

ON THE EXTREMAL SOLUTIONS OF SEMILINEAR ELLIPTIC PROBLEMS

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We investigate here the properties of extremal solutions for semilinear elliptic equation $-\Delta u = \lambda f(u)$ posed on a bounded smooth domain of \mathbb{R}^n with Dirichlet boundary condition and with f exploding at a finite positive value a .

1. Introduction

We consider the following semilinear elliptic problem:

(P_λ)

$$\begin{aligned} -\Delta u &= \lambda f(u) && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\lambda > 0$, $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain and f satisfies the following condition:

(H) f is a C^2 positive nondecreasing convex function on $[0, \infty)$ such that

$$\lim_{t \rightarrow +\infty} \frac{f(t)}{t} = +\infty. \tag{1.2}$$

It is well known that under this condition (H) , there exists a critical positive value $\lambda^* \in (0, \infty)$ for the parameter λ such that the following holds.

(C_1) For any $\lambda \in (0, \lambda^*)$, there exists a positive, minimal, classical solution $u_\lambda \in C^2(\bar{\Omega})$. The function u_λ is minimal in the following sense: for every solution u of (P_λ) , we have $u_\lambda \leq u$ on Ω . In addition, the function $\lambda \mapsto u_\lambda$ is increasing and $\lambda_1(-\Delta - \lambda f'(u_\lambda)) > 0$, for example, for any $\varphi \in H_0^1(\Omega) \setminus \{0\}$,

$$\lambda \int_{\Omega} f'(u_\lambda) \varphi^2 dx < \int_{\Omega} |\nabla \varphi|^2 dx. \tag{1.3}$$

(C_2) For any $\lambda > \lambda^*$, there exists no classical solution for (P_λ) .

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When λ tends to λ^* ,

$$u^* = \lim_{\lambda \rightarrow \lambda^*} u_\lambda \quad (1.4)$$

always exists by the monotonicity of u_λ . In [3], Brezis et al. have introduced a notion of weak solution as follows: we say u is a weak solution for (P_λ) if $u \in L^1(\Omega)$, $u \geq 0$, $f(u)\delta \in L^1(\Omega)$ with $\delta(x) = \text{dist}(x, \partial\Omega)$, and

$$\int_{\Omega} u(-\Delta\xi)dx = \lambda \int_{\Omega} f(u)\xi dx, \quad (1.5)$$

for all $\xi \in C^2(\bar{\Omega})$, $\xi|_{\partial\Omega} = 0$. They then proved the following.

(C₃) u^* is always a weak solution of the problem (P_{λ^*}) , and for $\lambda > \lambda^*$ no solution exists even in the weak sense.

Later, Martel proved in [6] that u^* is the unique weak solution of (P_{λ^*}) , the so called extremal solution.

The typical examples are when the nonlinearity of f is either exponential $f(u) = e^u$ or power-like $f(u) = (1+u)^p$, $p > 1$ (see [4, 5, 7]). For $f(u) = e^u$, u^* is smooth when $n \leq 9$, if $n \geq 10$, $u^* = -2\ln|x|$ is the extremal solution on $B_1(0)$. When $f(u) = (1+u)^p$, if $n < n_p = 6 + 4(1 + \sqrt{p(p-1)})/(p-1)$, u^* is regular, and for $n \geq n_p$, $u^* = |x|^{-2/(p-1)} - 1$ is the extremal solution on $B_1(0)$. An immediate consequence is that with any $p > 1$ and $n \leq 10$, u^* is a smooth solution. It is natural to ask the following question: for small dimension n , is u^* always a classical solution for any function f satisfying (H) and any domain $\Omega \subset \mathbb{R}^n$? Nedev in [9] and Ye and Zhou in [10] had given some partial answers to this question.

THEOREM 1.1 [9]. *Suppose that f satisfies (H), then for $n = 2$ or 3 , u^* is always a classical solution. Moreover, when $n \geq 4$, $u^* \in L^q(\Omega)$, for any $q < n/(n-4)$ and $f(u^*) \in L^q(\Omega)$, for any $q < n/(n-2)$.*

THEOREM 1.2 [10]. *Let f verify (H), rewrite $f(t) = f(0) + te^{g(t)}$. Assume that there exists t_0 positive such that $t^2g'(t)$ is nondecreasing in $[t_0, \infty)$, then for any $\Omega \subset \mathbb{R}^n$ with $n \leq 9$, u^* is a classical solution.*

