

ON A CERTAIN FUNCTIONAL EQUATION IN THE ALGEBRA OF POLYNOMIALS WITH COMPLEX COEFFICIENTS

E. MUHAMADIEV

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Many analytical problems can be reduced to determining the number of roots of a polynomial in a given disc. In turn, the latter problem admits further reduction to the generalized Rauss-Hurwitz problem of determining the number of roots of a polynomial in a semiplane. However, this procedure requires complicated coefficient transformations. In the present paper we suggest a *direct* method to evaluate the number of roots of a polynomial with complex coefficients in a disc, based on studying a certain equation in the algebra of polynomials. An application for computing the rotation of plane polynomial vector fields is also given.

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1. Functional equations: basic properties of solutions

Let

$$f(z) = a_0 + a_1z + \cdots + a_nz^n, \quad a_n \neq 0, \quad (1.1)$$

$$F(z) = b_0 + b_1z + \cdots + b_nz^n + b_{n+1}z^{n+1}, \quad b_0 \neq 0, b_{n+1} \neq 0, \quad (1.2)$$

be polynomials with complex coefficients a_0, a_1, \dots, a_n and b_0, b_1, \dots, b_{n+1} of degree n and $n + 1$, respectively. Assume the polynomials f and F to satisfy the functional equation

$$(a + bz)f(z) + (c + dz)f^*(z) = F(z), \quad (1.3)$$

where a, b, c, d are certain complex numbers and the polynomial f^* is defined by

$$f^*(z) = \bar{a}_0z^n + \bar{a}_1z^{n-1} + \cdots + \bar{a}_n = z^n \overline{f\left(\frac{1}{\bar{z}}\right)}. \quad (1.4)$$

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Consider along with (1.3) the following functional equation:

$$g(z) \cdot f(z) = (\bar{a}z + \bar{b})F(z) - (c + dz)F^*(z), \quad (1.5)$$

where

$$g(z) = \alpha z^2 + \beta z + \bar{\alpha}, \quad \alpha = \bar{a}b - \bar{c}d, \quad \beta = |a|^2 + |b|^2 - |c|^2 - |d|^2, \quad (1.6)$$

$$F^*(z) = \bar{b}_0 z^{n+1} + \bar{b}_1 z^n + \dots + \bar{b}_{n+1}. \quad (1.7)$$

LEMMA 1.1. *If the polynomials f and F satisfy the functional equation (1.3), then they also satisfy the functional equation (1.5).*

Conversely, if at least one of the numbers α, β is different from zero and f, F satisfy (1.5), then they satisfy (1.3) as well.

Proof. Let polynomials (1.1) and (1.2) satisfy (1.3). By the definition of f^* and F^* , one has

$$(\bar{a}z + \bar{b})f^*(z) + (\bar{c}z + \bar{d})f(z) = F^*(z). \quad (1.8)$$

Multiplying (1.3) (resp., (1.8)) by $(\bar{a}z + \bar{b})$ (resp., by $(c + dz)$) and taking the difference of the obtained expressions, one obtains (1.5), where the coefficients α and β are defined by (1.6). Thus the first implication is established.

Conversely, assume that f and F satisfy (1.5), and at least one of α, β is different from zero. Since the coefficient β of g is real (cf. [4]), one has

$$z^2 \overline{g\left(\frac{1}{z}\right)} = g(z). \quad (1.9)$$

Therefore, it follows from (1.5) that

$$g(z)f^*(z) = (a + bz)F^*(z) - (\bar{c}z + \bar{d})F(z). \quad (1.10)$$

Multiplying (1.5) (resp., (1.10)) by $(a + bz)$ (resp., by $(c + dz)$) and summing up the obtained expressions, one arrives at the following equality:

$$g(z)[(a + bz)f(z) + (c + dz)f^*(z)] = g(z)F(z). \quad (1.11)$$

Since $g(z) \not\equiv 0$ and the algebra of polynomials does not contain zero divisors, it follows that f and F satisfy (1.3).

The lemma is completely proved. \square

Assume that the polynomials f and F satisfy (1.3). It follows from (1.1)–(1.3) that (1.3) is equivalent to the following system:

$$aa_k + c\bar{a}_{n-k} + ba_{k-1} + d\bar{a}_{n-k+1} = b_k, \quad k = 0, 1, \dots, n+1, \quad (1.12)$$

where we put $a_k = 0$ for $k < 0$ and $k > n$.

Similarly, (1.5) is equivalent to the following system:

$$\bar{\alpha}a_k + \beta a_{k-1} + \alpha a_{k-2} = \bar{a}b_{k-1} - d\bar{b}_{n-k+2} + \bar{b}b_k - c\bar{b}_{n-k+1}, \quad k = 0, 1, \dots, n+2, \quad (1.13)$$

where $a_k = b_k = 0$ for $k < 0$ and $a_k = b_{k+1} = 0$ for $k > n$.

Thus, under the assumption that a, b, c , and d satisfy the condition $|\alpha| + |\beta| > 0$, system (1.3) is equivalent to (1.12) as well as to (1.13).

Below we will list some properties of solutions to (1.3).

