)) &= \int_{B_1} |x|^{2\alpha} \eta^2 (h'(u))^2 |\nabla u|^2 + 2\int_{B_1} |x|^{2\alpha} \eta \phi(u) \nabla u \nabla \eta \\ &= \int_{B_1} |x|^{2\alpha} \eta^2 |\nabla (h(u))|^2 + 2\int_{B_1} |x|^{2\alpha} \eta \phi(u) \nabla u \nabla \eta.\end{aligned} \tag{3.8}$$

Since $|\nabla(\eta h(u))|^2 = \eta^2|\nabla(h(u))|^2 + h^2(u)|\nabla\eta|^2 + 2\eta h(u)\nabla(h(u))\nabla\eta$, by (3.7), we have

$$\begin{aligned}\int_{B_1} |x|^{2\alpha} \nabla u \nabla (\eta^2 \phi(u)) &= \int_{B_1} |x|^{2\alpha} |\nabla (\eta h(u))|^2 - \int_{B_1} |x|^{2\alpha} h^2(u) |\nabla \eta|^2 \\ &\quad - 2\int_{B_1} |x|^{2\alpha} \eta h(u) h'(u) \nabla u \nabla \eta + 2\int_{B_1} |x|^{2\alpha} \eta \phi(u) \nabla u \nabla \eta \\ &\ge \int_{B_1} |x|^{2\alpha} |\nabla (\eta h(u))|^2 - \int_{B_1} |x|^{2\alpha} h^2(u) |\nabla \eta|^2 \\ &\quad - 2\int_{B_1} |x|^{2\alpha} \eta |\phi(u) - h(u)h'(u)| |\nabla u \nabla \eta| \\ &\ge \int_{B_1} |x|^{2\alpha} |\nabla (\eta h(u))|^2 - \int_{B_1} |x|^{2\alpha} h^2(u) |\nabla \eta|^2 \\ &\quad - 2C_t \int_{B_1} |x|^{2\alpha} |\eta h(u) \nabla (h(u)) \nabla \eta|.\end{aligned} \tag{3.9}$$

Since

$$\begin{aligned}\int_{B_1} |x|^{2\alpha} |\eta h(u) \nabla (h(u)) \nabla \eta| &= \int_{B_1} |x|^{2\alpha} |(\nabla (\eta h(u)) - h(u) \nabla \eta) \nabla \eta| \, |h(u)| \\ &\le \int_{B_1} |x|^{2\alpha} |h(u) \nabla (\eta h(u)) \nabla \eta| + \int_{B_1} |x|^{2\alpha} |h(u)|^2 |\nabla \eta|^2 \\ &\le \frac{1}{2}\int_{B_1} |x|^{2\alpha} h^2(u) |\nabla \eta|^2 + \frac{1}{2}\int_{B_1} |x|^{2\alpha} |\nabla (\eta h(u))|^2 \\ &\quad + \int_{B_1} |x|^{2\alpha} |h(u)|^2 |\nabla \eta|^2,\end{aligned} \tag{3.10}$$

and by (3.9), we deduce that

$$\begin{aligned}&\int_{B_1} |x|^{2\alpha} \nabla u \nabla (\eta^2 \phi(u)) \\ &\quad \ge \int_{B_1} |x|^{2\alpha} |\nabla (\eta h(u))|^2 - \int_{B_1} |x|^{2\alpha} h^2(u) |\nabla \eta|^2 \\ &\qquad - 2C_t \left(\frac{1}{2}\int_{B_1} |x|^{2\alpha} h^2(u) |\nabla \eta|^2 + \frac{1}{2}\int_{B_1} |x|^{2\alpha} |\nabla (\eta h(u))|^2 + \int_{B_1} |x|^{2\alpha} |h(u)|^2 |\nabla \eta|^2\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{t}{2(t-1)}\int_{B_1}|x|^{2\alpha}|\nabla(\eta h(u))|^2 - (1+3C_t)\int_{B_1}|x|^{2\alpha}h^2(u)|\nabla\eta|^2 \\
&\geq \frac{Ct}{2(t-1)}\left(\int_{B_1}|x|^{2^*\alpha}|\eta h(u)|^{2^*}\right)^{2/2^*} - (1+3C_t)\int_{B_1}|x|^{2\alpha}h^2(u)|\nabla\eta|^2.
\end{aligned}
\tag{3.11}
$$

