ON σ-POROUS SETS IN ABSTRACT SPACES

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Received 18 December 2003

The main aim of this survey paper is to give basic information about properties and applications of *σ-porous sets in Banach spaces* (and some other infinite-dimensional spaces). This paper can be considered a partial continuation of the author's 1987 survey on porosity and σ-porosity and therefore only some results, remarks, and references (important for infinite-dimensional spaces) are repeated. However, this paper can be used without any knowledge of the previous survey. Some new results concerning σ-porosity in finite-dimensional spaces are also briefly mentioned. However, results concerning porosity (but not σ-porosity) are mentioned only exceptionally.

Contents

Abstract and Applied Analysis 2005:5 (2005) 509–534
DOI: 10.1155/AAA.2005.509

1. Introduction

The main aim of this paper is to give basic information about properties and applications of *σ-porous sets in Banach spaces* (and some other infinite-dimensional spaces). No attempt to cite all relevant papers was made. The paper can be considered a continuation of my 1987 survey [142, 143] and therefore only some results, remarks, and references are repeated. The present (partial) survey can be used without any knowledge of [142]. On the other hand, for those readers, who are interested in properties and applications of σ-porosity in $\mathbb{R}$ and $\mathbb{R}^n$, the knowledge of [142] is necessary. To those readers Sections 7 and 8 are addressed.

Results concerning porosity (but not σ-porosity) are discussed only exceptionally. In particular, important applications of lower porosity in the theory of quasiconformal mappings are not mentioned.

The notion of porosity of a subset E of a metric space X at a point $x \in X$ concerns the size of "pores in E" (i.e., balls or open sets of other type which are disjoint with E) near to x. A porous set $P \subset X$ is not only nowhere dense but it is small in a stronger sense: near to each point $x \in E$ there are pores in E which are big in some sense.

Porosity in $\mathbb{R}$ was used (under a different nomenclature) already by A. Denjoy in 1920. There are two main types of porosity (of a set at a point): the first ("upper porosity") is defined as an upper limit and the second ("lower porosity") as a lower limit (see Definition 2.1 below). Denjoy used the upper porosity (the symmetric upper porosity was the main notion for him).

In the present paper we concentrate on properties and applications of *σ-porous sets* (i.e., countable unions of porous sets). The interest in the notion of σ-porosity is motivated mainly by two facts:

(i) some interesting sets (of "singular points") are σ-porous,

(ii) the system of all σ-porous sets is (in most interesting metric spaces) a *proper* subsystem of the system of all first category (meager) sets. Moreover, in $\mathbb{R}^n$ it is a proper subsystem of the system of all first category Lebesgue null sets.

Thus σ-porous sets provide an important tool in the theory of exceptional sets.

Probably the first implicit application of σ-porous sets which can be found in [6] (cf. [142, page 344]) is due to Piateckii-Shapiro. But the theory of σ-porous sets was started in 1967 by Dolženko [28] who applied σ-porous sets (defined by upper porosity) in the theory of boundary behaviour of functions and who used for the first time the term "porous set." In the differentiation theory, σ-porous sets were used for the first time in 1978 [8] and in Banach space theory in 1984 [97].

In all these applications (and in most applications in real analysis) upper porosity is used and the term "σ-porous sets" was used for σ-upper porous sets. This terminology was used also in the survey [142], where σ-lower porous sets were treated only briefly under the name "σ-very porous sets."

Now there exists a number of papers using σ-lower porous sets which are also called simply σ-porous sets there. Since this natural but confusing situation will probably continue, I follow here the suggestion of D. Preiss to use the natural terms "σ-upper porous set" and "σ-lower porous set" when it is necessary to explain which type of porosity is used. This (or similar) terminology was already used, for example, in [113, page 93], [83, 84].

Now many porosity notions are considered in the literature and many others can be easily defined (cf. [112, 113, 130, 142]). Of course, most interesting are these kinds of porosity or σ-porosity which were used in an interesting theorem. Types of σ-porosity which were applied in infinite-dimensional spaces are discussed in Sections 3–5 below.

2. Basic properties of σ-upper porous sets and of σ-lower porous sets

In the following, we suppose that X is a fixed nonempty metric space. The open ball with center $x \in X$ and radius $r > 0$ will be denoted by $B(x,r)$; further put $B(x,0) := \emptyset$.

Definition 2.1. Let $M \subset X$, $x \in X$, and $R > 0$. Then define $\gamma(x,R,M)$ as the supremum of all $r \geq 0$ for which there exists $z \in X$ such that $B(z,r) \subset B(x,R) \setminus M$. Further define the upper porosity of M at x as

$$\overline{p}(M,x) := 2\limsup_{R\to 0+} \frac{\gamma(x,R,M)}{R}, \tag{2.1}$$

and the lower porosity of M at x as

$$\underline{p}(M,x) := 2\liminf_{R\to 0+} \frac{\gamma(x,R,M)}{R}. \tag{2.2}$$

Say that M is upper porous (lower porous, c-upper porous, c-lower porous) at x if $\overline{p}(M,x) > 0$ ($\underline{p}(M,x) > 0, \overline{p}(M,x) \geq c, \underline{p}(M,x) \geq c$).

Say that M is upper porous (lower porous, c-upper porous, c-lower porous) if M is upper porous (lower porous, c-upper porous, c-lower porous) at each point $y \in M$. Say that M is σ-upper porous (σ-lower porous) if it is a countable union of upper porous (lower porous) sets.

It is clear that each lower porous set is upper porous and each upper porous set is nowhere dense. Consequently *each σ-lower porous set is σ-upper porous and each σ-upper porous set is a first category set.*

In most papers *in real analysis, σ-porosity means σ-upper porosity* and σ-lower porous sets are sometimes called "σ-very porous sets."

On the other hand, in many recent papers (see, e.g., [23, 25, 26, 105]) concerning Banach (and other abstract) spaces *σ-porosity means σ-lower porosity*. In fact, in these papers σ-porosity is defined in a formally different but equivalent way: by the condition (ii) of the following well-known proposition.

Proposition 2.2. *Let X be a metric space and $A \subset X$. Then the following statements are equivalent.*

(i) *A is σ-lower porous.*

(ii) *$A = \bigcup_{n \in \mathbb{N}} P_n$, where each set $P := P_n$ has the following property:*

$$\exists \alpha > 0\ \exists r_0 > 0\ \forall x \in X\ \forall r \in (0, r_0)\ \exists y \in X : B(y, \alpha r) \subset B(x, r) \setminus P. \tag{2.3}$$

If moreover X is a normed linear space, (i) and (ii) are equivalent to

(iii) *$A = \bigcup_{n \in \mathbb{N}} P_n$, where each set $P := P_n$ has the following property:*

$$\exists \alpha > 0\ \forall x \in X\ \forall r > 0\ \exists y \in X : B(y, \alpha r) \subset B(x, r) \setminus P. \tag{2.4}$$

If X is a normed linear space, then (i) is equivalent to (iii) by [166, Lemma E]. Now suppose that $A \subset X$ is lower porous and put

$$A_k = \left\{ x \in A : \frac{\gamma(x, r, A)}{r} > \frac{1}{k} \text{ for each } 0 < r < \frac{1}{k} \right\}. \tag{2.5}$$

Clearly $A = \bigcup_{k \in \mathbb{N}} A_k$ and each $P = A_k$ satisfies the condition which we obtain from (2.3) writing $x \in P$ instead of $x \in X$. But it is easy to see that this condition is equivalent to (2.3) (see, e.g., [23, Proposition 3]). Thus (i) $\Rightarrow$ (ii); the opposite implication is obvious.