- (1) (a) The coefficients a, b, c, d along with the polynomial f determine the polynomial F uniquely.
- (b) If the polynomials f and F are defined by (1.1) and (1.2) and satisfy (1.3), then the coefficients a, b, c, d satisfy the conditions

$$|a| + |c| > 0, \quad |b| + |d| > 0. \quad (1.14)$$

- (c) If a collection (a, b, c, d, f, F) of numbers a, b, c, d , and polynomials f, F satisfy (1.3), then so is the collection $(\lambda a, \lambda b, \bar{\lambda}c, \bar{\lambda}d, f/\lambda, F)$ for any complex number $\lambda \neq 0$.

- (2) Given a polynomial F and numbers a, b, c , and d , satisfying $|\alpha| + |\beta| > 0$, there exists a unique f satisfying (1.3). Indeed, if $\alpha \neq 0$, then the first $n+1$ equations of system (1.13) completely determine the coefficients a_0, a_1, \dots, a_n of the polynomial f . If, however, $\alpha = 0$ and $\beta \neq 0$, then all the coefficients a_0, a_1, \dots, a_n of the polynomial f are completely determined by $n+1$ equations of system (1.13) starting with the second one.
- (3) It follows from (1.5) that the roots of $g(z)$ turn out to be the roots of $G(z) = (\bar{a}z + \bar{b})F(z) - (c + dz)F^*(z)$. Also, if $\alpha \neq 0$ and $\beta^2 \neq 4|\alpha|^2$, then $z_0, z_1 = 1/\bar{z}_0$, where

$$z_0 = \frac{-\beta + \sqrt{\beta^2 - 4|\alpha|^2}}{2\alpha}, \quad (1.15)$$

are the roots of $g(z)$. Therefore, F and the coefficients a, b, c, d are connected by the following relations:

$$(\bar{a}z_j + \bar{b})F(z_j) - (c + dz_j)F^*(z_j) = 0, \quad j = 0, 1. \quad (1.16)$$

If $\alpha \neq 0$ and $\beta^2 = 4|\alpha|^2$, then $z_1 = z_0$ is a multiple root of $g(z)$, and, therefore,

$$\begin{aligned} (\bar{a}z_0 + \bar{b})F(z_0) - (c + dz_0)F^*(z_0) &= 0, \\ (\bar{a}z_0 + \bar{b})F'(z_0) + \bar{a}F(z_0) - (c + dz_0)F^{*'}(z_0) - dF^*(z_0) &= 0. \end{aligned} \quad (1.17)$$

Finally, if $\alpha = 0$ and $\beta \neq 0$, then the linear function $g(z) = \beta z$ has the only root $z_0 = 0$. Hence it follows from (1.5) (see also (1.10)) that

$$\begin{aligned} \bar{b}F(0) - cF^*(0) &= 0, \\ \bar{d}F(0) - aF^*(0) &= 0. \end{aligned} \quad (1.18)$$

2. Functional equations: solubility conditions

Given the polynomial (1.2), consider the solubility problem for the functional equation (1.3) with respect to unknown coefficients a, b, c, d and a polynomial f . To treat the above problem, we will use the necessary conditions for the solubility of (1.3) given by (3.1)–(3.8) (depending on whether the root z_0 of $g(z)$ satisfies $0 < |z_0| < 1$, $|z_0| = 1$, or $z_0 = 0$).

Assume z_0 and z_1 to be given and consider system (1.16) with respect to unknown a, b, c, d . We will try to find a solution to (1.16) in such a way that (1.3) will have a solution with respect to f . Also, given z_0 , we will follow the same way regarding system (1.17).

To describe the solubility conditions for the functional equation (1.3), it is convenient to introduce the notion of a regular point.

A point z is called *regular* with respect to the polynomial F if the following conditions are satisfied:

$$\begin{aligned} F(z) \cdot F^*(z) &\neq 0, \quad |F(z)| \neq |F^*(z)|, \quad \text{for } |z| \neq 1, \\ (n+1)|F(z)|^2 &\neq 2\Re[\overline{F(z)}F'(z)], \quad \text{for } |z| = 1. \end{aligned} \quad (2.1)$$

Observe that the notion of a regular point is introduced with respect to the unit circle. It follows immediately from the definition of a regular point that if z_0 is regular, then so is $z_1 = 1/\bar{z}_0$ and vice versa.

According to the definition of polynomial F^* , the rational function

$$A(z) = \frac{F^*(z)}{F(z)} \quad (2.2)$$

satisfies the identity

$$\overline{A(z)} \cdot A\left(\frac{1}{\bar{z}}\right) \equiv 1. \quad (2.3)$$

In addition, $|A(z)| \neq 1$ for all regular points $z, |z| \neq 1$.

Assume that z_0 is a regular point of F , $A_0 = A(z_0)$, $A_1 = 1/\bar{A}_0$, and σ_0, σ_1 are arbitrary complex numbers. Consider the linear system

$$\begin{aligned} \bar{a}z_0 + \bar{b} &= \sigma_0 A_0, & \bar{a} + \bar{b}\bar{z}_0 &= \sigma_1 A_1, \\ c + dz_0 &= \sigma_0, & c\bar{z}_0 + d &= \sigma_1, \end{aligned} \quad (2.4)$$

with unknown a, b, c , and d . It should be pointed out that any solution to system (2.4) is also a solution to (1.16) for $z_0 \neq 0$, as well as a solution to (1.18) for $z_0 = 0$.