By (3.6), we have

$$
\begin{aligned}
&\int_{B_1}|x|^{2^*\alpha}u^{2^*-1}\eta^2\phi(u) + \int_{B_1}|x|^{2\alpha}u\eta^2\phi(u) \\
&\quad\leq \frac{t^2}{4(t-1)}\int_{B_1}|x|^{2^*\alpha}|u|^{2^*-2}|\eta h(u)|^2 + \frac{t^2}{4(t-1)}\int_{B_1}|x|^{2\alpha}|\eta h(u)|^2 \\
&\quad\leq \frac{t^2}{4(t-1)}\left(\int_{\eta\neq 0}|x|^{2^*\alpha}|u|^{2^*}\right)^{(2^*-2)/2^*}\left(\int_{B_1}|\eta h(u)|^{2^*}\right)^{2/2^*} \\
&\qquad + \frac{t^2}{4(t-1)}\int_{B_1}|x|^{2\alpha}|\eta h(u)|^2.
\end{aligned}
\tag{3.12}
$$

Notice that u is a solution of (3.3), by (3.11) and (3.12) we have

$$
\begin{aligned}
&\left(\int_{B_1}|x|^{2^*\alpha}|\eta h(u)|^{2^*}\right)^{2/2^*} \\
&\quad\leq \frac{t}{2C}\left(\int_{\eta\neq 0}|x|^{2^*\alpha}|u|^{2^*}\right)^{(2^*-2)/2^*}\left(\int_{B_1}|x|^{2^*\alpha}|\eta h(u)|^{2^*}\right)^{2/2^*} \\
&\qquad + \frac{2(1+3C_t)(t-1)}{Ct}\int_{B_1}|x|^{2\alpha}h^2(u)|\nabla\eta|^2 + \frac{t}{2C}\int_{B_1}|x|^{2\alpha}|\eta h(u)|^2.
\end{aligned}
\tag{3.13}
$$

Choose r_1 small enough such that $(t/2C)(\int_{\eta\neq 0}|x|^{2^*\alpha}|u|^{2^*})^{(2^*-2)/2^*} < 1/2$. Notice that $2(1+3C_t)(t-1)/t < 8$ (since $0 < C_t < 1$ and $t > 2$) and $|\nabla\eta| < 2/(r_1 - r_2)$, from (3.13) we have

$$
\left(\int_{B(\theta,r_2)}|x|^{2^*\alpha}|h(u)|^{2^*}\right)^{2/2^*} \leq \left(\frac{64}{C(r_1-r_2)^2} + \frac{t}{C}\right)\int_{B(\theta,r_1)}|x|^{2\alpha}h^2(u).
\tag{3.14}
$$

Choosing $2(N-2\alpha)/(N-2+2\alpha) > t_0 > 2$ and letting $k \to \infty$ in (3.14), we get

$$
\left(\int_{B(\theta,r_2)}|x|^{2^*\alpha}|u|^{2^*t_0/2}\right)^{2/2^*} \leq \left(\frac{64}{C(r_1-r_2)^2} + \frac{t_0}{C}\right)\int_{B(\theta,r_1)}|x|^{2\alpha}|u|^{t_0}.
\tag{3.15}
$$

