A STUDY OF THE INVERSE OF A FREE-SURFACE PROBLEM

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We prove the existence of an obstacle lying on the bottom of an infinite channel inducing a surface on the upper bound of the fluid domain. This problem is the inverse of the free-surface problem flow which has been studied by several authors. We use the implicit function theorem to establish the existence of the solution of the problem.

1. Introduction

This paper considers the problem of the existence of an obstacle lying on the bottom of a channel where the upper bound of the flow is known (see Figure 2.1). The flow is bidimensional, stationary, and irrotational. The fluid is inviscid and incompressible. The gravity is considered while the effects of the superficial tension are neglected. Several authors have studied the direct problem (the free-surface problem). It consists of the determination of the free-surface flow for a given obstacle lying on the bottom of the channel. Our aim is to study the inverse of this problem. In [2], Felici has studied the inverse problem in magneto hydrodynamic. The inverse problem is nonlinear just as the direct problem; the nonlinearity is due to the Bernoulli condition at the upper bound and the fact that the bottom is unknown.

The plan of this paper is as follows. In Section 2, we formulate the governing equations of the problem in the dimensionless form. In Section 3, we introduce the stream function in these equations. In Section 4, the problem is formulated as an equation for an operator on which we apply the implicit function theorem. For this, theorems and propositions are given. We achieve this work by a conclusion in Section 5.

2. The governing equations

We consider a steady two-dimensional flow of an ideal and incompressible fluid in a channel with a given upper bound created by an obstacle which is the principal unknown of our problem. We denote by Ω_b^γ the domain occupied by the fluid, where b is the equation of the obstacle and γ is the perturbation of the upper bound. We set

$$\Omega_b^\gamma = \{(x,y) \in \mathbb{R}^2 \mid -\infty < x < +\infty,\ b(x) < y < y_0 + \gamma(x)\}, \tag{2.1}$$

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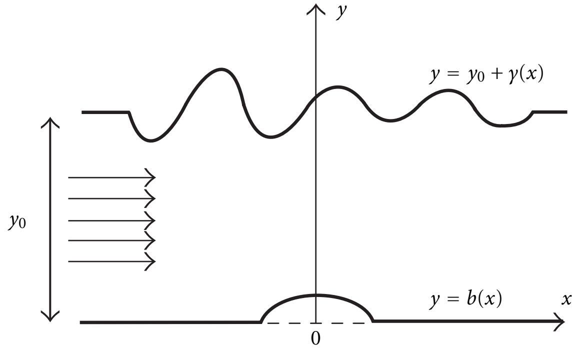

Figure 2.1

where (x, y) is a coordinate system in which x and y are, respectively, the horizontal and positive vertical directions, y_0 is the height of the unperturbed fluid. The function γ is a $C^2(\mathbb{R})$ function verifying $\lim_{|x|\to\infty} \gamma(x) = 0$. We look for a function b in the same space as γ with the conditions $0 \le b(x) < y_0 + \gamma(x)$ and $\lim_{|x|\to\infty} b(x) = 0$.

The problem is formulated as follows. Given a function $\gamma : \mathbb{R} \to \mathbb{R}$, find a function $b : \mathbb{R} \to \mathbb{R}$ and a vector field $\vec{u}$ (velocity of the fluid) such that the following hold.

Governing equations in Ω_b^γ.

$$\operatorname{div} \vec{u} = 0 \quad \text{in } \Omega_b^\gamma, \tag{2.2}$$

$$\operatorname{curl} \vec{u} = 0 \quad \text{in } \Omega_b^\gamma. \tag{2.3}$$

Equation (2.2) expresses the incompressibility of the fluid, (2.3) is given by the irrotationality of the flow.

Boundary conditions.

$$\vec{u} \cdot \vec{\nu} = 0 \quad \text{for } y = b(x), \tag{2.4}$$

$$\vec{u} \cdot \vec{\nu} = 0 \quad \text{for } y = y_0 + \gamma(x), \tag{2.5}$$

where $\vec{\nu}$ is the exterior normal to the boundary of Ω_b^γ. Equation (2.4) describes the impermeability of the flow at the bottom and (2.5) is a kinematic condition.

Conditions at infinity. We suppose that the flow is asymptotically uniform and horizontally far upstream and downstream of the obstacle. We then write

$$\lim_{|x|\to\infty} \vec{u}(x, y) = (u_0, 0). \tag{2.6}$$

Condition across the upper bound. The dynamic condition of continuity of the pressure across the upper bound is given by

$$\frac{\rho}{2} |\vec{u}|^2 + \rho g y = c, \tag{2.7}$$

where ρ is the density of the fluid, g is the downward acceleration due to the gravity, and c is a constant.

This equation is called the Bernoulli equation.