For $|z_0| < 1$ system, (2.4) has the unique solution

$$\begin{aligned} \bar{a}(1 - |z_0|^2) &= \sigma_1 A_1 - \bar{z}_0 \sigma_0 A_0, & \bar{b}(1 - |z_0|^2) &= \sigma_0 A_0 - \bar{z}_0 \sigma_1 A_1, \\ c(1 - |z_0|^2) &= \sigma_0 - z_0 \sigma_1, & d(1 - |z_0|^2) &= \sigma_1 - \bar{z}_0 \sigma_0. \end{aligned} \quad (2.5)$$

With the above $a, b, c,$ and d on hands, the coefficients α, β from formula (1.6) satisfy the equalities

$$\alpha(1 - |z_0|^2) = \bar{z}_0(|A_1|^2 - 1)(|\sigma_0 A_0|^2 - |\sigma_1|^2), \quad (2.6)$$

$$\beta(1 - |z_0|^2) = -(|A_1|^2 - 1)(|\sigma_0 A_0|^2 - |\sigma_1|^2). \quad (2.7)$$

THEOREM 2.1. *Let z_0 with $|z_0| < 1$ be a regular point of the polynomial F , and assume the numbers σ_0 and σ_1 to satisfy the condition*

$$|\sigma_0 A_0| \neq |\sigma_1|. \quad (2.8)$$

Let, further, $a, b, c,$ and d be defined by (2.5). Then the functional equation (1.3) admits a solution $f(z)$.

Proof. To begin with, consider the case $z_0 \neq 0$. It follows from (2.8) and (2.6) that $\alpha \neq 0$. Formulae (2.4) and (2.3) provide that z_0 and $z_1 = 1/\bar{z}_0$ are the roots of the polynomial $g(z)$ (1.6):

$$\begin{aligned} g(z_0) &= (a + bz_0)(\bar{a}z_0 + \bar{b}) - (c + dz_0)(\bar{c}z_0 + \bar{d}) = \bar{\sigma}_1 \bar{A}_1 \sigma_0 A_0 - \sigma_0 \bar{\sigma}_1 = \bar{\sigma}_1 \sigma_0 - \sigma_0 \bar{\sigma}_1 = 0, \\ g(z_1) &= z_1^2 \overline{g(z_0)} = 0. \end{aligned} \quad (2.9)$$

On the other hand, the numbers z_0 and z_1 are the roots of the polynomial

$$G(z) = (\bar{a}z + \bar{b})F(z) - (c + dz)F^*(z). \quad (2.10)$$

Therefore, by the Bezout theorem, the rational function

$$f(z) = \frac{G(z)}{g(z)} \quad (2.11)$$

is, in fact, a polynomial satisfying the functional equation (1.5). Now the statement of the theorem in the considered case follows from Lemma 1.1.

Assume now $z_0 = 0$. The condition (2.8) and equalities (2.6), (2.7) yield $\alpha = 0$ and $\beta \neq 0$, that is, $g(z) = \beta z$ is a linear function. In addition, $z = 0$ is a root of the polynomial $G(z)$. Therefore, $f(z) = G(z)/g(z)$ is a polynomial satisfying (1.5), and again the statement of the theorem in the considered case follows from Lemma 1.1.

Theorem 2.1 is completely proved. \square

Assume now that a regular point z_0 of the polynomial F belongs to the unit circle $|z| = 1$. Given numbers c and d , consider system (1.18) with unknown a, b . By solving system (1.18) one obtains

$$\bar{a}F^2 = \Delta \cdot c + (FF^* + z\Delta)d, \quad \bar{b}F^2 = (FF^* - z\Delta)c - z^2\Delta d, \quad (2.12)$$

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where

$$\begin{aligned} \Delta &= z^n \Delta_0, \quad \Delta_0 = (n+1)|F|^2 - 2\Re[\bar{F}F'z], \\ F &= F(z), \quad F' = F'(z), \quad F^* = F^*(z), \quad z = z_0. \end{aligned} \quad (2.13)$$

Take the coefficients a , b , c , and d satisfying (2.12) and define the polynomial $g(z)$ by means of formula (1.6). Formula (2.12) provides the following relations for α and β :

$$z\alpha|F|^4 = \Delta_0[|F|^2(|c|^2 - |d|^2) - \Delta_0|c + zd|^2], \quad (2.14)$$

$$\beta|F|^4 = 2\Delta_0[\Delta_0|c + zd|^2 - |F|^2(|c|^2 - |d|^2)]. \quad (2.15)$$

THEOREM 2.2. *Let $z_0, |z_0| = 1$ be a regular point of the polynomial F and let the numbers c , d satisfy the condition:*

$$\Delta_0|c + z_0d|^2 \neq |F(z_0)|^2(|c|^2 - |d|^2). \quad (2.16)$$

If the coefficients a , b , c , and d satisfy relation (2.12), then the functional equation (1.3) has the unique solution (a, b, c, d, f) .

