A NEW TOPOLOGICAL DEGREE THEORY FOR DENSELY DEFINED QUASIBOUNDED $(\widetilde{S}_+)$-PERTURBATIONS OF MULTIVALUED MAXIMAL MONOTONE OPERATORS IN REFLEXIVE BANACH SPACES

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Let X be an infinite-dimensional real reflexive Banach space with dual space X^* and $G \subset X$ open and bounded. Assume that X and X^* are locally uniformly convex. Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone and $C : X \supset D(C) \to X^*$ quasibounded and of type $(\widetilde{S}_+)$. Assume that $L \subset D(C)$, where L is a dense subspace of X, and $0 \in T(0)$. A new topological degree theory is introduced for the sum $T + C$. Browder's degree theory has thus been extended to densely defined perturbations of maximal monotone operators while results of Browder and Hess have been extended to various classes of single-valued densely defined generalized pseudomonotone perturbations C. Although the main results are of theoretical nature, possible applications of the new degree theory are given for several other theoretical problems in nonlinear functional analysis.

1. Introduction and preliminaries

In what follows, the symbol X stands for an infinite-dimensional real reflexive Banach space which has been renormed so that it and its dual X^* are locally uniformly convex. The symbol $\|\cdot\|$ stands for the norm of X, X^* and $J : X \to X^*$ is the normalized duality mapping. In what follows, "continuous" means "strongly continuous" and the symbol "$\to$" ("$\rightharpoonup$") means strong (weak) convergence.

The symbol $\mathbb{R}(\mathbb{R}_+)$ stands for the set $(-\infty,\infty)([0,\infty))$ and the symbols ∂D, $\overline{D}$ denote the strong boundary and closure of the set D, respectively. We denote by $B_r(0)$ the open ball of X or X^* with center at zero and radius $r > 0$.

For an operator $T : X \to 2^{X^*}$, we denote by $D(T)$ the effective domain of T, that is, $D(T) = \{x \in X : Tx \neq \emptyset\}$. We denote by $G(T)$ the graph of T, that is, $G(T) = \{(x,y) : x \in D(T),\ y \in Tx\}$. An operator $T : X \supset D(T) \to 2^{X^*}$ is called "monotone" if for every $x, y \in D(T)$ and every $u \in Tx$, $v \in Ty$, we have

$$\langle u - v, x - y\rangle \geq 0. \tag{1.1}$$

A monotone operator T is "maximal monotone" if $G(T)$ is maximal in $X \times X^*$ when $X \times X^*$ is partially ordered by inclusion. In our setting, a monotone operator T is maximal if

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and only if $R(T+\lambda J)=X^*$ for all $\lambda\in(0,\infty)$. If T is maximal monotone, then the operator $T_t\equiv(T^{-1}+tJ^{-1})^{-1}:X\to X^*$ is bounded, continuous (see Lemma 3.1 below), maximal monotone and such that $T_tx\rightharpoonup T^{\{0\}}x$ as $t\to 0^+$ for every $x\in D(T)$, where $T^{\{0\}}x$ denotes the element $y^*\in Tx$ of minimum norm, that is, $\|T^{\{0\}}x\|=\inf\{\|y^*\|:y^*\in Tx\}$. In our setting, this infimum is always attained and $D(T^{\{0\}})=D(T)$. Also, $T_tx\in TJ_tx$, where $J_t\equiv I-tJ^{-1}T_t:X\to X$ and satisfies $\lim_{t\to 0}J_tx=x$ for all $x\in\overline{\operatorname{co}D(T)}$, where $\operatorname{co}A$ denotes the convex hull of the set A. The operators T_t, J_t were introduced by Brézis et al. in [2]. For their basic properties, we refer the reader to [2] as well as Pascali and Sburlan [22, pages 128–130]. In our setting, the duality mapping J is single-valued and bicontinuous.

An operator $T:X\supset D(T)\to Y$, with Y another real Banach space, is "bounded" if it maps bounded subsets of $D(T)$ onto bounded sets. It is "compact" if it is continuous and maps bounded subsets of $D(T)$ onto relatively compact subsets of Y. It is "demicontinuous" ("completely continuous") if it is strong-weak (weak-strong) continuous on $D(T)$.

We say that an operator $C:X\supset D(C)\to X^*$ satisfies condition "(S_+)" if $\{x_n\}\subset D(C)$, $x_n\rightharpoonup x_0$, and

$$\limsup_{n\to\infty}\langle Cx_n,x_n-x_0\rangle\le 0 \tag{1.2}$$

imply $x_n\to x_0$.

We say that an operator $C:X\supset D(C)\to X^*$ satisfies condition "$(\widetilde{S}_+)$" if $\{x_n\}\subset D(C)$, $x_n\rightharpoonup x_0$, $Cx_n\rightharpoonup h_0^*$, and (1.2) imply $x_n\to x_0$, $x_0\in D(C)$, and $Cx_0=h_0^*$.

The following lemma can be found in Zeidler [29, page 915].

Lemma 1.1. *Let $T:X\supset D(T)\to 2^{X^*}$ be maximal monotone. Then the following are true:*

(i) $\{x_n\}\subset D(T)$, $x_n\to x_0$ *and* $Tx_n\ni y_n\rightharpoonup y_0$ *imply* $x_0\in D(T)$ *and* $y_0\in Tx_0$;

(ii) $\{x_n\}\subset D(T)$, $x_n\rightharpoonup x_0$ *and* $Tx_n\ni y_n\to y_0$ *imply* $x_0\in D(T)$ *and* $y_0\in Tx_0$.

From Lemma 1.1, we see that either one of (i) and (ii) implies that the graph $G(T)$ of the operator T is closed, that is, $G(T)$ is a closed subset of $X\times X^*$.

For facts involving monotone operators, and other related concepts, the reader is referred to Barbu [1], Brézis et al. [2], Browder [3], Pascali and Sburlan [22], Simons [26], Skrypnik [27], and Zeidler [29].

We cite the books of Browder [3], Lloyd [21], Petryshyn [23], Rothe [25], and Skrypnik [27], and the papers of Browder [4, 5], Kartsatos and Skrypnik [15, 16], and Zhang and Chen [30] as references to degree theories.

For recent results about ranges of operators, we refer the reader to Guan and Kartsatos [9], Guan et al. [10], Kartsatos [11], the authors [13, 16, 17, 18], Li and Huang [20], and Zhang and Chen [30].

Recent related eigenvalue problems can be found in the paper of the authors [14] and the paper of Li and Huang [20].

In Section 2, we summarize the construction of our recent degree theory for two densely defined mappings T, C, where T is at least single-valued and maximal monotone, and C satisfies an L-related quasiboundedness condition and an L-related generalized (S_+)-condition with respect to T. Here, L is a dense subspace of X.

Section 3 contains the construction of the new degree. We only assume that $0 \in D(T)$, $0 \in T(0)$, and $T : X \to 2^{X^*}$ is maximal monotone. Unlike [15], we do not assume that T is densely defined and conditions like (t_2)–(t_4) (see Section 2), which make T stronger than just maximal monotone. The operator $C : X \supset D(C) \to X^*$, with $L \subset D(C)$, is quasibounded, finitely continuous on subspaces of L (see (c_3) below) and satisfies condition $(\widetilde{S}_+)$. In [15], we assumed that C satisfies a quasiboundedness condition and a condition of type (S_+), which involve the operator T and the space L.

The basic characteristic of the new degree is that the maximal monotone operator T may be multivalued but not necessarily densely defined. We should note here that the new degree theory does not contain the theory developed in [15] as a special case. Although the two degree theories overlap for certain combinations of operators T, C, and the degree in [15] is used for the construction herein, they are generally different even in the important case of a single-valued maximal monotone operator T. The new degree theory is also a substantial extension of Browder's degree theory in [5]. Browder's perturbation term is defined on the closure of an open and bounded set in the space X. However, we should mention here that our degree definition uses the degree of the mapping $T_t + C$, which is constant for all small values of t. Such an approach was first used by Browder in [5]. Naturally, we have to show here that this homotopy function $T_t + C$ is admissible for our degree in [15]. This is the content of Theorem 3.3.

Naturally, every new degree theory is useful provided that it carries appropriate homotopies that can be used for the calculation of the degree. Theorem 4.3 contains a basic homotopy result. This result is used in Theorem 4.4(iii), of Section 4, in order to obtain a rather important homotopy that we have actually used in all the applications of the new mapping theorems of Section 6. Again, unlike the main homotopy that has been used for the degree of [15], the main feature of the above homotopy is that we no longer assume that the dense linear space L lies in both domains $D(T)$, $D(C)$. Such an assumption must be made for the degree in [15], and it precludes us from considering many simple affine homotopies of the type $H(t,\cdot) \equiv t(T + C_1) + (1-t)C_2$ for general maximal monotone operators T. Simply, under this assumption, the degree $d(H(t,\cdot),G,0)$ might not be well defined, although $0 \notin H(t,\cdot)(\partial G)$, $t \in [0,1]$, and both degrees $d(C_1,G,0)$, $d(C_2,G,0)$ are well defined. This, at times, is due to the fact that the mappings T, C_1 have domains that contain different dense subspaces L.

An index theory for densely defined operators and the degree developed in [15] can be found in the authors' paper [17].

Section 4 contains some basic properties of the new degree including two basic homotopies.

In Section 5, we extend some results of Browder and Hess [6] about generalized pseudomonotone operators to pairs of operators T, C covered by the new degree theory.

Further mapping theorems for the new degree are given in Section 6.

2. The degree for densely defined mappings T, C

We exhibit below, in a summary, the degree theory that was recently developed by the authors in [15]. In this degree theory, both operators T, C are densely defined, and $d(T + C, G, 0)$ comes from approximation by finite-dimensional Brouwer degrees. We note that the operator T is single-valued and the operator C satisfies two basic conditions (quasiboundedness and generalized (S_+)) involving the dense subspace $L \subset D(T) \cap D(C)$ of the space X as well as the operator T itself. This introduction is instructive in view of the degree theory that we are going to develop later in this paper.

Let L be a subspace of X and let $\mathcal{F}(L)$ be the set of all finite-dimensional subspaces of L. Consider a single-valued operator $T : X \supset D(T) \to X^*$ satisfying the following conditions:

(t_1) T is monotone, that is,

$$\langle Tu - Tv, u - v\rangle \geq 0 \tag{2.1}$$

for every $u, v \in D(T)$. Moreover,

$$L \subset D(T), \qquad \overline{L} = X; \tag{2.2}$$

(t_2) for every $(u_0, h_0^*) \in X \times X^*$ with

$$\langle Tu - h_0^*, u - u_0\rangle \geq 0 \quad \text{for } u \in L, \tag{2.3}$$

we have $u_0 \in D(T)$ and $Tu_0 = h_0^*$;

(t_3) for any $u_0 \in D(T)$, we have

$$\inf\{\langle Tv - Tu_0, v - u_0\rangle : v \in L\} = 0; \tag{2.4}$$

(t_4) for every $F \in \mathcal{F}(L)$, $v \in L$, the mapping $\sigma(F, v) : F \to \mathcal{R}$, defined by $\sigma(F, v)u = \langle Tu, v\rangle$ is continuous.

Note that the conditions (t_2), (t_3) are automatically satisfied by a maximal monotone operator T whose domain $D(T) = L$.

We also consider a second operator $C : X \supset D(C) \to X^*$ satisfying the following conditions:

(c_1)

$$L \subset D(C) \tag{2.5}$$

and C is quasibounded with respect to T, that is, for every number $S > 0$, there exists a number $K(S) > 0$ such that from the inequalities

$$\langle Tu + Cu, u\rangle \leq 0, \quad \|u\| \leq S,\ u \in L, \tag{2.6}$$

we have $\|Cu\| \leq K(S)$;

(c_2) the operator C satisfies the following generalized (S_+) condition with respect to T: for every sequence $\{u_n\} \subset L$ such that $u_n \rightharpoonup u_0$, $Cu_n \rightharpoonup h_0$ and

$$\limsup_{n\to\infty} \langle Cu_n, u_n - u_0\rangle \leq 0, \qquad \langle Tu_n + Cu_n, u_n\rangle \leq 0, \tag{2.7}$$

for some $u_0 \in X$, $h_0 \in X^*$, we have $u_n \to u_0$, $u_0 \in D(C)$ and $Cu_0 = h_0$;

(c_3) for every $F \in \mathcal{F}(L)$, $v \in L$, the mapping $c(F,v) : F \to \mathcal{R}$, defined by $c(F,v)(u) = \langle Cu, v\rangle$, is continuous.

In what follows, $G_F = G \cap F$.

Lemma 2.1. *Assume that $T : X \supset D(T) \to X^*$ satisfies (t_1)–(t_3), while $C : X \supset D(C) \to X^*$ satisfies (c_1), (c_2). Let G be a bounded open set in X such that*

$$Tu + Cu \neq 0, \quad u \in \partial G \cap D(T+C), \tag{2.8}$$

where $D(C+T) = D(T) \cap D(C)$. Then there exists a space $F_0 \in \mathcal{F}(L)$ such that for every space $F \in \mathcal{F}(L)$ such that $F_0 \subset F$, it holds that

$$Z(F_0,F) \equiv \{u \in \partial G_F \cap D(T+C) : \langle Tu+Cu, u\rangle \leq 0,\ \langle Tu+Cu, v\rangle = 0,\ v \in F_0\} = \varnothing. \tag{2.9}$$

Let $F \in \mathcal{F}(L)$ and let $v_1,\dots,v_k$ be a basis for F. We define a finite-dimensional mapping $(T+C)_F : F \to F$ by

$$(T+C)_F(u) = \sum_{i=1}^{k} \langle Tu+Cu, v_i\rangle v_i. \tag{2.10}$$

Theorem 2.2. *Assume that $T : X \supset D(T) \to X^*$ satisfies (t_1)–(t_4), while $C : X \supset D(C) \to X^*$ satisfies (c_1)–(c_3). Let G be a bounded open set in X such that (2.8) holds. Let $F_0 \in \mathcal{F}(L)$ be the space defined in Lemma 2.1. Then for every space $F \in \mathcal{F}(L)$ with $F_0 \subset F$, the following relation holds:*

$$\deg\big((T+C)_F, G_F, 0\big) = \deg\big((T+C)_{F_0}, G_{F_0}, 0\big), \tag{2.11}$$

where $(T+C)_F$ is the finite-dimensional mapping defined by (2.10), and deg denotes the Brouwer degree.

Definition 2.3 (degree for densely defined T, C). Assume that the operators T, C and the set G satisfy the conditions of Theorem 2.2. Then the degree $d(T+C,G,0)$ is defined by

$$d(T+C,G,0) = \deg\big((T+C)_{F_0}, G_{F_0}, 0\big), \tag{2.12}$$

where the operator $(T+C)_F$ is defined by (2.10), and F_0 is the finite-dimensional subspace of L determined by Lemma 2.1.

The basic properties of our degree can be found in [15]. We do need to exhibit the basic homotopy invariance property of this degree. It is contained in Theorem 2.5. Before we state it, we need certain facts and a definition.

Consider the one-parameter family of operators $M_t : X \supset D(M_t) \to X^*$, $t \in [0,1]$, satisfying the following conditions:

($m_t^{(1)}$) for every $t \in [0,1]$, the operator M_t satisfies conditions (t_1)–(t_3) above with the space L independent of t;

($m_t^{(2)}$) for every $v \in L$, the mapping $\mu(v) : [0,1] \to X^*$, defined by $\mu(v)(t) = M_t(v)$, is continuous;

($m_t^{(3)}$) for every $F \in \mathcal{F}(L)$, $v \in L$ the mapping $\tilde{m}(F,v) : F \times [0,1] \to \mathcal{R}$, defined by $\tilde{m}(F,v)(u,t) = \langle M_t u, v\rangle$, is continuous.