Dimensionless equations. Dimensionless variables are defined by referring all lengths to the quantity y_0, and all velocities to u_0. We put

$$\begin{aligned} \vec{u} &= u_0\vec{u}^*, \\ x &= y_0 x^*, \\ y &= y_0 y^*. \end{aligned} \tag{2.8}$$

The system (2.2)–(2.5) becomes

$$\operatorname{div} u^* = 0 \quad \text{in } \Omega_{b^*}^{\gamma^*}, \tag{2.9}$$

$$\operatorname{curl} u^* = 0 \quad \text{in } \Omega_{b^*}^{\gamma^*}, \tag{2.10}$$

$$u^* \cdot \vec{\nu} = 0 \quad \text{for } y^* = b^*(x^*), \tag{2.11}$$

$$u^* \cdot \vec{\nu} = 0 \quad \text{for } y^* = 1 + \gamma^*(x^*), \tag{2.12}$$

where

$$\begin{gathered} \Omega_{b^*}^{\gamma^*} = \{(x^*, y^*) \in \mathbb{R}^2 \mid -\infty < x^* < +\infty,\ b^*(x^*) < y^* < 1 + \gamma^*(x^*)\}, \\ b^*(x^*) = \frac{1}{y_0} b(y_0 x^*), \qquad \gamma^*(x^*) = \frac{1}{y_0}\gamma(y_0 x^*). \end{gathered} \tag{2.13}$$

The conditions at infinity become

$$\lim_{|x| \to \infty} u^*(x^*, y^*) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2.14}$$

The Bernoulli equation takes the form

$$\frac{F^2}{2} |u^*|^2 + 1 + \gamma^*(x^*) = c, \tag{2.15}$$

where $F = u_0/\sqrt{g y_0}$ is called the Froude number of the flow.

3. Formulation of the problem in a stream function

In the following, we write all the variables without the symbol $*$. The irrotationality and the incompressibility of the fluid lead us to define a harmonic stream function Ψ such that

$$\vec{u} = \begin{pmatrix} \dfrac{\partial \Psi}{\partial y} \\ -\dfrac{\partial \Psi}{\partial x} \end{pmatrix}. \tag{3.1}$$

Equation (2.11) will be written as

$$\begin{pmatrix} \dfrac{\partial \Psi}{\partial y} \\ -\dfrac{\partial \Psi}{\partial x} \end{pmatrix} \cdot \begin{pmatrix} b'(x) \\ -1 \end{pmatrix} = 0 \tag{3.2}$$

and becomes

$$b'(x)\frac{\partial \Psi}{\partial y} + \frac{\partial \Psi}{\partial x} = 0 \quad \text{in } y = b(x). \tag{3.3}$$

This is equivalent to

$$\frac{\partial \Psi}{\partial \tau} = 0 \quad \text{in } y = b(x). \tag{3.4}$$

In the same way, (2.12) gives

$$\frac{\partial \Psi}{\partial \tau} = 0 \quad \text{in } y = 1 + \gamma(x). \tag{3.5}$$

We deduce that Ψ is constant in $y = b(x)$ and $y = 1 + \gamma(x)$.

Thanks to the condition at infinity, we can evaluate the constant which appears in (2.7), and the values of Ψ at the bound of Ω_b^γ. In fact, at infinity we have

$$\lim_{|x| \to \infty} \Psi(x, y) = y + k. \tag{3.6}$$

Since the function Ψ is a stream function, we can choose $k = 0$.

Hence

$$\lim_{|x| \to \infty} \Psi(x, y) = y. \tag{3.7}$$

Replacing these limits in (2.15), we obtain

$$\frac{F^2}{2} + 1 = c. \tag{3.8}$$

Moreover, we deduce from (3.7) that

$$\begin{aligned} \Psi = 0 \quad &\text{in } y = b(x), \\ \Psi = 1 \quad &\text{in } y = 1 + \gamma(x). \end{aligned} \tag{3.9}$$

Then the stream function Ψ verifies

$$\Delta\Psi = 0 \quad \text{in } \Omega_b^\gamma, \tag{3.10}$$

$$\Psi = 0 \quad \text{in } y = b(x), \tag{3.11}$$

$$\Psi = 1 \quad \text{in } y = 1+\gamma(x), \tag{3.12}$$

$$\lim_{|x|\to\infty} \Psi(x,y) = y, \tag{3.13}$$

$$\frac{F^2}{2}|\nabla\Psi|^2(x,1+\gamma(x)) + \gamma(x) = \frac{F^2}{2} \quad \text{in } y = 1+\gamma(x). \tag{3.14}$$

Taking into account the condition (3.13), we can write

$$\Psi = y + \psi, \tag{3.15}$$

where ψ is the perturbation of the stream function.