The sets P satisfying (2.4) are called "globally very porous" in [142] and some other papers (this notion is clearly nontrivial in unbounded X only).

The Lebesgue density theorem easily implies that each σ-upper porous set $A \subset \mathbb{R}^n$ is of Lebesgue measure zero. Therefore the following fact is of basic importance for applications of σ-upper porous sets in finite-dimensional spaces.

Theorem 2.3. *There exists a closed nowhere dense set $F \subset \mathbb{R}^n$ of Lebesgue measure zero which is not σ-upper porous.*

There exist a number of different proofs of this *relatively deep result* (cf. [142] and Section 8.3 below). (Note that if $F \subset \mathbb{R}$ has the property from Theorem 2.3, then also $F^* = F \times \mathbb{R}^{n-1}$ has this property in $\mathbb{R}^n$; cf. [135, page 353] or [146].) We stress that the analogue of Theorem 2.3 for σ-lower porous sets (which is a much weaker theorem) *is an easy fact* (cf., e.g., Remark 2.8(i) below).

If X is an infinite-dimensional Banach space, then not only σ-upper porous sets but even *σ-lower porous sets need not be negligible in usual "measure senses"* (cf. Section 5.1 below). Therefore the following (relatively difficult) result [150] is important for papers which work with σ-porosity in infinite-dimensional spaces.

Theorem 2.4. *Let $X \neq \emptyset$ be a topologically complete metric space without isolated points. Then there exists a closed nowhere dense set $F \subset X$ which is not σ-upper porous.*

In fact, most papers which deal with σ-porosity in infinite-dimensional spaces use σ-lower porous sets. For motivation of these papers, the analogue of Theorem 2.4 for σ-lower porous sets is sufficient. This analogue (Proposition 2.7) is an easy fact (see the proof below).

In the case when X is a Banach space, the set F from Theorem 2.4 can be constructed in the following simple way: choose a nonzero functional $f \in X^*$ and a closed nowhere dense set $Z \subset \mathbb{R}$ of positive Lebesgue measure and put $F := f^{-1}(Z)$. This observation was presented in [142] with an argument which is correct in separable Banach spaces only. A more complicated argument, which works also in nonseparable spaces, is contained in [146].

It is obvious that if a set P satisfies the condition (2.3), then also its closure $\overline{P}$ satisfies this condition. Thus Proposition 2.2 implies the following easy well-known fact.

Proposition 2.5. *Let A be a σ-lower porous subset of a metric space X. Then A can be covered by countably many closed lower porous sets.*

This fact and the Baire theorem easily give the following.

Proposition 2.6. *Let (X,ρ) be a metric space and let F be a topologically complete subspace of X. Suppose there exists a set $A \subset F$ dense in F such that F is lower porous (in X) at no point $x \in A$. Then F is not a σ-lower porous subset of X.*

Proof. Suppose on the contrary that $F \subset \bigcup_{n\in\mathbb{N}} Z_n$, where each set Z_n is closed and lower porous. Since (F,ρ) is topologically complete, by Baire theorem there exists an open set $H \subset X$ such that $\emptyset \neq H \cap F \subset Z_n$. Choose $y \in A \cap H$. Since Z_n is lower porous, we obtain that F is lower porous at y, a contradiction with $y \in A$. □

The analogues of Propositions 2.5 and 2.6 for σ-upper porous sets do not hold. This is an important difference between upper porosity and lower porosity, which shows the main reason why the proofs that some "small" sets are not σ-lower porous are usually much easier than corresponding proofs for σ-upper porosity. Using Proposition 2.6, we can easily prove the following weaker version of Theorem 2.4.

Proposition 2.7. *Let $(X,\rho) \neq \emptyset$ be a topologically complete metric space with no isolated points. Then there exists a closed nowhere dense set $F \subset X$ which is not σ-lower porous.*

Proof. First observe that for each $z \in X$ there exists a set $M_z \subset X \setminus \{z\}$ such that $(M_z)' = \{z\}$ and M_z is not upper porous at z. (It is sufficient to choose in each set $\{x \in X : 1/(n+1) \leq \rho(x,z) < 1/n\}$ $(n \in \mathbb{N})$ a maximal $(1/n^2)$-discrete set M_n and put $M_z := \bigcup_{n\in\mathbb{N}} M_n$.)

Now we put $F_0 := \emptyset$ and we will construct inductively closed sets $F_1 \subset F_2 \subset \cdots$ and open sets $G_1 \supset G_2 \supset \cdots$ such that the following statements (in which we put $D_n = F_n \setminus F_{n-1}$) for each $n \in \mathbb{N}$ hold.

(i) $F_{n-1} = (D_n)'$.

(ii) D_n is upper porous at no point $x \in D_{n-1}$ if $n \geq 2$.

(iii) $D_n \subset G_n$.

(iv) For each $x \in F_n$, there exists a point $w \notin F_n \cup \overline{G_n}$ such that $\rho(x,w) < 1/n$.

Choose $a \in X$ and put $F_1 := \{a\}$. Choose $b \neq a$ with $\varepsilon := \rho(a,b) < 1$ and put $G_1 := B(a, \varepsilon/2)$. The conditions (i)–(iv) then clearly hold for $n = 1$.

Further suppose that $k \geq 2$, the sets $F_1, G_1, \ldots, F_{k-1}, G_{k-1}$ are defined, and (i)–(iv) hold for $n = k-1$. For each $z \in D_{k-1} = F_{k-1} \setminus F_{k-2}$ we can clearly choose $w_z \notin F_{k-1}$ such that $\varepsilon_z := \rho(z,w_z) < \min(3^{-1}\rho(z,F_{k-1}\setminus\{z\}),(2k)^{-1})$. Put

$$\begin{aligned} B_z &:= B\left(z, \frac{\varepsilon_z}{2}\right), \\ G_k &:= G_{k-1} \cap \bigcup_{z\in D_{k-1}} B_z, \\ F_k &:= F_{k-1} \cup \bigcup_{z\in D_{k-1}} (M_z \cap G_k). \end{aligned} \tag{2.6}$$

Clearly $D_k = F_k \setminus F_{k-1} = \bigcup_{z\in D_{k-1}}(M_z \cap G_k)$ and (i)–(iii) are satisfied for $n = k$. If $x \in D_k$, then $x \in B_z$ for some $z \in D_{k-1}$. Clearly $w_z \notin F_{k-1} \cup \overline{G_k} = F_k \cup \overline{G_k}$ and $\rho(z,w_z) < 3(4k)^{-1}$. Using also $F_{k-1} = (D_k)'$, we obtain (iv) for $n = k$.

Now put $F := \overline{\bigcup_{n\in\mathbb{N}} F_n}$. Since clearly $F \subset F_n \cup \overline{G_n}$ for each n, the condition (iv) easily implies that F is a closed nowhere dense set.

By (ii) F is upper porous at no point of the set $A := \bigcup F_n = \bigcup D_n$ which is dense in F. Consequently Proposition 2.6 implies that F is not σ-lower porous. □

Remark 2.8. (i) If X is separable and μ is a finite continuous (nonatomic) Borel measure on X, it is easy to modify the above construction to obtain $\mu(F) = 0$. Indeed, then D_n is countable and we can construct G_n so that $\mu(\overline{G_n}) < n^{-1}$.