Let $A_t : X \supset D(A_t) \to X^*$, $t \in [0,1]$, be a second one-parameter family of operators satisfying the following conditions:

($a_t^{(1)}$) for every $t \in [0,1]$, let $L \subset D(A_t)$ be as in conditions ($m_t^{(1)}$)–($m_t^{(3)}$), and let the family $\{A_t\}$ be uniformly quasibounded with respect to M_t, that is, for every $S > 0$, there exists $K(S) > 0$ such that

$$\langle M_t u + A_t u, u\rangle \le 0, \quad \|u\| \le S,\ u \in L,\ t \in [0,1], \tag{2.13}$$

implies the estimate $\|A_t u\| \le K(S)$;

($a_t^{(2)}$) for every pair of sequences $\{t_j\} \subset [0,1]$, $\{u_j\} \subset L$ such that $u_j \rightharpoonup u_0$, $A_{t_j} u_j \rightharpoonup h_0^*$, $t_j \to t_0$ and

$$\limsup_{j\to\infty} \langle A_{t_j} u_j, u_j - u_0\rangle \le 0, \qquad \langle M_{t_j} u_j + A_{t_j} u_j, u_j\rangle \le 0, \tag{2.14}$$

for some $t_0 \in [0,1]$, $u_0 \in X$, $h_0^* \in X^*$, we have $u_j \to u_0$, $u_0 \in D(A)$ and $A_{t_0} u_0 = h_0^*$;

($a_t^{(3)}$) for every $F \in \mathcal{F}(L)$, $v \in L$, the mapping $\tilde{a}(F,v) : F \times [0,1] \to \mathcal{R}$, defined by $\tilde{a}(F,v)(u,t) = \langle A_t u, v\rangle$, is continuous.

Definition 2.4. Let $M^{(i)} : X \supset D(M^{(i)}) \to X^*$, $A^{(i)} : X \supset D(A^{(i)}) \to X^*$, $i = 0,1$, satisfy conditions (t_1)–(t_4) and (c_1)–(c_3) above, with a common space L. We say that the operators $A^{(0)} + M^{(0)}$, $A^{(1)} + M^{(1)}$ are homotopic with respect to the open bounded set $D \subset X$ if there exist one-parameter families of operators $M_t : X \supset D(M_t) \to X^*$, $A_t : X \supset D(A_t) \to X^*$ satisfying conditions ($m_t^{(1)}$)–($m_t^{(3)}$) and ($a_t^{(1)}$)–($a_t^{(3)}$), respectively, and such that

$$\begin{gathered} M^{(i)} = M_i, \quad A^{(i)} = A_i, \quad i = 0,1, \\ M_t u + A_t u \ne 0, \quad u \in \partial D \cap D(M_t + A_t),\ t \in [0,1]. \end{gathered} \tag{2.15}$$

We also say in this case that $\{M_t + C_t\}$, $t \in [0,1]$, is an "admissible homotopy."

Theorem 2.5. *Assume that the operators $M^{(i)}$, $A^{(i)}$, $i = 0,1$, satisfy conditions (t_1)–(t_4) and (c_1)–(c_3), respectively. Assume that the operators $M^{(0)} + A^{(0)}$, $M^{(1)} + A^{(1)}$ are homotopic with respect to the bounded open set $G \subset X$. Then*

$$d(M^{(0)} + A^{(0)}, G, 0) = d(M^{(1)} + A^{(1)}, G, 0), \tag{2.16}$$

where the degree d is as in Definition 2.3.

Remark 2.6. It is important to mention here that our degree theory above was actually developed in [15] with S in place of 0 in the first inequality in (2.6) and (2.13). It can be seen that the present situation is sufficient for the development of our degree after a careful study of the construction in [15].

3. The construction of the new degree

We are now ready to state the hypotheses needed for our new degree. As above, L is a dense subspace of X carrying the family $\mathcal{F}(L)$. For the operator T, we assume the following:

($t1$) $T : X \supset D(T) \to 2^{X^*}$ is maximal monotone with $0 \in D(T)$ and $0 \in T(0)$.

For the operator C, we assume that

($c1$) $C : X \supset D(C) \to X^*$, with $L \subset D(C)$, is quasibounded, that is, for every $S > 0$, there exists $K(S) > 0$ such that $u \in D(C)$ with

$$\|u\| \le S, \qquad \langle Cu, u\rangle \le S \tag{3.1}$$

implies $\|Cu\| \le K(S)$;

($c2$) the operator C satisfies condition $(\widetilde{S}_+)$;

($c3$) for every $F \in \mathcal{F}(L)$, $v \in L$, the mapping $c(F,v) : F \to \mathcal{R}$, defined by $c(F,v)(u) = \langle Cu, v\rangle$, is continuous.

Condition ($c3$) here is the same as condition (c_3). It is included with a new symbol for convenience.

The following lemma is a new result that shows the continuity of the operator $(t,x) \to T_t x$ on $(0,\infty) \times X$. For $D(T)$ in place of X, this was shown differently in the paper [30].

Lemma 3.1. *Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone with $0 \in D(T)$ and $0 \in T(0)$. Then the mapping $(t,x) \to T_t x$ is continuous on $(0,\infty) \times X$.*

Proof. Fix $\delta > 0$. Let $\{x_n\} \subset X$, $\{t_n\} \subset [\delta, \infty)$ be such that $x_n \to x_0$ and $t_n \to t_0$. Let $y_n^* = T_{t_n} x_n$. Then, for some $z_n \in D(T)$ with $y_n^* \in T z_n$,

$$(T^{-1} + t_n J^{-1}) y_n^* \ni x_n = z_n + t_n J^{-1} y_n^*. \tag{3.2}$$

Using the monotonicity of the operator T and the condition $0 \in T(0)$, we get

$$\langle y_n^*, x_n\rangle = \langle y_n^*, z_n\rangle + t_n \langle y_n^*, J^{-1} y_n^*\rangle \ge \delta \|y_n^*\|^2, \tag{3.3}$$

which gives us the boundedness of the sequence $\{y_n^*\}$ and hence the boundedness of $\{z_n\}$ by (3.2). Since X, X^* are reflexive, we may assume that $y_n^* \rightharpoonup y_0^*$, $z_n \rightharpoonup z_0$ and $J^{-1} y_n^* \rightharpoonup y_0$. Using this and the monotonicity of the duality mapping $J^{-1} : X^* \to X = X^{**}$, we obtain

$$\lim_{n\to\infty} \langle y_n^* - y_0^*, x_n\rangle = 0, \qquad \liminf_{n\to\infty} \langle y_n^* - y_0^*, t_n J^{-1} y_n^*\rangle \ge 0. \tag{3.4}$$

The second inequality of (3.4) follows from

$$\langle y_n^* - y_0^*, t_n (J^{-1} y_n^* - J^{-1} y_0^*)\rangle \ge t_n (\|y_n^*\| - \|y_0\|)^2, \tag{3.5}$$

which implies

$$\langle y_n^* - y_0^*, t_n J^{-1} y_n^*\rangle \ge \langle y_n^* - y_0^*, t_n J^{-1} y_0^*\rangle. \tag{3.6}$$

From (3.2) and (3.4), we have

$$\limsup_{n\to\infty}\langle y_n^* - y_0^*, z_n\rangle \le \lim_{n\to\infty}\langle y_n^* - y_0^*, x_n\rangle + \limsup_{n\to\infty}\left[-\langle y_n^* - y_0^*, t_n J^{-1} y_n^*\rangle\right], \tag{3.7}$$

which says

$$\limsup_{n\to\infty}\langle y_n^* - y_0^*, z_n\rangle \le 0. \tag{3.8}$$

This yields

$$\limsup_{n\to\infty}\langle y_n^*, z_n\rangle \le \langle y_0^*, z_0\rangle. \tag{3.9}$$

Fix $\tilde{x} \in D(T)$, $\tilde{x}^* \in Tx$. Using the monotonicity of the operator T, we get

$$\langle y_n^* - \tilde{x}^*, z_n - \tilde{x}\rangle \ge 0, \tag{3.10}$$

or

$$\langle y_n^*, z_n\rangle \ge \langle y_n^*, \tilde{x}\rangle + \langle \tilde{x}^*, z_n\rangle - \langle \tilde{x}^*, \tilde{x}\rangle. \tag{3.11}$$

Passing to the limit as $n \to \infty$, we obtain

$$\liminf_{n\to\infty}\langle y_n^*, z_n\rangle \ge \langle y_0^*, \tilde{x}\rangle + \langle \tilde{x}^*, z_0\rangle - \langle \tilde{x}^*, \tilde{x}\rangle. \tag{3.12}$$

Inequalities (3.9) and (3.12) imply

$$\langle y_0^* - \tilde{x}^*, z_0 - \tilde{x}\rangle \ge 0. \tag{3.13}$$

Since the point $(\tilde{x}, \tilde{x}^*) \in G(T)$ is arbitrary, we have $z_0 \in D(T)$ and $y_0^* \in Tz_0$ by the maximal monotonicity of the operator T. Thus, we may take in (3.12) $\tilde{x} = z_0$ to arrive at

$$\liminf_{n\to\infty}\langle y_n^*, z_n\rangle \ge \langle y_0^*, z_0\rangle. \tag{3.14}$$

Now from (3.2), the first equality in (3.4), and (3.14), we obtain

$$\limsup_{n\to\infty}\langle y_n^* - y_0^*, t_n J^{-1} y_n^*\rangle \le 0. \tag{3.15}$$

Using the (S_+)-property of the operator J^{-1}, we obtain $y_n^* \to y_0^*$. Passing to the limit in (3.2) and taking into consideration that $y_0^* \in Tz_0$, we get

$$x_0 = z_0 + t_0 J^{-1} y_0^* \in \left(T^{-1} + t_0 J^{-1}\right) y_0^*. \tag{3.16}$$

Thus, $y_0^* = (T^{-1} + t_0 J^{-1})^{-1} x_0 = T_{t_0} x_0$, and the proof is complete. □

The following theorem will allow us to define the degree $d(T_t + C, G, 0)$ (see Theorem 3.3) provided that $0 \notin (T + C)(D(T) \cap D(C) \cap \partial G)$.

Theorem 3.2. *Assume that the operator T satisfies condition $(t1)$ and the operator C satisfies conditions $(c1)$–$(c3)$. Assume that $G \subset X$ is open and bounded and that*

$$0 \notin (T+C)\big(D(T)\cap D(C)\cap \partial G\big). \tag{3.17}$$

Then there exists $t_1 \in (0,\infty)$ such that $0 \notin (T_t+C)(D(C)\cap\partial G)$ for every $t\in(0,t_1]$, where $T_t = (T^{-1}+tJ^{-1})^{-1} : X \to X^*$, $t>0$.

Proof. Assume that (3.17) is true and that the conclusion is false. Then there exists a sequence $\{t_n\}$ such that $t_n \downarrow 0$, and a sequence $\{x_n\}\subset D(C)\cap\partial G$ such that

$$T_{t_n}x_n + Cx_n = 0. \tag{3.18}$$

Since G is bounded, we may assume that $x_n \rightharpoonup x_0 \in X$. Since $\{x_n\}$ is bounded and $\langle Cx_n, x_n\rangle \le 0$, because $\langle T_{t_n}x_n, x_n\rangle \ge 0$, we have by the quasiboundedness of C that $\{\|Cx_n\|\}$ is also bounded. We may thus assume that $Cx_n \rightharpoonup h^* \in X^*$. We claim that (3.18) implies (1.2). Assume that this is not true. Then there exists a subsequence of $\{x_n\}$, denoted again by $\{x_n\}$, such that

$$\lim_{n\to\infty}\langle Cx_n, x_n - x_0\rangle > 0. \tag{3.19}$$

This implies

$$\lim_{n\to\infty}\langle T_{t_n}x_n, x_n - x_0\rangle < 0. \tag{3.20}$$

We also have $T_{t_n}x_n \rightharpoonup -h^*$. Consequently, along with

$$\langle T_{t_n}x_n, x_n\rangle = \langle T_{t_n}x_n, x_n - x_0\rangle + \langle T_{t_n}x_n, x_0\rangle, \tag{3.21}$$

we obtain

$$\limsup_{n\to\infty}\langle T_{t_n}x_n, x_n\rangle < \limsup_{n\to\infty}\langle T_{t_n}x_n, x_0\rangle = \langle -h^*, x_0\rangle. \tag{3.22}$$

Let now $x \in D(T)$ and $x^* \in Tx$. Then, as in Browder [5, proof of Theorem 12],

$$\liminf_{n\to\infty}\langle T_{t_n}x_n, x_n\rangle \ge \liminf_{n\to\infty}\langle T_{t_n}x_n, x\rangle + \langle x^*, x_0 - x\rangle = \langle -h^*, x\rangle + \langle x^*, x_0 - x\rangle. \tag{3.23}$$

Thus, by (3.22),

$$\langle -h^*, x\rangle + \langle x^*, x_0 - x\rangle < \langle -h^*, x_0\rangle, \tag{3.24}$$

or

$$\langle -h^* - x^*, x_0 - x\rangle > 0. \tag{3.25}$$

Since (x,x^*) are arbitrary in the graph $G(T)$ and T is maximal monotone, we have $x_0 \in D(T)$ and $Tx_0 \ni -h^*$. Taking $x = x_0$ and $x^* = -h^*$ in (3.25), we obtain a contradiction. Consequently, (1.2) is true.

Using the fact that the operator C satisfies condition $(\widetilde{S}_+)$, we conclude that $x_n \to x_0$, $x_0 \in D(C) \cap \partial G$, and $Cx_0 = h^*$. This says $T_{t_n} x_n \rightharpoonup -h^*$ and

$$\lim_{n\to\infty} \langle T_{t_n} x_n, x_n - x_0 \rangle = -\lim_{n\to\infty} \langle Cx_n, x_n - x_0 \rangle = 0 \tag{3.26}$$

and implies, as in the argument starting with (3.20) above, that $x_0 \in D(T)$ and $Tx_0 \ni -h^*$. Consequently, $Tx_0 + Cx_0 \ni 0$ with $x_0 \in D(T) \cap D(C) \cap \partial G$, that is, a contradiction. The proof is complete. □

Theorem 3.3 below contains the fact that the degree $d(T_t + C, G, 0)$ is well defined and constant. The latter follows from the fact that the operator $T_t + C$ defines a homotopy which is admissible for all small $t > 0$, according to Definition 2.4.

Theorem 3.3. *Assume that the operator T satisfies condition $(t1)$ and the operator C satisfies conditions $(c1)$–$(c3)$. Assume that $G \subset X$ is open and bounded and that $0 \notin (T + C)(D(T) \cap D(C) \cap \partial G)$. Then if $t_1 \in (0,\infty)$ is as in Theorem 3.2, the degree $d(T_t + C, G, 0)$ is well defined and constant for every $t \in (0, t_1]$.*

Proof. We first note that $0 \in T(0)$ implies $T_t(0) = 0$, $t > 0$. In order to define the degree $d(T_t + C, G, 0)$, we need to show that the operators T_t, C satisfy the conditions (t_1)–(t_4) and (c_1)–(c_3), respectively. We know that T_t is maximal monotone and continuous. This takes care of (t_1) and (t_4).