Equations (3.10)–(3.14) will be written as

$$\begin{gathered} \Delta\psi = 0 \quad \text{in } \Omega_b^\gamma, \\ \psi(x,b(x)) = -b(x), \\ \psi(x,1+\gamma(x)) = -\gamma(x), \\ \lim_{|x|\to\infty} \psi(x,y) = 0, \\ \frac{F^2}{2}\left(|\nabla\psi|^2 + 2\frac{\partial\psi}{\partial y} + 1\right) + \gamma(x) = \frac{F^2}{2} \quad \text{in } y = 1+\gamma(x). \end{gathered} \tag{3.16}$$

4. Determination of the obstacle

We put

$$T(b,\gamma) = \frac{F^2}{2}\left[|\nabla\psi|^2(x,1+\gamma(x)) + 2\frac{\partial\psi}{\partial y}(x,1+\gamma(x))\right] + \gamma(x). \tag{4.1}$$

The problem can be formulated as follows. Given a function γ in a neighbourhood of zero in a space which will be defined later, find a function b in the same space and also in a neighbourhood of zero, such that

$$T(b,\gamma) = 0, \tag{4.2}$$

with ψ verifying (3.16).

For $b = \gamma = 0$, $\psi = 0$ verifies (3.16). So $T(0,0) = 0$.

To solve $T(b,\gamma) = 0$, we use the implicit function theorem in the neighbourhood of $(b,\gamma) = (0,0)$.

Consider the change of variables

$$\begin{gathered} \tilde{x} = x, \\ \tilde{y} = \frac{y - b(x)}{1+\gamma(x) - b(x)}. \end{gathered} \tag{4.3}$$

We transform the domain Ω_b^γ in the following infinite strip Q:

$$Q = \{(\tilde{x},\tilde{y}) \in \mathbb{R}^2 \mid -\infty < \tilde{x} < +\infty,\ 0 < \tilde{y} < 1\}. \tag{4.4}$$

We put $\psi(x,y) = \tilde{\psi}(\tilde{x},\tilde{y})$ then $\tilde{\psi}$ verifies

$$\begin{aligned} \Delta\tilde{\psi} + \mathcal{P}_b^\gamma\tilde{\psi} = 0 \quad & \text{in } Q, \\ \tilde{\psi}(\tilde{x},0) = -b(\tilde{x}), \quad & \tilde{x} \in \mathbb{R}, \\ \tilde{\psi}(\tilde{x},1) = -\gamma(\tilde{x}), \quad & \tilde{x} \in \mathbb{R}. \end{aligned} \tag{4.5}$$

$\mathcal{P}_b^\gamma$ is an operator defined by

$$\mathcal{P}_b^\gamma = a_1 \frac{\partial^2}{\partial\tilde{x}\partial\tilde{y}} + a_2 \frac{\partial^2}{\partial\tilde{y}^2} + a_3 \frac{\partial}{\partial\tilde{y}}, \tag{4.6}$$

where

$$\begin{aligned} a_1 &= 2\frac{\tilde{y}(b' - \gamma') - b'}{1+\gamma-b}, \\ a_2 &= \left(\frac{a_1}{2}\right)^2 - 1 + \frac{1}{(1+\gamma-b)^2}, \\ a_3 &= \frac{-1}{1+\gamma-b}\left[b'' + \tilde{y}(\gamma'' - b'')\right] + \frac{2}{(1+\gamma-b)^2}(\gamma' - b')\left[b' + \tilde{y}(\gamma' - b')\right]. \end{aligned} \tag{4.7}$$

The gradient operator becomes

$$\tilde{\nabla}_{b,\gamma} = \begin{pmatrix} \dfrac{\partial}{\partial\tilde{x}} + \dfrac{-b' - \tilde{y}(\gamma' - b')}{(1+\gamma-b)^2}\dfrac{\partial}{\partial\tilde{y}} \\ \dfrac{1}{1+\gamma-b}\dfrac{\partial}{\partial\tilde{y}} \end{pmatrix}. \tag{4.8}$$

Equation (4.2) will be written as

$$\gamma(\tilde{x}) + \frac{F^2}{2}\left[\left|\tilde{\nabla}_{b,\gamma}\tilde{\psi}\right|^2(\tilde{x},1) + \frac{2}{1+\gamma-b}\frac{\partial\tilde{\psi}}{\partial\tilde{y}}(\tilde{x},1)\right] = 0. \tag{4.9}$$

We will consider γ and b in the space

$$B_c^{2,\lambda}(\mathbb{R}) = \left\{ v \in C^{2,\lambda}(\mathbb{R}) \mid \sum_{0\le k\le 2} \sup_{x\in\mathbb{R}} e^{c|x|}\left|D_x^k v(x)\right| < \infty \right\} \tag{4.10}$$

and $\tilde{\psi}$ in the space

$$B_c^{2,\lambda}(\overline{Q}) = \left\{ v \in C^{2,\lambda}(\overline{Q}) \mid \sup_{k+l\le 2} \sup_{(\tilde{x},\tilde{y})\in\overline{Q}} e^{c|\tilde{x}|}\left|D_{\tilde{x}}^k D_{\tilde{y}}^l v\right| < \infty \right\}, \tag{4.11}$$

where $0 < \lambda < 1$ and $c > 0$. The choice of these spaces will appear evident later [1].

