A GENERALIZED σ-POROUS SET WITH A SMALL COMPLEMENT

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Received 29 October 2003

We show that in every Banach space, there is a g-porous set, the complement of which is of $\mathcal{H}^1$-measure zero on every C^1 curve.

1. Introduction

The notion of porosity is one of the notions of smallness which had appeared in a natural way in the theory of differentiation of functions. Porous and σ-porous sets play the role of the exceptional sets in many contexts. A survey as well as further references on this topic can be found in [4]. In finite-dimensional Banach spaces, the family of all σ-porous sets is a proper subfamily of the sets of both 1st category and Lebesgue-measure zero.

The question how big a porous or a σ-porous set can be in an infinite-dimensional Banach space is closely related to the existence of a point of Fréchet differentiability of a Lipschitz function. The simplest connection may be the easy observation that for a porous set M, the distance function $f(x) = \operatorname{dist}(x, M)$ is a Lipschitz function which is not Fréchet differentiable at any point of M. Another more important link is represented by a deep result in [1]. From this result it follows, in particular, that *every Lipschitz function on a separable Asplund space is Fréchet differentiable except for the points belonging to the union of a σ-porous set and a Γ-null set*. The concept of a Γ-null set is another and relatively new concept of smallness of a set. We will not give here the definition (it can be found in the just-mentioned paper [1]) since it will not be used here. We only remark that Γ-null sets form a σ-ideal of subsets and in a finite-dimensional space, the Γ-null sets are precisely the Lebesgue null sets.

There are spaces in which σ-porous sets are Γ-null. Among the classical Banach spaces, the space c_0 is such an example, see [1]. In those cases, the above-quoted statement gives a nice quantitative result saying that a Lipschitz function is Fréchet differentiable except for a Γ-null set. Unfortunately, the spaces ℓ_p, $1 < p < \infty$, do not belong to this class. The reason for that is an example published in [3]. It shows, though in an implicit way and with the help of the mean-value theorem from [1], that every space ℓ_p, $1 < p < \infty$, necessarily contains a σ-porous set which is not Γ-null. Nevertheless, one can still profit from the result about the points of differentiability unless the corresponding σ-porous set is so

Abstract and Applied Analysis 2005:5 (2005) 535–541
DOI: 10.1155/AAA.2005.535

big that its complement is Γ-null. A recent example in [2], however, makes clear that such a huge σ-porous set does exist. Namely, *there is a σ-porous set in a separable Hilbert space such that its complement intersects every three dimensional non degenerate C^1 surface in a set of measure zero*. This property is sufficient for a set to have a Γ-null complement. It is interesting to note that if we use two-dimensional surfaces instead of three-dimensional, the statement is false. For another result in [2], it was shown that given a porous set M in a separable Hilbert space, almost all points of a typical two-dimensional surface do not belong to M.

One of the most restrictive notions of smallness is the so-called unrectifiability. A set is called 1-purely unrectifiable if the intersection with any C^1 curve is of $\mathcal{H}^1$-measure zero. The space ℓ_1 contains a σ-porous set whose complement is 1-purely unrectifiable, see [2]. From what was said above, it is clear that such a big σ-porous set cannot occur in every (separable) Banach space, for example not in c_0. However, if we relax a bit the requirement for porosity, then it is possible to construct in every Banach space the "generalized" σ-porous set with 1-purely unrectifiable complement, see Theorem 2.2 below. An analogous result for standard σ-porous set holds true provided that we use only lines instead of all C^1 curves, see [3]: *in every separable Banach space, there is a σ-porous set the complement of which is of measure zero on every line.*

Although the generalized porous set does not seem to have a direct connection to differentiation of Lipschitz functions, we believe that the result presented in the next section may be of some interest. At least it answers the question which naturally appeared in this context: how big can a g-porous set be?

2. Main result

We start with the notion of a g-porous set. This concept is a direct generalization of the standard porosity. It can be found also in the above-mentioned survey paper [4] together with the more detailed treatment of various other notions of porous sets. In what follows, we denote by $B(x,r)$ the ball with centre x and radius r.

Definition 2.1. Let $g:[0,\infty)\to[0,\infty)$ be a continuous increasing function, and let $g(0)=0$. A set $M\subset X$ in a metric space X is called g-porous if for any $x\in M$,

$$\limsup_{s\to 0_+}\frac{g(\gamma(x,s,M))}{s}>0, \tag{2.1}$$

where $\gamma(x,s,M)=\sup\{r\geq 0 \mid B(y,r)\cap M=\varnothing,\ \operatorname{dist}(x,y)\leq s\}$.