To show (t_2), fix $t > 0$ and let $(u_0, h_0^*) \in X \times X^*$ be such that

$$\langle T_t u - h_0^*, u - u_0 \rangle \geq 0, \quad u \in L. \tag{3.27}$$

Since L is dense in X and T_t is continuous, it follows easily that this inequality holds for all $u \in X$. Since T_t is maximal monotone, this says that $u_0 \in D(T_t) = X$ and $T_t u_0 = h_0^*$. Thus, (t_2) is true.

To show (t_3), we fix $t > 0$ and note that $u_0 \in X$ and

$$\inf \{ \langle T_t v - T_t u_0, v - u_0 \rangle : v \in L \} > 0 \tag{3.28}$$

imply, by the continuity of T_t and the density of L in X, that the same inequality is true for $v \in X$. This however is false because one such v is the element u_0.

We now show that C satisfies (c_1), (c_2), and (c_3) is identical to $(c3)$ with T_t in place of T. To see that (c_1) is satisfied, it suffices to observe that, in view of $c1$, the first inequality of (2.6) is true without the term Tu because $\langle Tu, u \rangle \geq 0$. To see that (c_2) is satisfied, it suffices to observe that $(c2)$ is stronger than (c_2).

It follows that the degree $d(T_t + C, G, 0)$, $t \in (0, t_1]$, is well defined. Now, fix the point $t_0 \in (0, t_1)$ and let $\lambda(t) \equiv t t_0 + (1-t) t_1$, $t \in [0,1]$. Since t_0 is picked arbitrarily in $(0, t_1)$, in order to show that this degree is constant on $(0, t_1]$, it suffices to show that $\{T_{\lambda(t)} + C\}$, $t \in [0,1]$, is an admissible homotopy in the sense of Definition 2.4 with $M_t = T_{\lambda(t)}$ and $A_t = C$.

To this end, we observe first that $(m_t^{(1)})$ is satisfied by what we saw above.

To see that $(m_t^{(2)})$ is true, we observe that for every $v \in L$, actually for every $v \in X \supset L$, and every $\bar{t} \in [0,1]$, we have

$$\lim_{t \to \bar{t}} T_{\lambda(t)} v = T_{\lambda(\bar{t})} v \tag{3.29}$$

by Lemma 3.1 above.

To show $(m_t^{(3)})$, fix $\bar{t} \in [0,1]$, $F \in \mathcal{F}(L)$, $u \in F$ and let $\{u_n\} \subset F$, $\{t_n\} \subset [0,1]$ be such that $t_n \to \bar{t}$ and $u_n \to u$. Then we have

$$\lim_{n \to \infty} T_{\lambda(t_n)} u_n = T_{\lambda(\bar{t})} u \tag{3.30}$$

by Lemma 3.1 above, which shows our assertion.

As far as $A_t = C$ is concerned, we have already checked the validity of $(a_t^{(1)})$ and $(a_t^{(3)})$. To show $(a_t^{(2)})$, we observe that $A_{t_j} = C$ and that the assumptions on C in it are stronger than those of $(\widetilde{S}_+)$.

Thus, $\{T_{\lambda(t)} + C\}$, $t \in [0,1]$, is an admissible homotopy. □

Definition 3.4 (degree for $(\widetilde{S}_+)$-perturbations C). Assume that the operators T, C and the set G satisfy the conditions of Theorem 3.3. Assume that $0 \notin (T+C)(D(T) \cap D(C) \cap \partial G)$. Then the new degree $d(T+C,G,0)$ is defined by

$$d(T+C,G,0) = d(T_t + C,G,0), \quad t \in (0,t_1], \tag{3.31}$$

where t_1 is as in the conclusion of Theorem 3.2. We also set

$$d(T+C,G,p^*) \equiv d(T+C-p^*,G,0), \qquad d(T+C,\emptyset,p^*) = 0 \tag{3.32}$$

for every $p^* \in X^*$ with $p^* \notin (T+C)(D(T) \cap D(C) \cap \partial G)$.

Remark 3.5. We note that in the above definition the operator $C - p^*$ satisfies all the assumptions $(c1)$–$(c3)$. Thus, the degree $d(T+C-p^*,G,0)$ is well defined. To see, in particular, that $C - p^*$ is quasibounded, let $\|u\| \le S$ and $\langle Cu - p^*, u\rangle \le S$, where S is a positive constant. Then

$$\|u\| \le S_1, \qquad \langle Cu, u\rangle \le S_1, \tag{3.33}$$

where

$$S_1 = S(\|p^*\| + 1). \tag{3.34}$$

Thus, the quasiboundedness of C implies $\|Cu\| \le K(S_1)$ and $\|Cu - p^*\| \le K(S_1) + \|p^*\| \equiv \widetilde{K}(S)$, where $\widetilde{K}(S)$ is now the quasiboundedness constant for $C - p^*$.

We should also point out that the degree $d(J,G,0)$ is well defined if $0 \notin J(\partial G)$, which is equivalent to $0 \notin \partial G$. We are allowed to take $C = \varepsilon J$, $\varepsilon > 0$ (or $C = \varepsilon J_\psi$, with J_ψ defined in Section 5), in Definition 3.4. However, we are not allowed to have $C = 0$ there. This is due to the fact that $C = 0$ does not satisfy the $(\widetilde{S}_+)$-condition.

4. Basic properties of the new degree

We are now going to establish (see Theorem 4.3 below) a homotopy property of the new degree. This property is used in Theorem 4.4(iii) in order to establish a more concrete and useful homotopy.

We consider the one-parameter family of operators $T^\tau : X \supset D(T^\tau) \to 2^{X^*}$, $\tau \in [0,1]$, satisfying the following conditions:

($t_\tau^{(1)}$) for each τ, the operator T^τ is maximal monotone, $0 \in D(T^\tau)$ and $0 \in T^\tau(0)$;

($t_\tau^{(2)}$) given sequences $\{\tau_n\}$, $\{u_n\}$, $\{v_n^*\}$, $n = 1,2,\ldots$, such that $u_n \in D(T^{\tau_n})$, $v_n^* \in T^{\tau_n}u_n$, $\tau_n \to \tau_0$, $u_n \rightharpoonup u_0$, $v_n^* \rightharpoonup v_0^*$, and

$$\limsup_{n\to\infty} \langle v_n^*, u_n\rangle \le \langle v_0^*, u_0\rangle, \tag{4.1}$$

for some $u_0 \in X$, $v_0^* \in X^*$, we have

$$u_0 \in D(T^{\tau_0}), \qquad v_0^* \in T^{\tau_0}u_0, \qquad \langle v_n^*, u_n\rangle \longrightarrow \langle v_0^*, u_0\rangle. \tag{4.2}$$

The condition $t_\tau^{(2)}$ was introduced by Browder [5] and was called "generalized pseudomonotonicity" condition.

The following lemma was proved by Browder in [5, Proposition 1(iv)].

Lemma 4.1. *Assume that the family of operators $\{T^\tau\}$ satisfies the condition ($t_\tau^{(1)}$). Then the condition ($t_\tau^{(2)}$) is equivalent to the following condition:*

($t_\tau^{(3)}$) *let $\{\tau_n\} \subset [0,1]$ be such that $\tau_n \to \tau_0$ and let $x \in D(T^{\tau_0})$, $x^* \in T^{\tau_0}x$, then there exist sequences $x_n \in D(T^{\tau_n})$, $x_n^* \in T^{\tau_n}x_n$ such that $x_n \to x$ and $x_n^* \to x^*$.*

Let $C^\tau : X \supset D(C^\tau) \to X^*$, $C^{\tau_n} \subset D(C^\tau)$, $\tau \in [0,1]$, be a second one-parameter family of operators satisfying the following conditions:

($c_\tau^{(1)}$) the family $\{C^\tau\}$ is "uniformly quasibounded", that is, for every $S > 0$, there exists $K(S) > 0$ such that

$$\langle C^\tau u, u\rangle \le 0, \quad \|u\| \le S, \quad \text{for some } \tau \in [0,1],\ u \in D(C^\tau), \tag{4.3}$$

imply the estimate $\|C^\tau u\| \le K(S)$;

($c_\tau^{(2)}$) for every pair of sequences $\{\tau_n\} \in [0,1]$, $\{u_n\} \subset L$ such that $u_n \rightharpoonup u_0$, $C^{\tau_n}u_n \rightharpoonup h^*$, $\tau_n \to \tau_0$, and

$$\limsup_{n\to\infty} \langle C^{\tau_n}u_n, u_n - u_0\rangle \le 0, \qquad \langle C^{\tau_n}u_n, u_n\rangle \le 0, \tag{4.4}$$

for some $\tau_0 \in [0,1]$, $u_0 \in X$, $h^* \in X^*$, we have $u_n \to u_0$, $u_0 \in D(C^{\tau_0})$ and $C^{\tau_0}u_0 = h^*$;

($c_\tau^{(3)}$) for every $F \in \mathcal{F}(L)$, $v \in L$, the mapping $\tilde{c}(F,v) : F \times [0,1] \to \mathcal{R}$ defined by $\tilde{c}(F,v)(u,\tau) = \langle C^\tau u, v\rangle$, is continuous.

Definition 4.2. Let $T^{(i)} : X \supset D(T^{(i)}) \to 2^{X^*}$, $C^{(i)} : X \supset D(C^{(i)}) \to X^*$, $i = 0,1$, satisfy conditions $(t1)$, $(c1)$–$(c3)$ of Section 3, respectively, with the space L independent of i. We say that the operators $T^{(0)}+C^{(0)}$, $T^{(1)}+C^{(1)}$ are "homotopic" with respect to the open bounded set $G\subset X$ if there exist one-parameter families of operators $T^\tau : X\supset D(T^\tau)\to 2^{X^*}$, $C^\tau : X\supset D(C^\tau)\to X^*$, $\tau\in[0,1]$, satisfying conditions $(t_\tau^{(1)})$, $(t_\tau^{(2)})$ and $(c_\tau^{(1)})$–$(c_\tau^{(3)})$, respectively, and such that

$$T^{(i)}=T^i,\quad C^{(i)}=C^i,\quad i=0,1, \tag{4.5}$$

$$T^\tau u+C^\tau u\neq 0,\quad u\in D(T^\tau)\cap D(C^\tau)\cap\partial G,\ \tau\in[0,1]. \tag{4.6}$$

When the operators T^τ, C^τ are as above, we also say that the mapping $H(\tau,x)\equiv(T^\tau+C^\tau)x$ is an "admissible homotopy."

Theorem 4.3. *Assume that the operators $T^{(i)}$, $C^{(i)}$, $i=0,1$ satisfy conditions $(t1)$, $(c1)$–$(c3)$ of Section 3, respectively. Assume that the operators $T^{(0)}+C^{(0)}$, $T^{(1)}+C^{(1)}$ are homotopic with respect to the bounded open set $G\subset X$. Then if T^τ, C^τ are as in Definition 4.2, it holds that*

$$d(T^\tau+C^\tau,G,0)=d(T^{(0)}+C^{(0)},G,0)=d(T^{(1)}+C^{(1)},G,0),\quad \tau\in[0,1], \tag{4.7}$$

where the degrees are well defined according to Definition 3.4.

Proof. We let $T_t^\tau\equiv(T^{\tau-1}+tJ^{-1})^{-1}:X\to X^*$, $t>0$. We will show the existence of $\tilde t_1>0$ such that

$$0\notin(T_t^\tau+C^\tau)(D(C^\tau)\cap\partial G),\quad (t,\tau)\in(0,\tilde t_1]\times[0,1]. \tag{4.8}$$

We assume that the contrary is true. Then there exist sequences $\{t_n\}\subset(0,\infty)$, $\{\tau_n\}\subset[0,1]$, $\{u_n\}\subset D(C^{\tau_n})\cap\partial G$ such that $t_n\to 0$, $\tau_n\to\tau_0\in[0,1]$, $u_n\rightharpoonup u_0\in X$, and

$$T_{t_n}^{\tau_n}u_n+C^{\tau_n}u_n=0. \tag{4.9}$$

Since $\langle T_{t_n}^{\tau_n}u_n,u_n\rangle\ge 0$, we have $\langle C^{\tau_n}u_n,u_n\rangle\le 0$ and from the uniform quasiboundedness of $\{C^\tau\}$ follows the boundedness of $\{\|C^{\tau_n}u_n\|\}$. We may assume that $C^{\tau_n}u_n\rightharpoonup h^*\in X^*$. We are going to show that

$$\limsup_{n\to\infty}\langle C^{\tau_n}u_n,u_n-u_0\rangle\le 0. \tag{4.10}$$

Assume that this is not true. Then we may also assume that $\{\tau_n\}$, $\{u_n\}$ are such that

$$\lim_{n\to\infty}\langle C^{\tau_n}u_n,u_n-u_0\rangle>0. \tag{4.11}$$

By virtue of (4.9), this implies

$$\lim_{n\to\infty}\langle T_{t_n}^{\tau_n}u_n,u_n-u_0\rangle<0. \tag{4.12}$$

Denote $g_n^*=T_{t_n}^{\tau_n}u_n$. Then there exists $w_n\in D(T^{\tau_n})$ such that

$$u_n=t_nJ^{-1}g_n^*+w_n,\quad g_n^*\in T^{\tau_n}w_n. \tag{4.13}$$

Now, let $(x,x^*) \in G(T^{\tau_0})$. Then, by Lemma 4.1, there exist $x_n \in D(T^{\tau_n})$, $x_n^* \in T^{\tau_n}x_n$ such that $x_n \to x$, $x_n^* \to x^*$. By the monotonicity of T^{τ_n}, we obtain

$$0 \le \langle g_n^* - x_n^*, u_n - t_n J^{-1} g_n^* - x_n \rangle. \tag{4.14}$$

Noting that $g_n^* \rightharpoonup -h^*$, we obtain from (4.14)

$$\liminf_{n\to\infty} \langle T_{t_n}^{\tau_n} u_n, u_n \rangle \ge \langle -h^*, x \rangle + \langle x^*, u_0 - x \rangle. \tag{4.15}$$

This and (4.12) imply

$$\langle -h^* - x^*, u_0 - x \rangle > 0. \tag{4.16}$$

Since $x \in D(T^{\tau_0})$, $x^* \in T^{\tau_0}x$ are otherwise arbitrary and T^{τ_0} is maximal monotone, we have from (4.16) $u_0 \in D(T^{\tau_0})$ and $-h^* \in T^{\tau_0}u_0$. Taking $x = u_0$ in (4.16), we obtain a contradiction. Consequently, (4.10) is true.

Using the condition $(c_\tau^{(2)})$, we conclude that $u_n \to u_0$, $u_0 \in D(C^{\tau_0}) \cap \partial G$, and $C^{\tau_0}u_0 = h^*$. This says

$$\lim_{n\to\infty} \langle T_{t_n}^{\tau_n} u_n, u_n - u_0 \rangle = -\lim_{n\to\infty} \langle C^{\tau_n} u_n, u_n - u_0 \rangle = 0. \tag{4.17}$$

Repeating the argument that we carried out above starting with (4.12), we obtain from (4.16) (with "$>$" replaced by "$\ge$") $u_0 \in D(T^{\tau_0})$, $-h^* \in T^{\tau_0}u_0$. Thus, $0 \in T^{\tau_0}u_0 + C^{\tau_0}u_0$ with $u_0 \in D(T^{\tau_0}) \cap D(C^{\tau_0}) \cap \partial G$, and we have a contradiction with (4.6). The proof of (4.8) is complete.