(ii) It is not difficult to modify the construction to obtain a closed F which is upper porous and is not σ-lower porous.

(iii) Using the Baire theorem as in the proof of Proposition 2.6, we easily see that F from the above proof is not even "σ-closure porous" (see Section 3.3 below for definition).

The following proposition is a useful tool if we work with σ-upper porous sets. (It easily follows from results of [135]; for a direct proof see [84, page 249].) No analogous result for σ-lower porous sets holds.

Proposition 2.9. *Let X be a metric space and $0 < c < 1$. Then each σ-upper porous set A is a countable union of c-upper porous sets.*

3. Subsystems of the system of σ-upper porous sets

3.1. σ-directionally porous sets. Let X be a normed linear space. We say that $A \subset X$ is *directionally porous at a point* $x \in X$ if there exist $c > 0$, $u \in X$ with $\|u\| = 1$, and a sequence $\lambda_n \to 0$ of positive real numbers such that $B(x + \lambda_n u, c\lambda_n) \cap A = \emptyset$. The notions of *directionally porous* sets and of *σ-directionally porous* sets are defined in the obvious way.

These notions naturally appear in some questions concerning Gâteaux differentiability of Lipschitz functions. For example, it is easy to see that if $A \subset X$ is directionally porous, then the distance function $f(x) = \operatorname{dist}(x, A)$ (which is Lipschitz) is Gâteaux differentiable at no point of A. Therefore, if X is a separable Banach space, then the well-known infinite-dimensional Rademacher theorem (see [9, page 155]) easily implies that each σ-directionally porous set $A \subset X$ is Aronszajn (equivalent to Gaussian, cf. Section 5.1 below) null. From the same reason A is also Γ-null (it follows from the "Γ-null version of Rademacher theorem": [71, Theorem 2.5]). Moreover, A belongs even to the σ-ideal $\tilde{\mathcal{A}}$ (it follows from [100, Theorem 12]: an infinite-dimensional version of Rademacher theorem which is stronger than the two theorems mentioned above).

If X has finite dimension, then a simple compactness argument shows that directional porosity coincides with porosity. The notion of σ-directionally porous sets found an interesting application in [100] (cf. Section 6.2 below).

3.2. σ-strongly porous sets. The notion of σ-strong porosity is quite natural and on the real line was already applied (cf. [142]). However, I do not know of its application in infinite-dimensional spaces. Thus, we note here only that strong (upper) porosity of a set A in a normed linear space at x means simply that $\overline{p}(A, x) = 1$. On the other hand, in a general metric space (X, ρ) another definition is natural (cf. [84, Remark 1.2(ii)]). Namely it is natural to say that A is *strongly porous* at x if $x \notin \overline{A}$ or there exists a sequence of balls $B(c_n, r_n)$ such that $c_n \to x$, $B(c_n, r_n) \cap A = \emptyset$, and $r_n/\rho(x, c_n) \to 1$.

3.3. σ-closure porous sets. Following [112], we say that a subset A of a metric space X is *closure porous* if its closure $\overline{A}$ is upper porous (equivalently, A is upper porous at all points $x \in X$). There are several results in the literature (cf., e.g., [47, 158]) which say that a set A is σ-closure porous. In some cases (as mentioned in an unpublished and different version of [112]) these results can be strengthened: A is even σ-lower porous. If this is possible in all cases, the notion of σ-closure porous sets is not too interesting.

3.4. Other systems. In $\mathbb{R}$ the notion of σ-symmetrically porous sets found interesting applications (see [36, 149] and Section 8.2). A generalization of symmetric porosity to general metric spaces ("shell porosity") is studied in [130]. For the notion of σ-$\langle g \rangle$-porous sets (which was probably never applied in abstract spaces) see [142].

4. Subsystems of the system of σ-lower porous sets

4.1. σ-c-lower porous sets. It is a probably well-known (although possibly not published) fact that in "metric spaces interesting for applications" (at least in Banach spaces) for each $0 < c < c^* < 1$, there exists a σ-c-lower porous set P which is not σ-c^*-lower porous (in a Banach space X it easily follows from [121, Lemma 3.7(a)] on Cantor sets $K(\theta)$ in $\mathbb{R}$; it is sufficient to put $P := f^{-1}(K(\theta))$ for a suitable θ and $f \in X^*$). Consequently there exists a σ-lower porous set which is σ-c-lower porous for no $0 < c < 1$. Thus results which assert that an interesting set is not only σ-lower porous but even σ-c-lower porous can be of some interest. Other similar systems can be obtained fixing $0 < \alpha < 1$ in (2.3) or (2.4).

4.2. Cone (angle) small sets and ball small sets. The notion of a cone small set (or an angle small set) was used in [98] (cf. [92]), [77, 145].

Definition 4.1. Let X be a Banach space. If $x^* \in X^*$, $x^* \neq 0$, and $0 \leq \alpha < 1$, define (the α-cone)

$$C(x^*,\alpha) = \{x \in X : \alpha\|x\| \cdot \|x^*\| < (x,x^*)\}. \tag{4.1}$$

A set $M \subset X$ is said to be α-cone porous at $x \in X$ if there exists $R > 0$ such that for each $\varepsilon > 0$ there exist $z \in B(x,\varepsilon)$ and $0 \neq x^* \in X^*$ such that

$$M \cap B(x,R) \cap (z + C(x^*,\alpha)) = \emptyset. \tag{4.2}$$

A subset of X is said to be α-cone porous if it is α-cone porous at all its points; σ-α-cone porous sets are defined in the obvious way. A set is said to be *cone small* if it is σ-α-cone porous for each $0 < \alpha < 1$.

If we write in (4.2) $M \cap (z + C(x^*,\alpha)) = \emptyset$, then we obtain (instead of the notion of a cone small set) the notion of an *angle small set* (cf. [92, 98]). If X is separable, then it is easy to see that the notions of cone smallness and angle smallness coincide. It is not true in nonseparable Hilbert spaces [55]. Clearly each cone small set is σ-lower porous.

In [98] also the following notion of a *ball small set* was defined, which is in Hilbert spaces clearly stronger than the notion of a cone small set. This notion was used as a useful tool for construction of counterexamples (implicitly in [76], explicitly in [29]; cf. Section 6.3). On the other hand, it seems that there is no result in the literature, which asserts that an interesting set of singular points is ball small.

Definition 4.2. Let X be a Banach space and let $r > 0$. Say that $A \subset X$ is r-ball porous if for each $x \in A$ and $\varepsilon \in (0,r)$ there exists $y \in X$ such that $\|x - y\| = r$ and $B(y, r - \varepsilon) \cap A = \emptyset$. Say that $A \subset X$ is ball small if it can be written in the form $A = \bigcup_{n=1}^{\infty} A_n$, where each A_n is r_n-ball porous for some $r_n > 0$.

4.3. Sets covered by surfaces of finite codimension and σ-cone-supported sets. Some sets of singular points in a separable Banach space X appear to be small in a very strong sense: they can be covered by countably many Lipschitz hypersurfaces (i.e., surfaces of codimension 1). Sets with this property were used in $\mathbb{R}^2$ (under a different but equivalent definition) by W. H. Young (under the name "ensemble ridée") and by H. Blumberg (under the name "sparse set") (cf. [139, page 294]). They were used in $\mathbb{R}^n$, for example, (implicitly) by Erdös [32] and in infinite-dimensional spaces (possibly for the first time) in [136, 137].