On the other hand, Brezis and Vazquez have given a characterization of unbounded extremal solutions in $H_0^1(\Omega)$ as follows: if $v \in H_0^1(\Omega)$ is an unbounded weak solution of (P_λ) with $\lambda > 0$ and satisfying the stability condition

$$\lambda \int_{\Omega} f'(v)\varphi^2 dx \leq \int_{\Omega} |\nabla\varphi|^2 dx, \quad \forall \varphi \in C_1(\bar{\Omega}), \varphi|_{\partial\Omega} = 0; \quad (1.6)$$

then $\lambda = \lambda^*$ and $v = u^*$. They remarked also that there exist unbounded weak solutions which satisfy (1.6), but do not belong to $H_0^1(\Omega)$, and which are not extremal solutions.

In this paper, we investigate some similar problems with f exploding at a finite positive value a . More precisely, let f satisfy the following condition:

(H') f is a C^1 positive, nondecreasing, convex function on $[0, a)$ with $a \in (0, \infty)$ and

$$\lim_{t \rightarrow a^-} f(t) = +\infty. \quad (1.7)$$

We consider the following problem:

(E_λ)

$$\begin{aligned} -\Delta u &= \lambda f(u) \quad \text{in } \Omega, \\ u &\in (0, a] \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.8}$$

By the work of Mignot and Puel (see [7]), we have always a critical value $\lambda^* \in (0, \infty)$ such that for any $\lambda \in (0, \lambda^*)$, there exists a positive, minimal, classical solution $u_\lambda \in C^2(\bar{\Omega})$, that is, $u_\lambda < a$ in $(\bar{\Omega})$ and for $\lambda > \lambda^*$, no classical solution exists. The aim of this work is to study the propriety of the solution of (E_λ) at the extremal value $\lambda = \lambda^*$ and to prove the nonexistence of weak solution when $\lambda > \lambda^*$. We define that ω is a weak solution of (E_λ) , if $\omega \in L^1(\Omega, [0, a])$ such that $f(\omega)\delta \in L^1(\Omega)$, and for all $\zeta \in C^2(\bar{\Omega})$, with $\zeta = 0$ on $\partial\Omega$,

$$-\int_{\Omega} \omega \Delta \zeta = \lambda \int_{\Omega} f(\omega) \zeta. \tag{1.9}$$

Similarly, we say that ω is a weak supersolution of (E_λ) , if $\omega \in L^1(\Omega, [0, a])$, such that $(\Delta\omega)\delta \in L^1(\Omega)$, and for all $\zeta \in C^2(\bar{\Omega})$, $\zeta \geq 0$ with $\zeta = 0$ on $\partial\Omega$,

$$-\int_{\Omega} \omega \Delta \zeta \geq \lambda \int_{\Omega} f(\omega) \zeta. \tag{1.10}$$

Our main results are the following.

THEOREM 1.3. *Given f satisfying (H') , if $\lambda > \lambda^*$, then there is no weak solution of (E_λ) .*

THEOREM 1.4. *The function $u^* = \lim_{\lambda \rightarrow \lambda^*} u_\lambda$ is the unique weak solution of (E_{λ^*}) . Moreover, for any $\varphi \in C^1(\bar{\Omega})$ with $\varphi = 0$ on $\partial\Omega$,*

$$\lambda^* \int_{\Omega} f'(u^*) \varphi^2 dx \leq \int_{\Omega} |\nabla \varphi|^2 dx. \tag{1.11}$$

THEOREM 1.5. *Assume that $v \in H_0^1(\Omega)$ is a weak solution of (E_λ) for some $\lambda > 0$, assume also that $\sup_{\Omega}(v) = a$ and*

$$\lambda \int_{\Omega} f'(v) \varphi^2 dx \leq \int_{\Omega} |\nabla \varphi|^2 dx, \tag{1.12}$$

for all $\varphi \in C^1(\bar{\Omega})$, $\varphi = 0$ on $\partial\Omega$, then $\lambda = \lambda^$ and $v = u^*$.*

2. Proof of Theorem 1.3

In fact, Theorem 1.3 is deduced from a general result, which is the following proposition.

PROPOSITION 2.1. *Given g satisfying (H') , if there exists a weak solution ω of*

$$\begin{aligned} -\Delta \omega &= g(\omega) \quad \text{in } \Omega, \\ \omega &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

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then, for any $\varepsilon \in (0, 1)$, there exists a classical solution ω_ε of

$$\begin{aligned} -\Delta \omega_\varepsilon &= (1 - \varepsilon)g(\omega_\varepsilon) && \text{in } \Omega, \\ \omega_\varepsilon &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (2.2)$$

For the proof of this result, we need the following lemmas which are proved in [3].

