AN EXAMPLE FOR A ONE-PARAMETER NONEXPANSIVE SEMIGROUP

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We give one example for a one-parameter nonexpansive semigroup. This example shows that there exists a one-parameter nonexpansive semigroup $\{T(t) : t \geq 0\}$ on a closed convex subset C of a Banach space E such that $\lim_{t\to\infty} \|(1/t)\int_0^t T(s)x\,ds - x\| = 0$ for some $x \in C$, which is not a common fixed point of $\{T(t) : t \geq 0\}$.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively.

A family $\{T(t) : t \geq 0\}$ of mappings on C is called a one-parameter nonexpansive semigroup on a subset C of a Banach space E if the following hold:

(sg1) for each $t \geq 0$, $T(t)$ is a nonexpansive mapping on C, that is,

$$\left\|T(t)x - T(t)y\right\| \leq \|x - y\| \quad \forall x, y \in C; \tag{1.1}$$

(sg2) $T(0)x = x$ for all $x \in C$;

(sg3) $T(s+t) = T(s) \circ T(t)$ for all $s, t \geq 0$;

(sg4) for each $x \in C$, the mapping $t \mapsto T(t)x$ is continuous.

We know that $\{T(t) : t \geq 0\}$ has a common fixed point under the assumption that C is weakly compact convex and E has the Opial property; see [3, 4, 5, 6, 8, 10, 12] and other works.

Convergence theorems for one-parameter nonexpansive semigroups are proved in [1, 2, 9, 11, 13, 15] and other works. For example, Baillon and Brezis in [2] proved the following theorem; see also [16, page 80].

Theorem 1.1 (Baillon and Brezis [2]). *Let C be a bounded closed convex subset of a Hilbert space E and let $\{T(t) : t \geq 0\}$ be a one-parameter nonexpansive semigroup on C. Then, for any $x \in C$,*

$$\frac{1}{t}\int_0^t T(s)x\,ds \tag{1.2}$$

converges weakly to a common fixed point of $\{T(t) : t \geq 0\}$ as $t \to \infty$.

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Also, Suzuki and Takahashi in [15] proved the following.

Theorem 1.2 (Suzuki and Takahashi [15]). *Let C be a compact convex subset of a Banach space E and let $\{T(t): t \geq 0\}$ be a one-parameter nonexpansive semigroup on C. Let $x_1 \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_{n+1} = \frac{\alpha_n}{t_n}\int_0^{t_n} T(s)x_n\,ds + (1-\alpha_n)x_n \tag{1.3}$$

for $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1]$ and $\{t_n\} \subset (0,\infty)$ satisfy the following conditions:

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1, \qquad \lim_{n\to\infty} t_n = \infty, \qquad \lim_{n\to\infty} \frac{t_{n+1}}{t_n} = 1. \tag{1.4}$$

Then $\{x_n\}$ converges strongly to a common fixed point z_0 of $\{T(t): t \geq 0\}$.

The following theorem plays a very important role in the proof of Theorem 1.2.

Theorem 1.3 (Suzuki and Takahashi [15]). *Let C be a compact convex subset of a Banach space E. Let $\{T(t): t \geq 0\}$ be a one-parameter nonexpansive semigroup on C. Then for $z \in C$, the following are equivalent:*

(i) *z is a common fixed point of $\{T(t): t \geq 0\}$;*
(ii)

$$\liminf_{t\to\infty} \left\| \frac{1}{t}\int_0^t T(s)z\,ds - z \right\| = 0 \tag{1.5}$$

holds.

Recently, Suzuki proved in [14] the following result similar to Theorem 1.3. This theorem also plays a very important role in the proof of the existence of some nonexpansive retraction onto the set of common fixed points.

Theorem 1.4 (Suzuki [14]). *Let E be a Banach space with the Opial property and let C be a weakly compact convex subset of E. Let $\{T(t): t \geq 0\}$ be a one-parameter nonexpansive semigroup on C. Then for $z \in C$, the following are equivalent:*

(i) *z is a common fixed point of $\{T(t): t \geq 0\}$;*
(ii) *formula (1.5) holds;*
(iii) *there exists a subnet of a net*

$$\left\{ \frac{1}{t}\int_0^t T(s)z\,ds \right\} \tag{1.6}$$

in C converging weakly to z.