Proof. It follows from (1.6) and (2.15)–(3.1) that $\beta = -2\alpha z_0 \neq 0$. Since the coefficient β is real and $z_0\bar{z}_0 = 1$, one obtains the equality $\bar{\alpha} = \alpha z_0^2$, that is, $g(z) \equiv \alpha(z - z_0)^2$. At the same time, z_0 is a multiple root to the polynomial $G(z) = (\bar{a}z + \bar{b})F(z) - (c + dz)F^*(z)$. Therefore, the rational function $f(z) = G(z)/g(z)$ is, in fact, a polynomial satisfying the functional equation (1.5). To complete the proof of Theorem 2.1 it remains to apply Lemma 1.1. The theorem follows. \square

Combining Theorems 2.1 and 2.2 with property (2) of solutions to (1.3) one can effectively compute the numbers a , b , c , d and the coefficients a_0, a_1, \dots, a_n of the polynomial f . Indeed, assume, for instance, that $z = 0$ is a regular point of the polynomial F . Then the regularity condition for the point $z = 0$ along with condition (2.8) take the form

$$b_0 \cdot b_{n+1} \neq 0, \quad |b_0| \neq |b_{n+1}|, \quad |cb_{n+1}| \neq |db_0|; \quad (2.17)$$

also, the equalities (2.6) and (2.7) take the form

$$\bar{a}b_{n+1} = d\bar{b}_0, \quad \bar{b}b_0 = c\bar{b}_{n+1}. \quad (2.18)$$

In this case the leading coefficient α of the polynomial $g(z)$, that is determined by a , b , c , d , is equal to zero. Hence, in order to determine unknown coefficients a_0, a_1, \dots, a_n from system (1.13), one has

$$\begin{aligned} 0 &= \bar{b}b_0 - c\bar{b}_{n+1}, \\ \beta a_0 &= \bar{a}b_0 - d\bar{b}_{n+1} + \bar{b}b_1 - c\bar{b}_n, \\ \beta a_{k-1} &= \bar{a}b_{k-1} - d\bar{b}_{n-k+2} + \bar{b}b_k - c\bar{b}_{n-k+1}, \quad k = 2, \dots, n, \\ \beta a_n &= \bar{a}b_n - d\bar{b}_1 + \bar{b}b_{n+1} - c\bar{b}_0, \\ 0 &= \bar{a}b_{n+1} - d\bar{b}_0. \end{aligned} \quad (2.19)$$

Observe that the first and the last equations in (2.19) coincide with (2.18). According to (2.17) and (2.18), one obtains the following relation for the coefficient β of the polynomial $g(z)$:

$$\beta = \left(|b_{n+1}|^2 - |b_0|^2 \right) \left(\frac{|c|^2}{|b_0|^2} - \frac{|d|^2}{|b_{n+1}|^2} \right) \neq 0. \quad (2.20)$$

Thus the coefficients a_0, a_1, \dots, a_n of the polynomial f can be uniquely determined from system (2.19).

It follows from Theorems 2.1 and 2.2 that the existence of a regular point for the polynomial F is a sufficient condition for the existence of a solution to the functional equation (1.3).

It turns out that the existence of a regular point for the polynomial F is intimately connected to the linear (in)dependence of the polynomials F and F^* in the complex linear space of (complex) polynomials. To be more precise, *there exists a regular point for F if and only if F and F^* are linearly independent*. This statement is a direct consequence of the following lemma.

LEMMA 2.3. *The following conditions are equivalent:*

- (a) *F and F^* are linearly independent in the complex linear space of (complex) polynomials;*
- (b) *the identity $|F(z)| \equiv |F^*(z)|$ is satisfied;*
- (c) *the identity*

$$2 \operatorname{Re} [\overline{F(w)} F'(w) w] \equiv (n+1) |F(w)|^2 \quad \forall |w| = 1 \quad (2.21)$$

is satisfied.

Proof. Assume (a) is satisfied: $F^* = C \cdot F$ for some nonzero complex number C . Then, according to the definition of the polynomial F^* , one has the following equality for the coefficients b_0, b_{n+1} : $\bar{b}_0 = C b_{n+1}$, $\bar{b}_{n+1} = C b_0$, from which it follows that $|C| = 1$, and therefore, $|F^*(z)| \equiv |F(z)|$.

Thus (a) implies (b).

Assume, further, $|F(z)| \equiv |F^*(z)|$. Applying to this identity the change of coordinates $z = rw$, $r \geq 0$, $|w| = 1$ and using the definition of F^* , one obtains

$$r^{2n+2} F(r^{-1}w) \overline{F(r^{-1}w)} \equiv F(rw) \overline{F(rw)}. \quad (2.22)$$

Differentiating the last identity with respect to the real argument r at the point $r = 1$ we obtain (c).

Thus (b) implies (c).

Finally, assume that (c) is satisfied and show that F and F^* are linearly dependent. Consider the function $\overline{F(w)}/F(w)$ on the unit circle $|w| = 1$, where $F(w) \neq 0$, and show that this function has a continuous extension over the unit circle. Assume that F vanishes at some point w_0 belonging to the unit circle. Then we have the following representation: $F(z) = (z - w_0)^k F_1(z)$, where $k \geq 1$ is an integer, $F_1(w_0) \neq 0$.