We fix $t_0 \in (0, \tilde{t}_1]$ and introduce the operator $M^\tau = T_{t_0}^\tau$. We need to check that conditions $(m_t^{(1)})$–$(m_t^{(3)})$ are satisfied for the operator M^τ and conditions $(a_t^{(1)})$–$(a_t^{(3)})$ are satisfied for the operator C^τ. Then the assertion of the theorem will follow immediately from Theorem 2.5.

Condition $(m_t^{(1)})$. Conditions (t_1)–(t_4) have already been checked in the proof of Theorem 3.3 for the operator satisfying the condition $t1$.

Condition $(m_t^{(2)})$. We have to show that for every $u \in X$, the mapping $\tau \to M^\tau u$ is continuous. Consider the sequence $\{\tau_n\} \subset [0,1]$ such that $\tau_n \to \tau_0$. Let $v_n^* = M^{\tau_n}u$. Then there exists $w_n \in D(T^{\tau_n})$ such that

$$u = t_0 J^{-1} v_n^* + w_n, \quad v_n^* \in T^{\tau_n} w_n. \tag{4.18}$$

Using the monotonicity of the operator T^{τ_n} and the condition $0 \in T^{\tau_n}(0)$, we get

$$\langle v_n^*, u \rangle = t_0 \langle v_n^*, J^{-1} v_n^* \rangle + \langle v_n^*, w_n \rangle \ge t_0 \|v_n^*\|^2, \tag{4.19}$$

which yields the boundedness of the sequence $\{v_n^*\}$ and hence the sequence $\{w_n\}$. Thus, we may assume that

$$v_n^* \rightharpoonup v_0^*, \qquad J^{-1}v_n^* \rightharpoonup v_0, \qquad w_n \rightharpoonup w_0, \tag{4.20}$$

for some $v_0, w_0 \in X$, $v_0^* \in X^*$.

From (4.18), (4.20), we obtain

$$\limsup_{n\to\infty} \langle v_n^*, w_n\rangle = \limsup_{n\to\infty} \left[\langle v_n^* - v_0^*, u\rangle - t_0\langle v_n^* - v_0^*, J^{-1}v_n^*\rangle + \langle v_0^*, w_n\rangle\right] \le \langle v_0^*, w_0\rangle. \tag{4.21}$$

Using the condition $(t_\tau^{(2)})$, we get from (4.21) $v_0^* \in T^{\tau_0}w_0$, $\langle v_n^*, w_n\rangle \to \langle v_0^*, w_0\rangle$. From this and (4.18), we have

$$\lim_{n\to\infty} \langle v_n^* - v_0^*, t_0 J^{-1}v_n^*\rangle = \lim_{n\to\infty} \langle v_n^* - v_0^*, u - w_n\rangle = 0. \tag{4.22}$$

This implies $v_n^* \to v_0^*$ by the (S_+)-property of the operator J^{-1}.

Passing to the limit in (4.18), we get $u = t_0 J^{-1}v_0^* + w_0$, which gives $v_0^* = T_{t_0}^{\tau_0} u = M^{\tau_0}u$, and the proof of the condition $(m_t^{(2)})$ is complete.

Condition $(m_t^{(3)})$. The proof of this condition goes as in the case of condition $(m_t^{(2)})$. It is therefore omitted.

Condition $(a_t^{(1)})$. Let $u \in L$ be such that

$$\langle M^\tau u + C^\tau u, u\rangle \le 0, \quad \|u\| \le S,\ \tau \in [0,1]. \tag{4.23}$$

Then condition $c_\tau^{(1)}$ and the fact that $\langle M^\tau u, u\rangle \ge 0$ imply $\|C^\tau u\| \le K(S)$. This proves condition $a_t^{(1)}$.

Conditions $(a_t^{(2)})$, $(a_t^{(3)})$ follow immediately from conditions $(c_\tau^{(2)})$, $(c_\tau^{(3)})$, respectively.

This ends the proof of the theorem. □

An important homotopy is included in the statement of Theorem 4.4(iii) below.

Let the mapping $\phi : \mathcal{R}_+ \to \mathcal{R}_+$ be such that $\phi(0) = 0$ and if $r_n > 0$, $n = 1, 2, \ldots$, satisfies

$$\lim_{n\to\infty} \phi(r_n) = 0, \tag{4.24}$$

then $r_n \to 0^+$. We say that the operator $C : X \to X^*$ belongs to the class Γ_ϕ if there exists a function ϕ, as above, such that $\langle Cx, x\rangle \ge \phi(\|x\|)$, $x \in X$.

Theorem 4.4. *Assume that the operator T satisfies $(t1)$ and the operator C satisfies $(c1)$–$(c3)$. Let G be an open and bounded subset of X. Let d denote the degree mapping defined in Definition 3.4. Then, the following statements are true.*

(i) *If $0 \in G$, then, for every $\lambda > 0$,*

$$d(T + \lambda J, G, 0) = 1. \tag{4.25}$$

If $0 \notin J(\overline{G})$,

$$d(J, G, 0) = 0. \tag{4.26}$$

(ii) *If* $p^* \notin (T+C)(D(T) \cap D(C) \cap \partial G)$ *and* $d(T+C, G, p^*) \neq 0$ *then there exists* $x \in D(T) \cap D(C) \cap G$ *such that* $(T+C)x \ni p^*$.

(iii) *If* $0 \in G$, *then the degree* $d(H(t,\cdot), G, 0)$ *is well defined and invariant under homotopies of the type*

$$H(t,x) \equiv t(T+C_1)x + (1-t)C_2 x, \quad t \in [0,1], \tag{4.27}$$

provided that $0 \notin H(t,\cdot)(\partial G)$, $t \in [0,1]$. *Here,* C_1 *satisfies c1–c3 and* $C_2 : X \to X^*$ *is bounded, demicontinuous, of type* (S_+), *and belongs to the class* Γ_ϕ, *for some function* $\phi : \mathcal{R}_+ \to \mathcal{R}_+$. *In particular,* $d(T+C_1, G, 0) = d(C_2, G, 0)$.

(iv) *The degree* $d(H(t,\cdot), G, 0)$ *is invariant under homotopies of the type*

$$H(t,x) \equiv (T+C)x - y^*(t), \quad t \in [0,1], \tag{4.28}$$

where $y^* : [0,1] \to X^*$ *is a continuous curve. Here,* $0 \notin H(t,\cdot)(\partial G)$, $t \in [0,1]$.

(v) *If* G_1, G_2 *are open and bounded sets in* X *such that* $G_1 \cap G_2 = \emptyset$, $\overline{G} = \overline{G}_1 \cup \overline{G}_2$, *and* $0 \notin (T+C)(D(T+C) \cap \partial G_i)$, $i = 1,2$, *then*

$$d(T+C, G, 0) = d(T+C, G_1, 0) + d(T+C, G_2, 0). \tag{4.29}$$

Proof. Property (i) is a well-known property of the degree mapping which goes back to Skrypnik in 1973 (see [27]). In fact, (4.25) follows from the fact that

$$d(T+\lambda J, G, 0) = \lim_{s \downarrow 0} d(T_s + \lambda J, G, 0) = 1 \tag{4.30}$$

because the operator $T_s + \lambda J$ is demicontinuous, bounded, strictly monotone (and thus one-to-one), and satisfies $\langle T_s x + \lambda J x, x \rangle \geq 0$, $x \in \partial G$ (cf. Browder [4, Theorem 3(iv)]). Equation (4.26) follows from the same Browder reference as well.

To show (ii), assume that $p^* \notin (T+C)(D(T) \cap D(C) \cap \partial G)$ and $d(T+C, G, p^*) \neq 0$. Then if t_1 is as in Theorem 3.2, with C replaced by $C - p^*$, we have $d(T_t + C - p^*, G, 0) \neq 0$ for all $t \in (0, t_1]$. This implies that for each $t \in (0, t_1]$, there exists $x_t \in D(C) \cap G$ such that

$$T_t x_t + C x_t = p^*. \tag{4.31}$$

Thus, given a sequence $\{\bar{t}_n\} \subset (0, t_1]$ such that $\bar{t}_n \downarrow 0$, there exists a sequence $\{x_n\} \subset D(C) \cap G$ such that

$$T_{\bar{t}_n} x_n + C x_n = p^*. \tag{4.32}$$

Since $\{x_n\}$ is bounded, we may assume that $x_n \rightharpoonup x_0 \in X$. Since

$$\langle C x_n - p^*, x_n \rangle = -\langle T_{\bar{t}_n} x_n, x_n \rangle \leq 0, \tag{4.33}$$

we can use the quasiboundedness of $C - p^*$ to conclude, without any loss of generality, that $Cx_n - p^* \rightharpoonup h^* \in X^*$. We can now show, as in the proof of Theorem 3.2, that

$$\limsup_{n\to\infty} \langle Cx_n - p^*, x_n - x_0 \rangle \le 0. \tag{4.34}$$

Using the condition $(\widetilde{S}_+)$, we see that $x_n \to x_0$, $x_0 \in D(C)$ and $Cx_0 - p^* = h^*$. Consequently, working as with (3.20) in the proof of Theorem 3.2 (we now have (3.26) instead of (3.20)), we obtain $x_0 \in D(T)$ and $Tx_0 + Cx_0 \ni p^*$.

(iii) We set $T^\tau = \tau T$, $C^\tau = \tau C_1 + (1-\tau)C_2$, and verify first that the family of operators C^τ satisfies conditions $(c_\tau^{(1)})$–$(c_\tau^{(3)})$.

To show $c_\tau^{(1)}$, let $S > 0$, $u \in D(C^\tau)$ be such that

$$\langle C^\tau u, u \rangle = \langle \tau C_1 u + (1-\tau)C_2 u, u \rangle \le 0, \quad \|u\| \le S,\ \tau \in [0,1]. \tag{4.35}$$

If $\tau = 0$, then $\|C^\tau u\| = \|C_2 u\| \le M$ by the boundedness of C_2 on the ball $\overline{B_S(0)}$. If $\tau \in (0,1]$, then $\langle C_1 u, u \rangle \le 0$, the quasiboundedness of C_1 and the boundedness of C_2 imply $\|C^\tau u\| = \|\tau C_1 u + (1-\tau)C_2 u\| \le K(S) + M = \widetilde{K}(S)$, where $K(S)$ is the quasiboundedness constant of C_1 and $\widetilde{K}(S)$ is the uniform quasiboundedness constant of C^τ. This finishes the proof of $(c_\tau^{(1)})$.

To show $(c_\tau^{(2)})$, assume that for a pair of sequences $\{\tau_n\} \subset [0,1]$, $\{u_n\} \subset L$, we have $u_n \rightharpoonup u_0$, $C^{\tau_n} u_n \rightharpoonup h_0^*$, $\tau_n \to \tau_0$, and

$$\limsup_{n\to\infty} \langle C^{\tau_n} u_n, u_n - u_0 \rangle \le 0, \qquad \langle C^{\tau_n} u_n, u_n \rangle \le 0, \tag{4.36}$$

for some $\tau_0 \in [0,1]$, $u_0 \in X$, $h_0^* \in X^*$. We observe first that the above assumptions are true for any subsequences $\{\tau_{n_k}\}$, $\{u_{n_k}\}$ as well.

We first consider the case $\tau_0 = 0$. Assume that $\tau_n = 0$ for all large n. Then $C^{\tau_n} u_n = C_2 u_n \rightharpoonup h_0^*$ and the first inequality of (4.36), along with the (S_+)-property of C_2, implies $u_n \to u_0$ and $C^{\tau_n} u_n \rightharpoonup C^0 u_0 = C_2 u_0 = h_0^*$.

Now, assume that there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $\tau_{n_k} = 0$ for all k. Then $C^{\tau_{n_k}} u_{n_k} = C_2 u_{n_k} \rightharpoonup h_0^*$ and the first inequality of (4.36), which holds for $\{n_k\}$ instead of $\{n\}$, along with the (S_+)-property of C_2, implies $u_{n_k} \to u_0$, $C^{\tau_{n_k}} u_{n_k} \rightharpoonup C^0 u_0 = C_2 u_0 = h_0^*$. Let, for another subsequence $\{n_j\}$ of $\{n\}$, $\tau_{n_j} > 0$ for all j. Then the second inequality of (4.36) says that

$$\langle \tau_{n_j} C_1 u_{n_j} + (1 - \tau_{n_j}) C_2 u_{n_j}, u_{n_j} \rangle \le 0 \tag{4.37}$$

and implies easily the boundedness of $\{\|C_1 u_{n_j}\|\}$ and $|\tau_{n_j} \langle C_1 u_{n_j}, u_{n_j} \rangle| \to 0$. Then the first inequality of (4.36), along with the (S_+)-property of C_2, implies again $u_{n_j} \to u_0$ and

$$C^{\tau_{n_j}} u_{n_j} = \tau_{n_j} C_1 u_{n_j} + (1 - \tau_{n_j}) C_2 u_{n_j} \rightharpoonup C^0 u_0 = C_2 u_0 = h_0^*. \tag{4.38}$$

It is evident from the above analysis that $u_n \to u_0$, $u_0 \in D(C^{\tau_0}) = D(C^0) = D(C_2) = X$ and $C^0 u_0 = h_0^*$. We are thus done with the case $\tau_0 = 0$.

For the case $\tau_0 > 0$, we set

$$a_n \equiv \tau_n \langle C_1 u_n, u_n - u_0 \rangle, \qquad b_n \equiv (1 - \tau_n) \langle C_2 u_n, u_n - u_0 \rangle \tag{4.39}$$

and assume that $\tau_n > 0$. We know from (4.36) that

$$\limsup_{n \to \infty} (a_n + b_n) \le 0, \tag{4.40}$$

with both sequences $\{a_n\}$, $\{b_n\}$ bounded. We also know that C_2 is bounded and that (4.36) and the quasiboundedness of C_1 imply the boundedness of $\{C_1 u_n\}$ as well. Thus, we may assume that $C_1 u_n \rightharpoonup h_1^*$ and $C_2 u_n \rightharpoonup h_2^*$. We also observe that there exists a subsequence $\{n_k\}$ of $\{n\}$ such that one of the inequalities

$$\limsup_{k \to \infty} a_{n_k} \le 0, \qquad \limsup_{k \to \infty} b_{n_k} \le 0 \tag{4.41}$$

holds true.

We assume first that $\tau_0 \in (0,1)$. We also assume that the first inequality of (4.41) is true. Then

$$\limsup_{k \to \infty} \tau_{n_k} \langle C_1 u_{n_k}, u_{n_k} - u_0 \rangle \le 0 \tag{4.42}$$

and $\tau_{n_k} \to \tau_0 > 0$ imply

$$\limsup_{k \to \infty} \langle C_1 u_{n_k}, u_{n_k} - u_0 \rangle \le 0. \tag{4.43}$$

The assumption $(\widetilde{S}_+)$ for C_1 implies that $u_{n_k} \to u_0$, $u_0 \in D(C_1)$, and $C_1 u_0 = h_1^*$.

Since C_2 is demicontinuous, $C_2 u_0 = h_2^*$. Thus,

$$\begin{aligned} C^{\tau_{n_k}} u_{n_k} &= \tau_{n_k} C_1 u_{n_k} + (1 - \tau_{n_k}) C_2 u_{n_k} \rightharpoonup \tau_0 C_1 u_0 + (1 - \tau_0) C_2 u_0 \\ &= \tau_0 h_1^* + (1 - \tau_0) h_2^* = h^*. \end{aligned} \tag{4.44}$$

It follows that $u_0 \in D(C^{\tau_0})$ and $C^{\tau_0} u_0 = h^*$.