So, it is a natural problem whether or not the conclusion of Theorems 1.3 and 1.4 holds in general. In this paper, we give one example concerning Theorems 1.3 and 1.4. This example shows that there exists a one-parameter nonexpansive semigroup $\{T(t): t \geq 0\}$

on a closed convex subset C of a Banach space E such that

$$\lim_{t\to\infty}\left\|\frac{1}{t}\int_0^t T(s)x\,ds - x\right\| = 0 \tag{1.7}$$

for some $x \in C$, which is not a common fixed point of $\{T(t) : t \geq 0\}$. That is, our answer of the problem is negative.

2. Example

We give one example concerning Theorems 1.3 and 1.4. See also [7, Example 3.7].

Example 2.1. Put $\Omega = \{-1\} \cup [0,\infty)$, let E be the Banach space consisting of all bounded continuous functions on Ω with supremum norm, and define a subset C of E by

$$C = \left\{x \in E : \begin{array}{l} 0 \leq x(u) \leq 1 \text{ for } u \in \Omega, \\ |x(u_1) - x(u_2)| \leq |u_1 - u_2| \text{ for } u_1, u_2 \in [0,\infty) \end{array}\right\}. \tag{2.1}$$

Define a nonexpansive semigroup $\{T(t) : t \geq 0\}$ as follows. For $t \in [0,1]$, define

$$(T(t)x)(u) = \begin{cases} x(u), & \text{if } u = -1, \\ x(u-t), & \text{if } u \geq t, \\ x(0) - t + u, & \text{if } 0 \leq u \leq t, \\ & 1 - \alpha_x(1-t+u) \leq x(0) - t + u, \\ x(0) + t - u, & \text{if } 0 \leq u \leq t, \\ & 1 - \alpha_x(1-t+u) \geq x(0) + t - u, \\ 1 - \alpha_x(1-t+u), & \text{if } 0 \leq u \leq t, \\ & |1 - \alpha_x(1-t+u) - x(0)| \leq t - u, \end{cases} \tag{2.2}$$

where

$$\alpha_x(1-t+u) = \sup\{x(s) : s \in \{-1\} \cup [1-t+u,\infty)\}. \tag{2.3}$$

For $t \in (1,\infty)$, there exist $m \in \mathbb{N}$ and $t' \in [0,1/2)$ satisfying $t = m/2 + t'$. Define $T(t)$ by

$$T(t) = T\left(\frac{1}{2}\right)^m \circ T(t'). \tag{2.4}$$

Then $0 \in C$ is not a common fixed point of $\{T(t) : t \geq 0\}$ and

$$\lim_{t\to\infty}\left\|\frac{1}{t}\int_0^t T(s)0\,ds - 0\right\| = 0 \tag{2.5}$$

holds.

Before proving Example 2.1, we need some lemmas.

Lemma 2.2. *The following hold:*

(i) $|\alpha_x(u_1) - \alpha_x(u_2)| \leq |u_1 - u_2|$ *for* $x \in C$ *and* $u_1, u_2 \in [0, \infty)$;

(ii) $|\alpha_x(u) - \alpha_y(u)| \leq \|x - y\|$ *for* $x, y \in C$ *and* $u \in [0, \infty)$.

Proof. We first show (i). Without loss of generality, we may assume $u_1 < u_2$. For $s \in [u_1, u_2]$, we have $|x(s) - x(u_2)| \leq |s - u_2|$ and hence

