PROPERTIES OF TYPICAL BOUNDED CLOSED CONVEX SETS IN HILBERT SPACE

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For a nonempty separable convex subset X of a Hilbert space $\mathbb{H}(\Omega)$, it is typical (in the sense of Baire category) that a bounded closed convex set $C \subset \mathbb{H}(\Omega)$ defines an m-valued metric antiprojection (farthest point mapping) at the points of a dense subset of X, whenever m is a positive integer such that $m \leq \dim X + 1$.

1. Introduction

Baire category techniques are known to be a powerful tool in the investigation of the convex sets. Their use, which goes back to the fundamental contribution of Klee [17], has permitted to discover several interesting unexpected properties of convex sets (see Gruber [14], Schneider [23], Zamfirescu [25]). A survey of this area of research and additional bibliography can be found in [15, 27].

In the present paper, we consider some geometric properties of typical (in the sense of the Baire categories) nonempty bounded closed convex sets contained in a separable real Hilbert space. It will be shown in the typical case, for a closed convex and bounded set C and an integer m, that there is a dense subset D of the Hilbert space $\mathbb{H}$ such that the farthest point mapping generated by C is precisely m-valued at the points of D. A result of this type was recently obtained in [5], for typical nonempty compact convex sets. However, the approach of [5] cannot be adopted here for, in absence of compactness, the antiprojection mapping could have empty images. To overcome this difficulty we will use some ideas from [28], developed in the framework of the metric projections.

Throughout, $\mathbb{H}(\Omega)$ is a Hilbert space over the field of real numbers $\mathbb{R}$ whose elements are mappings $x : \Omega \to \mathbb{R}$ with countably many nonzero values and convergent sums $\sum_{\omega\in\Omega} x_\omega^2$. We often prefer to denote $\mathbb{H}(\Omega)$ by $\mathbb{H}$. It is assumed always in the paper that $\dim \mathbb{H}(\Omega) \geq 2$. As usual, the inner product and the norm are denoted by $\langle\cdot,\cdot\rangle$ and $|\cdot|$.

For a nonempty bounded set $M \subset \mathbb{H}$, the function $f(x,M) = \sup\{|x-z| : z \in M\}$ is the farthest distance function, and the set-valued mapping

$$Q(x,M) = \{y \in M : |x-y| = f(x,M)\} \tag{1.1}$$

is called *metric antiprojection* or *farthest point mapping*.

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Let $\mathscr{C}$ be the complete metric space of all nonempty bounded closed convex subsets of $\mathbb{H}$, endowed with the Hausdorff metric χ for sets. The cardinal number of a set I is denoted by $\operatorname{card} I$ and $\mathbb{N}$ stands for the set of the natural numbers.

With every m in $\mathbb{N}$ and every $C \subset \mathbb{H}$ associate the sets

$$L^m(C) = \{x \in \mathbb{H} : \operatorname{card} Q(x,C) = m\}, \tag{1.2}$$

which are called *m-locus of the metric antiprojection generated by C*.

For $X \subset \mathbb{H}$ and $C \in \mathscr{C}$, the following sets are the loci generated by C in X:

$$L_X^m(C) = X \cap L^m(C). \tag{1.3}$$

We will also make use of the metric projection (or the nearest point mapping) defined for nonempty $M \subset \mathbb{H}$ by

$$P(x,M) = \{y \in M : |x-y| = d(x,M)\}, \tag{1.4}$$

with distance function $d(x,M) = \inf\{|x-z| : z \in M\}$.

Investigations on typical properties of antiprojections' loci reflect the research done for metric projections (see, e.g., [3, 18, 20, 24, 26]). In [2, 12], results of the following type are proved: for any bounded and closed subset M of a uniformly convex or locally uniformly convex Banach space X, the set $L^1(M)$ is residual in X, and $Q(\cdot,M)$ is continuous at $x \in L^1(M)$. Zamfirescu [26] proved for metric projections that typically the complement of $L^1(M)$ is dense. Further development for either metric projections or antiprojections is to be found in, for example, [6, 7, 8, 9, 14, 28, 29, 30, 31].

A set $C \subset \mathbb{H}$ is called *everywhere continual* in the set $X \subset \mathbb{H}$ if $\operatorname{card}(C \cap X \cap U) \geq \mathbf{c}$ for every open set $U \subset \mathbb{H}$ such that $X \cap U \neq \varnothing$. The letter $\mathbf{c}$ denotes the cardinal number of the continuum.

The following main result is to be established.

Theorem 1.1. *Let X be a nonempty separable convex subset of the Hilbert space $\mathbb{H}(\Omega)$ and let $m \in \mathbb{N}$ satisfy $\dim X \geq m-1$. Then there exists a residual subset $\mathscr{R}$ of $\mathscr{C}$ such that for every $C \in \mathscr{R}$ the locus $L_X^m(C)$ is dense in X. Moreover, if $\dim X \geq m$, then there exists a residual set $\mathscr{R}_{\mathbf{c}} \subset \mathscr{C}$ such that for $C \in \mathscr{R}_{\mathbf{c}}$ the set $L_X^m(C)$ contains an everywhere continual in X subset at each point of which $Q(\cdot,C)$ is upper semicontinuous.*

2. Notation

A topological space X is called Baire space if the intersection of every countable family of open and dense subsets of X is dense in X. A set $R \subset X$ is residual if it contains some dense G_δ subset of X. If R is residual in X, we say with some abuse of the formal logic that any element $x \in R$ is a typical element of X.