If the second inequality of (4.41) is true, and $\tau_0 \in (0,1)$, then

$$\limsup_{k \to \infty} \langle C_2 u_{n_k}, u_{n_k} - u_0 \rangle \le 0 \tag{4.45}$$

and the (S_+)-property of C_2 imply $u_{n_k} \to u_0 \in X$. Since C_2 is demicontinuous, $C_2 u_0 = h_2^*$. Since C_1 satisfies condition $(\widetilde{S}_+)$ and the first inequality in (4.41) is true again, we have $u_0 \in D(C_1)$ and $C_1 u_0 = h_1^*$. The rest of the proof for this case follows exactly as above. It is therefore omitted.

We now assume that $\tau_0 = 1$. Then since

$$(1 - \tau_n) \left| \langle C_2 u_n, u_n - u_0 \rangle \right| \le (1 - \tau_n) \|C_2 u_n\| \|u_n - u_0\| \longrightarrow 0, \tag{4.46}$$

(4.36) implies

$$\limsup_{n\to\infty} \langle C_1u_n, u_n - u_0\rangle \le 0. \tag{4.47}$$

Again, the $(\widetilde{S}_+)$-property of C_1 says that $u_n \to u_0 \in D(C_1)$ and $C_1u_0 = h_1^*$. Thus, $C_2u_0 = h_2^*$, and the situation repeats itself as above.

Property $c_\tau^{(3)}$ follows immediately from the corresponding property of C_1 and the demicontinuity of C_2.

We will now prove that $d(H(t,\cdot),G,0)$ does not depend on $t \in (0,1]$. We fix $t_0 \in (0,1)$ and consider the homotopy

$$H_1(t,x) = \left[t_0(1-t)+t\right](T+C_1)x + (1-t)(1-t_0)C_2x. \tag{4.48}$$

All the conditions of Theorem 4.3 are satisfied for $H_1(t,x)$. For example, condition $t_\tau^{(2)}$ follows immediately from Lemma 4.1. Conditions $(c_\tau^{(1)})$–$(c_\tau^{(3)})$ follow from the properties of the operator C^τ established above. Using Theorem 4.3, we have

$$d(H(t,\cdot),G,0) = d(H(1,\cdot),G,0) \tag{4.49}$$

for $t_0 \le t \le 1$ and, consequently, the last equality is true for $0 < t \le 1$ since the number t_0 is arbitrary.

Now, we need to prove the equality

$$d(H(0,\cdot),G,0) = \lim_{t\to 0} d(H(t,\cdot),G,0). \tag{4.50}$$

We set $T_t^\tau = (T^{\tau^{-1}} + tJ^{-1})^{-1}$. We will establish the existence of a number $\delta > 0$ such that

$$T_t^\tau x + C^\tau x \ne 0 \tag{4.51}$$

for $x \in \mathfrak{D}(C^\tau)\cap \partial G$, $0 < t \le \delta$, $0 \le \tau \le \delta$. Assume that the contrary is true. Then there exist sequences $\tau_n \in (0,1)$, $t_n \in (0,\infty)$, $x_n \in \partial G$ such that $\tau_n \to 0$, $t_n \to 0$, and

$$T_{t_n}^{\tau_n} x_n + C^{\tau_n} x_n = 0. \tag{4.52}$$

The other possible case of $\tau_n = 0$, for some n, can be easily discarded. Using the monotonicity of the operator T_t^τ, the equality $T_t^\tau 0 = 0$ and the inequality $\langle C_2x, x\rangle \ge 0$, we obtain from (4.52)

$$\langle C_1x_n, x_n\rangle \le 0. \tag{4.53}$$

The quasiboundedness of the operator C_1 implies the boundedness of the sequence $\{C_1x_n\}$. Now, we have from (4.52)

$$\limsup_{n\to\infty} \langle C_2x_n, x_n\rangle \le 0, \tag{4.54}$$

which contradicts the fact that the operator C_2 belongs to the class Γ_ϕ. It follows that $T_t^\tau x + C^\tau x \ne 0$ for $x \in D(C^\tau)\cap\partial G$, $0 < t \le \delta$, $0 \le \tau \le \delta$.

Consider the homotopy

$$H_2(t,x) = T_\delta^{\delta t}x + C^{\delta t}x. \tag{4.55}$$

We need to prove that this homotopy satisfies all the conditions of Theorem 2.5.

We will check the conditions $(m_t^{(2)})$, $(m_t^{(3)})$. It is sufficient to establish the continuity of $T_\delta^{\delta t}x$ with respect to t, x. Define the mapping

$$J^t \equiv I - \delta J^{-1}T_\delta^{\delta t} : X \longrightarrow X, \tag{4.56}$$

where $J : X \to X^*$ is the duality mapping. Then

$$J(x - J^t x) = \delta T_\delta^{\delta t}x \in \delta T^{\delta t}J^t x = \delta^2 tTJ^t x \tag{4.57}$$

is true for $t \in [0,1]$, $x \in X$. Let $t_n \in [0,1]$, $x_n \in X$ be such that $t_n \to t_0$, $x_n \to x_0$. From (4.57), we obtain the existence of $y_n^* \in TJ^{t_n}x_n$, $y_0^* \in TJ^{t_0}x_0$ such that

$$\delta^2 t_n y_n^* = J(x_n - J^{t_n}x_n), \qquad \delta^2 t_0 y_0^* = J(x_0 - J^{t_0}x_0). \tag{4.58}$$

Using this, the monotonicity of the operator T and the assumptions $0 \in D(T)$, $0 \in T(0)$, we have

$$\begin{aligned} \|x_n - J^{t_n}x_n\|^2 &= \langle J(x_n - J^{t_n}x_n), x_n - J^{t_n}x_n\rangle \\ &= \delta^2 t_n\langle y_n^*, x_n - J^{t_n}x_n\rangle \\ &\le \langle J(x_n - J^{t_n}x_n), x_n\rangle, \end{aligned} \tag{4.59}$$

which implies the boundedness of the sequence $\{J^{t_n}x_n\}$.

From (4.58) and the monotonicity of the operator T, we get

$$\begin{aligned} &\langle J(x_n - J^{t_n}x_n) - J(x_0 - J^{t_0}x_0), J^{t_n}x_n - J^{t_0}x_0\rangle \\ &\quad = \delta^2\langle t_n y_n^* - t_0 y_0^*, J^{t_n}x_n - J^{t_0}x_0\rangle \\ &\quad \ge \delta^2(t_n - t_0)\langle y_0^*, J^{t_n}x_n - J^{t_0}x_0\rangle. \end{aligned} \tag{4.60}$$

From this inequality, the boundedness of the sequence $\{J^{t_n}x_n\}$, $t_n \to t_0$, and $x_n \to x_0$, we obtain

$$\lim_{n\to\infty}\langle Jz_n - Jz_0, z_n - z_0\rangle = 0, \tag{4.61}$$

where $z_n = x_n - J^{t_n}x_n$, $z_0 = x_0 - J^{t_0}x_0$. This and a well-known property of the duality mapping imply $z_n \to z_0$ and, consequently, $J^{t_n}x_n \to J^{t_0}x_0$. Then $T_\delta^{\delta t_n}x_n \to T_\delta^{\delta t_0}x_0$ by virtue of (4.57), and the desired continuity of the mapping $T_\delta^{\delta t}$ has been established.

Using Definition 3.4, the properties of the operator C^τ established above and Theorem 2.5, we obtain the equality (4.50) and the proof of part (iii) is complete.

The proof of part (iv) follows simply from Theorem 4.3 and it is therefore omitted.

(v) Let $T_t : X \to X^*$ be as in Theorem 3.2. As in the proof of Theorem 3.2, we establish the existence of $t_1 \in (0,\infty)$ such that

$$0 \notin (T_t + C)(D(C) \cap \partial G_i), \quad i = 1,2, \qquad 0 \notin (T_t + C)(D(C) \cap \partial G), \tag{4.62}$$

$t \in (0,t_1]$. Using the additivity property of the degree of the operator $T_t + C$, which follows simply from our construction in [15], we have

$$d(T_t + C, G, 0) = d(T_t + C, G_1, 0) + d(T_t + C, G_2, 0), \quad t \in (0,t_1]. \tag{4.63}$$

Assertion (v) follows from this and Definition 3.4. □

5. Extending results of Browder and Hess

We denote by J_ψ the duality mapping with gauge function ψ. The function $\psi : \mathcal{R}_+ \to \mathcal{R}_+$ is continuous, strictly increasing and such that $\psi(0) = 0$ and $\psi(r) \to \infty$ at $r \to \infty$. This mapping J_ψ is continuous, bounded, surjective, strictly and maximal monotone, and satisfies condition (S_+). Also, $\langle J_\psi x, x\rangle = \psi(\|x\|)\|x\|$ and $\|J_\psi x\| = \psi(\|x\|)$, $x \in X$. Thus, $J_\psi \in \Gamma_\phi$, where $\phi(r) = \psi(r)r$. For these facts, we refer to Petryshyn [23, pages 32-33 and 132]. Petryshyn used in [23, Lemma 2.5] the separability of X in order to get a convergent subsequence of a bounded sequence $\{J_\psi x_j\}$ there. However, the separability of X is not needed in our setting because of the Eberlein-Smulyan theorem about reflexive spaces. For the property $d(J_\psi, G, 0) = 1$, for any bounded open set G containing zero, see Lemma 5.10 below.

The following proposition shows how we can solve an important approximate problem for the operator $T + C$. This approximate problem, inclusion (5.3) below, can be used in a variety of problems in nonlinear analysis, that is, problems of solvability, existence of eigenvalues, ranges of sums, invariance of domain, bifurcation, and so forth.

Proposition 5.1. *Assume that the operator T satisfies $(t1)$ and the operator C satisfies $(c1)$–$(c3)$. Let G be an open and bounded subset of X with $0 \in G$. Assume that $(H(t,\cdot))(\partial G) \not\ni p^*$, $t \ni [0,1]$, where*

$$H(t,x) \equiv t(T + C - p^* + \varepsilon J_\psi)x + (1-t)J_\psi x, \tag{5.1}$$

$p^* \in X^*$ is fixed, and ε is a positive constant. Then the degree $d(H(t,\cdot), G, 0)$ is well defined and

$$d(H(t,\cdot), G, 0) = \textit{constant}, \quad t \in [0,1]. \tag{5.2}$$

In particular, the inclusion

$$Tx + Cx + \varepsilon J_\psi x \ni p^* \tag{5.3}$$

is solvable in G.

Proof. The conclusion of this proposition follows from (i)–(iii) of Theorem 4.4. In fact, one may take here $C_1 = C - p^* + \varepsilon J_\psi$ and $C_2 = J_\psi$. Then the homotopy invariance in (iii) of Theorem 4.4 says that (5.2) is true. This says that

$$d(T + C - p^* + \varepsilon J_\psi, G, 0) = d(J_\psi, G, 0) = 1, \tag{5.4}$$

by Theorem 4.4(i), because $0 \in G$. Finally, Theorem 4.4(ii) implies (5.3). □

We need the following definition from Browder and Hess [6].

Definition 5.2. An operator $C : X \supset D(C) \to 2^{X^*}$ is called "generalized pseudomonotone" if for every sequence $(x_n, y_n^*) \subset G(C)$ such that

$$x_n \rightharpoonup x_0, \qquad y_n^* \rightharpoonup y_0^*, \qquad \limsup_{n\to\infty} \langle y_n^*, x_n - x_0 \rangle \le 0, \tag{5.5}$$

for some $x_0 \in X$, $y_0^* \in X^*$, it holds that $x_0 \in D(C)$, $y_0^* \in Cx_0$, and $\langle y_n^*, x_n \rangle \to \langle y_0^*, x_0 \rangle$.

An operator $C : X \supset D(C) \to 2^{X^*}$ is called "coercive" if there exists a function $\phi : \mathcal{R}_+ \to \mathcal{R}$ such that $\phi(r) \to \infty$ as $r \to \infty$ and

$$\langle y^*, x \rangle \ge \phi(\|x\|)\|x\|, \quad (x, y^*) \in G(C). \tag{5.6}$$

An operator $C : X \supset D(C) \to 2^{X^*}$ is called "smooth" if it is bounded, coercive, maximal monotone, and has effective domain $D(T) = X$.

A generalized pseudomonotone operator $C : X \supset D(C) \to 2^{X^*}$ is called "regular" if $R(T + C) = X^*$ for every smooth operator T.

The operator $T + C$ in our degree theory (as well as in Proposition 5.1 and Theorem 5.4 below) is generalized pseudomonotone. This is included in the following lemma.

Lemma 5.3. *Let T satisfy condition $(t1)$. Let $C : X \supset D(C) \to X^*$ be generalized pseudomonotone and satisfy $(c1)$, $(c3)$. Then the operator $T + C$ is generalized pseudomonotone.*

Proof. Our assertion follows from Theorem 1 of Browder and Hess [6, page 260]. In fact, it suffices to notice that T is generalized pseudomonotone (see [6, Proposition 2, page 257]) and such that $\langle u, x \rangle \ge 0$ for all $(x, u) \in G(T)$, while C is generalized pseudomonotone and quasibounded ("strongly quasibounded" according to Browder and Hess [6]). □

However, we cannot replace, within our methodology, the operator $T + C$ by a single multivalued generalized pseudomonotone operator, because we have no degree theory, as yet, for such mappings.

In order to demonstrate the applicability of our new degree theory, we give below an existence theorem concerning single-valued and densely defined generalized pseudomonotone perturbations. This result uses the homotopy function of Proposition 5.1, where the condition $(\widetilde{S}_+)$ for the operator C is actually replaced by the weaker assumption of generalized pseudomonotonicity and does not follow from any of the results of Browder and Hess [6]. A related result is in [10, Theorem 2.1].

Theorem 5.4 (existence). *Let T satisfy $(t1)$. Let $C : X \supset D(C) \to X^*$ satisfy $(c1)$, $(c3)$ and be generalized pseudomonotone. Assume that there exist a constant $Q > 0$ and $\beta : [Q,\infty) \to \mathcal{R}_+$, with $\beta(r) \to 0$ as $r \to \infty$, such that: for every $x \in D(T) \cap D(C)$ with $\|x\| \geq Q$ and every $u \in Tx$, it holds that*

$$\langle u + Cx, x\rangle \geq -\beta(\|x\|)\psi(\|x\|)\|x\|, \tag{5.7}$$

where ψ is a gauge function. Then, for every $\varepsilon > 0$, $R(T + C + \varepsilon J_\psi) = X^*$.

If, in addition,

$$\liminf_{\substack{x\in D(T)\cap D(C)\\ \|x\|\to\infty}} \frac{|Tx + Cx|}{\psi(\|x\|)} > 0, \tag{5.8}$$

then $R(T + C) = X^*$.