Remark 4.1. (i) The space $B_c^{m,\lambda}(\overline{Q})$ defined by

$$B_c^{m,\lambda}(\overline{Q}) = \left\{ v \in C^{m,\lambda}(\overline{Q}) \mid \sup_{k+l\le m} \sup_{(\widetilde{x},\widetilde{y})\in\overline{Q}} e^{c|\widetilde{x}|} \left| D_{\widetilde{x}}^k D_{\widetilde{y}}^l v \right| < \infty \right\} \tag{4.12}$$

provided with the norm

$$\begin{aligned} \|v\|_{m,c,\lambda} = & \sum_{k+l\le m} \sup_{(\widetilde{x},\widetilde{y})\in\overline{Q}} e^{c|\widetilde{x}|} \left| D_{\widetilde{x}}^k D_{\widetilde{y}}^l v \right| \\ & + \sup_{k+l=m} \sup_{(\widetilde{x},\widetilde{y})\neq(\widetilde{x}',\widetilde{y}')} \frac{\left| D_{\widetilde{x}}^k D_{\widetilde{y}}^l v(\widetilde{x},\widetilde{y}) - D_{\widetilde{x}}^k D_{\widetilde{y}}^l v\left(\widetilde{x}',\widetilde{y}'\right) \right|}{\left[\left(\widetilde{x}-\widetilde{x}'\right)^2 + \left(\widetilde{y}-\widetilde{y}'\right)^2 \right]^{\lambda/2}} \end{aligned} \tag{4.13}$$

is a Banach algebra.

(ii) The space $B_c^{m,\lambda}(\mathbb{R})$ defined by

$$B_c^{m,\lambda}(\mathbb{R}) = \left\{ v \in C^{m,\lambda}(\mathbb{R}) \mid \sum_{0\le k\le m} \sup_{x\in\mathbb{R}} e^{c|x|} \left| D_x^k v(x) \right| < \infty \right\} \tag{4.14}$$

provided with the norm

$$\|v\|_{m,c,\lambda} = \sum_{0\le k\le m} \sup_{x\in\mathbb{R}} e^{c|x|} \left| D_x^k v \right| + \sup_{\substack{(x,x')\in\mathbb{R}^2 \\ x\neq x'}} \frac{\left| D_x^m v(x) - D_x^m v(x') \right|}{\left| x - x' \right|^{\lambda}} \tag{4.15}$$

is also a Banach algebra.

Now we are able to state the main result of this section.

Theorem 4.2. *There exists $\widetilde{c} > 0$ such that for all λ, $0 < \lambda < 1$, and all c, $0 < c < \widetilde{c}$, there exists a neighbourhood $\mathcal{V}$ of zero in $B_c^{2,\lambda}(\mathbb{R})$ such that problem (3.16) has a unique solution ψ, where $\widetilde{\psi}$ belongs to $B_c^{2,\lambda}(\overline{Q})$ and there exists a mapping $g : \mathcal{V} \to B_c^{2,\lambda}(\mathbb{R})$ of class $\mathcal{C}^1$ such that $b = g(\gamma)$.*

This theorem is equivalent to the next one.

Theorem 4.3. *There exist $\widetilde{c} > 0$, and an open ball $\mathcal{B}$ of radius r_0 centered at the origin of $B_c^{2,\lambda}(\mathbb{R}) \times B_c^{2,\lambda}(\mathbb{R})$, where $c \in]0,\widetilde{c}[$, and $0 < \lambda < 1$, there exists a neighbourhood $\mathcal{V}_\gamma$ of zero in $B_c^{2,\lambda}(\mathbb{R})$, there exists a mapping*

$$g : \mathcal{V}_\gamma \longrightarrow B_c^{2,\lambda}(\mathbb{R}) \tag{4.16}$$

of class $\mathcal{C}^1$, such that $\{\forall (b,\gamma) \in \mathcal{B}, T(b,\gamma) = 0\}$ is equivalent to $\{\gamma \in \mathcal{V}_\gamma, b = g(\gamma)\}$.

Proof. In the next subsections, we will verify the hypothesis of the implicit function theorem. □

4.1. Differentiability of the operator T with respect to (b,γ). To study the differentiability of T with respect to b and γ, we use the following results.