A point-to-set mapping $F : X \to Y$, where X and Y are topological spaces, is upper semicontinuous (u.s.c.) (resp., lower semicontinuous (l.s.c.)) at $x_0 \in X$ provided for every open set $U \supset F(x_0)$ (resp., $U \cap F(x_0) \neq \varnothing$), there exists an open set $V \subset X$, $x_0 \in V$, such that $F(x) \subset U$ (resp., $F(x) \cap U \neq \varnothing$) whenever $x \in V$.

For a subset M of $\mathbb{H}$, $\operatorname{int} M$, $\overline{M}$, $\operatorname{span} M$, $\operatorname{co} M$, and $\operatorname{diam} M$ stand for the interior, closure, span, convex hull, and diameter of M, respectively, The set $\operatorname{usc} Q_M$ consists of all points $x \in \mathbb{H}$ at which $Q(x,M)$ is upper semicontinuous. In some cases, for an arbitrary set-valued mapping F, we write $\operatorname{usc} F$ and $\operatorname{lsc} F$ to denote the sets of upper semicontinuity and lower semicontinuity, respectively, of F.

Let θ be the origin of $\mathbb{H}$. The closed line segment with end-points $x, y \in \mathbb{H}$ is denoted by $[x,y]$. If $M_i \subset \mathbb{H}$ and $\lambda_i \in \mathbb{R}$ for $i = 1,2$, then $\lambda_1 M_1 + \lambda_2 M_2 = \{\lambda_1 x_1 + \lambda_2 x_2 : x_1 \in M_1, x_2 \in M_2\}$.

$\mathrm{B}[x,r]$ is the closed ball centered at $x \in \mathbb{H}$ with radius $r > 0$, while $\mathrm{B}(x,r)$ (resp., $\mathrm{S}(x,r)$) is the open ball (resp., the sphere) with the same center and radius. B is the closed unit ball of $\mathbb{H}$ and S is the unit sphere.

The balls in other metric spaces are denoted in a different way, that is, by B with a subscript indicating the space or without a subscript whenever there is no ambiguity. For instance, $B(C,r)$ (resp., $B[C,r]$) stands for the open (resp., closed) ball, centered at $C \in \mathscr{C}$ with radius $r > 0$.

Suppose $M_1, \dots, M_m$ are nonempty subsets. Denote the set of equidistant points from all M_i with respect to the farthest distance by

$$\tau(M_i)_{i=1}^m = \{x \in \mathbb{H} : f(x,M_i) = f(x,M_j),\ i,j = 1,\dots,m\}. \tag{2.1}$$

Obviously $\tau(M_i)_{i=1}^m$ is a closed set in $\mathbb{H}$ and whenever nonempty, it is a complete metric space under the metric induced by $|\cdot|$.

In the sequel by Hausdorff topology (of $\mathscr{C}$) we mean the topology of $\mathscr{C}$ generated by the Hausdorff distance χ. It is well known that $(\mathscr{C},\chi)$ is a complete metric space [16].

3. Topological facts

This small section contains some auxiliary results which are possibly not formulated in full generality but in a most suitable way for our purposes. The following are well-known theorems of Fort, Kuratowski and Ulam, Alexandroff and Urysohn, and Brouwer and Miranda.

Theorem 3.1 (Fort [13]). *Suppose X and Y are complete metric spaces, Y is separable, and $F : X \to Y$ is an upper semicontinuous set-valued mapping with nonempty compact images. Then F is lower semicontinuous on a dense G_δ subset of X.*

Theorem 3.2 (Kuratowski and Ulam, see [22]). *Suppose X and Y are complete metric spaces, Y is separable, and R is a dense G_δ subset of the product space $X \times Y$. Then there exists a dense G_δ subset R_X of X such that for every $x \in R_X$, the set $\{y \in Y : (x,y) \in R\}$ is dense G_δ subset of Y.*

Theorem 3.3 (Alexandroff and Urysohn, see [1]). *Let A be a dense G_δ subset of a nonempty metrizable compactum K, and assume K has no isolated points. Then* $\operatorname{card} A \geq \mathbf{c}$.

Theorem 3.4 (Brouwer and Miranda, see [4, 21]). *Let $I_r \subset \mathbb{R}^n$ be a bounded polyhedron of the form $\{x \in \mathbb{R}^n : |\langle v_i, x\rangle| \leq r,\ i = 1,\dots,n\}$ where $r > 0$, and $v_1,\dots,v_n$ are linearly independent vectors. For $i = 1,\dots,n$, let $A_i^{\pm} = \{x \in I_r : \langle v_i, x\rangle = \pm r\}$ and let $g_i : I_r \to \mathbb{R}$ be continuous* *functions such that* $g_i(x) < 0$ *if* $x \in A_i^-$, $g_i(x) > 0$ *if* $x \in A_i^+$. *Then there exists a point* $\hat{x} \in I_r$ *such that* $g_i(\hat{x}) = 0$ *for all* $i = 1, \dots, n$.

There are also several topological lemmas.

LEMMA 3.5 ([19], cf. [9]). *Suppose* X *is a complete metric space and* $R \subset X$ *satisfies the following property: for every* $x \in X$ *and every* $\eta > 0$ *there are* $y \in X$ *and* $\kappa > 0$ *such that* $B[y,\kappa] \subset B(x,\eta)$ *and* $R \cap B[y,\kappa]$ *is residual in* $B[y,\kappa]$ *with respect to the relative topology. Then* R *is a residual subset of* X.

LEMMA 3.6 [28]. *Let* X *and* Y *be complete metric spaces and let* $F : X \to Y$ *be an upper semicontinuous set-valued mapping with nonempty compact images. Then the set*

$$\Lambda' = \{(x,y) \in X \times Y : y \in F(x),\ F \text{ is l.s.c. at } x\} \tag{3.1}$$

is a Baire space with respect to the relative topology induced by the topology of $X \times Y$.