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The conditions

$$\lim_{w \rightarrow w_0} \frac{\bar{w} - \bar{w}_0}{w - w_0} = -\frac{1}{w_0^2}, \quad |w| = 1, \quad (2.23)$$

yield

$$\lim_{w \rightarrow w_0} \frac{\overline{F(w)}}{F(w)} = (-1)^k w_0^{-2k} \frac{\overline{F_1(w_0)}}{F_1(w_0)}, \quad |w| = 1, \quad (2.24)$$

providing the continuity of the function $\overline{F(w)}/F(w)$.

Differentiating the function $w^{n+1}\overline{F(w)}/F(w)$, $w = \exp(it)$ with respect to t at the points where $F(w) \neq 0$ and using condition (c), one obtains

$$\frac{d}{dt} \frac{w^{n+1}\overline{F(w)}}{F(w)} = iw^{n+1} \left\{ \frac{(n+1)\overline{F(w)} - \overline{F'(w)}w}{F(w)} - \frac{\overline{F(w)}F'(w)w}{F^2(w)} \right\} = 0. \quad (2.25)$$

The above equality along with the continuity of the function $w^{n+1}\overline{F(w)}/F(w)$ on the unit circle yield that the latter function is, in fact, constant, that is $w^{n+1}\overline{F(w)} = C \cdot F(w)$ or, equivalently, $F^*(w) = C \cdot F(w)$, $|w| = 1$. Therefore, the coefficients of the polynomials $C \cdot F$ and F^* coincide. The linear dependence of the polynomials F and F^* is established and the proof of Lemma 2.3 is complete. \square

3. An algorithm for computing the number of roots in the unit circle

In what follows we will be interested in the case when the coefficients a, b, c, d satisfy the following additional condition:

$$|a + bw| \geq |c + dw|, \quad |w| = 1. \quad (3.1)$$

LEMMA 3.1. *Linear functions $a + bz$, $c + dz$ satisfy condition (3.1) if and only if the numbers $\alpha = \bar{a}b - \bar{c}d$, $\beta = |a|^2 + |b|^2 - |c|^2 - |d|^2$ satisfy the inequality*

$$2|\alpha| \leq \beta. \quad (3.2)$$

Proof. The equality

$$|a + bz|^2 - |c + dz|^2 = |a|^2 + 2\Re(\bar{a}bz) + |b|^2|z|^2 - |c|^2 - 2\Re(\bar{c}dz) - |d|^2|z|^2 \quad (3.3)$$

yields, for $z = e^{it}$, $t \in [0, 2\pi]$,

$$|a + bz|^2 - |c + dz|^2 = \beta + 2\Re[\alpha e^{it}]. \quad (3.4)$$

Combining this with the equality

$$\min_t \Re[\alpha e^{it}] = -|\alpha|, \quad (3.5)$$

one obtains the equivalence of conditions (3.1) and (3.2). The lemma is proved. \square

Observe that Lemma 3.1 allows one to verify effectively the validity of condition (3.1) for the coefficients a, b, c, d determined by regular points of the polynomial F .

LEMMA 3.2. *Let $z_0, |z_0| \leq 1$, be a regular point of the polynomial F . Assume that the numbers a, b, c, d are determined by equalities (2.5) with*

$$(|A_0| - 1)(|\sigma_0 A_0| - |\sigma_1|) > 0, \quad |z_0| < 1, \quad (3.6)$$

or that they satisfy (2.12) with

$$\Delta_0 \left[\Delta_0 |c + z_0 d|^2 - |F(z_0)|^2 (|c|^2 - |d|^2) \right] > 0, \quad |z_0| = 1. \quad (3.7)$$

Then a, b, c, d satisfy (3.1).

The statement following below provides an important property of solution (a, b, c, d, f) to the functional equation (1.3).

THEOREM 3.3. *Assume that the polynomial F does not contain roots on the unit circle $|z| = 1$. Suppose, further, that the coefficients a, b, c, d satisfy condition (3.1) and*

$$|ad - bc| + \beta > 0, \quad \beta = |a|^2 + |b|^2 - |c|^2 - |d|^2. \quad (3.8)$$

Then the polynomial f as well as any polynomial of the parameterized family

$$G_\lambda(z) = (a + bz)f(z) + \lambda(c + dz)f^*(z), \quad 0 \leq \lambda \leq 1, \quad (3.9)$$

does not contain roots on the unit circle $|z| = 1$.

Proof. Arguing indirectly, one obtains the existence of numbers $w, |w| = 1$, and $\lambda \in [0, 1]$ such that

$$G_\lambda(w) = (a + bw)f(w) + \lambda(c + dw)f^*(w) = 0. \quad (3.10)$$

Since $\bar{w}w = 1$, one has

$$f^*(w) = \overline{w^n f\left(\frac{1}{\bar{w}}\right)} = w^n \overline{f(w)}, \quad (3.11)$$

from which it follows that

$$(a + bw)f(w) + \lambda(c + dw)w^n \overline{f(w)} = 0. \quad (3.12)$$

By condition,

$$G_1(w) = (a + bw)f(w) + (c + dw)w^n \overline{f(w)} = F(w) \neq 0, \quad (3.13)$$

therefore, $f(w) \neq 0$ and $0 \leq \lambda < 1$. Now, using the equality (3.12), we obtain

$$|a + bw| = \lambda |c + dw|, \quad (3.14)$$

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from which it follows (see Lemma 3.1) that $|a + bw| = 0$, $|c + dw| = 0$. The latter equalities yield $|a| = |b|$, $|c| = |d|$, $ad - bc = 0$ that contradicts condition (3.8) and the result follows. \square

Denote by $\kappa(F)$ the number of roots of the polynomial F (counted according to their multiplicity) belonging to the open unit disc $|z| < 1$. As noted, the computing $\kappa(F)$ maybe a reduction to the generalized Rauss-Hurwitz problem of determining the number of roots of a polynomial in a semiplane (see, for instance, [1–3]). The above results allow us to construct an iterative process for computing $\kappa(F)$, namely, the following theorem.