LEMMA 2.2. *Given $g \in L^1(\Omega, \delta(x)dx)$, there exists a unique $v \in L^1(\Omega)$ which is a weak solution of*

$$\begin{aligned} -\Delta v &= g && \text{in } \Omega, \\ v &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (2.3)$$

where $\|v\|_{L^1} \leq C\|g\|_{L^1(\Omega, \delta(x)dx)}$, for some C constant independent of g . In addition, if $g \geq 0$ a.e. in Ω , then $v \geq 0$ a.e. in Ω .

LEMMA 2.3. *Assume $g(0) > 0$ and set*

$$h(u) = \int_0^u \frac{ds}{g(s)}, \quad (2.4)$$

for all $0 \leq u \leq a$. Let \tilde{g} be a C^1 positive function on $[0, a)$ such that $\tilde{g} \leq g$ and $\tilde{g}' \leq g'$. Set

$$\tilde{h}(u) = \int_0^u \frac{ds}{\tilde{g}(s)}, \quad \Phi(u) = \tilde{h}^{-1}(h(u)), \quad (2.5)$$

for all $u \in [0, a]$. Then,

- (i) $\Phi(0) = 0$ and $0 \leq \Phi(u) \leq u$ for all $0 \leq u \leq a$,
- (ii) Φ is increasing, concave, and $\Phi'(u) \leq 1$ for all $0 \leq u \leq a$,
- (iii) $h(a) < \infty$ and $\Phi(a) < a$, if $\tilde{g} \not\equiv g$ in $[0, a]$.

Proof. It is easy to see that (i) and (iii) hold. We prove (ii), in fact $\Phi'(u) = \tilde{g}(\Phi(u))/g(u) > 0$, and

$$\Phi''(u) = \frac{g(u)\tilde{g}'(\Phi(u))\Phi'(u) - \tilde{g}(\Phi(u))g'(u)}{g(u)^2} = \frac{\tilde{g}(\Phi(u))(\tilde{g}'(\Phi(u)) - g'(u))}{g(u)^2}. \quad (2.6)$$

Since $\tilde{g}'(\Phi(u)) \leq g'(\Phi(u)) \leq g'(u)$, it follows that Φ is concave, which completes the proof. \square

Proof of Proposition 2.1 and Theorem 1.3. Choosing $\tilde{g} = (1 - \varepsilon)g$ in Lemma 2.3 and denote by $v = \Phi(\omega)$, where ω is the weak solution of (2.1) and using an approximating

argument for ω , we get

$$\begin{aligned} - \int_{\Omega} v \Delta \zeta &= - \int_{\Omega} \Phi(\omega) \Delta \zeta = - \int_{\Omega} \Delta \Phi(\omega) \zeta = - \int_{\Omega} [\Phi'(\omega) \Delta \omega + \Phi''(\omega) |\nabla \omega|^2] \zeta \\ &\geq \int_{\Omega} \Phi'(\omega) g(\omega) \zeta = \int_{\Omega} \bar{g}(\Phi(\omega)) \zeta = \int_{\Omega} (1 - \varepsilon) g(v) \zeta \end{aligned} \quad (2.7)$$

for any $\zeta \in C^1(\bar{\Omega})$, $\zeta \geq 0$ with $\zeta = 0$ on $\partial\Omega$. Hence, v is a weak supersolution of (2.2). The result of Proposition 2.1 follows by standard barrier method as follows. We define a sequence $(\omega_k)_{k \geq 0}$ by

$$\begin{aligned} -\Delta \omega_{k+1} &= (1 - \varepsilon) g(\omega_k) \quad \text{in } \Omega, \\ \omega_{k+1} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (2.8)$$

for $k \in \mathbb{N}$, with $\omega_0 = v$. Using Lemma 2.2, it is easy to check that $\omega_k \geq \omega_{k+1} \geq 0$, for all $k \in \mathbb{N}$, so the sequence ω_k is nonincreasing and converges in $L^1(\Omega)$ to a weak solution u of (2.2). Since $\sup_{\Omega}(u) \leq \sup_{\Omega}(v) < a$, u is a classical solution, Proposition 2.1 is proved. Theorem 1.3 is deduced by taking $g = \lambda f$ in Proposition 2.1. For any $\lambda > \lambda^*$, let $\varepsilon \in (0, 1)$ such that $\lambda^* < (1 - \varepsilon)\lambda < \lambda$, since there is no classical solution of

$$\begin{aligned} -\Delta \omega_{\varepsilon} &= (1 - \varepsilon) \lambda f(\omega_{\varepsilon}) \quad \text{in } \Omega, \\ \omega_{\varepsilon} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (2.9)$$

it follows by Proposition 1.3 that there is no weak solution of (E_{λ}) . \square