A topological phenomenon allows us to project orthogonally residual subsets of the graph of an l.s.c. mapping onto residual subsets of its domain. The following lemma can be derived from results in [11]. A detailed proof is contained in [28].

LEMMA 3.7. *Under the assumptions of the previous lemma, let also* Σ *be a residual subset of* Λ' *in the relative topology of* Λ'. *Then the orthogonal projection* π *along* Y *maps* Σ *onto a residual subset of* X.

4. Lemmas

LEMMA 4.1. *Let* $C_0 \in \mathscr{C}$, $s, r \in \mathbb{R}$, $0 < s < r$, *and* $a \in \mathbb{R}$. *Let* $C_0 \subset \mathrm{B}[a,s]$, *and let* $Y = \{y_1, \dots, y_m\}$ *be a nonempty subset of* $\mathrm{S}(a,r)$. *Put* $C' = \overline{\mathrm{co}}(Y \cup C_0)$. *Then for every* $\varepsilon \in (0, r-s)$ *there exists* $\delta > 0$ *such that* $d(x,Y) < \varepsilon$ *whenever* $x \in C'$ *and* $|x - a| > r - \delta$.

Proof. With no loss of generality assume $a = \theta$ and $r = 1$. Put $\delta = \varepsilon^2/8$ and take $x \in C' \setminus (1-\delta)\mathrm{B}$. Denote $u = x/|x|$ and consider the slice $\mathrm{Sl}_\delta(u) = \{z \in \mathrm{B} : \langle u, z\rangle > 1 - \delta\}$. In order to complete the proof it suffices to show that $d(x,Y) \geq \varepsilon$ implies $C' \subset \mathrm{B} \setminus \mathrm{Sl}_\delta(u)$.

For $z \in \mathrm{Sl}_\delta(u)$,

$$|u - z|^2 = 1 - 2\langle u, z\rangle + |z|^2 < 2\delta = \frac{\varepsilon^2}{4}, \tag{4.1}$$

that is, $|u - z| < \varepsilon/2$. Hence

$$d(z,Y) \geq d(x,Y) - |x - u| - |u - z| > \frac{\varepsilon}{2} - \delta > 0, \tag{4.2}$$

which entails $\mathrm{Sl}_\delta(u) \cap Y = \varnothing$. Since $\mathrm{Sl}_\delta(u) \cap \mathrm{B} = \varnothing$, by the choice of δ, then $\mathrm{Sl}_\delta(u) \cap (C_0 \cup Y) = \varnothing$. However, the set $\mathrm{B} \setminus \mathrm{Sl}_\delta(u)$ is closed and convex, so it contains C'. The proof is completed. □

LEMMA 4.2. *Under the assumptions of the previous lemma for every* $\varepsilon \in (0, r-s)$ *there exists* $\delta_0 \in (0, \varepsilon/2)$ *such that for every* $\delta \in (0, \delta_0]$, *every* $C \in B(C', \delta)$, *and every* $x \in \mathrm{B}(\theta, \delta)$, *the following hold:*

$$C \setminus \mathrm{B}(x, f(x,C) - 4\delta) \subset \bigcup_{j=1}^{m} \mathrm{B}[y_j, \varepsilon], \tag{4.3}$$

$$\forall j \quad (\mathrm{B}[y_j, \varepsilon] \cap C) \setminus \mathrm{B}(x, f(x,C) - 4\delta) \neq \varnothing, \tag{4.4}$$

which actually means that for $j = 1, \dots, m$ *there are nonempty closed sets* $M_j(x,C) \subset \mathrm{B}[y_j, \varepsilon]$ *such that*

$$C \setminus \mathrm{B}(x, f(x,C) - 4\delta) = \bigcup_{j=1}^{m} M_j(x,C). \tag{4.5}$$

Proof. There is no loss of generality in assuming $a = \theta$. Take $\varepsilon \in (0, r - s)$. As a consequence of Lemma 4.1, there exists $\delta' > 0$ such that

$$C' \setminus (r - \delta')\mathrm{B} \subset \bigcup_{j=1}^{m} \mathrm{B}\left(y_j, \frac{\varepsilon}{2}\right). \tag{4.6}$$

Let now $\delta_0 = \min\{\delta'/8, \varepsilon/2\}$. Take $\delta \in (0, \delta_0]$, $C \in B(C', \delta)$, $x \in \mathrm{B}(\theta, \delta)$, and suppose $z \in C \setminus \mathrm{B}(x, f(x,C) - 4\delta)$. There is $u \in C'$ satisfying $|u - z| \leq \delta$. It is easy to check

$$r - 2\delta < f(x,C) < r + 2\delta. \tag{4.7}$$

Then

$$r - 6\delta < f(x,C) - 4\delta \leq |z - x| \leq |z - u| + |u| + |x| < |u| + 2\delta, \tag{4.8}$$

that is, $|u| > r - 8\delta_0 \geq r - \delta'$ and (4.6) implies $d(u, Y) < \varepsilon/2$.

Further,

$$d(z, Y) \leq |z - u| + d(u, Y) < \delta + \frac{\varepsilon}{2} \leq \varepsilon, \tag{4.9}$$

which verifies (4.3).