THEOREM 4.4. *There exist $\tilde{c} > 0$ and $\lambda \in]0,1[$ such that for all $c \in]0,\tilde{c}[$, there exists an open ball $\mathcal{B}$ of radius $r_0 > 0$, centered at the origin in $B_c^{2,\lambda}(\mathbb{R}) \times B_c^{2,\lambda}(\mathbb{R})$ such that whenever $(b,\gamma) \in \mathcal{B}$, the following statement holds.*

The problem

$$\begin{aligned} \Delta\psi &= 0 \quad \text{in } \Omega_b^\gamma, \\ \psi(x,1+\gamma(x)) &= -\gamma(x), \quad x \in \mathbb{R}, \\ \psi(x,b(x)) &= -b(x), \quad x \in \mathbb{R}, \end{aligned} \tag{4.17}$$

has a unique solution ψ such that

(a) *$\tilde{\psi}$, the transform of ψ by (4.3), is in $B_c^{2,\lambda}(\overline{Q})$;*

(b) *the mapping $S : (b,\gamma) \mapsto \tilde{\psi}$ is continuously differentiable from $\mathcal{B}$ into $B_c^{2,\lambda}(\overline{Q})$.*

To prove this theorem, we use the following proposition which has been proved in [1].

PROPOSITION 4.5. *Let the boundary value problem*

$$\begin{aligned} \Delta v &= b_1 \quad \text{in } Q, \\ v(\tilde{x},1) &= b_2(\tilde{x}), \quad \tilde{x} \in \mathbb{R}, \\ v(\tilde{x},0) &= b_3(\tilde{x}), \quad \tilde{x} \in \mathbb{R}, \end{aligned} \tag{4.18}$$

where $(b_1,b_2,b_3) \in B_c^{0,\lambda}(\overline{Q}) \times B_c^{2,\lambda}(\mathbb{R}) \times B_c^{2,\lambda}(\mathbb{R})$, then there exists $\tilde{c} > 0$ such that whenever $0 < c < \tilde{c}$, problem (4.18) has a unique solution $v \in B_c^{2,\lambda}(\overline{Q})$. Furthermore, the solution map $(b_1,b_2,b_3) \mapsto v$ is a topological isomorphism between the corresponding spaces.

Proof of Theorem 4.4. (a) We want to prove that

$$\begin{aligned} \mathcal{A}_b^\gamma : B_c^{2,\lambda}(\overline{Q}) &\longrightarrow B_c^{0,\lambda}(\overline{Q}) \times B_c^{2,\lambda}(\mathbb{R}) \times B_c^{2,\lambda}(\mathbb{R}) = \mathcal{Y}, \\ v &\longmapsto (\Delta v + \mathcal{P}_b^\gamma v, v(\cdot,0), v(\cdot,1)) \end{aligned} \tag{4.19}$$

is an isomorphism. For this we prove by Proposition 4.5 that the operator

$$\mathcal{A} = \mathcal{A}_0^0 : v \longmapsto (\Delta v, v(\cdot,0), v(\cdot,1)) \tag{4.20}$$

is an isomorphism and that

$$\|\mathcal{A} - \mathcal{A}_b^\gamma\|_{\mathcal{L}(B_c^{2,\lambda}(\overline{Q}),\mathcal{Y})} \leq k, \tag{4.21}$$

where $k > 0$. Then the operator $\mathcal{A}_b^\gamma$ is also an isomorphism from $B_c^{2,\lambda}(\overline{Q})$ to $\mathcal{Y}$ for small b and γ. For more details, see [1] where the proof is complete.

(b) To prove that $S : (b,\gamma) \mapsto \tilde{\psi}$ is continuously differentiable from an open ball $\mathcal{B}$ of radius $r_0 > 0$ centered at the origin of $B_c^{2,\lambda}(\mathbb{R}) \times B_c^{2,\lambda}(\mathbb{R})$, we write S as follows:

$$S(b,\gamma) = (S_2 \circ S_1(b,\gamma))\mathcal{F}(b,\gamma) = \tilde{\psi}, \tag{4.22}$$

where

$$
\begin{gathered}
S_1 : \mathcal{B}(0,r_0) \longrightarrow \mathcal{L}(B_c^{2,\lambda}(\overline{Q}),\mathcal{Y}),\\
(b,\gamma) \longmapsto \mathcal{A}_b^{\gamma},\\
S_2 : I\,\mathrm{som}\,(B_c^{2,\lambda}(\overline{Q}),\mathcal{Y}) \longrightarrow I\,\mathrm{som}\,(\mathcal{Y},B_c^{2,\lambda}(\overline{Q})),\\
L \longmapsto L^{-1},\\
\mathcal{F}(b,\gamma) = \big(0,-b(\widetilde{x}),-\gamma(\widetilde{x})\big) = \mathcal{A}_b^{\gamma}\widetilde{\psi} \in \mathcal{Y}.
\end{gathered}
\tag{4.23}
$$

The differentiability of $\widetilde{\psi}$ is given by the differentiability of S_1, S_2, and $\mathcal{F}(b,\gamma)$, and these results have been proved in [1]. □