$$x(s) \leq x(u_2) + |s - u_2| \leq \alpha_x(u_2) + |u_1 - u_2|. \tag{2.6}$$

For $s \in [u_2, \infty)$, we have

$$x(s) \leq \alpha_x(u_2) \leq \alpha_x(u_2) + |u_1 - u_2|. \tag{2.7}$$

Hence,

$$\alpha_x(u_1) \leq \alpha_x(u_2) + |u_1 - u_2| \tag{2.8}$$

holds. Since $\alpha_x(u_2) \leq \alpha_x(u_1)$, we obtain

$$|\alpha_x(u_1) - \alpha_x(u_2)| \leq |u_1 - u_2|. \tag{2.9}$$

We next show (ii). For each $\varepsilon > 0$, there exists $s \in \{-1\} \cup [u, \infty)$ satisfying $x(s) > \alpha_x(u) - \varepsilon$. We have

$$\alpha_x(u) - \alpha_y(u) \leq x(s) + \varepsilon - y(s) \leq \|x - y\| + \varepsilon. \tag{2.10}$$

Since ε is arbitrary, we have $\alpha_x(u) - \alpha_y(u) \leq \|x - y\|$. Similarly we obtain $\alpha_y(u) - \alpha_x(u) \leq \|x - y\|$ and hence $|\alpha_x(u) - \alpha_y(u)| \leq \|x - y\|$. □

Lemma 2.3. *Fix* $x \in C$, $t \in [0,1]$, *and* u_1, u_2 *with* $0 \leq u_1 \leq u_2 \leq t$. *Then the following hold:*

(i) $1 - \alpha_x(1 - t + u_1) < (T(t)x)(u_2) - u_2 + u_1$ *implies that* $(T(t)x)(u_1) = x(0) - t + u_1$ *and* $(T(t)x)(u_2) = x(0) - t + u_2$;

(ii) $1 - \alpha_x(1 - t + u_1) > (T(t)x)(u_2) + u_2 - u_1$ *implies that* $(T(t)x)(u_1) = x(0) + t - u_1$ *and* $(T(t)x)(u_2) = x(0) + t - u_2$;

(iii) $|1 - \alpha_x(1 - t + u_1) - (T(t)x)(u_2)| \leq u_2 - u_1$ *implies that* $(T(t)x)(u_1) = 1 - \alpha_x(1 - t + u_1)$.

Remark 2.4. One and only one of the assumptions (i), (ii), and (iii) holds.

Proof. We first prove (i). We assume that $1 - \alpha_x(1 - t + u_2) > x(0) - t + u_2$. Then by the definition of $T(t)$,

$$(T(t)x)(u_2) = \min\{x(0) + t - u_2,\ 1 - \alpha_x(1 - t + u_2)\}. \tag{2.11}$$

So, we have

$$(T(t)x)(u_2) - u_2 + u_1 \leq 1 - \alpha_x(1 - t + u_2) - u_2 + u_1 \leq 1 - \alpha_x(1 - t + u_1) \tag{2.12}$$

by Lemma 2.2. This is a contradiction. Therefore we obtain $1-\alpha_x(1-t+u_2) \le x(0)-t+u_2$. Hence $(T(t)x)(u_2) = x(0)-t+u_2$. Since

$$1-\alpha_x(1-t+u_1) < (T(t)x)(u_2) - u_2 + u_1 = x(0) - t + u_1, \tag{2.13}$$

we have $(T(t)x)(u_1) = x(0)-t+u_1$. Similarly, we can prove (ii). We finally prove (iii). We assume that $1-\alpha_x(1-t+u_1) < x(0)-t+u_1$. Then by Lemma 2.2, we have

$$\begin{aligned} 1-\alpha_x(1-t+u_2) &\le 1-\alpha_x(1-t+u_1) + u_2 - u_1 \\ &< x(0)-t+u_1+u_2-u_1 = x(0)-t+u_2. \end{aligned} \tag{2.14}$$

Hence $(T(t)x)(u_2) = x(0)-t+u_2$. So,

$$(T(t)x)(u_2) - (1-\alpha_x(1-t+u_1)) > (x(0)-t+u_2) - (x(0)-t+u_1) = u_2 - u_1. \tag{2.15}$$

This is a contradiction. Therefore we obtain $1-\alpha_x(1-t+u_1) \ge x(0)-t+u_1$. Similarly we can prove that $1-\alpha_x(1-t+u_1) \le x(0)+t-u_1$. Hence $(T(t)x)(u_1) = 1-\alpha_x(1-t+u_1)$. □