Definition 4.3. Let X be a Banach space and $n \in \mathbb{N}$, $1 \leq n < \dim X$. Say that $A \subset X$ is a *Lipschitz surface (a d.c. surface) of codimension n* if there exist an n-dimensional linear space $F \subset X$, its topological complement E, and a Lipschitz mapping (a d.c. mapping) $\varphi : E \to F$ such that $A = \{x + \varphi(x) : x \in E\}$.

Note that, since F is finite dimensional, φ is a delta-convex (d.c.) mapping [134] if and only if $y^* \circ \varphi$ is a d.c. function (i.e., the difference of two continuous convex functions) for each $y^* \in F^*$ (or equivalently, for each $y^* \in F^*$ from a fixed basis of F^*).

A Lipschitz surface (d.c. surface) of codimension 1 is said to be a *Lipschitz hypersurface (d.c. hypersurface)*, respectively. The σ-ideals of sets which can be covered by countably many Lipschitz surfaces (d.c. surfaces) of codimension n will be denoted by $\mathcal{L}^n(X)$ ($\mathcal{DC}^n(X)$), respectively.

We note that the notion of sets from $\mathcal{DC}^n(X)$ (also for $n > 1$) was applied, for example, in [131, 138, 139] (cf. Sections 6.1 and 6.3 below) and that sets from $\mathcal{L}^1(X)$ ($\mathcal{DC}^1(X)$) are called "sparse" (d.c. sparse) in [139].

Suppose that X is separable and $n < \dim X$. Then it is easy to see that $\mathcal{DC}^n(X) \subset \mathcal{L}^n(X)$ and that this inclusion is proper (see [139, page 295] for $n = 1$). Clearly each set $A \in \mathcal{L}^1(X)$ is σ-lower porous. Moreover, it is clearly also σ-directionally porous; in particular it is both Aronszajn (equivalent to Gauss) null and Γ-null.

Further, the obvious inclusion $\mathcal{L}^n(X) \subset \mathcal{L}^{n-1}(X)$ ($n > 1$) is proper (it follows in the case $\dim X < \infty$ easily from the theory of Hausdorff measures and for $\dim X = \infty$ from [53], cf. Section 5.3 below).

The following notion of σ-cone-supported sets which in nonseparable spaces naturally generalizes the notion of sets from $\mathcal{L}^1(X)$ ("sparse sets") was applied in [52, 77, 145] (cf. Sections 6.1 and 6.3 below).

Definition 4.4. If X is a Banach space, $v \in X$, $\|v\| = 1$, and $0 < c < 1$, then define the cone $A(v,c) := \bigcup_{\lambda>0} \lambda \cdot B(v,c)$. Say that $M \subset X$ is *cone-supported* if for each $x \in M$ there exist $r > 0$ and a cone $A(v,c)$ such that $M \cap (x + A(v,c)) \cap B(x,r) = \emptyset$. The notion of a *$\sigma$-cone-supported* set is defined in the usual way.

If X is separable, it is easy to show that the system $\mathcal{L}^1(X)$ coincides with the system of all σ-cone-supported sets (cf. [137, Lemma 1]). Each σ-cone-supported set is clearly both σ-lower porous and σ-directionally porous.

4.4. σ-porous sets in bimetric spaces. The following definition of ("lower") porosity in a "bimetric space" (X,ρ_1,ρ_2) was defined and used in [162] (see also [1, 2, 103, 106, 108, 163]).

Definition 4.5. If $\rho_1 \leq \rho_2$ are metrics on a set $X \neq \emptyset$, then $P \subset X$ is said to be *porous (with respect to the pair (ρ_1,ρ_2))* if there exist $\alpha > 0$, $r_0 > 0$ such that for each $r \in (0,r_0)$ and each $x \in X$ there exists $y \in X$ for which $\rho_2(x,y) \leq r$ and $B_{\rho_1}(y,\alpha r) \cap P = \emptyset$.

Note that if P is porous with respect to the pair (ρ_1,ρ_2), then (2.3) clearly holds for P both in (X,ρ_1) and in (X,ρ_2).

4.5. HP-small sets. The notion of HP-small sets was defined, studied, and applied in [65] (cf. Section 6.5 below).

Definition 4.6. A subset A of a Banach space X is said to have property $\mathrm{HP}_{(c)}$ $(0 < c \le 1)$ if for every $0 < c' < c$ and $r > 0$ there exist $K > 0$ and a sequence of balls $\{B_i\} = \{B(y_i, c'r)\}$ with $\|y_i\| \le r$, $i \in \mathbb{N}$, such that for every $x \in X$,

$$\operatorname{card}\{i \in \mathbb{N} : (x + B_i) \cap A \neq \emptyset\} \le K. \tag{4.3}$$

If A is a countable union of sets with property HP $_{(c)}$ $(0 < c \le 1)$, then A is *HP-small* (with porosity constant c).

Each HP-small set is clearly σ-lower porous and HP-small sets with porosity constant 1 are even ball small. Moreover, each HP-small set is Haar null (H in HP is for Haar and P is for porous).

5. Further properties of the above systems

5.1. Smallness in the sense of measure. In this subsection we suppose that X is a *separable infinite-dimensional* Banach space. Then there is no nonzero σ-finite translation invariant (or quasi-invariant) measure on X (cf. [9, pages 130, 143]) and therefore there is no natural generalization of the Lebesgue measure on X. However, there are important (translation invariant) notions of "null sets" in X (which generalize the notion of Lebesgue null sets) that have interesting applications. A Borel set $A \subset X$ is called *Gauss null* set if $\mu(A) = 0$ for every nondegenerate Gaussian measure on X. Gauss null sets coincide with *Aronszajn null* sets (which have a more elementary definition) and also with "cube null sets" (cf. [9, page 163]). A bigger important translation invariant system is formed by *Haar null* sets (defined by P. R. Christensen) (cf. [9, page 126]).

Quite recently a new (translation invariant) notion of Γ-null sets (which is noncomparable with the above two notions) was defined and applied in [71] (cf. [70]) in a sophisticated way which "combines category and measure." Roughly speaking, a Baire metric space Γ of Radon measures on X is defined and a Borel set $A \subset X$ is said to be Γ-*null* if $\{\mu \in \Gamma : \mu(A) = 0\}$ is residual in Γ.

We will use these three notions of nullness *also for non-Borel sets*: A is said to be null if it is contained in a Borel null set.

Remember (see Section 3.1), that each σ-directionally porous set (and therefore also each set from $\mathcal{L}^1(X)$) is both Gauss null and Γ-null.

Consider now a closed nowhere dense convex set $\emptyset \neq C \subset X$. Using (a geometrical form of) the Hahn-Banach theorem, it is easy to verify that C is both ball small, (even r-ball porous for each $r > 0$) and cone small (even 0-cone porous); in particular C is lower porous. This implies that each compact set $K \subset X$ is lower porous, ball small, and cone small, since $C := \overline{\operatorname{conv} K}$ is convex and compact (and therefore closed nowhere dense). Consequently for each nonzero Radon measure μ on X, there exists a (compact) set of positive measure which is lower porous, ball small, and cone small. In particular, there exists a (compact) set which is not Gauss (equivalent to Aronszajn) null and is lower porous, ball small, and cone small.