Theorem 4.6. *Under the hypothesis of Theorem 4.4, the operator T is continuously Fréchet differentiable on $\mathcal{B}$.*

Proof. With the new variables $\widetilde{x}$, $\widetilde{y}$, the operator $T(b,\gamma)$ takes the form

$$
T(b,\gamma) = \gamma(\widetilde{x}) + \frac{F^2}{2}\left[\,|\,\widetilde{\nabla}_{b,\gamma}\widetilde{\psi}\,|^2(\widetilde{x},1) + \frac{2}{1+\gamma-b}\frac{\partial\widetilde{\psi}}{\partial\widetilde{y}}(\widetilde{x},1)\right],
\tag{4.24}
$$

which gives

$$
\begin{aligned}
T(b,\gamma) &= \gamma(\widetilde{x}) + \frac{F^2}{2}\left\{\left[\frac{\partial\widetilde{\psi}}{\partial\widetilde{x}}(\widetilde{x},1) - \frac{\gamma'}{(1+\gamma-b)^2}\frac{\partial\widetilde{\psi}}{\partial\widetilde{y}}(\widetilde{x},1)\right]^2 + \frac{1}{(1+\gamma-b)^2}\left(\frac{\partial\widetilde{\psi}}{\partial\widetilde{y}}(\widetilde{x},1)\right)^2\right.\\
&\qquad\qquad\left. + \frac{2}{1+\gamma-b}\frac{\partial\widetilde{\psi}}{\partial\widetilde{y}}(\widetilde{x},1)\right\}\\
&= \gamma(\widetilde{x}) + \frac{F^2}{2}\left\{\left[\frac{\partial\widetilde{\psi}}{\partial\widetilde{x}}(\widetilde{x},1) + \lambda_1(b,\gamma)\frac{\partial\widetilde{\psi}}{\partial\widetilde{y}}(\widetilde{x},1)\right]^2 + \lambda_2^2(b,\gamma)\left(\frac{\partial\widetilde{\psi}}{\partial\widetilde{y}}(\widetilde{x},1)\right)^2\right.\\
&\qquad\qquad\left. + 2\lambda_2(b,\gamma)\frac{\partial\widetilde{\psi}}{\partial\widetilde{y}}(\widetilde{x},1)\right\}
\end{aligned}
\tag{4.25}
$$

with

$$
\lambda_1(b,\gamma) = -\frac{\gamma'}{(1+\gamma-b)^2}, \qquad \lambda_2(b,\gamma) = \frac{1}{1+\gamma-b}.
\tag{4.26}
$$

We have shown that $\widetilde{\psi}$ is continuously differentiable with respect to b and γ. Moreover, it is evident that $\lambda_1(b,\gamma)$ and $\lambda_2(b,\gamma)$ are continuously differentiable with respect to b and γ. We deduce that $T(b,\gamma)$ is continuously Fréchet differentiable with respect to b and γ. □

4.2. Expression of $(\partial T/\partial b)(0,0)$. In the last subsection, we have seen that $\widetilde{\psi}$ is the solution of the problem

$$\begin{aligned} \Delta\widetilde{\psi} + \mathcal{P}_b^\gamma \widetilde{\psi} = 0 \quad \text{in } Q, \\ \widetilde{\psi}(\widetilde{x},1) = -\gamma(\widetilde{x}), \quad \widetilde{x} \in \mathbb{R}, \\ \widetilde{\psi}(\widetilde{x},0) = -b(\widetilde{x}), \quad \widetilde{x} \in \mathbb{R}. \end{aligned} \tag{4.27}$$

Note that $\mathcal{P}_0^0\widetilde{\psi} = 0$ in Q and $\widetilde{\psi}_{|b=\gamma=0} = 0$ in Q. Let $h \in B_c^{2,\lambda}(\mathbb{R})$. We put $\gamma = 0$ in the system above, we derive with respect to b in the direction h, and we evaluate the derivative at $b = 0$. We put

$$w_h = \frac{\partial\widetilde{\psi}}{\partial b}\,|_{b=\gamma=0} \cdot h. \tag{4.28}$$

We obtain

$$\begin{aligned} \Delta w + \mathcal{P}_0^0 w + \frac{\partial}{\partial b}(\mathcal{P}_b^0)_{|b=0}\widetilde{\psi}_{|b=\gamma=0} \cdot h = 0 \quad \text{in } Q, \\ w(\widetilde{x},1) = 0 \quad \widetilde{x} \in \mathbb{R}, \\ w(\widetilde{x},0) = -h(\widetilde{x}) \quad \widetilde{x} \in \mathbb{R}, \end{aligned} \tag{4.29}$$

and there follows Theorem 4.7.