Proof of Example 2.1. It is clear that C is closed and convex. We first prove that $T(t)x \in C$ for all $t \in [0,1]$ and $x \in C$. It is clear that

$$\begin{aligned} 0 \le (T(t)x)(-1) = x(-1) \le 1, \\ 0 \le (T(t)x)(u) = x(u-t) \le 1 \end{aligned} \tag{2.16}$$

for $u \in [t,\infty)$. For $u \in [0,t]$, since $0 \le 1-\alpha_x(1-t+u) \le 1$, $x(0)-t+u \le x(0) \le 1$ and $x(0)+t-u \ge x(0) \ge 0$, we have $0 \le (T(t)x)(u) \le 1$. Fix $u_1, u_2 \in [0,\infty)$ with $u_1 < u_2$. In the case when $t \le u_1$, we have

$$\begin{aligned} |(T(t)x)(u_1) - (T(t)x)(u_2)| &= |x(u_1-t) - x(u_2-t)| \\ &\le |(u_1-t)-(u_2-t)| = |u_1-u_2|. \end{aligned} \tag{2.17}$$

In the case when $u_2 \le t$, by Lemma 2.3, it is easily proved that $|(T(t)x)(u_1) - (T(t)x)(u_2)| \le |u_1-u_2|$. In the case when $u_1 \le t \le u_2$, we have

$$\begin{aligned} &|(T(t)x)(u_1) - (T(t)x)(u_2)| \\ &\quad \le |(T(t)x)(u_1) - (T(t)x)(t)| + |(T(t)x)(t) - (T(t)x)(u_2)| \\ &\quad \le |u_1-t| + |t-u_2| = |u_1-u_2|. \end{aligned} \tag{2.18}$$

Therefore we have shown that $T(t)x \in C$ for $t \in [0,1]$ and $x \in C$. By the definition of $\{T(t) : t \ge 0\}$, we have $T(t)x \in C$ for all $t \in [0,\infty)$ and $x \in C$. We next show that $\{T(t) : t \ge 0\}$ is a one-parameter nonexpansive semigroup on C.

(sg1) Fix $t \in [0,1]$, and $x, y \in C$. We will prove that

$$|(T(t)x)(u) - (T(t)y)(u)| \le \|x-y\| \quad \forall u \in \Omega. \tag{2.19}$$

We have

$$\left|(T(t)x)(-1)-(T(t)y)(-1)\right| = \left|x(-1)-y(-1)\right| \leq \|x-y\|. \tag{2.20}$$

For $u \geq t$, we have

$$\left|(T(t)x)(u)-(T(t)y)(u)\right| = \left|x(u-t)-y(u-t)\right| \leq \|x-y\|. \tag{2.21}$$

Fix u with $0 \leq u \leq t$. In the case when $1-\alpha_x(1-t+u) \leq x(0)-t+u$ and $1-\alpha_y(1-t+u) \leq y(0)-t+u$, we have

$$\begin{aligned}\left|(T(t)x)(u)-(T(t)y)(u)\right| &= \left|(x(0)-t+u)-(y(0)-t+u)\right| \\ &= \left|x(0)-y(0)\right| \leq \|x-y\|.\end{aligned} \tag{2.22}$$

In the case when $1-\alpha_x(1-t+u) \leq x(0)-t+u$ and $1-\alpha_y(1-t+u) > y(0)-t+u$, we have

$$(T(t)y)(u) = \min\{1-\alpha_y(1-t+u),\ y(0)+t-u\} \geq y(0)-t+u. \tag{2.23}$$

Hence,

$$\begin{aligned}(T(t)x)(u)-(T(t)y)(u) &\leq (x(0)-t+u)-(y(0)-t+u) = x(0)-y(0) \leq \|x-y\|, \\ (T(t)y)(u)-(T(t)x)(u) &\leq (1-\alpha_y(1-t+u))-(1-\alpha_x(1-t+u)) \\ &= \alpha_x(1-t+u)-\alpha_y(1-t+u) \leq \|x-y\|\end{aligned} \tag{2.24}$$

hold. Therefore (2.19) holds. Similarly we can prove (2.19) in the other cases. On the other hand, we have

$$\begin{aligned}\|T(t)x-T(t)y\| &\geq \sup\{\left|(T(t)x)(u)-(T(t)y)(u)\right| : u \in \{-1\} \cup [t,\infty)\} \\ &= \sup\{\left|x(u)-y(u)\right| : u \in \Omega\} = \|x-y\|.\end{aligned} \tag{2.25}$$