THEOREM 3.4. *Assume that the polynomial F does not have roots on the unit circle $|z| = 1$. If the polynomials F and F^* are linearly dependent, then*

$$\kappa(F) = \frac{n+1}{2}, \quad (3.15)$$

otherwise

$$\kappa(F) = \kappa(G_0), \quad (3.16)$$

where $G_0(z) = (a + bz)f(z)$ and the collection (a, b, c, d, f) satisfying (3.1), (3.8) is a solution to the functional equation (1.3).

Proof. Assume the polynomials F and F^* to be linearly dependent, that is $F^*(z) \equiv CF(z)$. Then the following presentation takes place:

$$F(z) = b_{n+1}(z - z_1)^{\alpha_1} \cdots (z - z_m)^{\alpha_m} \left(z - \frac{1}{\bar{z}_1}\right)^{\alpha_1} \cdots \left(z - \frac{1}{\bar{z}_m}\right)^{\alpha_m}, \quad (3.17)$$

where

$$|z_s| < 1, \quad s = 1, \dots, m, \quad 2(\alpha_1 + \cdots + \alpha_m) = n + 1. \quad (3.18)$$

From this it follows that the number $n + 1$ is even and

$$\kappa(F) = \alpha_1 + \cdots + \alpha_m = \frac{(n+1)}{2}. \quad (3.19)$$

Assume now the polynomials F and F^* to be linearly independent. Then, by Lemma 2.3, F admits a regular point z_0 , $|z_0| \leq 1$. Therefore, by Lemmas 1.1, 2.3, and 3.2, there exists a collection (a, b, c, d, f) satisfying conditions (3.1), (3.8) and being a solution to the functional equation (1.3). Combining Theorem 3.3 and the Rouché theorem one obtains $\kappa(F) = \kappa(G_0)$, where $G_0(z) = (a + bz)f(z)$.

The proof of Theorem 3.4 is complete. \square

It is easy to see that the number $\kappa(G_0)$ satisfies the equality

$$\kappa(G_0) = \kappa(f) + \varepsilon, \quad (3.20)$$

where $\varepsilon = 1$ for $|a| < |b|$ and $\varepsilon = 0$, otherwise. A simple argument shows that if the numbers a, b, c, d are determined by a regular point z_0 , then ε can be evaluated according to

the formulae

$$\varepsilon = \frac{1 + \operatorname{sign}(|F^*(z_0)| - |F(z_0)|)}{2}, \quad |z_0| < 1, \quad \varepsilon = \frac{1 - \operatorname{sign}(\Delta_0)}{2}, \quad |z_0| = 1. \quad (3.21)$$

Thus, under the assumptions of Theorems 2.1 and 2.2, one has

$$\kappa(F) = \frac{1 + \operatorname{sign}(|F^*(z_0)| - |F(z_0)|)}{2} + \kappa(f), \quad |z_0| < 1, \quad (3.22)$$

$$\kappa(F) = \frac{1 - \operatorname{sign}(\Delta_0)}{2} + \kappa(f), \quad |z_0| = 1. \quad (3.23)$$

Formulae (3.22) and (3.23) give rise to a recurrent procedure for the computation of κ . Indeed, they allow one to compute $\kappa(F)$, where F is a polynomial of degree $n + 1$, based on $\kappa(f)$, where f is a polynomial of degree n and its coefficients are completely determined by coefficients of F .

Observe that if $z_0 = 0$, then formula (3.22) takes the form

$$\kappa(F) = \frac{1 + \operatorname{sign}(|b_{n+1}| - |b_0|)}{2} + \kappa(f). \quad (3.24)$$

Here the coefficients of f can be determined from system (2.19).

4. Criterion for the absence of roots on the unit circle

Given the coefficients of the polynomials F and F^* , one can construct the following $(2n + 2) \times (2n + 2)$ matrix:

$$M_F = \begin{pmatrix} b_0 & b_1 & \cdot & b_{n+1} & \cdot & 0 \\ \bar{b}_{n+1} & \bar{b}_n & \cdot & \bar{b}_0 & \cdot & 0 \\ 0 & b_0 & \cdot & b_n & \cdot & 0 \\ 0 & \bar{b}_{n+1} & \cdot & \bar{b}_0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & b_1 & \cdot & b_{n+1} \\ 0 & 0 & \cdot & \bar{b}_{n+1} & \cdot & \bar{b}_0 \end{pmatrix}. \quad (4.1)$$