Theorem 4.7. *Let $h \in B_c^{2,\lambda}(\mathbb{R})$. Then $w = w_h$ is the unique solution of the problem*

$$\begin{aligned} \Delta w = 0 \quad \textit{in } Q, \\ w(\widetilde{x},1) = 0 \quad \widetilde{x} \in \mathbb{R}, \\ w(\widetilde{x},0) = -h(\widetilde{x}) \quad \widetilde{x} \in \mathbb{R}. \end{aligned} \tag{4.30}$$

For the proof of this theorem, we use Proposition 4.5.

Now we can evaluate $(\partial T/\partial b)(0,0)$. We have

$$T(b,0) = \frac{F^2}{2}\left\{\left[\frac{\partial\widetilde{\psi}}{\partial\widetilde{x}}(\cdot,1)\right]^2 + \frac{1}{(1-b)^2}\left(\frac{\partial\widetilde{\psi}}{\partial\widetilde{y}}(\cdot,1)\right)^2 + \frac{2}{1-b}\frac{\partial\widetilde{\psi}}{\partial\widetilde{y}}(\cdot,1)\right\}. \tag{4.31}$$

We derive with respect to b in the direction h at $b = 0$. Taking into account the fact that $\widetilde{\psi}_{|b=\gamma=0} = 0$ in Q, we obtain

$$\begin{aligned} \frac{\partial T}{\partial b}(0,0) \cdot h &= F^2\left\{\frac{\partial}{\partial b}\left(\frac{\partial\widetilde{\psi}}{\partial\widetilde{y}}(\cdot,1)_{|b=\gamma=0}\right) \cdot h\right\} \\ &= F^2\left\{\frac{\partial}{\partial\widetilde{y}}\left(\frac{\partial\widetilde{\psi}}{\partial b}(\cdot,1)_{|b=\gamma=0}\right) \cdot h\right\}. \end{aligned} \tag{4.32}$$

Keeping the same notation as in Theorem 4.7, we denote $w_h = (\partial\widetilde{\psi}/\partial b)(\cdot,1)\,|_{b=\gamma=0}\,\cdot h$. Then we can write

$$\frac{\partial T}{\partial b}(0,0)\cdot h = F^2\frac{\partial w_h}{\partial \widetilde{y}}. \tag{4.33}$$

4.3. Inversibility of $(\partial T/\partial b)(0,0)$. We recall that the operator $(\partial T/\partial b)(0,0)$ is defined by

$$\begin{gathered} B_c^{2,\lambda}(\mathbb{R}) \longrightarrow B_c^{1,\lambda}(\mathbb{R}), \\ h \longmapsto F^2\frac{\partial w_h}{\partial \widetilde{y}}(\cdot,1), \end{gathered} \tag{4.34}$$

where w_h verifies the problem

$$\begin{gathered} \Delta w = 0 \quad \text{in } Q, \\ w(\widetilde{x},1) = 0, \quad \widetilde{x}\in\mathbb{R}, \\ w(\widetilde{x},0) = -h(\widetilde{x}), \quad \widetilde{x}\in\mathbb{R}. \end{gathered} \tag{4.35}$$

We show here the injectivity of the operator$(\partial T/\partial b)(0,0)$.

Let h_1 and h_2 be given in $B_c^{2,\lambda}(\mathbb{R})$ such that

$$\frac{\partial T}{\partial b}(0,0)h_1 = \frac{\partial T}{\partial b}(0,0)h_2. \tag{4.36}$$

It is equivalent to write

$$F^2\frac{\partial w_{h_1}}{\partial \widetilde{y}} = F^2\frac{\partial w_{h_2}}{\partial \widetilde{y}}, \tag{4.37}$$

where w_{h_i} is the solution of the problem

$$\begin{gathered} \Delta w = 0 \quad \text{in } Q, \\ w(\widetilde{x},1) = 0, \quad \widetilde{x}\in\mathbb{R},\ i=1,2, \\ w(\widetilde{x},0) = -h_i(\widetilde{x}), \quad \widetilde{x}\in\mathbb{R}. \end{gathered} \tag{4.38}$$

We put

$$w = w_{h_1} - w_{h_2}. \tag{4.39}$$

Then w verifies

$$\begin{gathered} \Delta w = 0 \quad \text{in } Q, \\ w(\widetilde{x},0) = h_2 - h_1, \quad \widetilde{x}\in\mathbb{R}, \\ w(\widetilde{x},1) = 0, \quad \widetilde{x}\in\mathbb{R}, \\ \frac{\partial w}{\partial \widetilde{y}}(\widetilde{x},1) = 0, \quad \widetilde{x}\in\mathbb{R}. \end{gathered} \tag{4.40}$$

This problem has a solution if and only if $h_2 - h_1 = 0$, and this solution is $w = 0$ (Holmgren theorem, e.g., [3]).

Then we have $h_1 = h_2$ and the operator $(\partial T/\partial b)(0,0)$ is injective from $B_c^{2,\lambda}(\mathbb{R})$ to $B_c^{1,\lambda}(\mathbb{R})$.