A set which is a countable union of g-porous sets is called a σ-g-porous set.

Every g-porous set is a nowhere dense set and, consequently, σ-g-porous set is a set of the 1st category. If $g(t)=t$, then g-porosity becomes the standard porosity. Notice also that if $g'_+(0)$ is finite and nonzero (or, more generally, $0<\underline{D}_+g(0)\leq\overline{D}_+g(0)<\infty$), then a g-porous set is again a porous set only. If we allow $g'_+(0)=+\infty$, then we get a type of sets which are in general bigger than porous sets. The next theorem deals exactly with such types of g-porous sets.

The symbols $\mathcal{H}^1$ and $\mathcal{L}^1$ denote one-dimensional Hausdorff and Lebesgue measures, respectively.

Theorem 2.2. *Let X be a Banach space and let $g : [0,\infty) \to [0,\infty)$ be a continuous, increasing function with $g(0) = 0$ and*

$$\limsup_{s \to 0_+} \frac{g(s)}{s} = \infty. \tag{2.2}$$

Then there is a σ-g-porous set $M \subset X$ such that the complement $X \setminus M$ meets every C^1 curve in a set of $\mathcal{H}^1$-measure zero. In short, $X \setminus M$ is purely 1-unrectifiable.

Remark 2.3. It is useful to realize the following equivalent reformulation of the conclusion of Theorem 2.2. If $\varphi : \mathbb{R} \to X$ is any C^1 curve, then

$$\mathcal{H}^1\big(\varphi(\mathbb{R}) \cap (X \setminus M)\big) = 0 \tag{2.3}$$

if and only if

$$\mathcal{L}^1\{t \in \mathbb{R} \mid \varphi(t) \notin M,\ \varphi'(t) \neq 0\} = 0. \tag{2.4}$$

This is an immediate consequence of the formula for the length of a curve as follows:

$$\begin{aligned} \mathcal{H}^1\big(\varphi(\mathbb{R}) \cap (X \setminus M)\big) &= \int_{\{t \mid \varphi(t) \notin M\}} \|\varphi'(t)\| \, d\mathcal{L}^1(t) \\ &= \int_{\{t \mid \varphi(t) \notin M,\ \varphi'(t) \neq 0\}} \|\varphi'(t)\| \, d\mathcal{L}^1(t). \end{aligned} \tag{2.5}$$

We prove first a simple lemma.

Lemma 2.4. *Let $\varphi : \mathbb{R} \to X$ and let $v \in X$, $v \neq 0$, be such that $\varphi(t) - vt$ is a K-Lipschitz mapping with $0 \leq K < \|v\|$.*

(i) *For any $s, t \in \mathbb{R}$,*

$$\big(\|v\| - K\big)|s - t| \leq \big\|\varphi(s) - \varphi(t)\big\| \leq \big(\|v\| + K\big)|s - t|. \tag{2.6}$$

(ii) *If $P \subset X$ is a nonempty set, then*

$$\mathcal{H}^1\big(\varphi(\mathbb{R}) \cap P\big) \leq \frac{\|v\| + K}{\|v\| - K} \operatorname{diam} P. \tag{2.7}$$

Remark 2.5. The assumption that $\varphi(t) - vt$ is a K-Lipschitz mapping is a weakening of a more geometrically apparent condition which one can use in the case of C^1 mapping, namely, $\|\varphi'(t) - v\| \leq K$. Indeed, let $s, t \in \mathbb{R}$, $s \leq t$, be arbitrary. Then the mean-value theorem gives

$$\begin{aligned} \big\|\varphi(s) - vs - (\varphi(t) - vt)\big\| &\leq \sup\big\{\|\varphi'(\theta) - v\| \mid \theta \in (s,t)\big\}|s - t| \\ &\leq K|s - t|. \end{aligned} \tag{2.8}$$

Proof. (i) Since $\varphi(t) - vt$ is K-Lipschitz, we have

$$\begin{aligned} \|\varphi(s) - \varphi(t)\| &\le \|\varphi(s) - vs - \varphi(t) - vt\| + \|vs - vt\| \\ &\le K|s-t| + \|v\||s-t| = (\|v\| + K)|s-t|. \end{aligned} \tag{2.9}$$

Similarly, we estimate the expression $\|\varphi(s) - \varphi(t)\|$ from below, which gives (i).