The case of Haar nullness is more complicated. First, note that each compact set in X is Haar null [9, page 128]. If X is not reflexive, then [81] there exists a convex closed nowhere dense (and thus lower porous, ball small, and cone small) set $C \subset X$ which is not Haar null. But such C does not exist if X is reflexive ([80]; cf. [78], and [9, page 130] for the case of a superreflexive X).

By [96] (see [9, page 152]) each separable infinite-dimensional X can be decomposed into two Borel subsets $X = A \cup B$ so that the intersection of A with any line in X has (one-dimensional) measure zero (and therefore A is Gauss null) and B is a countable union of closed upper porous sets. This immediately implies that there exists a closed upper porous set in X which is not Haar null.

If X is superreflexive, then there exists a continuous convex function f (even an equivalent norm) on X such that the set of Fréchet differentiable points of f is Gauss null ([79]; cf. [9, page 157] for $X = \ell_2$). Using the result of [98] (cf. Section 6.1 below), we have that there exists a cone small (and thus σ-lower porous) set in X that has a Gauss null complement, and consequently is not Haar null. The paper [76] implicitly contains the fact that in X there exists a ball small set which has a Gauss null complement (and thus is not Haar null) if $X = \ell_2$ (see also [29]). The same decomposition result for more general X (e.g., for $X = l_p$, $1 < p < \infty$) is implicitly contained in an old version of [79] (Institutsbericht Nr. 534, Universität Linz, 1997) and was independently (for slightly more general spaces) proved in [30].

By [71], in $X = \ell_2$ (and also in $X = l_p$, $1 < p < \infty$) there exists a σ-upper porous set A which is not Γ-null (even A with Γ-null complement [72]) but each σ-upper porous set in $X = c_0$ is Γ-null (see Section 6.2 for consequences of this latter fact). From [98, Theorem 2] and [71, Corollary 3.11], it immediately follows that each ball small set in $X = \ell_2$ is Γ-null.

5.2. Descriptive properties. Each σ-lower porous set is contained in a σ-lower porous F_σ set (see Proposition 2.5 above) and analogous results can be easily proved also for a number of systems mentioned above.

Each σ-upper porous set is contained in a σ-upper porous $G_{\delta\sigma}$ set (it is an easy fact, cf. [44] for subsets of $\mathbb{R}$). (A quite analogous result holds also for σ-directionally porous sets in separable Banach spaces; it easily follows from [99, Lemma 4.3].) On the other hand, there exists [170] a σ-upper porous subset of $\mathbb{R}$ which is contained in no σ-upper porous $F_{\sigma\delta}$ set.

Each Suslin non-σ-upper porous set in a topologically complete metric space contains a closed non-σ-upper porous subset [171]. A simpler (nonconstructive) proof in separable locally compact metric spaces (which works also for $\langle g \rangle$-porosity and symmetric porosity in $\mathbb{R}$) is contained in [172]. A proof of the corresponding result for lower porosity is much simpler [155].

The (constructive) proof of [171] is based on a rather complicated but very general method of construction of non-σ-upper porous sets which was used in [169, 170, 171] to solve a number of difficult problems (cf. Section 7.1 below). In particular, it was proved in [171] that the system of all compact σ-upper porous subsets of a compact space X is a coanalytic non-Borel subset of the "hyperspace" of all compact subsets of X. A simpler proof of this result (which works also for $\langle g \rangle$-porosity, strong porosity, and symmetric porosity in $\mathbb{R}$) is contained in [154], where also a "lower porous" version is proved.

Recall that, in proofs that a (small) set is not σ-lower porous, the Baire theorem can be used (cf. Section 2). The case of σ-upper porous sets is usually much more difficult. The tool which is usually used in this case (and can be considered as a substitute for the Baire theorem) is "Foran's lemma," which works with "Foran systems" (or "non-σ-porosity families," cf. [142]) of closed [142, 154] or G_δ [146] sets. (In fact these papers contain different versions of Foran's lemma with very similar proofs.) We stress that Foran's lemma holds for all "abstract porosities" (called "porosity relations" or "V-porosities" in [142] and "porosity-like point-set relations" in [154]) and can be therefore applied to most systems considered above (cf. proof of [88, Proposition 5.6] for application to the system $\mathscr{L}^1(X)$).

5.3. Other properties. A complete characterization (in an arbitrary metric space) of σ-lower porous sets using a version of the Banach-Mazur game is proved in [166].

Results on σ-upper porous sets in the product $X \times Y$ of metric spaces are proved in [99, 146]. For example, it is shown in [99] that no reasonable classical form of "Fubini-type theorems" can hold for σ-upper porosity (even in the plane). However, a weak relevant result [99, Theorem 3.8] is proved. It implies that each Borel σ-upper porous set $M \subset X \times Y$ has a Borel decomposition $M = A \cup B$ such that all sections $A^y (y \in Y)$ and $B_x (x \in X)$ are σ-upper porous in X and in Y, respectively.

In [99] also nontrivial properties of σ-directionally porous sets (one of which was applied in [100]) were proved.

The following result (which answers a question suggested by D. Preiss) is proved in [53].

Proposition 5.1. *Let X be a separable infinite-dimensional space, $A \in \mathscr{L}^n(X)$, $Y \subset X$ a closed linear space of codimension $k < n$, and $\pi : X \to Y$ a linear projection. Then $\pi(A)$ is a first category set in the space Y.*

Moreover, $\pi(A)$ is also a Gauss null set in Y [153].

Proposition 5.1 easily implies that the inclusions $\mathscr{L}^n(X) \subset \mathscr{L}^{n-1}(X)$ $(n > 1)$ are proper.

6. Applications

6.1. Differentiability of convex functions. If f is a function on (a subset of) a Banach space, we will denote by $NF(f)$ ($NG(f)$) the set of all points of the domain of f at which f is not Fréchet (Gâteaux) differentiable.

Probably the first paper which works with σ-porosity in an infinite-dimensional space was [97], where it was proved that $NF(f)$ is σ-upper porous whenever f is a continuous convex function on a Banach space X which is separable and Asplund (i.e., X^* is separable). This result was improved in [98] ($NF(f)$ is even cone small) and generalized in [145] (if X is an arbitrary Asplund space, then $NF(f) = A \cup B$, where A is cone small and B is cone-supported; in particular $NF(f)$ is σ-lower porous). These results are obtained in [98, 145] as corollaries of corresponding theorems (on noncontinuity points) of an arbitrary monotone operator $T : X \to X^*$. (For an analogue for accretive operators see [133, page 49].)

It seems that no complete characterization of the σ-ideal $\mathcal{I}$ generated by sets of the form $NF(f)$ (where f is continuous and convex on X) is known even for $X = \ell_2$. However, by [98] the inclusions $\mathrm{BS} \subset \mathcal{I} \subset \mathrm{CS}$ hold, where BS and CS are the systems of all ball small and cone small sets in ℓ_2, respectively. Since BS and CS seem to be rather close together, these inclusions give rather good estimates of smallness of sets from $\mathcal{I}$ in ℓ_2.

On the other hand, a simple complete characterization of smallness of sets of the form $NG(f)$ in every separable Banach space is known ([138], cf. [9, Theorem 4.20]): for a set $A \subset X$, there exists a continuous convex function f on X such that $A \subset NG(f)$ if and only if A can be covered by countably many d.c. hypersurfaces (i.e., if $A \in \mathcal{DC}^1(X)$, cf. Section 4.3 above).