Concerning (4.4), for each $j = 1, \dots, m$ there exists $z_j \in C$ such that $|z_j - y_j| < \delta < \varepsilon$. Then by (4.7),

$$f(x,C) - 2\delta < r = |y_j| \leq |y_j - z_j| + |z_j - x| + |x| < |z_j - x| + 2\delta, \tag{4.10}$$

whence $f(x,C) - 4\delta < |z_j - x|$ for $j = 1, \dots, m$. The proof is completed. □

Lemma 4.3. *Under the assumptions of Lemma 4.1, there exists* $\delta_0 > 0$ *such that for* $\delta \in (0, \delta_0]$, $C \in B(C', \delta)$, $x \in \mathrm{B}(a, \delta_0)$, *and* $j \in \{1, \dots, m\}$ *the following inequality holds:*

$$\left| f(x, C \cap \mathrm{B}[y_j, \delta]) - |x - y_j| \right| \leq \chi(C, C'). \tag{4.11}$$

Proof. Assume without any loss of generality $a = \theta$ and $r = 1$. Let $\varepsilon' > 0$ be chosen so that the sets $s\mathrm{B}$ and $\mathrm{B}[y_j,\varepsilon']$ for $j = 1,\dots,m$ be pairwise disjoint. Find $\gamma > 0$ such that

$$\mathrm{Sl}_\gamma(y_j) = \{z \in \mathrm{B} : \langle y_j,z\rangle \geq 1-\gamma\} \subset \mathrm{B}(y_j,\varepsilon'). \tag{4.12}$$

Find next, $\varepsilon > 0$ so that in turn $\mathrm{B}(y_j,2\varepsilon) \subset \mathrm{Sl}_\gamma(y_j)$, $j = 1,\dots,m$.

According to Lemma 4.2 there exists $\delta_0 \in (0,\varepsilon/3)$ such that (4.3) and (4.4) hold whenever $\delta \in (0,\delta_0)$, $C \in B(C',\delta)$, and $x \in \mathrm{B}(\theta,\delta)$. Now, the inequality

$$|x-y_j| \leq f(x,C\cap \mathrm{B}[y_j,\delta]) + \chi(C,C'), \quad j = 1,\dots,m \tag{4.13}$$

is obviously fulfilled. In order to verify

$$f(x,C\cap \mathrm{B}[y_j,\delta]) \leq |x-y_j| + \chi(C,C'), \quad j = 1,\dots,m, \tag{4.14}$$

fix j and take $y \in C \cap \mathrm{B}[y_j,\delta]$. For every $\sigma \in (0,\varepsilon/3)$ there exists $y_\sigma \in \mathrm{co}(C_0 \cup Y)$ with $|y-y_\sigma| < \chi(C,C')+\sigma$. Obviously, $y_\sigma \in \mathrm{B}(y_j,\varepsilon)$.

Our next goal is to prove

$$|x-y_\sigma| \leq |x-y_j|. \tag{4.15}$$

If $y_\sigma \neq y_j$ and y_σ is given in the form

$$y_\sigma = \lambda_0 y_0 + \sum_{i=1}^{m}\lambda_i y_i, \quad y_0 \in C_0, \qquad \sum_{i=0}^{m}\lambda_i = 1, \quad \lambda_i \geq 0,\ i = 0,\dots,m, \tag{4.16}$$

then for $y'_\sigma = (1-\lambda_j)^{-1}(\lambda_0 y_0 + \sum_{i=1,i\neq j}^{m}\lambda_i y_i)$ we have $y_\sigma \in [y_j,y'_\sigma]$. If $y'_\sigma \notin \mathrm{B}(x,f(x,C')-4\delta)$ then, according to Lemma 4.2 for the case $C = C'$, $y'_\sigma \in \cup_{i=1}^{m}\mathrm{B}[y_i,\varepsilon]$. However, $y'_\sigma \notin \mathrm{B}[y_j,\varepsilon]$ because this ball is strongly separated from $\overline{\mathrm{co}}(C_0 \cup Y \setminus \{y_j\})$ by a hyperplane orthogonal to y_j. Hence, there is unique i, $i \neq j$, such that $y'_\sigma \in \mathrm{B}[y_i,\varepsilon]$. In that case there exists $y''_\sigma \in G = \{z \in \mathrm{B} : \langle y_i,z\rangle = \gamma\}$ satisfying $y_\sigma \in [y_j,y''_\sigma]$. The hyperplane segment $G \cap C'$ which is contained in $\mathrm{B}(y_i,\varepsilon')$ is also a subset of $\mathrm{B}(x,f(x,C')-4\delta)$, due to Lemma 4.2, since it does not meet the set $Y+\varepsilon\mathrm{B}$.

Thus, either $y'_\sigma \in \mathrm{B}(x,f(x,C')-4\delta)$ or there is another point y''_σ from that ball such that $y_\sigma \in [y_j,y''_\sigma]$. In both cases y_σ belongs to a line segment with one end-point in $\mathrm{B}(x,f(x,C')-4\delta)$ and the other one being y_j. On the other hand, it is easy to check

$$\mathrm{B}(x,f(x,C')-4\delta) \subset \mathrm{B}(x,|x-y_j|), \tag{4.17}$$

whence (4.15) follows.

Therefore,

$$|x-y| \leq |x-y_\sigma| + |y_\sigma - y| < |x-y_j| + \chi(C,C') + \sigma \tag{4.18}$$

for arbitrary $\sigma \in (0,\varepsilon/2)$, which implies (4.14). The proof is completed. □

LEMMA 4.4. *Suppose* (C_n) *is a sequence in* $\mathscr{C}$, $\lim C_n = C_0$, *and for some* $y \in \mathbb{H}$ *and* $r > 0$, $C_n \cap \mathrm{B}(y,r) \neq \varnothing$ *for* $n = 0,1,\ldots$. *Then,*

$$\lim \chi(C_n \cap \mathrm{B}(y,r), C_0 \cap \mathrm{B}(y,r)) = 0. \tag{4.19}$$

Proof. Let $\varepsilon > 0$ be arbitrary, and $\sigma > 0$ is chosen so that both σ and $(\sigma^2 + 2r\sigma)^{1/2}$ are smaller than $\varepsilon/2$. There is $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and every $x \in C_0 \cap \mathrm{B}(y,r)$ there exists $x_n \in C_n$ with $|x - x_n| < \sigma$. If x_n is not already in $\mathrm{B}(y,r)$, then there are $z_n \in C_n \cap \mathrm{B}(y,r)$ and $v_n \in (x_n, z_n) \cap \mathrm{S}(y,r)$.