By Lemma 3.1, we know that $(\int_{B_1}|x|^{2\alpha}|u|^{t_0})^{2/t_0} \leq \int_{B_1}|x|^{2\alpha}|\nabla u|^2 < \infty$. Combining (3.15), we get that

$$
\int_{B_1}|x|^{2^*\alpha}|u|^{2^*t_0/2} < \infty.
\tag{3.16}
$$

Since

$$\int_{B_1} |x|^{2\alpha}\nabla u \nabla(\phi(u)) = \int_{B_1} |x|^{2\alpha} |\nabla(h(u))|^2 \geq \left(\int_{B_1} |x|^{2^*\alpha} |h(u)|^{2^*}\right)^{2/2^*},$$

$$\begin{aligned}
&\int_{B_1} |x|^{2^*\alpha} u^{2^*-1}\phi(u) + \int_{B_1} |x|^{2\alpha} u\phi(u) \\
&\quad\leq \frac{t^2}{4(t-1)} \int_{B_1} |x|^{2^*\alpha}|u|^{2^*-2}|h(u)|^2 + \frac{t^2}{4(t-1)} \int_{B_1} |x|^{2\alpha}|h(u)|^2 \\
&\quad\leq \frac{t^2}{4(t-1)} \left(\int_{B_1} |x|^{2^*\alpha}|u|^{2^* t_0/2}\right)^{2(2^*-2)/2^* t_0} \left(\int_{B_1} |x|^{2^*\alpha}|h(u)|^q\right)^{2/q} \\
&\qquad+ \frac{t^2}{4(t-1)} \int_{B_1} |x|^{2\alpha}|h(u)|^2 \\
&\quad\leq \frac{t^2}{4(t-1)} \left(\int_{B_1} |x|^{2^*\alpha}|u|^{2^* t_0/2}\right)^{2(2^*-2)/2^* t_0} \left(\int_{B_1} |x|^{2^*\alpha}|h(u)|^q\right)^{2/q} \\
&\qquad+ \frac{t^2}{4(t-1)} \left(\int_{B_1} |x|^{(2\alpha-2^*\alpha/q)q'}\right)^{1/q'} \left(\int_{B_1} |x|^{2^*\alpha}|h(u)|^q\right)^{2/q},
\end{aligned} \tag{3.17}$$

where $q = 2 \cdot 2^* t_0/((t_0-2)2^*+4)$ and $2/q + 1/q' = 1$, we can deduce that if $\epsilon > 0$ small enough and $t_0 \in (2, 2+\epsilon)$, then $(2\alpha - 2^*\alpha/q)q' > -2$. Thus $(\int_{B_1} |x|^{(2\alpha-2^*\alpha/q)q'})^{1/q'} < \infty$. Let $C' = (\int_{B_1} |x|^{2^*\alpha}|u|^{2^* t_0/2})^{2(2^*-2)/2^* t_0} + (\int_{B_1} |x|^{(2\alpha-2^*\alpha/q)q'})^{1/q'}$, then by (3.17), we have

$$\left(\int_{B_1} |x|^{2^*\alpha}|h(u)|^{2^*}\right)^{2/2^*} \leq \frac{C' t^2}{4(t-1)} \left(\int_{B_1} |x|^{2^*\alpha}|h(u)|^q\right)^{2/q}. \tag{3.18}$$

Letting $k \to \infty$, we get

$$|u|_{2^* t/2, 2^*\alpha} \leq \left(\frac{C' t^2}{4(t-1)}\right)^{1/t} |u|_{qt/2, 2^*\alpha}, \tag{3.19}$$

where $|u|_{l,2^*\alpha} := (\int_{B_1} |x|^{2^*\alpha}|u|^l)^{1/l}$.

Choose $t_1 = (2^*/q)^n$, $n = 1, 2, \ldots$. Then by (3.19) we have

$$|u|_{2^* t_n/2, 2^*\alpha} \leq \prod_{i=1}^{n} \left(\frac{C' t_i^2}{4(t_i-1)}\right)^{1/t_i} |u|_{2^*/2, 2^*\alpha}. \tag{3.20}$$

Letting $n \to \infty$, we deduce that $u \in L^\infty(B_1)$. Thus there is $C_2 > 0$ such that $v(x) \leq C_2|x|^\alpha$.

Since $\operatorname{div}(|x|^{2\alpha}\nabla u) \leq 0$, by [4, Lemma 4.2], we have $u(x) \geq C'' > 0$ for $x \in B_\delta$. So, there is $C_1 > 0$ such that $u(x) \geq C_1|x|^\alpha$ for $x \in B_\delta$. □

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