A complete characterization ("F_σ sets from $\mathcal{DC}^1(\mathbb{R}^n)$") of the sets $NG(f)$ $(= NF(f))$ for convex functions f on $\mathbb{R}^n$ is given in [90].

In particular, $NG(f)$ is σ-cone-supported if X is separable. The same holds also for some nonseparable X: if X is Asplund or X^* is strictly convex (i.e., rotund) [145], or if X is a *GSG* space [52].

Moreover, if f is a continuous convex function on a separable Banach space X, then the set of points $x \in X$ at which $\dim(\partial f(x)) \geq n$ belongs to $\mathcal{DC}^n(X)$ [138]. (This result gives via Proposition 5.1 an alternative proof of [94, Theorem 1.3].) It is not known whether the same holds if we consider an arbitrary monotone operator $T : X \to \exp X^*$ instead of $T := \partial f$. In this case the result holds with $\mathcal{L}^n(X)$ instead of $\mathcal{DC}^n(X)$ [136]; for more precise results see [131, 132].

The authors of [21] proved that special convex integral functionals are weak Hadamard differentiable except on a σ-lower porous set.

6.2. Differentiability of Lipschitz functions. (For an interesting detailed survey on differentiability of Lipschitz functions see [70].)

The famous result of Preiss [95] says that each real Lipschitz function on an Asplund Banach space X is Fréchet differentiable at all points of a dense uncountable subset of X. The natural question arises, whether also an "almost everywhere" version of Preiss' theorem exists. It was clear for a long time that the notion of a σ-upper porous set is related to this question. Indeed, if $A \subset X$ is an upper porous set, then the distance function d_A is Fréchet differentiable at no point of A. Moreover, if X is separable and $A \subset X$ is σ-upper porous, then there exists a Lipschitz function f on X such that $A \subset NF(f)$ (see [96] or [9, page 159] for a σ-closure porous A and [56] for the general case).

Recently [71] an almost everywhere version of Preiss' theorem was proved for $X = c_0$ (and for its subspaces and some other special spaces) using σ-upper porosity as one of the important auxiliary notions in the proof. In fact, the main result of [71] is the following first theorem on Fréchet differentiability of general Lipschitz mappings between infinite-dimensional Banach spaces (see Section 5.1 above for some information concerning Γ-null sets).

Theorem 6.1. *Let Y be a Banach space having the Radon-Nikodým property. Then each Lipschitz mapping $f : c_0 \to Y$ is Fréchet differentiable at all points except those which belong to a Γ-null set (and therefore at all points of a dense uncountable set).*

One important ingredient of the proof of this deep theorem is the fact that each σ-upper porous subset of $X = c_0$ is Γ-null. Unfortunately, this is not true [71] for $X = \ell_2$. As concerns real functions, the following result is proved in [71]: if X^* is separable and f is a real Lipschitz function on X, then $NF(f) = A \cup B$, where A is σ-upper porous and B is Γ-null.

This result cannot imply any almost everywhere version of Preiss' theorem in ℓ_2, since [72] there exists a decomposition $\ell_2 = A \cup B$, where A is σ-upper porous and B is Γ-null. Moreover, this decomposition shows that no "sense of nullness" weaker than Γ-nullness can be used for an almost everywhere version of Preiss' theorem in ℓ_2 (if such a version exists).

If f is a Lipschitz function on a Banach space X with a separable X^*, then the set of all points $x \in X$ at which f is Fréchet subdifferentiable but is not Fréchet differentiable is σ-upper porous [140].

In the study of Gâteaux differentiability of Lipschitz functions, the notion of σ-directionally porous sets appears to be important. As already stated in Section 3.1, if A is directionally porous, then $A \subset NG(d_A)$. Further, if X is separable and $A \subset X$ is σ-directionally porous, then there exists [100] a Lipschitz function f on X such that $A \subset NG(f)$. Moreover, using properties of σ-directionally porous sets, it was proved in [100] that each Lipschitz mapping from a separable Banach space X to a Banach space with the Radon-Nikodým property is Gâteaux differentiable at all points except those belonging to a set $A \in \tilde{\mathscr{A}}$. This version of infinite-dimensional Rademacher theorem is an improvement of Aronszajn's ([5], cf. [9]) version, since $\tilde{\mathscr{A}}$ is a proper subsystem of the system of all Aronszajn (equivalent to Gauss) null sets [100, page 18]. In the proof of this theorem, the following result was used.

Theorem 6.2. *If f is a Lipschitz mapping from a separable Banach space X to a Banach space Y, then the following implication holds at each point $x \in X$ except a σ-directionally porous set: if the one-sided directional derivative $f'_+(x,u)$ exists for all vectors u from a set $U_x \subset X$ whose linear span is dense in X, then f is Gâteaux differentiable at x.*

If we write above $U_x = X$, then we can write [88] "except a set from $\mathscr{L}^1(X)$ (and even more)" instead of "except a σ-directionally porous set."

If $f : X \to \mathbb{R}$ is Lipschitz and X is superreflexive (or separable), then f has an "intermediate derivative" at all points except a σ-lower porous set (or a σ-directionally porous set); see [151] (or [100]), respectively. An analogous but weaker result (on the existence of a "weak Dini subgradient") was proved in [147] in the case of X which admits a uniformly Gâteaux differentiable norm. Other related results are contained in [10, 100].

Some "σ-porous" results on Fréchet or Gâteaux differentiability of distance functions (which are closely connected with properties of metric projections discussed in Section 6.3) can be found in [139, pages 302–303], [140, page 408], [141], [77, page 106], and [147, page 331].

6.3. Approximation in Banach spaces. Let X be a Banach space and let $\emptyset \neq F \subset X$ be a closed set. Let $P_F : X \to \exp X$ be the metric projection on F (the best approximation mapping). Consider the "ambiguous locus" $A(F) := \{x \in X : \operatorname{card}(P_F(x)) \geq 2\}$, the set $E(F) = \{x \in X : P_F(x) \neq \emptyset\}$ of points which have a nearest point in F, the set $C(F) := \{x \in E(F) \setminus A(F) : P_F$ is upper semicontinuous at $x\}$, and the set $W(F)$ of those points $x \in X$ at which the minimization problem "$\|x - y\| \to \min$, $y \in F$" is well posed (i.e., $P_F(x)$ is a singleton $\{y\}$ and $y_n \to y$ whenever $y_n \in F$ and $\|y_n - x\| \to \|y - x\|$). Note that [43] $W(F) = C(F)$ if X has Fréchet differentiable and uniformly Gâteaux differentiable norm and X^* has Fréchet differentiable norm (in particular, if X is a Hilbert space).

The set $A(F)$ is frequently σ-cone-supported and therefore both σ-lower porous and σ-directionally porous. This result for $X = \mathbb{R}^n$ is implicitly contained in the proof of [32]. If X is a separable Hilbert space, then $A(F)$ always belongs [139] even to the σ-ideal $\mathcal{DC}^1(X)$ (i.e., it can be covered by countably many d.c. hypersurfaces, cf. Section 4.3) and this σ-ideal is the smallest one (this follows from [66]). Each $A(F)$ is σ-cone-supported whenever X is separable and strictly convex [137, 139], or X is strictly convex and has a uniformly Fréchet differentiable norm [77].