Hence we have shown that

$$\|T(t)x-T(t)y\| = \|x-y\| \tag{2.26}$$

for $t \in [0,1]$ and $x,y \in C$. So, by the definition of $\{T(t) : t \geq 0\}$, (2.26) holds for all $t \in [0,\infty)$ and $x,y \in C$.

(sg2) It is clear that $T(0)$ is the identity mapping on C.

(sg3) Fix $t_1,t_2 \in [0,1/2]$ and $x \in C$. We will prove that

$$(T(t_1) \circ T(t_2)x)(u) = (T(t_1+t_2)x)(u) \quad \forall u \in \Omega. \tag{2.27}$$

We have

$$(T(t_1) \circ T(t_2)x)(-1) = (T(t_2)x)(-1) = x(-1) = (T(t_1+t_2)x)(-1). \tag{2.28}$$

For $u \geq t_2$, we have

$$(T(t_1+t_2)x)(t_1+u) = x((t_1+u)-(t_1+t_2)) = x(u-t_2) = (T(t_2)x)(u). \tag{2.29}$$

For $u \in [0,t_2]$, since $t_1+u \leq t_1+t_2$, $1-\alpha_x(1-t_2+u) = 1-\alpha_x(1-(t_1+t_2)+(t_1+u))$, $x(0)-t_2+u = x(0)-(t_1+t_2)+(t_1+u)$, and $x(0)+t_2-u = x(0)+(t_1+t_2)-(t_1+u)$, the two definitions of $(T(t_1+t_2)x)(t_1+u)$ and $(T(t_2)x)(u)$ coincide. Therefore

$$(T(t_1+t_2)x)(t_1+u) = (T(t_2)x)(u). \tag{2.30}$$

So, for $u \geq t_1$,

$$\begin{aligned}(T(t_1)\circ T(t_2)x)(u) &= (T(t_2)x)(u-t_1)\\ &= (T(t_1+t_2)x)(t_1+(u-t_1))\\ &= (T(t_1+t_2)x)(u).\end{aligned} \tag{2.31}$$

Fix u with $0 \leq u \leq t_1$. Then we have

$$\begin{aligned}1-\alpha_{T(t_2)x}(1-t_1+u) &= 1-\sup\{(T(t_2)x)(s) : s \in \{-1\}\cup[1-t_1+u,\infty)\}\\ &= 1-\max\{x(-1),\ \sup\{x(s-t_2) : s\in[1-t_1+u,\infty)\}\}\\ &= 1-\alpha_x(1-t_1-t_2+u).\end{aligned} \tag{2.32}$$

In the case when $1-\alpha_{T(t_2)x}(1-t_1+u) < (T(t_2)x)(0)-t_1+u$, we have

$$(T(t_1)\circ T(t_2)x)(u) = (T(t_2)x)(0)-t_1+u. \tag{2.33}$$

Since

$$\begin{aligned}1-\alpha_x(1-t_1-t_2+u) = 1-\alpha_{T(t_2)x}(1-t_1+u) &< (T(t_2)x)(0)-t_1+u\\ &= (T(t_1+t_2)x)(t_1)-t_1+u,\end{aligned} \tag{2.34}$$

we have

$$\begin{aligned}(T(t_1+t_2)x)(u) &= x(0)-t_1-t_2+u,\\ (T(t_1+t_2)x)(t_1) &= x(0)-t_1-t_2+t_1 = x(0)-t_2\end{aligned} \tag{2.35}$$

by Lemma 2.3. So,

$$\begin{aligned}(T(t_1)\circ T(t_2)x)(u) &= (T(t_2)x)(0)-t_1+u = (T(t_1+t_2)x)(t_1)-t_1+u\\ &= x(0)-t_2-t_1+u = (T(t_1+t_2)x)(u).\end{aligned} \tag{2.36}$$