(ii) If $\operatorname{diam} P = \infty$, the statement is clear. Assume $\operatorname{diam} P < \infty$. We denote

$$\begin{aligned} t_0 &= \inf\{t \in \mathbb{R} \mid \varphi(t) \in P\}, \\ t_1 &= \sup\{t \in \mathbb{R} \mid \varphi(t) \in P\}. \end{aligned} \tag{2.10}$$

From already proved part (i), we infer that both t_0 and t_1 are finite and that

$$(\|v\| - K)\,|t_1 - t_0| \le \|\varphi(t_1) - \varphi(t_0)\| \le \operatorname{diam} P. \tag{2.11}$$

Hence, using also the fact that $\mathcal{H}^1(\varphi([t_0,t_1])) \le (\|v\| + K)\,|t_1 - t_0|$, we obtain

$$\begin{aligned} \mathcal{H}^1(\varphi(\mathbb{R}) \cap P) &= \mathcal{H}^1(\varphi([t_0,t_1]) \cap P) \\ &\le (\|v\| + K)\,|t_1 - t_0| \\ &\le \frac{\|v\| + K}{\|v\| - K} \operatorname{diam} P. \end{aligned} \tag{2.12}$$

□

Proof of Theorem 2.2. Consider the family $\mathcal{D}$ of all 1-discrete sets in X. (We recall that a set D is called λ-discrete if $\|d_1 - d_2\| \ge \lambda$ whenever d_1, d_2 are two distinct elements in D.) By Zorn's lemma, $\mathcal{D}$ contains a maximal element D. Further, the assumption (2.2) implies that there is a sequence $(s_n) \subset \mathbb{R}$, $s_n \searrow 0$, such that

$$r_n \equiv \frac{s_n}{g(s_n)} \searrow 0, \qquad \sum_{n=1}^{\infty} r_n < \infty. \tag{2.13}$$

Denote

$$\mathcal{B}_n = \bigcup \{B(d,r_n) \mid d \in D\}, \qquad M_n = X \setminus \bigcup_{k \ge n} g(s_k)\mathcal{B}_k. \tag{2.14}$$

We show that M_n is g-porous. Let $x \in M_n$, and let $k \ge n$. By the maximality of D, there is $d \in D$ such that

$$\left\| \frac{x}{g(s_k)} - d \right\| < 1, \quad \text{i.e.,} \quad \|x - g(s_k)d\| < g(s_k). \tag{2.15}$$

The ball $B(g(s_k)d, s_k) = g(s_k)B(d,r_k) \subset g(s_k)\mathcal{B}_k$ is obviously contained in the complement of M_n. Hence

$$\gamma(x, g(s_k), M_n) \ge s_k. \tag{2.16}$$

The ratio of the g-porosity can be now estimated from below:

$$\begin{aligned}\limsup_{s\to 0_+}\frac{g(\gamma(x,s,M_n))}{s} &\geq \limsup_{k\to\infty}\frac{g(\gamma(x,g(s_k),M_n))}{g(s_k)}\\ &\geq \limsup_{k\to\infty}\frac{g(s_k)}{g(s_k)}=1,\end{aligned}\tag{2.17}$$

and the g-porosity of M_n is proved.

We put $M=\bigcup_n M_n$. Clearly, M is a σ-g-porous set.

Let $\varphi:\mathbb{R}\to X$ be any C^1 curve. In view of Remark 2.3, we finish the proof by showing that the set

$$\{t\in\mathbb{R}\mid \varphi(t)\notin M,\ \varphi'(t)\neq 0\}\tag{2.18}$$

has $\mathcal{L}^1$-measure zero. To this end, let t_0 be an arbitrary point from this set and let $v=\varphi'(t_0)$. Then v is a nonzero vector. By continuity of φ', there is a compact interval I containing t_0 in the interior and a number $0\leq K<\|v\|$ verifying

$$\|\varphi'(t)-v\|<K,\quad t\in I.\tag{2.19}$$

Since

$$\begin{aligned}\varphi(I)\cap(X\setminus M) &= \varphi(I)\cap\bigcap_{n=1}^{\infty}(X\setminus M_n)\\ &= \varphi(I)\cap\bigcap_{n=1}^{\infty}\Big(\bigcup_{k\geq n}g(s_k)\mathcal{B}_k\Big)\\ &\subset \varphi(I)\cap\Big(\bigcup_{k\geq n}g(s_k)\mathcal{B}_k\Big)\\ &= \bigcup_{k\geq n}(\varphi(I)\cap g(s_k)\mathcal{B}_k)\end{aligned}\tag{2.20}$$

for any $n\in\mathbb{N}$, it follows that

$$\mathcal{H}^1(\varphi(I)\cap(X\setminus M))\leq\sum_{k\geq n}\mathcal{H}^1(\varphi(I)\cap g(s_k)\mathcal{B}_k).\tag{2.21}$$