Proof. We fix $p^* \in X^*$, $\varepsilon > 0$, and consider the problem

$$Tx + Cx + \varepsilon J_\psi x \ni p^*. \tag{5.9}$$

As in Proposition 5.1, we consider the homotopy inclusion

$$H(t,x) \equiv t(Tx + Cx - p^* + \varepsilon J_\psi x) + (1-t)J_\psi x \ni 0, \quad t \in [0,1], \tag{5.10}$$

and apply Theorem 4.4(iii). To this end, we need to show first that the operator $U = C + \varepsilon J_\psi - p^*$ satisfies the conditions $(c1)$–$(c3)$, and the operator J_ψ satisfies the conditions on C_2 in Theorem 4.4(iii). The latter is obviously true. Also, it is evident that the operator U satisfies $c1$, $c3$. To show that U satisfies $c2$, assume that $x_n \rightharpoonup x_0$, $Ux_n \rightharpoonup h^*$, and

$$\limsup_{n\to\infty} \langle Ux_n, x_n - x_0\rangle \leq 0, \tag{5.11}$$

for some $x_0 \in X$, $h^* \in X^*$. Since $\{J_\psi x_n\}$ is bounded, we may assume that $Cx_n \rightharpoonup h_1^*$. We show that $x_n \to x_0$, $x_0 \in D(U)$ and $Ux_0 = h^*$. Since $\langle p^*, x_n - x_0\rangle \to 0$, (5.11) implies

$$\limsup_{n\to\infty} \langle Cx_n + \varepsilon J_\psi x_n, x_n - x_0\rangle \leq 0. \tag{5.12}$$

Using the monotonicity of J_ψ, we get

$$\begin{aligned}\langle Cx_n, x_n - x_0\rangle &\leq \langle Cx_n + \varepsilon J_\psi x_n - \varepsilon J_\psi x_0, x_n - x_0\rangle \\ &= \langle Cx_n + \varepsilon J_\psi x_n, x_n - x_0\rangle - \langle \varepsilon J_\psi x_0, x_n - x_0\rangle,\end{aligned} \tag{5.13}$$

which, along with (5.12), gives

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0\rangle \leq 0. \tag{5.14}$$

Since C is generalized pseudomonotone, we obtain $x_0 \in D(C)$, $Cx_0 = h_1^*$ and $\langle Cx_n, x_n\rangle \to \langle Cx_0, x_0\rangle$. Thus,

$$\begin{aligned}\lim_{n\to\infty} \langle Cx_n, x_n - x_0\rangle &= \lim_{n\to\infty} \langle Cx_n, x_n\rangle - \lim_{n\to\infty} \langle Cx_n, x_0\rangle \\ &= \langle Cx_0, x_0\rangle - \langle Cx_0, x_0\rangle = 0.\end{aligned} \tag{5.15}$$

Using this in (5.12), we get

$$\limsup_{n\to\infty} \langle J_\psi x_n, x_n - x_0\rangle \le 0. \tag{5.16}$$

Since J_ψ is of type (S_+), we have $x_n \to x_0$ and $J_\psi x_n \to J_\psi x_0$. Consequently, $x_0 \in D(U) = D(C)$ and $Ux_0 = Cx_0 - p^* + \varepsilon J_\psi x_0 = h^*$. It follows that $c2$ is satisfied.

We now show that all possible solutions of the inclusion (5.10) are bounded by a constant which is independent of $t \in [0,1]$. To this end, assume that there exists a sequence $\{t_m\} \subset [0,1]$ and a sequence $\{x_m\} \subset D(H(t_m,\cdot))$ such that $\|x_m\| \to \infty$ as $m \to \infty$. If there exists a subsequence $\{t_{m_k}\}$ of $\{t_m\}$ such that $t_{m_k} = 0$, $k = 1,2,\ldots$, then $x_{m_k} = 0$ for all k, which contradicts $\|x_{m_k}\| \to \infty$ as $k \to \infty$. We may thus assume that $t_m > 0$, $m = 1,2,\ldots$. Then, $D(H(t_m,\cdot)) = D(C) \cap D(T)$ and

$$t_m(Tx_m + Cx_m - p^* + \varepsilon J_\psi x_m) + (1 - t_m)J_\psi x_m \ni 0 \tag{5.17}$$

or, for some $u_m \in Tx_m$,

$$t_m[u_m + (Cx_m - p^*)] + [1 - t_m(1-\varepsilon)]J_\psi x_m = 0. \tag{5.18}$$

By our hypothesis, assuming that $\|x_m\| \ge Q$ for all m, we find

$$\begin{aligned}\langle u_m + Cx_m - p^*, x_m\rangle &\ge -\langle p^*, x_m\rangle - \beta(||x_m||)\psi(||x_m||)||x_m|| \\ &\ge -\left[\frac{||p^*||}{\psi(||x_m||)} + \beta(||x_m||)\right]\psi(||x_m||)||x_m|| \\ &= -\widetilde{\beta}(||x_m||)\psi(||x_m||)||x_m||,\end{aligned} \tag{5.19}$$

where $\widetilde{\beta}(\|x_m\|) \to 0$ as $m \to \infty$. Using this along with (5.18), we obtain

$$\begin{aligned}\varepsilon\psi(||x_m||)||x_m|| &\le [1 - t_m(1-\varepsilon)]\psi(||x_m||)||x_m|| \\ &\le -t_m\langle u_m + Cx_m - p^*, x_m\rangle \\ &\le t_m\widetilde{\beta}(||x_m||)\psi(||x_m||)||x_m||.\end{aligned} \tag{5.20}$$

This says that $\varepsilon \le \widetilde{\beta}(\|x_n\|) \to 0$ as $m \to \infty$, that is, a contradiction. Thus, there exists a number $r > 0$ such that all possible solutions of (5.10) lie in the ball $B_r(0)$. Consequently, no solution of (5.10) lies in $\partial B_r(0)$, and the degree mapping $d(H(t,\cdot), B_r(0), 0)$ is well defined. By the homotopy invariance property of this degree (Theorem 4.4(iii)), we obtain

$$d(T+C+\varepsilon J_\psi, B_r(0), 0) = d(T+C+\varepsilon J_\psi, B_r(0), p^*) = d(J_\psi, B_r(0), 0) = 1. \tag{5.21}$$

By (ii) of Theorem 4.4, the inclusion (5.9) is solvable for every $\varepsilon > 0$.

Let x_n be a solution of

$$Tx + Cx + \left(\frac{1}{n}\right) J_\psi x \ni p^*. \tag{5.22}$$

We assume that (5.8) holds and show that the sequence $\{x_n\}$ is bounded. To this end, assume that there exists a subsequence of $\{x_n\}$, denoted again by $\{x_n\}$, such that $\|x_n\| \to \infty$. Then there exists $\alpha > 0$ such that

$$\liminf_{n\to\infty} \frac{|Tx_n + Cx_n|}{\psi(\|x_n\|)} \geq \liminf_{\substack{x\in D(T)\cap D(C)\\ \|x\|\to\infty}} \frac{|Tx+Cx|}{\psi(\|x\|)} = \alpha. \tag{5.23}$$

However, for some $u_n \in Tx_n$, we have

$$\begin{gathered} \|u_n + Cx_n\| = \left\|p^* - \left(\frac{1}{n}\right) J_\psi x_n\right\| \leq \left(\frac{1}{n}\right)\psi(\|x_n\|) + \|p^*\|, \\ \alpha \leq \liminf_{n\to\infty} \frac{|Tx_n + Cx_n|}{\psi(\|x_n\|)} \leq \liminf_{n\to\infty} \frac{\|u_n + Cx_n\|}{\psi(\|x_n\|)} \leq \liminf_{n\to\infty}\left[\frac{1}{n} + \frac{\|p^*\|}{\psi(\|x_n\|)}\right] = 0, \end{gathered} \tag{5.24}$$

that is, a contradiction.

Since $\{x_n\}$ bounded, there exists a sequence $u_n \in Tx_n$ such that

$$\lim_{n\to\infty} (u_n + Cx_n) = p^*. \tag{5.25}$$

Now, we may assume that $x_n \rightharpoonup x_0$. Since

$$\limsup_{n\to\infty} \langle u_n + Cx_n, x_n - x_0\rangle = \lim_{n\to\infty} \langle p^*, x_n - x_0\rangle = 0 \tag{5.26}$$

and the operator $T+C$ is generalized pseudomonotone (see Lemma 5.3), we have $x_0 \in D(T+C)$ and $Tx_0 + Cx_0 \ni p^*$. The proof is complete. □

As Kartsatos has noted in [11, page 1673], neither one of the conditions (5.7), (5.8) is sufficient for the surjectivity of the operator $T+C$ under the rest of the assumptions of Theorem 5.4. The simple counterexamples of [11] hold true here as well.

From the proof of the above theorem, we have the following useful lemma.

Lemma 5.5. *Fix ψ as in the definition of J_ψ, $p^* \in X^*$ and $\lambda > 0$. If $C : X \supset D(C) \to X^*$ is generalized pseudomonotone and satisfies $(c1)$, $(c3)$, then the operator $U = C + \lambda J_\psi - p^*$ satisfies $(c1)$–$(c3)$. In particular, U satisfies condition $(\widetilde{S}_+)$.*

As a special case of the above theorem, we obtain the following single-valued extension of Theorem 5 of Browder and Hess [6, page 273]. In [6], it was assumed that the operator C is coercive.

Corollary 5.6. *Let $C : X \supset D(C) \to X^*$ satisfy $(c1)$, $(c3)$ and be generalized pseudomonotone. Assume that there exist a constant $Q > 0$ and a function $\beta : \mathcal{R}_Q \to \mathcal{R}_+$, with $\beta(r) \to 0$ as $r \to \infty$, such that for every $x \in D(C)$ with $\|x\| \geq Q$, it holds that*

$$\langle Cx, x\rangle \geq -\beta(\|x\|)\psi(\|x\|)\|x\|, \tag{5.27}$$

where ψ is a gauge function. Then, for every $\varepsilon > 0$, $R(C + \varepsilon J_\psi) = X^*$.

If, in addition,

$$\liminf_{\substack{x \in D(C) \\ \|x\| \to \infty}} \frac{\|Cx\|}{\psi(\|x\|)} > 0, \tag{5.28}$$

then $R(C) = X^*$.

Another corollary of Theorem 5.4 and its proof is the following.

Corollary 5.7. *Assume that T satisfies $(t1)$ and $C : X \supset D(C) \to X^*$ satisfies $(c1)$, $(c3)$ and is generalized pseudomonotone. Assume that*

(a) *there exist constants $k > 0$, $Q > 0$ such that*

$$\langle u + Cx, x\rangle \geq -k\|x\|, \quad x \in D(T) \cap D(C),\ u \in Tx,\ \|x\| \geq Q; \tag{5.29}$$

(b) *$(T + C)^{-1}$ is bounded.*

Then $R(T + C) = X^*$.

Proof. We observe first that (5.29) implies

$$\langle Tx + Cx, x\rangle \geq -\frac{k}{\|x\|}\|x\|^2 = -\beta(\|x\|)\|x\|^2, \quad x \in D(T) \cap D(C),\ \|x\| \geq Q, \tag{5.30}$$

where $\beta(r) = k/r$. Thus, (5.7) is true with $\psi(r) = r$. Consequently, Theorem 5.4 implies $R(T + C + \varepsilon J) = X^*$, that is, given any $p^* \in X^*$, the inclusion

$$Tx + Cx + \varepsilon Jx \ni p^* \tag{5.31}$$

is solvable for every $\varepsilon > 0$. Here, $J_\psi = J$, that is, J_ψ is the normalized duality mapping. We fix $p^* \in X^*$ and consider a solution x_n of the inclusion

$$Tx_n + Cx_n + \left(\frac{1}{n}\right)Jx_n \ni p^*. \tag{5.32}$$

To show that $\{x_n\}$ is bounded, we assume that the contrary is true. Then, without any loss of generality, we may also assume that $\|x_n\| \geq Q$, $n = 1, 2, \ldots$. Then (5.32) says, for some $y_n^* \in Tx_n$,

$$y_n^* + Cx_n + \left(\frac{1}{n}\right)Jx_n = p^*, \tag{5.33}$$

and (5.29) implies

$$-k\|x_n\| + \left(\frac{1}{n}\right)\|x_n\|^2 \leq \|p^*\|\|x_n\|, \tag{5.34}$$

or

$$\left(\frac{1}{n}\right)||x_n|| \leq ||p^*|| + k. \tag{5.35}$$

This and (5.33) imply in turn

$$||y_n^* + Cx_n|| \leq 2||p^*|| + k. \tag{5.36}$$

Using our assumption (b), we obtain now that the sequence $\{\|x_n\|\}$ is bounded, that is, a contradiction. We may thus assume that $x_n \rightharpoonup x_0$. Hence,

$$\lim_{n\to\infty} \langle y_n^* + Cx_n, x_n - x_0 \rangle = \lim_{n\to\infty} \left\langle -\left(\frac{1}{n}\right)Jx_n + p^*, x_n - x_0 \right\rangle = 0. \tag{5.37}$$

Since the operator $T + C$ is generalized pseudomonotone (see Lemma 5.3), we can conclude that $x_0 \in D(T + C)$ and $p^* \in (T + C)x_0$. It follows that $p^* \in R(T + C)$ and the proof is complete. □

Corollary 5.7 is related to Theorem 7 of Browder and Hess [6, page 282]. In that theorem, the operator C is multivalued and coercive. If C is coercive in Corollary 5.7, then both conditions (a), (b) in it are trivially satisfied because $T + C$ is also coercive. When C is coercive in Corollary 5.7, then this corollary is also related to [6, Theorem 5]. In that theorem, T is the zero operator and C is multivalued, "weakly quasibounded" (i.e., for every $S > 0$, there exists $K(S) > 0$ such that: $(x, y^*) \in G(C)$ with $\|x\| \leq S$ and $\langle y^*, x\rangle \leq S\|x\|$ imply $\|y^*\| \leq K(S)$) generalized pseudomonotone, and such that $L \subset D(C)$ and a condition like $c3$ is satisfied. However, unlike our simple degree-theoretic argument, the proof of Theorem 5 in [6] is about 5 pages long (cf. [6, pages 273–279]).

We now consider the solvability of a Leray-Schauder type of problem.

Theorem 5.8 (Leray-Schauder condition). *Let T satisfy $(t1)$ and let $C : X \supset D(C) \to X^*$ satisfy $(c1)$, $(c3)$ and be generalized pseudomonotone. Assume, further, that there exists an open, bounded, and convex set $G \subset X$ containing zero and such that the inclusion*

$$Tx + Cx \ni \lambda Jx \tag{5.38}$$

has no solution $x \in D(T + C) \cap \partial G$ for any $\lambda \leq 0$. Then the inclusion $Tx + Cx \ni 0$ has a solution $x \in D(T + C) \cap G$.