The matrix M_F coincides with the Sylvester matrix of F and F^* (up to a permutation of its lines). Therefore, $\det(M_F)$ coincides (up to a sign) with the resultant $R(F, F^*)$ of the polynomials F and F^* . Let $z_s, s = 1, \dots, n + 1$ be all the roots of F (counted according to their multiplicities). By condition, $F(0) = b_0 \neq 0$, therefore, all the roots are different from zero. By definition of the polynomial F^* , the numbers $\bar{z}_s^{-1}, s = 1, \dots, n + 1$, are roots of F^* . Hence (cf. [5]), $R(F, F^*)$ can be represented as follows:

$$R(F, F^*) = \bar{b}_0^{n+1} b_{n+1}^{n+1} \prod_{s,t} \left(z_s - \frac{1}{\bar{z}_t} \right). \quad (4.2)$$

Formula (4.2) gives rise to the following criteria for the polynomial F to have no roots on the unit circle.

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THEOREM 4.1. *Let $\det(M_F)$ be different from zero. Then F does not have roots on the unit circle $|z| = 1$.*

Assume that a collection $(a, b, 1, 1, f)$ of the numbers $a, b, c = d = 1$ and a polynomial f satisfy the functional equation (1.3) and the condition $\alpha \equiv \bar{a}b - 1 = 0$. Using a, b define the following square matrix of order $2n + 2$:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ -\bar{b} & 1 & -\bar{a} & 1 & \cdot & 0 & 0 & 0 & 0 \\ 1 & -a & 1 & -b & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & -\bar{b} & 1 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -a & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & -\bar{b} & 1 & -\bar{a} & 1 \\ 0 & 0 & 0 & 0 & \cdot & 1 & -a & 1 & -b \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (4.3)$$

It is easy to see that $\det(P)$ satisfies the following condition:

$$\det P = (-1)^{n-1} \beta^{n-1} (1 - |a|^2), \quad (4.4)$$

where $\beta = |a|^2 + |b|^2 - 2$.

Let M_f be a square matrix of order $2n$ determined by the polynomial f . Using the matrices M_F, M_f , and P one can express a connection between the coefficients of polynomials F and f , given in system (2.19), by the following matrix equality:

$$PM_F = \begin{pmatrix} b_0 & b_1 & b_2 & \cdot & & 0 \\ 0 & & & & & 0 \\ 0 & & \beta M_f & & & 0 \\ \cdot & & & & & \cdot \\ 0 & & \cdot & \bar{b}_2 & \bar{b}_1 & \bar{b}_0 \end{pmatrix}. \quad (4.5)$$

Consider the matrix written in the right-hand side of (4.5) more intently: (i) its first and last lines coincide with the corresponding lines of the matrix M_F ; (ii) the remaining entries (except for zeros related to the first and the last column) are filled out by the elements of the matrix $\beta \cdot M_f$. Therefore, equality (4.5) connects $\det(M_F)$ and $\det(M_f)$ as follows.

THEOREM 4.2. *Let $c = d = 1$ and assume that the numbers a, b along with coefficients a_0, \dots, a_n of the polynomial f satisfy (2.19), $\beta \neq 0$. Then one has*

$$\det M_F = (-1)^{n+1} (|b_{n+1}|^2 - |b_0|^2) \beta^n \det M_f. \quad (4.6)$$

Proof. Combining equality (4.5), the above formula for $\det(P)$ with the standard determinant properties yields

$$\det M_F = \frac{|b_0|^2 \det(\beta M_f)}{\det P} = (-1)^{n+1} (|b_{n+1}|^2 - |b_0|^2) \beta^n \det M_f. \quad (4.7)$$

Assume a collection $(a, b, 1, 1, f, F)$ of the numbers $a, b, c = d = 1$ and polynomials f, F to satisfy the functional equation (1.3) and the condition $\alpha \equiv \bar{a}b - 1 = 0$. Consider a sequence of collections $(a^k, b^k, 1, 1, F^k)$ of numbers $a^k, b^k, c^k = d^k = 1$ and polynomials

$$F^k(z) = b_{0k} + b_{1k}z + \cdots + b_{kk}z^k, \quad b_{0k} \neq 0, \quad b_{kk} \neq 0, \quad (4.8)$$

of degree k , satisfying the functional equation

$$(a^k + b^k z)F^k(z) + (1+z)(F^k)^*(z) = F^{k+1}(z), \quad k = n, n-1, \dots, 1, \quad (4.9)$$

and the condition

$$\alpha^k \equiv \overline{a^k} b^k - 1 = 0, \quad (4.10)$$

where $a^n = a, b^n = b, F^n = f, F^{n+1} = F$. By Theorem 2.1, if $z = 0$ is a regular point of the polynomial

$$F^{k+1}(z) = b_{0k+1} + b_{1k+1}z + \cdots + b_{k+1,k+1}z^{k+1}, \quad (4.11)$$

that is,

$$b_{0k+1} \cdot b_{k+1,k+1} \neq 0, \quad |b_{0k+1}| \neq |b_{k+1,k+1}|, \quad (4.12)$$

then the system of (4.9), (4.10) has the unique solution (a^k, b^k, F^k) . Moreover,

$$\overline{a^k} = \frac{\overline{b_{0k+1}}}{b_{k+1,k+1}}, \quad \overline{b^k} = \frac{1}{a^k}. \quad (4.13)$$