By Remark 2.5, the mapping $\varphi(t)-vt$ is K-Lipschitz on I. Thus we can apply Lemma 2.4(ii) to each ball $B(g(s_k)d,s_k)$ which meets the arc $\varphi(I)$ and we get

$$\mathcal{H}^1(\varphi(I)\cap B(g(s_k)d,s_k))\leq\frac{\|v\|+K}{\|v\|-K}2s_k.\tag{2.22}$$

Now let $n\in\mathbb{N}$ be large enough to guarantee that $1\geq g(s_k)\geq 3s_k$ for all $k\geq n$. (This is possible due to (2.13).) Let $k\geq n$ and let N_k denote the number of balls $B(g(s_k)d,s_k)$, $d\in D$, intersecting the arc $\varphi(I)$. Since D is 1-discrete, the distance between any two distinct points from $g(s_k)D$ is at least $g(s_k)$. Hence the part of the arc $\varphi(I)$ between two consecutive balls has measure at least $g(s_k)-2s_k$. So we infer that

$$(N_k-1)\big(g(s_k)-2s_k\big) \leq \mathcal{H}^1\big(\varphi(I)\setminus g(s_k)\mathcal{B}_k\big). \tag{2.23}$$

The mapping φ itself is $(\|v\|+K)$-Lipschitz on I by Lemma 2.4(i). So

$$\mathcal{H}^1\big(\varphi(I)\setminus g(s_k)\mathcal{B}_k\big) \leq \mathcal{H}^1\big(\varphi(I)\big) \leq (\|v\|+K)\mathcal{L}^1(I). \tag{2.24}$$

Combining (2.23) and (2.24), we obtain an upper estimate for the number N_k of balls meeting $\varphi(I)$:

$$N_k \leq 1+\frac{(\|v\|+K)\mathcal{L}^1(I)}{g(s_k)-2s_k}. \tag{2.25}$$

With the help of (2.22), we get

$$\begin{aligned}
\mathcal{H}^1\big(\varphi(I)\cap g(s_k)\mathcal{B}_k\big) &= \sum_{d\in D}\mathcal{H}^1\big(\varphi(I)\cap B\big(g(s_k)d,s_k\big)\big)\\
&\leq N_k\frac{\|v\|+K}{\|v\|-K}2s_k\\
&\leq \left[1+\frac{(\|v\|+K)\mathcal{L}^1(I)}{g(s_k)-2s_k}\right]\frac{\|v\|+K}{\|v\|-K}2s_k\\
&\leq \left[1+\frac{(\|v\|+K)\mathcal{L}^1(I)}{2/3}\right]\frac{\|v\|+K}{\|v\|-K}\,\frac{2s_k}{g(s_k)} \leq Cr_k,
\end{aligned} \tag{2.26}$$

where the constant C depends only on $\mathcal{L}^1(I)$ and K. Consequently, in view of (2.21),

$$\mathcal{H}^1\big(\varphi(I)\cap(X\setminus M)\big) \leq C\sum_{k\geq n} r_k. \tag{2.27}$$

This is true for all n sufficiently big. So we conclude that the interval I possesses the property

$$\mathcal{H}^1\big(\varphi(I)\cap(X\setminus M)\big)=0, \quad \text{i.e.,} \quad \mathcal{L}^1\{t\in I \mid \varphi(t)\notin M,\ \varphi'(t)\neq 0\}=0. \tag{2.28}$$

By separability of $\mathbb{R}$, the set $\{t\in\mathbb{R}\mid \varphi(t)\notin M,\ \varphi'(t)\neq 0\}$ can be covered by countably many of such intervals I. This fact, together with (2.28), finishes the proof. □

Acknowledgment

The author was supported by Grants GAČR 201/04/0090 and MSM 210000010.

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