It is easy to check $|x - v_n| < \sigma + (\sigma^2 + 2r\sigma)^{1/2} < \varepsilon$, and to find another point $w_n \in (x_n, z_n) \cap \mathrm{B}(y,r)$ which is sufficiently close to v_n such that $|x - w_n| < \varepsilon$. Certainly, $w_n \in C_n$.

Thus $C_n \cap \mathrm{B}(y,r) + \mathrm{B}(\theta,\varepsilon) \supset C_0 \cap \mathrm{B}(y,r)$, whenever $n \geq n_0$. Similarly, one proves $C_0 \cap \mathrm{B}(y,r) + \mathrm{B}(\theta,\varepsilon) \supset C_n \cap \mathrm{B}(y,r)$ for $n \geq n_0$, whence

$$\chi(C_n \cap \mathrm{B}(y,r), C_0 \cap \mathrm{B}(y,r)) < \varepsilon, \quad n \geq n_0. \tag{4.20}$$

□

5. Main construction

PROPOSITION 5.1. *Suppose X is a nonempty separable convex subset of* $\mathbb{H}$, *and let* $m \in \mathbb{N}$ *be such that*

$$\dim X \geq m. \tag{5.1}$$

Given $C_0 \in \mathscr{C}$, $a \in X$, $\varepsilon > 0$, *there are* $C' \in \mathscr{C}$, $\kappa > 0$,

$$B[C', \kappa] \subset B(C_0, \varepsilon), \tag{5.2}$$

and a residual subset $\mathscr{R}(a,\varepsilon) \subset B[C',\kappa]$ *with respect to the relative topology in* $B[C',\kappa]$ *induced by the Hausdorff metric* χ *such that*

$$\operatorname{card}\left(L_X^m(C) \cap \mathrm{B}(a,\varepsilon) \cap \operatorname{usc} Q_C\right) \geq \mathbf{c} \tag{5.3}$$

whenever $C \in \mathscr{R}(a,\varepsilon)$, *that is, the m-locus of* $C \in \mathscr{R}(a,\varepsilon)$ *meets the set* $X \cap \mathrm{B}(a,\varepsilon)$ *at continuum points at which* $Q(\cdot, C)$ *is upper semicontinuous.*

Proof. It is no loss of generality to assume that $a = \theta$, $f(\theta, C_0) > 2\varepsilon$, and that there exists an m-dimensional subspace H such that θ belongs to the relative interior of the set $X' = H \cap X$, that is, for some $\varepsilon' > 0$ the following inclusion holds:

$$H \cap \mathrm{B}(\theta, \varepsilon') \subset X'. \tag{5.4}$$

In case $m = 1$, the proof is more simple and follows the scheme of the general case. Assume $m > 1$.

Put $f_0 = f(\theta, C_0)$ and find points $y_1, y_2, \ldots, y_m \in \mathbb{H}$ such that

$$y_j \in \mathrm{S}\left(\theta, f_0 + \frac{\varepsilon}{3}\right), \quad d(y_j, C_0) < \frac{\varepsilon}{2}, \quad j = 1, \ldots, m. \tag{5.5}$$

Certainly,

$$d(y_j, C_0) \geq \frac{\varepsilon}{3}, \quad j = 1,\dots,m. \tag{5.6}$$

Denote by π the orthogonal projection onto H. Let $Y = \{y_1,\dots,y_m\}$ and $v_j = \pi(y_{j+1} - y_1)$ for $j = 1,\dots,m-1$. Take $v_0 \in H \setminus \operatorname{span} V$ and designate

$$V = \{v_1,\dots,v_{m-1}\}, \qquad \widetilde{V} = \{v_0\} \cup V. \tag{5.7}$$

It is assumed also, which is no loss of generality, that the elements of Y have already been chosen in such a way that the elements of $\widetilde{V}$ are linearly independent, that is, $\operatorname{span} \widetilde{V} = H$.

Put

$$C' = \overline{\operatorname{co}}(C_0 \cup Y). \tag{5.8}$$

According to Lemma 4.3 there exists $\varepsilon'' > 0$ such that for every $\delta \in (0, \varepsilon'']$, $C \in \mathrm{B}[C', \delta]$, $x \in \mathrm{B}(\theta, \delta)$, and every $j = 1,\dots,m$ the inequality (4.11) holds.