Unfortunately, we cannot prove the surjectivity of this operator from $B_c^{2,\lambda}(\mathbb{R})$ to $B_c^{1,\lambda}(\mathbb{R})$. Nevertheless, we have the surjectivity from $B_c^{2,\lambda}(\mathbb{R})$ on its image by $(\partial T/\partial b)(0,0)$ denoted by $\mathcal{I}$. We do not know how to characterize $\mathcal{I}$; then it will be interesting to exhibit an explicit subspace $\mathcal{K}$ of $\mathcal{I}$, which will be consistent enough. Let $K = [-a, +a] \subset \mathbb{R}$. We consider the subspace $\mathcal{K}$ of the function $g \in B_c^{2,\lambda}(\mathbb{R})$ which can be extended to the complex space as an analytic function $\zeta \mapsto g(\zeta)$ such that, for every integer $m \in \mathbb{N}$, there is a constant c_m such that for all $\zeta = \xi + i\eta$,

$$|g(\zeta)| \leq c_m (1 + |\zeta|)^{-m} \exp(2\pi a|\eta|). \tag{4.41}$$

We want to prove that $\mathcal{K} \subset \mathcal{I}$, that is, for each $g \in \mathcal{K}$, there exists $h \in B_c^{2,\lambda}(\mathbb{R})$ such that $(\partial T/\partial b)(0,0) \cdot h = g$.

We search for $w \in B_c^{2,\lambda}(Q)$ such that

$$\begin{aligned} \Delta w = 0 \quad & \text{in } Q, \\ w(\tilde{x}, 1) = 0, \quad & \tilde{x} \in \mathbb{R}, \\ \frac{\partial w}{\partial \tilde{y}}(\tilde{x}, 1) = g, \quad & \tilde{x} \in \mathbb{R}. \end{aligned} \tag{4.42}$$

For $g \in \mathcal{K}$, we have $w \in \mathcal{S}'$; applying the Fourier transformation to the last problem, we find

$$\begin{aligned} \frac{\partial^2 \hat{w}}{\partial \tilde{y}^2} &= \xi^2 \hat{w}, \\ \hat{w}(\xi, 1) &= 0, \\ \frac{\partial \hat{w}}{\partial \tilde{y}}(\xi, 1) &= \hat{g}(\xi). \end{aligned} \tag{4.43}$$

This implies that

$$\hat{w}(\xi, y) = \hat{g}(\xi) \frac{\operatorname{sh} \xi(\tilde{y} - 1)}{\xi}. \tag{4.44}$$

From Paley-Wiener theorem, we have $\hat{g} \in \mathcal{C}^\infty(\mathbb{R})$ and $\operatorname{supp} \hat{g} \subset K$.

Let $\theta \in D(\mathbb{R})$, with $\theta = 1$ in a neighbourhood of K. Because $\operatorname{supp} \hat{g} \subset K$, one can write

$$\begin{aligned} \hat{w}(\xi, \tilde{y}) &= \hat{g}(\xi) \frac{\operatorname{sh} \xi(\tilde{y} - 1)}{\xi} = \hat{g}(\xi)\theta(\xi) \frac{\operatorname{sh} \xi(\tilde{y} - 1)}{\xi}, \\ w(\tilde{x}, \tilde{y}) &= g(\tilde{x}) * \bar{\mathcal{F}}\left(\theta(\xi) \frac{\operatorname{sh} \xi(\tilde{y} - 1)}{\xi}\right) = \int_{\mathbb{R}} g(\tilde{x} - z) N(z, \tilde{y})\, dz, \end{aligned} \tag{4.45}$$

where

$$N(z,\tilde{y}) = \bar{\mathcal{F}}\left(\theta(\xi)\frac{\operatorname{sh}\xi(\tilde{y}-1)}{\xi}\right)(z,\tilde{y}). \tag{4.46}$$

It is easy to verify that $w \in B_c^{2,\lambda}(Q)$. While putting $w_{|y=0} = h$, one has the result, that is, $h \in B_c^{2,\lambda}(\mathbb{R})$ and so $\mathcal{K} \subset \mathcal{I}$. We have then proved the inversibility of the operator $(\partial T/\partial b)(0,0)$ from $B_c^{2,\lambda}(\mathbb{R})$ to $B_c^{2,\lambda}(\mathbb{R})$.

5. Conclusion

In this work, we have shown that for a given upper bound $y = y_0 + \gamma(x)$ where γ is in the space $B_c^{2,\lambda}(\mathbb{R})$, there exists a bottom $y = b(x)$ in the same space, which has created the perturbation $\gamma(x)$ at the upper bound. This result is local in a neighbourhood of $(b,\gamma) = (0,0)$ for all Froude numbers $F > 0$.

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