3. Proof of Theorem 1.4

We know that u^* is the increasing limit of classical solution u_{λ} with positive first eigenvalue, that is, for any $\varphi \in C^1(\bar{\Omega})$ with $\varphi = 0$ on $\partial\Omega$,

$$\lambda \int_{\Omega} f'(u_{\lambda}) \varphi^2 dx \leq \int_{\Omega} |\nabla \varphi|^2 dx. \quad (3.1)$$

Passing to the limit, the inequality (1.11) holds. To prove the uniqueness, we will in fact also prove a slightly stronger result.

PROPOSITION 3.1. *Let $v \in L^1(\Omega, [0, a])$ be a weak supersolution of (E_{λ^*}) , then $v = u^*$.*

Proof. We proceed in two steps. First, we show that v is a weak solution of (E_{λ^*}) . Next, we prove that if $v \neq u^*$, then we obtain a contradiction.

Step 1. Suppose that v is not a weak solution of (E_{λ^*}) , then we can assume that there exists $\beta > 0$ and $\xi_0 \in C^2(\bar{\Omega})$, $\xi_0 \geq 0$, with $\xi_0|_{\partial\Omega} = 0$ such that

$$- \int_{\Omega} v \Delta \xi_0 = \lambda^* \int_{\Omega} f(v) \xi_0 + \beta, \quad (3.2)$$

it follows that there exists a nonnegative measure $\mu \neq 0$, with $\mu\delta$ bounded on Ω , such that

$$- \int_{\Omega} v \Delta \xi = \int_{\Omega} (\lambda^* f(v) + \mu) \xi, \quad (3.3)$$

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for all $\xi \in C^2(\bar{\Omega})$ with $\xi|_{\partial\Omega} = 0$. Consider φ and χ , the solutions of

$$\begin{aligned} -\Delta\varphi &= \mu & \text{in } \Omega, & \quad \varphi = 0 & \text{on } \partial\Omega, \\ -\Delta\chi &= 1 & \text{in } \Omega, & \quad \chi = 0 & \text{on } \partial\Omega. \end{aligned} \quad (3.4)$$

By $\mu \neq 0$, it follows from the properties of the Laplacian that there exists $\varepsilon > 0$ such that $\varepsilon\chi \leq \varphi$. Set $z = v + \varepsilon\chi - \varphi \leq v$. Then, since f is nondecreasing,

$$-\int_{\Omega} z\Delta\xi = \int_{\Omega} (\lambda^* f(v) + \varepsilon)\xi \geq \int_{\Omega} (\lambda^* f(z) + \varepsilon)\xi, \quad (3.5)$$

for all $\xi \in C^2(\bar{\Omega})$, $\xi \geq 0$, with $\xi|_{\partial\Omega} = 0$. This means that z is a weak supersolution for $-\Delta\omega = g(\omega)$, where $g(v) = \lambda^* f(v) + \varepsilon$. Using the proof of Proposition 2.1 and Lemma 2.3 with $\bar{g}(v) = \lambda^* f(v) + \varepsilon/2$, we can get a classical solution v_1 of

$$\begin{aligned} -\Delta v_1 &= \lambda^* f(v_1) + \left(\frac{\varepsilon}{2}\right) & \text{in } \Omega, \\ v_1 &= 0 & \text{on } \partial\Omega. \end{aligned} \quad (3.6)$$

Moreover, there exists $\alpha > 0$, such that $2\alpha v_1 \leq \varepsilon\chi$. Set $z = v_1 + \alpha v_1 - (\varepsilon/2)\chi$. It is clear that $0 < z \leq v_1$ and z satisfies $-\Delta z \geq (1 + \alpha)\lambda^* f(v_1) \geq (1 + \alpha)\lambda^* f(z)$ in Ω . Thus, the classical barrier method gives a solution of $(E_{(1+\alpha)\lambda^*})$, which contradicts then the definition of λ^* , so v is a solution of (E_{λ^*}) .