Denote $\varepsilon''' = 4^{-1} \min\{|y_i - y_j| : i, j = 1,\dots,m,\ i \neq j\}$ and

$$\varepsilon_0 = \min\left\{\varepsilon', \varepsilon'', \varepsilon''', \frac{\varepsilon}{6}\right\}. \tag{5.9}$$

In view of Lemma 4.2 there exists a number δ, from now on it is fixed,

$$0 < \delta < \frac{\varepsilon_0}{4}, \tag{5.10}$$

such that for every $C \in B(C', \delta)$ and $x \in \mathrm{B}(\theta, \delta)$ there are nonempty closed sets $M_j(x, C)$ for $j = 1,\dots,m$ such that

$$C \setminus \mathrm{B}(x, f(x, C) - 4\delta) = \bigcup_{j=1}^{k} M_j(x, C), \quad M_j(x, C) \subset \mathrm{B}[y_j, \varepsilon_0]. \tag{5.11}$$

Now for $r > 0$ define $\widetilde{I}_r = \{x \in H : |\langle v, x\rangle| \leq r,\ v \in \widetilde{V}\}$. Since $\widetilde{V}$ is a basis for H, then for each $r > 0$ the set $\widetilde{I}_r$ is bounded and $\lim \operatorname{diam} \widetilde{I}_r = 0$. Having also in mind (5.4), fix $r \in \mathbb{R}$ such that

$$\widetilde{I}_r \subset X \cap \mathrm{B}(\theta, \delta). \tag{5.12}$$

The following functions are defined on H:

$$\gamma_j(x) = |x - y_1| - |x - y_{j+1}|, \quad j = 1,\dots,m-1. \tag{5.13}$$

Denoting

$$A_j^{\pm} = \{x \in \widetilde{I}_r : \langle v_j, x\rangle = \pm r\}, \quad j = 1,\dots,m-1, \tag{5.14}$$

observe that for any $j = 1,\dots,m-1$ the sign of $\gamma_j(\cdot)$ is equal to the sign of the inner product $\langle v_j,\cdot\rangle$. This is a consequence of the equalities

$$|x-y_1|^2 - |x-y_{j+1}|^2 = 2\langle y_{j+1}-y_1,x\rangle = \langle v_j,x\rangle, \tag{5.15}$$

since

$$\langle y_{j+1}-y_1-v_j,x\rangle = 0 \quad \text{for } x\in H,\ j=1,\dots,m-1. \tag{5.16}$$

Thus $\gamma_j(\cdot)$ take opposite sign values on the corresponding faces A_j^- and A_j^+.

Since $A_j^\pm$ are compact, $\gamma_j(x)$ attains minimal and maximal values on them for each $j=1,\dots,m-1$. Denote for $j=1,\dots,m-1$,

$$-\alpha_j = \max\{\gamma_j(x) : x\in A_j^-\}, \qquad \beta_j = \min\{\gamma_j(x) : x\in A_j^+\}, \tag{5.17}$$

Choose $\kappa > 0$ such that

$$\kappa < \delta, \tag{5.18}$$

$$\kappa \le \frac{1}{3}\min\{\alpha_j,\beta_j : j=1,\dots,m-1\}. \tag{5.19}$$

By (5.8), (5.5), (5.10), and (5.18), for $C\in B[C',\kappa]$ we have

$$\chi(C,C_0) \le \chi(C,C') + \chi(C',C_0) < \kappa + \frac{\varepsilon}{2} < \varepsilon, \tag{5.20}$$

thus verifying the inclusion (5.2).

For every $C\in B[C',\kappa]$ and $j=1,\dots,m$, define $N_j(C) = C\cap \mathrm{B}[y_j,\varepsilon_0]$. Notice that (4.11) can be rewritten in the following way:

$$\big|f(x,N_j(C)) - |x-y_j|\big| \le \kappa, \quad j=1,\dots,m. \tag{5.21}$$

Consider the following functions defined on $H\cap \mathrm{B}(\theta,\delta)$:

$$\gamma_j(x,C) = f(x,N_1(C)) - f(x,N_{j+1}(C)), \quad j=1,\dots,m-1. \tag{5.22}$$

Let $t\in[-r,r]$ be fixed. Denoting $L_t = \{x\in H : \langle v_0,x\rangle = t\}$, and $I_r(t) = \widetilde{I}_r\cap L_t$, we apply the Brouwer-Miranda theorem to the set $I_r(t)$ and the functions defined by (5.22). In order to verify the boundary, conditions take, for instance, $x\in A_j^-\cap L_t$ for $j=1,\dots,m-1$, and then, having in mind (5.19) and (5.21),

$$\gamma_i(x,C) \le |x-y_1| - |x-y_{i+1}| + 2\kappa \le -\alpha_i + 2\kappa < 0. \tag{5.23}$$

Analogously, for $x\in A_i^+\cap L_t$,

$$\gamma_i(x,C) \ge |x-y_1| - |x-y_{i+1}| - 2\kappa \ge \beta_i - 2\kappa > 0. \tag{5.24}$$

Therefore, for every $t \in [-r,r]$ there exists at least one point $\hat{x}_t \in I_r(t)$ at which all functions in (5.22) vanish, that is, $\hat{x}_t \in \tau(N_j(C))_{j=1}^m$. Now, according to (5.11) and (5.18) for every $x \in \mathrm{B}(\theta,\delta)$ and $C \in B[C',\kappa]$, we have $M_j(x,C) \subset N_j(C)$, $j = 1,\dots,m$, and then

$$Q(x,C) \subset \bigcup_{j=1}^{m} N_j(C), \tag{5.25}$$

Thus for every $C \in B[C',\kappa]$ and $t \in [-r,r]$ there is $\hat{x}_t \in I_r(t)$ such that for $j = 1,\dots,m$,

$$f(\hat{x}_t,C) = f(\hat{x}_t,N_j(C)). \tag{5.26}$$

In order to prove that the antiprojections $Q(\cdot,C)$ are actually m-valued and u.s.c. at "many points around θ," we make further considerations involving topological lemmas from Section 3. Introduce complete metric spaces $\mathcal{T} = B[C',\kappa] \times [-r,r]$ and $\mathcal{S} = \mathcal{T} \times \widetilde{I}_r$ with the box metric ρ on the products, that is, for $(C,t,x),(D,s,y) \in \mathcal{S}$,

$$\rho((C,t,x),(D,s,y)) = \max\{\chi(C,D),|t-s|,|x-y|\}. \tag{5.27}$$

Define a set-valued mapping $G : \mathcal{T} \to \widetilde{I}_r$ by

$$G(C,t) = I_r(t) \cap \tau(N_j(C))_{j=1}^m. \tag{5.28}$$

The images of G are nonempty, and by the continuity of the farthest distance function they are closed, hence compact as $\widetilde{I}_r$ is compact.