If X^* is separable, the norm on X is uniformly Fréchet differentiable, and the dual norm is Fréchet differentiable (in particular if X is a separable Hilbert space), then [141] the set $X \setminus C(F) = X \setminus W(F)$ is cone small (and therefore σ-lower porous). The set $X \setminus W(F)$ is σ-lower porous whenever X is a Hilbert (possibly nonseparable) space [22] or, more generally, X is uniformly convex [25]. The result of [22] was improved and that of [141] "almost generalized" in [77]: if the norm on X is uniformly Fréchet differentiable and the dual norm is Fréchet differentiable, then $X \setminus C(F) = X \setminus W(F)$ is the union of a cone small set and a cone-supported set. The papers [22, 25] contain also corresponding results for farthest points in F. For related results (using σ-lower porous sets) see [24, 69].

If X is a separable Banach space and $A \subset X$ is r-ball porous, then there exist a closed set $F \subset X$ and a set $S \in \mathcal{L}^1(X)$ such that $(A \setminus S) \cap E(F) = \emptyset$. Since a ball small set need not be Haar null in $X = \ell_p$, $p > 1$, we obtain that $X \setminus E(F)$ need not be Haar null in these spaces [29, 30].

"Approximation properties" of typical (in the sense of σ-porosity) closed subsets of a Banach space are considered in [64, 106, 109] (the case of closed convex sets). For a related result in complete hyperbolic spaces see [108].

6.4. Well-posed optimization problems and related (variational) principles. The well-known Deville-Godefroy-Zizler smooth variational principle is improved in [26].

A topological space X and a Banach space $(Y, \|\cdot\|_Y)$ of bounded continuous functions on X is considered. If X and Y satisfy some simple conditions, then for each proper bounded from below lower semicontinuous function $f : X \to \mathbb{R} \cup \{\infty\}$, the set T of all "perturbations" $g \in Y$ for which $f + g$ attains strict minimum (i.e., the minimization problem "$(f + g)(x) \to \min$, $x \in X$" is well posed) is not only residual, but $Y \setminus T$ is even σ-lower porous.

Subsequently related (variational) principles (on σ-lower porosity of ill-posed problems) were proved in [58, 75, 162, 165]. These principles were then applied to more concrete optimization problems; for example, in the calculus of variations [165] or in the optimal control theory [162].

For a related paper concerning only existence (without well posedness) see [163].

6.5. Properties typical in the sense of σ-porosity. These results usually improve older results on typical properties in the sense of category: they assert that a set of functions (mappings, sets) having a concrete property is not only residual, but has even σ-porous complement in a considered space of functions (mappings, sets).

Classical results (of Banach, Mazurkiewicz, and Jarník) on nondifferentiability of typical $f \in C[0,1]$ were improved (using σ-lower porous sets) in [4, 47, 111]. In [65] joint improvements of some of these "porosity results" and results of [57] ("on Haar nullness") were proved. For example, the set of all $f \in C[0,1]$ which have a finite one-sided approximative derivative at a point is HP-small (with porosity constant 1).

Related results for functions of n-variables and functions on Banach spaces are contained in [45] and [73, page 1192], respectively.

Level sets of typical (in the sense of σ-lower porosity) $f \in C[0,1]$ are investigated in [46].

Typical properties (in the sense of σ-porosity) of elements of "hyperspaces" (of closed, compact, or convex sets) are investigated in several papers. For example, a typical (in the sense of σ-closure porosity) bounded closed subset of a Banach space is strongly (upper) porous [156].

A typical (in the sense of σ-closure porosity) closed convex body $C \subset \mathbb{R}^n$ is smooth and strictly convex [157]. The case of strict convexity was improved and generalized in [164]. Namely, a typical (in the sense of σ-lower porosity) closed convex subset of a Hilbert space is strictly convex in a strong sense. Analogous results on strict convexity in some classes of convex functions defined on a convex subset of a pre-Hilbert space are also proved in [164]. For other results on convex sets see [61, 158] and papers cited in Section 8.7.

A typical (in the sense of σ-lower porosity) nonexpansive mapping on a bounded closed convex subset of a Banach space has a fixed point [23]; moreover, it is contractive [105]. For related results see [18, 86].

Typical properties (in the sense of σ-lower porosity or some stronger sense) of sequences from some (Fréchet) sequential metric spaces are studied, for example, in [27, 68, 74, 120, 125]. For related result in a metric space of measurable functions see [173].

6.6. Other applications. A version of Vitali's covering theorem (which "leaves uncovered" an upper porous set) can be found in [122] (with a proof which is correct at least in Banach spaces).

The papers [1, 2, 102, 103, 104, 110] consider metric spaces of "iterative minimization processes" which can be applied to minimization problems for convex (or Lipschitz) functions on Banach spaces. It is shown that, under suitable conditions, all these processes ("descent methods"), except those belonging to a σ-lower porous set, do their job. More specifically [104], for a given Lipschitz convex function f on a Banach space X, a space of descent methods is defined as a subset (equipped with a natural metric) of vector fields $V : X \to X$ for which (the one-sided derivative) $f'_+(x, V(x)) \leq 0$ for each $x \in X$.

For other applications see [3, 19, 89, 107, 118].

6.7. General remarks. The most interesting application of σ-porosity is contained in [71] where a deep important theorem (see Section 6.2 above, Theorem 6.1) was proved.

This theorem is formulated without any porosity notion and the notion of σ-upper porous sets is an important tool in the proof. An interesting application of σ-directionally porous sets is contained in [100] (cf. Section 6.2 above).

A natural question is whether "supergeneric results" (i.e., results which say that a set of singular points is not only of the first category, but belongs to a smaller class of sets) using some types of σ-porosity have frequently interesting *immediate consequences* which are formulated without any porosity notion. The aim of [148] is to show that this is the case if a "supergeneric result" works with σ-cone-supported sets. The reason is that *there are interesting sets of the first category which are not σ-cone-supported*, for example, the set of all increasing real analytic functions in $C[0,1]$. Thus the supergeneric result of [138] (cf. Section 6.1) implies that for each continuous convex function F on $C[0,1]$, there exists an increasing real analytic function $x \in C[0,1]$ such that F is Gâteaux differentiable at x; but this result does not follow from the Mazur generic result on Gâteaux differentiability of convex functions. Similarly the supergeneric result of [145] (cf. Section 6.1) gives that each continuous convex function on $l_2(\Gamma)$ (Γ arbitrary) is Gâteaux differentiable at a point from $l_2(\Gamma) \cap l_1(\Gamma)$. On the other hand, I do not know natural interesting subsets of Banach spaces which are of the first category but are not σ-lower porous.

7. About the author's previous survey

7.1. On questions mentioned in [142]**.** Most of these questions were already answered.

(i) Question 3.2 (Dolženko's question concerning boundary behaviour of functions) was answered in [168].

(ii) The questions from [142, Remark 4.18(b) and (c)] were answered in [170]. In particular, there exists an upper porous set $P \subset \mathbb{R}$ which is contained in no σ-upper porous $F_{\sigma\delta}$ set.

(iii) Question 4.20 which asked whether each Borel non-σ-upper porous set contains a closed non-σ-upper porous subset was settled in [171] (cf. Section 5.2 above).

(iv) Question 4.31 was answered in the negative in [84]: if a Radon measure (e.g., on $\mathbb{R}^n$) is null on all strongly porous sets, then it is null on all upper porous sets.