Step 2. Clearly, $v \geq u_\lambda$ for any $\lambda < \lambda^*$, hence $v \geq u^*$. Suppose that $v \neq u^*$, take $\Psi = f(v) - f(u^*) \geq 0$, it is clear that $\Psi \delta \in L^1(\Omega)$. We have then $\Psi \neq 0$, because otherwise $f(v) = f(u^*)$ a.e. on Ω , and Lemma 2.2 will give $v = u^*$ a.e. on Ω . Let g be the weak solution of

$$\begin{aligned} -\Delta g &= \Psi & \text{in } \Omega, \\ g &= 0 & \text{on } \partial\Omega. \end{aligned} \quad (3.7)$$

By the maximum principle, we have $g \geq c\delta$ on Ω for some $c > 0$. Hence,

$$-\int_{\Omega} (v - u^* - \lambda^* g)\Delta\xi = 0, \quad (3.8)$$

for all $\xi \in C^2(\bar{\Omega})$, with $\xi|_{\partial\Omega} = 0$. We obtain by Lemma 2.2 that $v - u^* = \lambda^* g \geq \lambda^* c\delta$ a.e. on Ω , set $Z = (v + u^*)/2$, then

$$-\int_{\Omega} Z\Delta\xi = \frac{\lambda^*}{2} \int_{\Omega} (f(v) + f(u^*))\xi = \lambda^* \int_{\Omega} (f(Z) + h)\xi > \lambda^* \int_{\Omega} f(Z)\xi \quad (3.9)$$

for all $\xi \in C^2(\bar{\Omega})$, $\xi \geq 0$, with $\xi|_{\partial\Omega} = 0$, where h is given by

$$h = \frac{1}{2}(f(v) + f(u^*)) - f\left(\frac{v + u^*}{2}\right) = \frac{1}{2} \int_{u^*}^v ds \int_{(s+u^*)/2}^s f''(\sigma) d\sigma. \quad (3.10)$$

Clearly, $h\delta \in L^1(\Omega)$. Suppose first that $h \equiv 0$, then $f''(\sigma) = 0$ if $\sigma \in [u^*, v]$, hence $f(\sigma) = f(0) + f'(0)\sigma$ on $\cup_{x \in \Omega} [u^*(x), v(x)] = [0, \sup_{\Omega} v]$, since $v > u^*$ in Ω . Then, if $\sup_{\Omega} v = a$, we obtain a contradiction by (1.7), and if $\sup_{\Omega} v < a$, both u^* and v are classical solutions of a linear problem with $f(t) = A + Bt$ for which the uniqueness is known (see, for instance, [8]). If $h \neq 0$, it follows that Z is a strict supersolution of (E_{λ^*}) and we obtain also a contradiction by Step 1. \square

4. Proof of Theorem 1.5

Suppose that $\lambda < \lambda^*$. We observe that by a density argument, the inequality (1.12) holds for every $\Phi \in H_0^1(\Omega)$. Taking $\Phi = v - u_{\lambda}$ in (1.12), we get

$$\lambda \int_{\Omega} f'(v)(v - u_{\lambda})^2 dx \leq \int_{\Omega} |\nabla(v - u_{\lambda})|^2 dx = \lambda \int_{\Omega} [f(v) - f(u_{\lambda})](v - u_{\lambda}) dx, \quad (4.1)$$

that is,

$$\lambda \int_{\Omega} [f(v) - f(u_{\lambda}) - f'(v)(v - u_{\lambda})](v - u_{\lambda}) dx \geq 0. \quad (4.2)$$

Since f is convex and $v \geq u_{\lambda}$, we get $f(v) = f(u_{\lambda}) + f'(v)(v - u_{\lambda})$ a.e. on Ω . Hence, f must be linear in the interval $[u_{\lambda}(x), v(x)]$ for a.e. $x \in \Omega$. If $v > u_{\lambda}$, we get that f is linear in $\cup_x [u(x), v(x)] = [0, \sup_{\Omega} v] = [0, a]$, which contradicts (1.7). So, $v = u_{\lambda}$, as v is not a classical solution, we get a contradiction, so $\lambda = \lambda^*$. The similar argument with (1.11) shows that $v = u^*$.