Also, in view of Lemma 4.4, G is upper semicontinuous. To verify this, it is sufficient to notice that if (C_n,t_n) is a sequence in $\mathcal{T}$ convergent to some $(C,t) \in \mathcal{T}$, then the sequences $(N_j(C_n))$ converge to $N_j(C)$ for $j = 1,\dots,m$ with respect to the distance χ, while $I_r(t_n)$ obviously converges to $I_r(t)$. Having in mind the continuity of the farthest distance $f(\cdot,\cdot)$ with respect to both the arguments and the compactness of $\widetilde{I}_r$, it is possible for any sequence (u_n), $u_n \in G(C_n,t_n)$ to find a cluster point in $G(C,t)$.

The theorem of Fort implies that $\operatorname{lsc} G$ is a residual subset of $\mathcal{T}$. Denote by Λ the graph of G

$$\Lambda = \{(C,t,x) \in \mathcal{S} : x \in G(C,t)\}, \tag{5.29}$$

and by Λ' the "graph of lower semicontinuity"

$$\Lambda' = \{(C,t,x) \in \Lambda : (C,t) \in \operatorname{lsc} G\}. \tag{5.30}$$

According to Lemma 3.6, Λ' is a Baire space in the relative topology induced by the product topology of $\mathcal{T} \times I_r$.

Let for $n \in \mathbb{N}$,

$$\begin{aligned} \mathcal{U}_n = \{&(C,t,x) \in \Lambda' : \exists s,\ 0 < s < 4\delta, \\ &\operatorname{diam}(N_j(C) \setminus \mathrm{B}(x, f(x,N_j(C)) - s)) < n^{-1},\ j = 1,\dots,m\}. \end{aligned} \tag{5.31}$$

Claim. $\mathcal{U}_k$ contains open dense set in Λ'.

Take arbitrary $(D^0,t_0,x_0) \in \Lambda'$ and $\sigma > 0$. It has to be proved that $B_{\mathcal{G}}((D_0,t_0,x_0),\sigma) \cap \mathcal{U}_n$ contains an open subset of Λ'. Assume without any loss of generality that $D_0 \in B(C',\kappa)$. Indeed, due to the l.s.c. of G at (D_0,t_0) and, by [13], the fact that $\operatorname{lsc} G$ is residual in the open dense subset $B(C',\kappa) \times [-r,r]$ of $\mathcal{T}$, one can conclude that arbitrary close to (D_0,t_0,x_0) there are points (D,t,x) from Λ' with $D \in B(C',\kappa)$. Hence there is $\lambda \in (0,\kappa)$, $\lambda < \sigma/2$, such that $\chi(D_0,C') < \kappa - \lambda$.

Denoting $K_j = (\mathbb{H} \setminus \mathrm{B}[x_0, f(x_0,D_0)]) \cap \mathrm{B}(y_j,\varepsilon_0)$, $j = 1,\dots,m$, we have $K_j \neq \varnothing$. Certainly, there are $y \in \mathrm{B}(y_j,\varepsilon_0)$ satisfying $|y| > |y_j| + 3\varepsilon_0/4$ and if we assume $|x_0 - y| \le f(x_0,D_0)$ for all $y \in \mathrm{B}(y_j,\varepsilon_0)$, then (also having in mind (5.10) and (5.18))

$$|y| \le |y - x_0| + |x_0| < f(x_0,D_0) + \delta < |y_j| + \kappa + 2\delta < |y_j| + \frac{3\varepsilon_0}{4} \tag{5.32}$$

gives a contradiction.

Since

$$\inf\{d(z,N_j(D_0)) : z \in K_j\} = 0, \quad j = 1,\dots,m, \tag{5.33}$$

it is a matter of routine to find points $z_j \in K_j$, $j = 1,\dots,m$, and $\xi > 0$ such that

$$d(z_j,N_j(D_0)) < \lambda, \quad z_j \in \mathrm{S}(x_0, f(x_0,D_0) + \xi), \quad j = 1,\dots,m. \tag{5.34}$$

Consider the set $D' = \overline{\mathrm{co}}(D_0 \cup \{z_1,\dots,z_m\})$. Obviously, $Q(x_0,D') = \{z_1,\dots,z_m\}$.

Now, let $\mu > 0$, $\mu < \min\{\xi,\lambda,(2n)^{-1}\}$ be chosen so that $\mathrm{B}(z_j,\mu) \subset \mathrm{B}(y_j,\varepsilon_0)$ for all $j = 1,\dots,m$. Apply Lemma 4.2 with respect to the sets D_0, D' and the point x_0 (instead of C_0, C' and θ, resp.). There is $\eta \in (0,\mu/2)$ such that for every $D \in B[D',\eta]$ and $x \in \mathrm{B}(x_0,\eta)$ there are nonempty closed sets $M_j(x,D) \subset \mathrm{B}[z_j,\mu]$, $j = 1,\dots,m$, satisfying

$$D \setminus \mathrm{B}(x, f(x,D) - 4\eta) = \bigcup_{j=1}^{m} M_j(x,D). \tag{5.35}$$

Notice that $\operatorname{diam} M_j(x,D) \le 2\mu < n^{-1}$ and then for $s = 4\eta$,

$$\operatorname{diam}(N_j(D) \setminus \mathrm{B}(x, f(x,N_j(D)) - s)) < n^{-1}, \quad j = 1,\dots,m. \tag{5.36}$$

Also $B[D',\eta] \subset B[C',\kappa]$, and the claim easily follows from the lower semicontinuity of G applied for $x_0 \in G(D_0,t_0)$.