(v) A positive answer to Question 6.4 is given in [169]. There exists even an absolutely continuous function on $[0,1]$ whose graph is not σ-upper porous.

(vi) Question 6.7 was answered in the negative in [171]: there exists a closed U-set (a set of uniqueness for trigonometrical series) which is not σ-upper porous. For a simpler proof concerning the (weaker) corresponding result on U_0-sets, see [167].

7.2. On mistakes in [142] **and other remarks.** The author made the following mistakes.

(i) Lemma 2.29 and Proposition 2.30 of [142] hold but cannot be proved so easily as claimed (see notes on $f^{-1}(Z)$ before Proposition 2.5 above).

(ii) It is not true (as claimed) that [142, Definitions 2.45 and 2.47] give the same notions of globally porous sets on $\mathbb{R}$ (see [54]). However it is proved in [54] that the corresponding σ-ideals coincide.

(iii) Remark 2.54 of [142] ("superporous implies very porous") is not true in all metric spaces (but it is true, e.g., in normed linear spaces).

(iv) The definition of strong porosity suggested in [142, page 321] is not suitable in all metric spaces (cf. Section 3.2 above).

(v) By a mistake the following interesting Väisälä result of [127] (which is cited in [142] as a preprint) on images of porous sets was not stated in [142] (cf. [128] for an alternative proof).

If $A \subset \mathbb{R}^n$ is upper or lower porous and $f : A \to \mathbb{R}^n$ is an η-quasisymmetric mapping, then $f(A)$ is also upper or lower porous, respectively.

We stress that even for the special case of a bilipschitz mapping, this result is not easy for $n > 1$ (since the domain of f is not $\mathbb{R}^n$).

An essential (not the author's) printing error occured in Konyagin's proof in [142, page 342], see [143].

Kelar's results on "porosity topologies" mentioned in [142, Section 2.G] were published in [63] and the proof of [142, Proposition 4.14] can be found in [144]. It seems that the proof of [142, Theorem 6.2] was not published.

8. Some recent finite-dimensional results

8.1. Relations between some types of σ-porosity in $\mathbb{R}$. There exists a σ-upper porous set $A \subset \mathbb{R}$ which is not σ-symmetrically porous [39, 116]. Even there exists a σ-right upper porous set $A \subset \mathbb{R}$ which is not σ-left upper porous [87]. For a close connection between (two types of) globally porous sets in $\mathbb{R}$ and "uniformly closure porous" sets, see [54].

8.2. Applications and properties of σ-symmetrically porous sets. The notion of a σ-symmetrically porous set was applied, for example, in [34, 36, 37, 38, 149]. Structural properties of the class of σ-symmetrically porous sets were proved in [35, 39]. For properties of σ-symmetrically porous sets, see also [129]. Symmetric porosity and σ-symmetric porosity of symmetric Cantor sets are studied in [40, 41].

8.3. On T-measures. A nonzero Radon measure μ on $\mathbb{R}^n$ is said to be a T-measure (cf. [142]) if it is singular (i.e., orthogonal to the Lebesgue measure) and $\mu(A) = 0$ whenever $A \subset \mathbb{R}^n$ is σ-upper porous. Remember that the existence of a T-measure on $\mathbb{R}$ (proved in [126]) easily implies the important Theorem 2.3 (existence of a closed Lebesgue null non-σ-upper porous set in $\mathbb{R}^n$) which is a relatively deep fact.

Thus the following observation of Martio from a (1992) private letter to the author which shows that the existence of a T-measure on $\mathbb{R}$ and therefore also Theorem 2.3 easily follows from a well-known 1956 result of Beurling and Ahlfors [11] is very interesting. (Remember that Theorem 2.3 was stated in 1967 paper [28] and proved in [135].)

By [11] there exist an increasing homeomorphism $f : \mathbb{R} \to \mathbb{R}$ which is K-quasisymmetric (i.e., $K^{-1} \leq (f(x+t) - f(x))/(f(x) - f(x-t)) \leq K$ for all $x \in \mathbb{R}$ and $t > 0$) and a closed set C of Lebesgue measure zero such that $f(C)$ is of positive Lebesgue measure. It is easy to show that $A \subset \mathbb{R}$ is upper porous if and only if $f(A)$ is upper porous. This immediately implies that C is not σ-upper porous (which proves Theorem 2.3 for $n = 1$). Moreover, it is easy to see that the image measure $f^{-1}(\chi_{f(C)} \cdot \lambda)$ is a T-measure.

Besicovitch's density theorem implies (cf. [101]) that each singular doubling measure on $\mathbb{R}^n$ is a T-measure. For the existence of singular doubling measures on $\mathbb{R}^n$ see, for example, [62]; for other results on doubling measures and/or closely related (cf., e.g., [14]) quasisymmetric mappings which have connection with T-measures, see also [14, 15, 119] and references in these papers. In [15, 101] the existence of a singular doubling measure on $\mathbb{R}$ is easily deduced from a Kakutani theorem on infinite product measures.

We note also that "a weak form of the Fubini theorem for σ-upper porous sets" of [99] (see Section 5.3) immediately implies that the product of two T-measures (on $\mathbb{R}^n$ and $\mathbb{R}^k$) is a T-measure on $\mathbb{R}^{n+k}$. See also Section 7.1(iv) for a result related to T-measures.

8.4. Other results on porosity and measures. Results on Hausdorff dimension of lower porous sets in $\mathbb{R}^n$ are contained in [121] and [82, page 156] (cf. also [67]). The notion of (lower) porosity of measures was studied in [7, 31, 59]; upper porosity of measures is investigated in [83, 84]. Also [16] belongs to this subsection.

8.5. Trigonometrical series and porosity. In 1991, Šleich proved (see [152] for the proof) that each set of type $H^{(s)}$ is σ-upper porous (even σ-bilaterally (upper) porous). Remember that each set of type $H^{(s)}$ is a set of uniqueness (U-set) for trigonometrical series. The existence of a closed non-σ-porous U-set A was proved in [171]. Using Šleich's result, we see that A provides an example of a closed U-set which is not a countable union of sets from $\bigcup_{s\in\mathbb{N}} H^{(s)}$.

8.6. Set theoretical results. Results of this type (which concern cardinal characteristics of ideals related to porosity) are contained in [12, 114, 115, 117].

8.7. Miscellaneous applications. Results on cluster sets and boundary behaviour of functions are contained in [48, 51, 60, 85, 93, 123, 124, 168].

Results on differentiation of functions on $\mathbb{R}$ are contained in [33, 34, 42, 149].

Results on typical (in the sense of category) properties of sets (functions, mappings) concerning σ-porosity (porosity) in finite-dimensional spaces can be found in [49, 50, 158, 159, 160, 161].

The notion of a σ-upper porous set was used in [13] in "one-dimensional dynamics" and in [56] in the study of gradient maps of differentiable functions on $\mathbb{R}^n$.

Further results can be found in [17, 20, 91].

Acknowledgments

The work in this paper was partially supported by the Grant GAČR 201/03/0931 from the Grant Agency of Czech Republic and by the Grant MSM 113200007 from the Czech Ministry of Education. I thank Jana Björn for information on papers concerning doubling measures. I also thank Miroslav Zelený, Jan Kolář, Simeon Reich, and an anonymous referee for valuable comments.

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