5. Application

Now, we consider a special case $f(u) = 1/(1 - u)^p$ with $p > 0$ and $\Omega = B_1(0)$, this problem was studied by Brauner and Nicolaenko in [1, 2]. When $p = 1$, this equation appears as a limit of some problem of disruption in biochemistry; it allows then to justify some phenomenon in kinetic enzymatic and the kinetic of reactors associated to some limit coat. For $n \geq 2$, we know an explicit weak solution

$$U(x) = 1 - |x|^{2/(p+1)}, \quad (5.1)$$

which is obviously in $H_0^1(\Omega)$, it corresponds to the parameter value

$$\lambda^{\sharp}(n, p) = \frac{2}{p+1} \left(n - \frac{2p}{p+1} \right) > 0. \quad (5.2)$$

The linearized operator is

$$L_{\sharp}\Phi = -\Delta\Phi - \frac{2p}{p+1} \left(n - \frac{2p}{p+1} \right) \frac{\Phi}{r^2}, \quad (5.3)$$

where $r = |x|$. By Theorem 1.5, U is the extremal solution if and only if for any $\Phi \in H_0^1(\Omega)$,

$$\frac{2p}{p+1} \left(n - \frac{2p}{p+1} \right) \int_B \frac{\Phi^2}{r^2} \leq \int_B |\nabla \Phi|^2 dx. \quad (5.4)$$

Thanks to Hardy's inequality, this holds if and only if (see [4])

$$\frac{2p}{p+1} \left(n - \frac{2p}{p+1} \right) \leq H = \frac{(n-2)^2}{4}. \quad (5.5)$$

Thus, we have the following proposition.

PROPOSITION 5.1. *For any $p > 0$, let*

$$n_0(p) = \frac{2}{p+1} \left[(3p+1) + 2\sqrt{p(p+1)} \right]. \quad (5.6)$$

Then,

- (i) *if $n \geq n_0(p)$, $u^*(x) = 1 - |x|^{2/(p+1)}$, and $\lambda^* = \lambda^\#$;*
- (ii) *if $n < n_0(p)$, $\lambda^* > \lambda^\#$ and u^* is smooth.*

Proof. By an easy computation, we have that $n \geq n_0(p)$ is equivalent to (5.5), so (i) is proved by Theorem 1.5. The proof of (ii) is given in [7]. \square

We remark that when p tends to 0, $n_0(p)$ tends to 2. So, for any $n \geq 3$, we can meet some nonlinearities f (by choosing appropriate p) such that the extremal solution is no longer classical, this fact is different from the situation for $a = \infty$, if we compare with the results in [9, 10]. Thus, a natural question is raised, for f satisfying (H') and Ω bounded smooth domain in \mathbb{R}^2 , do we have always that u^* is a classical solution?

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References

- [1] C.-M. Brauner and B. Nicolaenko, *Sur une classe de problèmes elliptiques non linéaires*, C. R. Acad. Sci. Paris Sér. A-B **286** (1978), no. 21, A1007–A1010 (French).
- [2] ———, *Sur des problèmes aux valeurs propres non linéaires qui se prolongent en problèmes à frontière libre*, C. R. Acad. Sci. Paris Sér. A-B **288** (1979), no. 2, A125–A127 (French).
- [3] H. Brezis, T. Cazenave, Y. Martel, and A. Ramiandrisoa, *Blow up for $u_t - \Delta u = g(u)$ revisited*, Adv. Differential Equations **1** (1996), no. 1, 73–90.
- [4] H. Brezis and J. L. Vázquez, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Complut. Madrid **10** (1997), no. 2, 443–469.
- [5] M. G. Crandall and P. H. Rabinowitz, *Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems*, Arch. Ration. Mech. Anal. **58** (1975), no. 3, 207–218.
- [6] Y. Martel, *Uniqueness of weak extremal solutions of nonlinear elliptic problems*, Houston J. Math. **23** (1997), no. 1, 161–168.

- [7] F. Mignot and J.-P. Puel, *Sur une classe de problèmes non linéaires avec non linéarité positive, croissante, convexe*, Comm. Partial Differential Equations **5** (1980), no. 8, 791–836 (French).
- [8] P. Mironescu and V. D. Rădulescu, *The study of a bifurcation problem associated to an asymptotically linear function*, Nonlinear Anal. **26** (1996), no. 4, 857–875.
- [9] G. Nedev, *Regularity of the extremal solution of semilinear elliptic equations*, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), no. 11, 997–1002.
- [10] D. Ye and F. Zhou, *Boundedness of the extremal solution for semilinear elliptic problems*, Commun. Contemp. Math. **4** (2002), no. 3, 547–558.

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