Finally, put $\mathcal{U} = \cap_{n=1}^{\infty} \mathcal{U}_n$. The set $\mathcal{U}$ is residual in Λ' and by Lemma 3.7 is orthogonally projected on a residual subset $\mathcal{V}$ of $\mathcal{T}$. Thus for every $(C,t) \in \mathcal{V}$ there is $x(t) \in G(C,t)$ such that

$$x(t) \in I_r(t) \cap \tau(N_j(C))_{j=1}^{m}, \tag{5.37}$$

and all mappings $Q(\cdot,N_j(C))$ for $j = 1,\dots,m$ are single-valued and upper semicontinuous at $x(t)$. Apply the Kuratowski-Ulam theorem to the product space $\mathcal{T}$ to show the existence of a residual subset $\mathcal{R}(\theta,\varepsilon)$ of $B[C',\kappa]$ such that, in view of Alexandroff-Urysohn theorem, (5.37) is satisfied for a continuum of reals $t \in [-r,r]$, whenever $C \in \mathcal{R}(\theta,\varepsilon)$. Thus (5.3) is verified and the proof is completed. $\square$

Proposition 5.2. *If in Proposition 5.1 the condition (5.1) is replaced by the following one:*

$$\dim X = m - 1, \tag{5.38}$$

then in the conclusion (5.3) is to be replaced by

$$C_X^m(C) \cap \mathrm{B}(a,\varepsilon) \cap \operatorname{usc} Q_C \neq \varnothing \quad \textit{for } C \in \mathcal{R}(a,\varepsilon). \tag{5.39}$$

Proof. In case $m = 1$, the set X is a singleton and it is a matter of routine to have a direct proof of the fact that there exists a residual subset of a ball $B[C',\kappa] \subset \mathscr{C}$ such that both (5.2) and (5.39) are fulfilled.

The proof in the general case differs slightly from the proof of the previous proposition. For instance, the theorem of Brouwer-Miranda is applied with respect to the set $I_r = \{x \in H : |\langle v,x\rangle| \leq r,\ v \in V\}$ instead of $\widetilde{I}_r$, and $H = \operatorname{span} V$. $\square$

6. Proof of Theorem 1.1

Suppose $\dim H \geq m$. By Proposition 5.1 and by Lemma 3.5 for every $a \in X$ and every $\varepsilon > 0$, there is a residual subset $\mathcal{R}(a,\varepsilon)$ of $\mathcal{B}$ such that

$$\operatorname{card}\left(C_X^{n,m}(C) \cap \mathrm{B}(a,\varepsilon) \cap \operatorname{usc} Q_C\right) \geq \mathbf{c} \quad \text{for } C \in \mathcal{R}(a,\varepsilon). \tag{6.1}$$

Let $\{a_1,a_2,\dots\}$ be a countable dense set in X. Put

$$\mathcal{R} = \bigcap_{n=1}^{\infty} \mathcal{R}(a_n,n^{-1}). \tag{6.2}$$

For every $C \in \mathcal{R}$ the locus $L_X^m(C)$ intersects an arbitrary nonempty open subset U of X at a set containing at least continuum points of upper semicontinuity of $Q(\cdot,C)$, that is, the set $L_X^m(C) \cap \operatorname{usc} Q_C$ is everywhere continual in X.

If $\dim H = m - 1$, then by Proposition 5.2 and Lemma 3.5 again, for typical $C \in \mathscr{C}$ the set $L_X^m(C) \cap \operatorname{usc} Q_C$ is dense in X. The theorem is proved.

Corollary 6.1. *If X is a nonempty convex subset of $\mathbb{H}(\Omega)$ and Ω is a finite set, that is, $\mathbb{H}(\Omega) = \mathbb{R}^k$, $k \in \mathbb{N}$, then for a typical convex compact K of $\mathbb{R}^k$ the loci $L_X^m(K)$ in X partition X into a finite sequence of dense sets such that*

(i) *$L_X^1(K)$ is dense G_δ,*

(ii) *$L_X^m(K)$ are everywhere continual, for $1 < m \leq \dim X$,*

(iii) *$L_X^m(K)$ are dense, whenever $m = \dim X + 1$.*

Proof. (i) If $m = 1$ and $\dim X \geq 1$, then the mapping $Q(\cdot,M)$ is u.s.c. with nonempty images, and it is single-valued on a dense subset of X. It follows from a known result, for instance [29], that it is single-valued on a dense G_δ subset of X.

To prove (ii) and (iii), recall a result from [14], proved in another setting but adaptable for the present purpose, which states that $L_X^m(K) = \varnothing$ for typical $K \in \mathscr{C}$ whenever $m > \dim X + 1$. Apply Proposition 5.1 for establishing (ii), and Proposition 5.2 for (iii). $\square$

COROLLARY 6.2. *Suppose* Ω *is a countable set, that is,* $\mathbb{H}(\Omega)$ *is a separable Hilbert space. Then for every* $m \in \mathbb{N}$ *there exists a residual subset* $\mathcal{R}^m$ *of* $\mathcal{C}$ *such that* $L^m(C)$ *is dense in* $\mathbb{H}(\Omega)$ *whenever* $C \in \mathcal{R}^m$.

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