Similarly, we can prove that $(T(t_1)\circ T(t_2)x)(u) = (T(t_1+t_2)x)(u)$ in the cases when $1-\alpha_{T(t_2)x}(1-t_1+u) > (T(t_2)x)(0)+t_1-u$ and $|1-\alpha_{T(t_2)x}(1-t_1+u)-(T(t_2)x)(0)| \leq t_1-u$. Therefore $T(t_1)\circ T(t_2) = T(t_1+t_2)$. So, we have, for $t \in [1/2,1)$,

$$T(t) = T\left(\frac{1}{2}\right)\circ T\left(t-\frac{1}{2}\right), \qquad T(1) = T\left(\frac{1}{2}\right)\circ T\left(\frac{1}{2}\right)\circ T(0). \tag{2.37}$$

Fix $t_1, t_2 \in [0,\infty)$. Then there exist $m_1, m_2 \in \mathbb{N} \cup \{0\}$ and $t_1', t_2' \in [0,1/2)$ satisfying $t_1 = m_1/2 + t_1'$ and $t_2 = m_2/2 + t_2'$. We have

$$\begin{aligned} T(t_1) \circ T(t_2) &= T\left(\frac{1}{2}\right)^{m_1} \circ T(t_1') \circ T\left(\frac{1}{2}\right)^{m_2} \circ T(t_2') = T\left(\frac{1}{2}\right)^{m_1+m_2} \circ T(t_1') \circ T(t_2') \\ &= T\left(\frac{1}{2}\right)^{m_1+m_2} \circ T\left(\min\left\{t_1'+t_2', \frac{1}{2}\right\}\right) \circ T\left(\max\left\{0, t_1'+t_2'-\frac{1}{2}\right\}\right) \\ &= T(t_1+t_2). \end{aligned} \tag{2.38}$$

(sg4) For $x \in C$ and $t \in [0,\infty)$, we have

$$\begin{aligned} \|T(t)x - x\| &= \sup\{|(T(t)x)(u) - x(u)| : u \in [0,\infty)\} \\ &= \sup\{|(T(t)x)(u) - (T(t)x)(t+u)| : u \in [0,\infty)\} \\ &\leq \sup\{|u-(t+u)| : u \in [0,\infty)\} = t. \end{aligned} \tag{2.39}$$

Therefore we obtain

$$\|T(t_1)x - T(t_2)x\| = \|T(|t_1 - t_2|)x - x\| \leq |t_1 - t_2| \tag{2.40}$$

for $x \in C$ and $t_1, t_2 \in [0,1]$. Therefore $T(\cdot)x$ is continuous for all $x \in C$.

We prove that

$$\bigcap_{t\geq 0} F(T(t)) = \left\{v_s : s \in \left[0, \frac{1}{2}\right]\right\} \cup \left\{w_s : s \in \left[0, \frac{1}{2}\right]\right\}, \tag{2.41}$$

where

$$\begin{aligned} v_s(u) &= \begin{cases} 1-s & \text{if } u = -1, \\ s & \text{if } u \in [0,\infty), \end{cases} \\ w_s(u) &= \begin{cases} s & \text{if } u = -1, \\ \dfrac{1}{2} & \text{if } u \in [0,\infty). \end{cases} \end{aligned} \tag{2.42}$$

Fix $s \in [0,1/2]$ and $t \in [0,1]$. Then we have

$$\begin{aligned} |1 - \alpha_{v_s}(1-t+u) - v_s(0)| &= |1-(1-s)-s| = 0 \leq t-u, \\ |1 - \alpha_{w_s}(1-t+u) - w_s(0)| &= \left|1 - \frac{1}{2} - \frac{1}{2}\right| = 0 \leq t-u, \end{aligned} \tag{2.43}$$

for $u \in [0,t]$. So

$$\begin{aligned} (T(t)v_s)(u) &= 1 - \alpha_{v_s}(1-t+u) = s = v_s(u), \\ (T(t)w_s)(u) &= 1 - \alpha_{w_s}(1-t+u) = \frac{1}{2} = w_s(u). \end{aligned} \tag{2.44}$$