Proof. We consider again the homotopy equation

$$H(t, x) \equiv t(Tx + Cx + \varepsilon Jx) + (1 - t)Jx \ni 0. \tag{5.39}$$

It is obvious, by our assumption, that (5.39) has no solution $x \in \partial G$ for $t = 1$. This is also true for $t = 0$ because $Jx = 0$ implies $x = 0 \notin \partial G$. We now assume that for some $t \in (0, 1)$, the inclusion (5.39) has a solution $x \in \partial G$. Then

$$Tx + Cx + \left[\left(\frac{1}{t} - 1\right) + \varepsilon\right]Jx \ni 0, \tag{5.40}$$

which contradicts our assumption about (5.38). Thus, by Proposition 5.1, the inclusion

$$Tx + Cx + \varepsilon Jx \ni 0 \tag{5.41}$$

is solvable in G for every $\varepsilon > 0$. Letting $x_n \in G$ solve

$$Tx + Cx + \left(\frac{1}{n}\right)Jx \ni 0, \tag{5.42}$$

and assuming, without any loss of generality, that $x_n \rightharpoonup x_0$, we obtain, for some $y_n^* \in Tx_n$,

$$\lim_{n\to\infty} \langle y_n^* + Cx_n, x_n - x_0\rangle = \lim_{n\to\infty} \left\langle -\left(\frac{1}{n}\right)Jx_n, x_n - x_0\right\rangle = 0. \tag{5.43}$$

Using again the fact that $T + C$ is generalized pseudomonotone (see Lemma 5.3), we obtain $x_0 \in D(T + C)$ and $0 \in Tx_0 + Cx_0$. Obviously, $x_0 \in \overline{\operatorname{co} G} = \overline{G}$, but $x_0 \notin \partial G$ because of our assumption on (5.38). The proof is complete. □

The problem in Theorem 5.8 was solved first by de Figueiredo [7] and then by Browder and Hess [6] for single multivalued pseudomonotone operators C with $D(C) = X$ and regular generalized pseudomonotone operators C, respectively. The set G in these references was $B_r(0)$. It was also assumed in [6] that the operator C satisfies $\langle u, x\rangle \geq -k\|x\|$, for every $x \in D(C)$, $u \in Cx$, where k is a fixed positive constant. The authors of [6, 7] used Rockafellar's mapping from [24]:

$$T_r x = \begin{cases} \{0\}, & \text{if } \|x\| < r, \\ \lambda Jx, & \lambda \geq 0, \text{ if } \|x\| = r, \end{cases} \tag{5.44}$$

which is maximal monotone and quasibounded because $\operatorname{int} D(T_r) \neq \emptyset$ (cf. [6, Proposition 14]). Thus, in [6], the operator $T_r + C$ is regular and generalized pseudomonotone. This allows the solvability of the problem $T_r x + Cx + \lambda Jx \ni 0$ in $\overline{B_r(0)}$ and, eventually, the solvability of $T_r x + Cx \ni 0$. Also, in [7] the operator $T_r + C$ is shown to be surjective via a different method of proof. Kenmochi extended this result in [19, Theorem 22] by considering a more general boundary condition on a closed convex subset of X instead of the ball $\overline{B_r(0)}$. The reader is also referred to Kenmochi [19] for other results involving the class of operators of type (M), which is more general than the class of pseudomonotone mappings.

Corollary 5.9. *Let $T : X \supset D(T) \to 2^{X^*}$ satisfy $(t1)$ and $C : X \supset D(C) \to X^*$ satisfy $(c1)$, $(c3)$ and be generalized pseudomonotone. Assume, further, that there exists an open, bounded and convex set $G \subset X$ containing zero and such that for every $x \in D(T + C) \cap \partial G$ and every $u^* \in Tx$ we have*

$$\langle u^* + Cx\rangle \geq 0. \tag{5.45}$$

Then the inclusion $Tx + Cx \ni 0$ has a solution $x \in D(T + C) \cap G$.

Proof. It suffices to note that (5.38) is impossible for $\lambda \leq 0$. □

It is actually possible to replace $J_\psi x$ by $J_\psi(x-x_0)$ in various homotopies provided that $x_0 \in G$. In this case, we do not need $0 \in G$. In fact, our assertion will become obvious from the following lemma.

Lemma 5.10. *Let G be a bounded open subset of X and fix $x_0 \in G$ and constants $\mu_1 > 0$, $\mu_2 > 0$. Let*

$$H(t,x) \equiv t\mu_1 J_\psi(x-x_0) + (1-t)\mu_2(J_\psi x - J_\psi x_0), \quad (t,x) \in [0,1] \times X, \tag{5.46}$$

for a gauge function $\psi : \mathcal{R}_+ \to \mathcal{R}_+$. Then the degree mapping $d(H(t,\cdot),G,0)$ is well defined and constant on $[0,1]$. In particular,

$$d(\mu_1 J_\psi(\cdot - x_0),G,0) = d(\mu_2(J_\psi - J_\psi x_0),G,0) = 1. \tag{5.47}$$

Proof. Since the mappings $\mu_1 J_\psi(\cdot - x_0)$, $\mu_2(J_\psi - J_\psi x_0)$ are continuous, bounded and satisfy (S_+), they also satisfy $(\widetilde{S}_+)$ and the degree $d(H(t,\cdot),G,0)$ is well defined, provided that $0 \notin H(t,\partial G)$, for $t \in [0,1]$. Assume that this last assertion is not true. Then, for some $\widetilde{x} \in \partial G$, $H(t,\widetilde{x}) = 0$. If $t = 0$ or $t = 1$, we obtain $\widetilde{x} = x_0$, which contradicts $\partial G \cap \operatorname{int} G = \emptyset$. Let $t \in (0,1)$. Then

$$\begin{aligned}
\mu_1\psi(\|x-x_0\|)\|\widetilde{x}-x_0\| &= \langle \mu_1 J_\psi(\widetilde{x}-x_0),\widetilde{x}-x_0\rangle \\
&= -\left(\frac{1}{t}-1\right)\langle \mu_2(J_\psi\widetilde{x}-J_\psi x_0),\widetilde{x}-x_0\rangle \\
&\le -\mu_2\left(\frac{1}{t}-1\right)(\psi(\|\widetilde{x}\|)-\psi(\|x_0\|))(\|\widetilde{x}\|-\|x_0\|),
\end{aligned} \tag{5.48}$$

which says $t = 1$ and $\widetilde{x} = x_0$, that is, a contradiction again. It follows that the mapping $H(t,x)$ is an admissible homotopy for our degree. Thus, $d(H(t,\cdot),G,0)$ is well defined and constant for all $t \in [0,1]$. In particular,

$$d(\mu_1 J_\psi(\cdot - x_0),G,0) = d(\mu_2(J_\psi - J_\psi x_0),G,0) = d(\mu_2 J_\psi,G,\mu_2 J_\psi x_0) = 1. \tag{5.49}$$

In fact, to show that $d(\mu(J_\psi - J_\psi x_0),G,0) = 1$, we first observe that we can consider instead the translated mapping $\widetilde{J}_\psi x = \mu(J_\psi(x+x_0) - J_\psi x_0)$ on the translated set $\widetilde{G} = G - x_0$. We do this because we now have $0 \in \widetilde{G}$ and $\widetilde{J}_\psi(0) = 0$. Another way of saying this is to consider the mapping $g(x) = x - x_0$ and the degree $d(J_\psi g^{-1} - J_\psi x_0, g(G), 0)$, where $g^{-1}(x) = x + x_0$. Since g is a homeomorphism on X with all the desirable properties, the mapping $d(fg^{-1},g(G),0)$ is another degree mapping on the demicontinuous, bounded, and (S_+)-mappings $f : \overline{G} \to X^*$. Since this degree is unique (cf. Browder [5]), we must have

$$d(\mu(J_\psi - J_\psi x_0),G,0) = d(\mu_2(J_\psi g^{-1} - J_\psi x_0),g(G),0) = 1, \tag{5.50}$$

by Browder [4, Theorem 3(iv)], because $0 \in g(G)$, $0 \in \mu_2(J_\psi g^{-1} - J_\psi x_0)(g(G))$, and $\mu_2(J_\psi g^{-1} - J_\psi x_0)$ is continuous and one-to-one on $\overline{g(G)} = \overline{G} - x_0$, and satisfies

$$\langle \mu_2((J_\psi g^{-1})(x) - J_\psi x_0),x\rangle \ge \mu_2(\psi(\|x+x_0\|) - \psi(\|x_0\|))(\|x+x_0\| - \|x_0\|) \ge 0 \tag{5.51}$$

for all $x \in \partial(g(G)) = \partial G - x_0$. The proof is finished. $\square$

The boundary condition (5.38) may be replaced, in view of Lemma 5.10, by the condition that $Tx + Cx \not\ni 0$, $x \in D(T+C) \cap \partial G$, and

$$\langle u + Cx, x\rangle > -\|u\|\|x - x_0\|, \quad x \in D(T+C) \cap \partial G,\ u \in Tx, \tag{5.52}$$

provided, again, that $x_0 \in G$ and the set G does not necessarily contain 0. In fact, this will follow trivially from the following lemma (cf., e.g., Guan [8, Theorem 3, page 14]) for the case of a single-valued mapping T.

Lemma 5.11. *Assume that $T : X \supset D(T) \to 2^{X^*}$ and let G be a bounded open subset of X. Assume that $0 \notin Tx$, $x \in D(T) \cap \partial G$. Then the following are equivalent:*

(a) *there exists $x_0 \in G$ such that $Tx \not\ni tJ(x - x_0)$, $x \in D(T) \cap \partial G$, $t < 0$;*
(b) *there exists $x_0 \in G$ such that $\langle u, x - x_0\rangle > -\|u\|\|x - x_0\|$, $x \in D(T) \cap \partial G$, $u \in Tx$.*

6. Further mapping theorems for the new degree

Another result that we prove here has to do with the establishment of necessary and sufficient conditions for an operator $T + C$ to have a zero in a given open and bounded set. Such conditions were given by Kartsatos in [12] for compact perturbations of maximal monotone operators. It is rather interesting to establish them for operators $T + C$ that are not necessarily satisfying any "infinite-dimensional" continuity assumption on their domains. Naturally, the Leray-Schauder boundary condition (see Theorem 5.8 and the remark preceding Lemma 5.10) plays an important role here.

Another familiar problem in nonlinear functional analysis is the problem of showing that, under certain conditions, there exists an open ball in the range of the operator $T + C$. For some recent results of this type, the reader is referred to Kartsatos and Skrypnik [13] and Yang [28]. Here, we give a solution to such a problem involving the sum $T + C$.

Finally, an invariance of domain result is given, Theorem 6.3, according to which the operator $T + C$ maps a relatively open set onto an open set in X^*.

Let $G \subset X$ be open. An operator $T : X \supset D(T) \to 2^{X^*}$ is called "locally monotone" on G if for every $x_0 \in D(T) \cap G$, there exists a ball $\overline{B_r(x_0)} \subset G$ such that T is monotone on $D(T) \cap \overline{B_r(x_0)}$.

Theorem 6.1 (equivalent conditions for the existence of zeros). *Let T satisfy $(t1)$ and let $C : X \supset D(C) \to X^*$ be generalized pseudomonotone and satisfy $(c1)$, $(c3)$. Assume that, for some open and bounded set $G \subset X$, the operator $T + C$ is locally monotone on G. Then the following conditions are equivalent:*

(a) $0 \in (T+C)(D(T) \cap G)$;
(b) *there exist $r > 0$ and $x_0 \in D(T+C) \cap G$ such that $\overline{B_r(x_0)} \subset G$ and*

$$\langle u^* + Cx, x - x_0\rangle \geq 0, \quad \textit{for every } (x, u^*) \in (D(T+C) \cap \partial B_r(x_0)) \times Tx; \tag{6.1}$$

(c) *there exist $r > 0$ and $x_0 \in D(T+C) \cap G$ such that $\overline{B_r(x_0)} \subset G$ and*

$$(T+C)x \not\ni \lambda J(x - x_0), \quad \textit{for every } (\lambda, x) \in (-\infty, 0) \times (D(T+C) \cap \partial B_r(x_0)). \tag{6.2}$$

Proof. Assume that $0 \in (T+C)(D(T+C) \cap G)$. Then there exists $x_0 \in D(T+C) \cap G$ such that $0 \in (T+C)x_0$. Since $T+C$ is locally monotone on G, there exists a ball $\overline{B_r(x_0)} \subset G$ such that $T+C$ is monotone on $D(T+C) \cap \overline{B_r(x_0)}$. Consequently,

$$\langle u^* + Cx, x - x_0 \rangle \geq 0, \quad \text{for every } (x, u^*) \in (D(T+C) \cap \partial B_r(0)) \times Tx. \tag{6.3}$$

It follows that (a)⇒(b).

To show that (b)⇒(c), assume that (b) holds and let $(T+C)x \ni \lambda J(x - x_0)$, for some $(\lambda, x) \in (-\infty, 0) \times (D(T+C) \cap \partial B_r(x_0))$. Then, for some $u^* \in Tx$,

$$0 \leq \langle u^* + Cx, x - x_0 \rangle = \lambda \langle J(x - x_0), x - x_0 \rangle = \lambda \|x - x_0\|^2 < 0. \tag{6.4}$$

This contradiction says that (b)⇒(c).

Let (c) hold. We consider the approximate problem

$$Tx + Cx + \left(\frac{1}{n}\right) J(x - x_0) \ni 0. \tag{6.5}$$

Taking into consideration the statement preceding Lemma 5.10, we see that Theorem 5.8 implies the solvability of (6.5) in $B_r(x_0)$ for any $n = 1, 2, \ldots$. We call x_n a solution of (6.5) lying in $B_r(x_0)$. We may assume that $x_n \rightharpoonup \tilde{x} \in \overline{B_r(x_0)}$. Then, for some $y_n^* \in Tx_n$, we have

$$y_n^* + Cx_n + \left(\frac{1}{n}\right) J(x_n - x_0) = 0. \tag{6.6}$$

Then

$$\lim_{n \to \infty} \langle y_n^* + Cx_n, x_n - \tilde{x} \rangle = -\lim_{n \to \infty} \left(\frac{1}{n}\right) \langle J(x_n - x_0), x_n - \tilde{x} \rangle = 0. \tag{6.7}$$

Since $y_n^* + Cx_n \to 0$ and $T+C$ is generalized pseudomonotone (see Lemma 5.3), we get $\tilde{x} \in D(T+C)$ and $0 \in (T+C)\tilde{x}$. However, $\tilde{x} \notin \partial G$ because $\overline{B_r(x_0)} \subset G$. Thus, $\tilde{x} \in D(T+C) \cap G$ and the proof is complete. □

Naturally, Theorem 6.1 is true if instead of the local monotonicity assumption, we assume that $T+C$ is monotone on the set $D(T+C) \cap G$.

It is easy to see that the assumption of local monotonicity cannot be deleted from Theorem 6.1, which is also true in the finite-dimensional case. In fact, let $Tx = x^3$, $Cx = -4x$ and $G = (-1, 1)$. Then $\tilde{x}_0 = 0 \in G$ is a zero of the operator $T+C$, which is not locally monotone on G. Now, let x_0 be any point in G. Let $B_r(x_0) \subset G$. Then $r < 1$ and $\partial B_r(x_0) = \{x_0 - r, x_0 + r\}$. Assume that

$$\langle Tx + Cx, x - x_0 \rangle = (x^3 - 4x)(x - x_0) \geq 0, \quad x \in \{x_0 - r, x_0 + r\}. \tag{6.8}$$

Then, for $x = x_0 - r$ and $x = x_0 + r$,

$$-r\left[(x_0 - r)^3 - 4(x_0 - r)\right] \geq 0, \qquad r\left[(x_0 + r)^3 - 4(x_0 + r)\right] \geq 0, \tag{6.9}$$

or

$$\left[(x_0 - r)^2 - 4\right](x_0 - r) \le 0, \qquad \left[(x_0 + r)^2 - 4\right](x_0 + r) \ge 0, \tag{6.10}$$

respectively. Since $|x_0 - r|^2$, $|x_0 + r|^2 < 4$, we actually obtain

$$x_0 - r \ge 0, \qquad x_0 + r \le 0, \tag{6.11}$$

which are incompatible. Consequently,

$$\langle Tx + Cx, x - x_0 \rangle \ge 0, \quad \forall x \in D(T + C) \cap \partial B_r(x_0), \tag{6.12}$$

is false for any $x_0 \in G$ and any closed ball $\overline{B_r(x_0)} \subset G$. It follows that (a) does not imply (b) in Theorem 6.1.