Hence, $T(t)v_s = v_s$ and $T(t)w_s = w_s$. Therefore, v_s and w_s are common fixed points of $\{T(t) : t \geq 0\}$. Conversely, we assume that $x \in C$ is a common fixed point of $\{T(t) : t \geq 0\}$. Put $s = x(0)$. Then we have

$$x(t+u) = (T(t)x)(t+u) = x(t+u-t) = x(u) \tag{2.45}$$

for all $u \in [0,\infty)$ and $t \in [0,1]$. So, $x(u) = x(0) = s$ hold for all $u \in [0,\infty)$. We also have

$$s = x(0) = (T(1)x)(0) = 1 - \alpha_x(1-1+0) = 1 - \alpha_x(0) = \min\{1 - x(-1),\ 1 - s\}. \tag{2.46}$$

Hence $x(-1) \leq 1 - s$ and $s \leq 1/2$. If $s = 1/2$, then $x = w_{x(-1)}$. If $s < 1/2$, then $x(-1) = 1 - s$ and hence $x = v_s$. Therefore we have shown (2.41).

Define a function f from $\mathbb{R}$ into $[0,1]$ by

$$f(u) = \begin{cases} 0, & \text{if } u \geq 0, \\ -u, & \text{if } -1 \leq u \leq 0, \\ u+2, & \text{if } -2 \leq u \leq -1, \\ 0, & \text{if } u \leq -2. \end{cases} \tag{2.47}$$

We finally show that

$$(T(t)0)(u) = \begin{cases} 0, & \text{if } u = -1, \\ f(u-t), & \text{if } u \in [0,\infty). \end{cases} \tag{2.48}$$

Fix $t \in [0,1]$ and $u \in [0,t]$. Then we have

$$1 - \alpha_0(1-t+u) = 1 \geq 0 + t - u \tag{2.49}$$

and hence

$$(T(t)0)(u) = 0 + t - u = t - u = f(u-t) \tag{2.50}$$

because $-1 \leq u - t \leq 0$. Therefore

$$(T(1)0)(s) = \begin{cases} 0, & \text{if } s = -1 \text{ or } s \geq 1, \\ 1-s, & \text{if } 0 \leq s \leq 1. \end{cases} \tag{2.51}$$

Since

$$\begin{aligned} 1 - \alpha_{T(1)0}(1-t+u) &= 1 - \big(1 - (1-t+u)\big) = 1 - t + u \\ &= (T(1)0)(0) - t + u, \end{aligned} \tag{2.52}$$

we have

$$\begin{aligned} (T(t+1)0)(u) &= (T(t) \circ T(1)0)(u) = (T(1)0)(0) - t + u \\ &= 1 - t + u = f(u - (1+t)). \end{aligned} \tag{2.53}$$

Therefore

$$(T(2)0)(s) = \begin{cases} 0, & \text{if } s = -1 \text{ or } s \geq 2, \\ 2 - s, & \text{if } 1 \leq s \leq 2, \\ s, & \text{if } 0 \leq s \leq 1. \end{cases} \tag{2.54}$$

Since

$$\left| 1 - \alpha_{T(2)0}(1 - t + u) - (T(2)0)(0) \right| = |1 - 1 - 0| = 0 \leq t - u, \tag{2.55}$$

we have

$$\begin{aligned} (T(t+2)0)(u) &= (T(t) \circ T(2)0)(u) \\ &= 1 - \alpha_{T(2)0}(1 - t + u) = 0 = f(u - (2+t)). \end{aligned} \tag{2.56}$$

Similarly, for $k \in \mathbb{N}$ with $k > 2$, we can prove

$$(T(t+k)0)(u) = 0 = f(u - (k+t)). \tag{2.57}$$

Therefore we have shown (2.48). So, (2.5) clearly holds. This completes the proof. □

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