In the following theorem, we obtain sufficient conditions for an open ball to lie in the range of the operator $T + C$.

Theorem 6.2 (balls in the range of $T + C$). *Let T satisfy* ($t1$), *and let $C : X \supset D(C) \to X^*$ be generalized pseudomonotone and satisfy* ($c1$), ($c3$). *Assume, further, that G is a bounded open subset of X, and that there exist a constant $r > 0$ and $z_0^* \in X^*$ such that*

$$\begin{gathered} \|z_0^*\| < r \le |Tx + Cx|, \quad x \in D(T + C) \cap \partial G, \\ \langle u + Cx - z_0^*, x \rangle \ge 0, \quad x \in D(T + C) \cap \partial G,\ u \in Tx. \end{gathered} \tag{6.13}$$

Then $B_r(0) \subset (T + C)(D(T + C) \cap \overline{\operatorname{co} G})$. If, moreover, G is convex, then $B_r(0) \subset (T + C)(D(T + C) \cap G)$ and $\overline{B_r(0)} \subset (T + C)(D(T + C) \cap \overline{G})$.

Proof. We fix $p^* \in B_r(0)$ and consider the approximate problem

$$Tx + Cx + \left(\frac{1}{n}\right) Jx \ni p^*. \tag{6.14}$$

We also consider the homotopy mappings

$$\begin{aligned} H_1(t,x) &\equiv t(Tx + Cx - z_0^*) + \left(\frac{1}{n}\right) Jx \\ &= t\left(Tx + Cx - z_0^* + \left(\frac{1}{n}\right) Jx\right) + (1 - t)\left(\frac{1}{n}\right) Jx, \\ H_2(t,x) &\equiv Tx + Cx + \left(\frac{1}{n}\right) Jx - t z_0^* - (1 - t) p^*. \end{aligned} \tag{6.15}$$

Following the proof of Theorem 1 in [13], we see that when n is sufficiently large, say $n \ge n_0$, both these homotopies are admissible and the degrees

$$d(H_1(t,\cdot), G, 0), \qquad d(H_2(t,\cdot), G, 0) \tag{6.16}$$

are well defined and constant for $t \in [0,1]$. However,

$$d(H_1(0,\cdot), G, 0) = d\left(\left(\frac{1}{n}\right) J, G, 0\right) = 1 \tag{6.17}$$

(see Lemma 5.10). It follows that

$$d(H_2(0,\cdot),G,0) = d(H_2(1,\cdot),G,0) = d(H_1(1,\cdot),G,0) = d(H_1(0,\cdot),G,0) = 1. \tag{6.18}$$

Thus, the inclusion (6.14) is solvable in G for all large n. We assume that this is true for all $n = 1,2,\dots,$ and consider a solution $x_n \in G$ of (6.14) for each n. We may also assume that $x_n \rightharpoonup x_0 \in \overline{\operatorname{co} G}$. Since, for some $y_n^* \in Tx_n$,

$$\begin{gathered}\lim_{n\to\infty}(y_n^* + Cx_n - p^*) = -\lim_{n\to\infty}\left[\left(\frac{1}{n}\right)Jx_n\right] = 0,\\ \lim_{n\to\infty}\langle y_n^* + Cx_n - p^*, x_n - x_0\rangle = -\lim_{n\to\infty}\left[\left(\frac{1}{n}\right)Jx_n\right] = 0,\end{gathered} \tag{6.19}$$

the generalized pseudomonotonicity of the operator $T + C - p^*$ (see Lemma 5.3 with C replaced by $C - p^*$) implies $x_0 \in D(T + C)$ and $p^* \in Tx_0 + Cx_0$. Consequently, $B_r(0) \subset (T + C)(D(T + C) \cap \overline{\operatorname{co} G})$, which finishes the proof of the first conclusion of the theorem.

If, in addition, G is convex, then

$$B_r(0) \subset (T + C)\big(D(T + C) \cap \overline{G}\big), \tag{6.20}$$

but the boundary of G is excluded from this inclusion because $p^* \in B_r(0)$ implies

$$\|u^* + Cx\| \geq |Tx + Cx| > \|p^*\|, \quad x \in D(T + C) \cap \partial G,\ u^* \in Tx. \tag{6.21}$$

Thus, $B_r(0) \subset (T + C)(D(T + C) \cap G)$. Also, since $\overline{G}$ is bounded and weakly closed, the generalized pseudomonotonicity of $T + C$ and Lemma 1.1 in Section 7 below imply that the set $(T + C)(D(T + C) \cap \overline{G})$ is closed. Thus, (6.20) implies

$$\overline{B_r(0)} \subset (T + C)\big(D(T + C) \cap \overline{G}\big). \tag{6.22}$$

The proof is complete. □

In the next theorem, we are using another application of our new degree to the solvability of an invariance of domain problem for the sum $T + C$.

A subset M of the space X^* is called "pathwise connected" if for every $x^*, y^* \in M$ there exists a continuous function $s : [0,1] \to M$ such that $s(0) = x^*$ and $s(1) = y^*$. The function s is called a "path."

A pathwise connected set M, associated with the norm topology, is connected.

Let $G \subset X$ be open and bounded. We say that the operator $T : X \supset D(T) \to 2^{X^*}$ is "locally injective" on the set $G \subset X$ if for every point $x_0 \in D(T) \cap G$, there exists a closed ball $\overline{B_r(x_0)} \subset G$ such that the operator T is injective on $D(T) \cap \overline{B_r(x_0)}$. If $G = X$, in the previous definition, we simply say that T is "locally injective."

Theorem 6.3 (invariance of domain). *Assume that T satisfies $(t1)$, while $C : X \supset D(C) \to X^*$ satisfies $(c1)$, $(c3)$ and is generalized pseudomonotone. For an open bounded set $G \subset X$, assume that $T + C + \varepsilon J$ is injective on $D(T + C) \cap \overline{G}$ for every $\varepsilon > 0$. Then*

(i) *if either (a) G is convex or (b) C satisfies c2, then for every pathwise connected set $M \subset X^*$ with*

$$((T+C)(D(T+C)\cap \partial G))\cap M=\varnothing,\\ ((T+C)(D(T+C)\cap G))\cap M\neq\varnothing, \tag{6.23}$$

it holds that $M \subset (T+C)(D(T+C)\cap G)$;

(ii) *if $T+C$ is locally injective on G, then the set $(T+C)(D(T+C)\cap G)$ is open.*

Proof. (i) Without loss of generality, we may assume that $C(0)=0$, $0\in M$, and $0\in G$. Fix a point $y^*\in M$ such that $y^*\neq 0$ and let $s(t)$, $t\in[0,1]$ be a path in M such that $s(0)=0$ and $s(1)=y^*$. We consider the two homotopy inclusions:

$$H_1(t,x)\equiv Tx+C_nx-s(t)\ni 0, \tag{6.24}$$

$$H_2(t,x)\equiv t(Tx+C_nx)+(1-t)Jx\ni 0, \tag{6.25}$$

where $C_n=C+(1/n)J$. We first show that (6.24) has no solution $x\in D(T+C)\cap\partial G$ for any $t\in[0,1]$. Assume that this is not true. Then we may also assume that there exist sequences $\{t_m\}\subset[0,1]$, $\{x_n\}\subset D(T+C)\cap\partial G$ such that $t_m\to t_0\in[0,1]$, $x_m\rightharpoonup x_0\in X$ and

$$Tx_m+C_nx_m\ni s(t_m), \tag{6.26}$$

or

$$y_m^*+C_nx_m=s(t_m), \tag{6.27}$$

for some $y_m^*\in Tx_m$. Since

$$\lim_{m\to\infty}\langle y_m^*+C_nx_m,x_m-x_0\rangle=\lim_{m\to\infty}\langle s(t_m),x_m-x_0\rangle=0 \tag{6.28}$$

and $y_m^*+C_nx_m\to s(t_0)$ and the operator C_n satisfies $(\widetilde{S}_+)$ and is quasibounded, we can repeat the proof of Theorem 7 in [13] to obtain $x_n\to x_0\in D(T+C)\cap\partial G$ and $s(t_0)\in Tx_0+C_nx_0$. This, however, is a contradiction to our assumption that the set M does not intersect the set $(T+C)(D(T+C)\cap\partial G)$. The quoted proof is for single-valued operators T, but it goes through for multivalued ones as well.

To show that (6.25) has no solutions $x\in\partial G$, we assume again that this is not the case and that $\{t_m\}\subset[0,1]$, $\{x_m\}\subset\partial G$ are such that $t_m\to t_0$, $x_m\rightharpoonup x_0\in X$.

$$t_m(y_m^*+C_nx_m)+(1-t_m)Jx_m\ni 0, \tag{6.29}$$

for some $y_m^*\in Tx_n$. From $\langle y_m^*+(1/n)Jx_m,x_m\rangle\geq 0$ and the quasiboundedness of C, we can now obtain the boundedness of the sequence $\{C_nx_m\}$, which implies the boundedness of $\{y_m^*\}$ as well. Consequently, we may also assume that $y_m^*\rightharpoonup y_0^*\in X^*$ and $C_nx_m\rightharpoonup h^*\in X^*$. If $t_m=0$ for some m, then (6.29) says that $0\in\partial G$, that is, a contradiction.

If $t_m = 1$, then we get a contradiction again by the injectivity of $T + C_n$ and the fact that $0 \in (T + C_n)(0)$. Thus, $t_m \in (0,1)$ for all m. Again, we can now repeat the relevant part of the proof of Theorem 7 in [13] in order to obtain a contradiction.

It follows that $H_1(t,x)$, $H_2(t,x)$ are admissible homotopies for our degree. As such, they have constant degrees. Thus, we have

$$d(H_1(1,\cdot),G,0) = d(H_1(0,\cdot),G,0) = d(H_2(1,\cdot),G,0) = 1. \tag{6.30}$$

This says that the inclusion

$$Tx + Cx + \left(\frac{1}{n}\right)Jx \ni y^* \tag{6.31}$$

is solvable in G for every $n = 1,2,\dots$. Let x_n solve this inclusion. Then, for some $y_n^* \in Tx_n$,

$$y_n^* + Cx_n + \left(\frac{1}{n}\right)Jx_n = y^*. \tag{6.32}$$

Since $\{x_n\}$ is bounded, we may assume that $x_n \rightharpoonup x_0$. We assume that (a) is true. Then $x_0 \in \overline{\operatorname{co} G} = \overline{G}$. We also have $y_n^* + Cx_n \to y^*$ and

$$\lim_{n\to\infty} \langle y_n^* + Cx_n, x_n - x_0\rangle = \lim_{n\to\infty} \left\langle -\left(\frac{1}{n}\right)Jx_n + y^*, x_n - x_0\right\rangle = 0. \tag{6.33}$$

Since $T + C$ is generalized pseudomonotone, we obtain $x_0 \in D(T + C) \cap \overline{G}$ and $y^* \in Tx_0 + Cx_0$. However, $x_0 \notin \partial G$ because of (6.23). Thus, $M \subset (T + C)(D(T + C) \cap G)$.

If (b) is true, we can use the fact that the operator C satisfies $(\widetilde{S}_+)$ along with the proof of Theorem 7 in [13] to arrive at the same conclusion.

(ii) We fix $y_0^* \in (T + C)(D(T + C) \cap G)$ with $y_0^* = (T + C)x_0$. By our assumption, there exists a ball $B_q(x_0) \subset X$ such that $\overline{B_q(x_0)} \subset G$ and the operator $T + C$ is injective on $D(T + C) \cap \overline{B_q(x_0)}$. We show that there exists $r > 0$ such that

$$((T + C)(D(T + C) \cap \partial B_q(x_0))) \cap B_r(y_0^*) = \varnothing, \tag{6.34}$$

where $B_r(y_0^*) \subset X^*$. Assume the contrary and let $r_n \downarrow 0$, $p_n^* \in B_{r_n}(y_0^*) \subset X^*$, $\{x_n\} \in D(T + C) \cap \partial B_q(x_0)$ be such that

$$y_n^* + Cx_n = p_n^*, \tag{6.35}$$

for some $y_n^* \in Tx_n$. Then we may assume that $x_n \rightharpoonup \widetilde{x} \in \overline{B_q(x_0)}$. Since $y_n^* + Cx_n \to y_0^*$ and

$$\limsup_{n\to\infty} \langle y_n^* + Cx_n, x_n - \widetilde{x}\rangle = \lim_{n\to\infty} \langle y_0^*, x_n - \widetilde{x}\rangle = 0, \tag{6.36}$$

we can use the generalized pseudomonotonicity of the operator $T + C$ to obtain $\widetilde{x} \in D(T + C)$ and $y_0^* \in T\widetilde{x} + C\widetilde{x}$. Naturally, $\widetilde{x} \notin \partial B_q(x_0)$ because we already have $y_0^* \in Tx_0 + Cx_0$ and the operator $T + C$ is injective on $D(T + C) \cap \overline{B_q(x_0)}$. Thus, (6.34) is true.

Using part (i) of the proof with $M = B_r(y_0^*)$ and the convex open set $G = B_q(x_0)$, we obtain

$$B_r(y_0^*) \subset (T+C)(D(T+C) \cap B_q(x_0)) \subset (T+C)(D(T+C) \cap G). \tag{6.37}$$

It follows that the set $(T+C)(D(T+C) \cap G)$ is open, and the proof is finished. □

Since every open set G is the union of bounded open subsets of it (i.e., open balls about its points lying in it), part (ii) of Theorem 6.3 is actually true for any open set G. We state this fact in the following corollary.

Corollary 6.4. *Assume that T satisfies $(t1)$, while $C : X \supset D(C) \to X^*$ satisfies $(c1)$, $(c3)$ and is generalized pseudomonotone. Then*

(i) *if $T+C$ is locally injective on an open set $G \subset X$, the set $(T+C)(D(T+C) \cap G)$ is open;*

(ii) *if $T+C$ is locally injective and $R(T+C)$ is closed, then $R(T+C) = X^*$.*

Proof. (ii) If $T+C$ is locally injective, then (i) implies that $R(T+C) = (T+C)(D(T+C) \cap X)$ is open. If $R(T+C)$ is also closed, then it must equal X^* because the only open and closed sets in a Banach space are the empty set and the space itself. □

7. Discussion

"Ranges of sums" problems can also be handled with our new degree theory in the spirit of the results of the paper [10]. However, Theorem 2.1 in that paper is a very general result for densely defined, (weakly) quasibounded, finitely continuous, and generalized pseudomonotone perturbations C of maximal monotone operators T. That result uses an approximation involving a duality mapping J_ϕ, where the gauge function ϕ is produced by the weak quasiboundedness property of the operator C.

In the proof of Theorem 6.2, we made use of the following lemma that can be found in [6, page 263]. Since the operator $T+C$ in our new degree theory is generalized pseudomonotone, it is useful to state explicitly this lemma here for future use.

Lemma 7.1. *Let $T : X \supset D(T) \to 2^{X^*}$ be generalized pseudomonotone. Let M be a bounded weakly closed subset of X. Then $T(D(T) \cap M)$ is closed. In particular, $T(D(T) \cap \overline{B_r(x_0)})$ is closed for every $x_0 \in X$ and every $r > 0$.*

The authors of the papers [20, 30] claim that if T_1, T_2 are maximal monotone, then so are the operators $T(t) \equiv tT_1 + (1-t)T_2$, $t \in [0,1]$. This is obviously not true in general.

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