This along with formula (3.24) justify the following relation:

$$\kappa(F) = \sum_{k=1}^{n+1} \frac{1 + \text{sign}(|b_{kk}| - |b_{0k}|)}{2}. \quad (4.14)$$

□

5. Application for computing the rotation of a plane vector field

(1) Consider a vector field $\Phi(x, y) = \{p(x, y), q(x, y)\}$, where

$$p(x, y) = \sum_{k,j} a_{kj} x^k y^j, \quad q(x, y) = \sum_{k,j} b_{kj} x^k y^j \quad (5.1)$$

are polynomials in real variables x and y with real coefficients. Assume that $\Phi(x, y) \neq 0, (x, y) \in S = \{(x, y) : x^2 + y^2 = 1\}$. We are interested in computing the rotation $\gamma(\Phi, S)$.

Recall the definition of rotation. Consider the complex presentation of the field Φ :

$$p + iq = \exp(i\theta(t)) |p + iq|, \quad p + iq = p(\cot t, \sin t) + iq(\cos t, \sin t), \quad t \in [0, 2\pi], \quad (5.2)$$

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where $\theta(t)$ is a continuous function. Then (cf. [4])

$$\gamma(\Phi, S) := \frac{1}{2\pi} (\theta(2\pi) - \theta(0)). \quad (5.3)$$

The following statement reduces the computation of rotation of a plane vector field to the computation of the number of roots in the unit disc of some polynomial.

LEMMA 5.1. *Given a (polynomial) plane vector field Φ , there exists a unique pair (m, F) , where m is an integer and F is a polynomial in complex variable with complex coefficients, $F(0) \neq 0$, satisfying the following condition:*

$$F(z) = z^m (p(x, y) + iq(x, y)), \quad z = x + iy, |z| = 1. \quad (5.4)$$

Proof. Take a polynomial $P(x, y) = p(x, y) + iq(x, y)$ in real variables x, y . The change of variables

$$(x, y) \rightarrow \left(\frac{1+z^2}{2z}, \frac{i(1-z^2)}{2z} \right) \quad (5.5)$$

determines the rational function in complex variable z :

$$R(z) = P\left(\frac{1+z^2}{2z}, \frac{i(1-z^2)}{2z} \right). \quad (5.6)$$

The function $R(z)$ can be represented in the form

$$R(z) = \frac{F(z)}{z^m}, \quad (5.7)$$

where $F(z)$ is a polynomial satisfying the condition $F(0) \neq 0$ and m is an integer. Since

$$x = \frac{1+z^2}{2z}, \quad y = \frac{i(1-z^2)}{2z}, \quad z = x + iy, |z| = 1, \quad (5.8)$$

the pair (m, F) satisfies (5.4).

To complete the proof of Lemma 5.1, it remains to establish the uniqueness of the pair satisfying (5.4). Suppose that (m_1, F_1) is another pair satisfying

$$F_1(z) = z^{m_1} (p(x, y) + iq(x, y)), \quad z = x + iy, |z| = 1, \quad (5.9)$$

and $F_1(0) \neq 0$. Assuming, without loss of generality, that $m_1 \geq m$, one obtains the following equalities for F_1 and $z^{m_1-m}F$:

$$F_1(z) = z^{m_1} P(x, y) = z^{m_1-m} z^m P(x, y) = z^{m_1-m} F(z) \quad (5.10)$$

for $z = x + iy, |z| = 1$. From this it follows that F_1 and $z^{m_1-m}F$ coincide. Further, by assumption, $F(0) \neq 0, F_1(0) \neq 0$, hence $m_1 = m$. Thus $F_1 = F$ and Lemma 5.1 is completely proved. \square

Let Φ and F be as in Lemma 5.1. Then $\gamma(\Phi, S)$ and $\kappa(F)$ satisfy the following relation:

$$\kappa(F) = m + \gamma(\Phi, S). \quad (5.11)$$

(2) Assume a field Φ to be given in a parametric form: $\Phi(t) = \{p(t), q(t)\}$, where $p(t), q(t)$ are real continuous 2π -periodic functions. Suppose $\Phi(t) \neq 0, t \in [0, 2\pi]$. The field Φ may be considered as the one defined on the unit circle S , by assigning to each point $x = \cos t, y = \sin t$ the vector $\{p(t), q(t)\}$. Therefore, the rotation $\gamma(\Phi, S)$ is correctly defined on S . Assuming the functions $p(t), q(t)$ to be smooth enough, one can assign to the field Φ the Fourier series of the complex function $P(t) = p(t) + iq(t)$:

$$P(t) = \sum_{k=-\infty}^{\infty} c_k \exp(ikt). \quad (5.12)$$

Since the series (5.12) converges uniformly and Φ does not vanish, there exists an integer N such that for all $t \in [0, 2\pi]$ the following estimate is true:

$$\left| P(t) - \sum_{k=-N}^N c_k \exp(ikt) \right| < |P(t)|. \quad (5.13)$$

Set $m := \max\{-k : |c_k| > 0, |k| \leq N\}$ and consider the polynomial

$$F(z) = \sum_{k=0}^{N+m} c_{k-m} z^k. \quad (5.14)$$

Using the same arguments as in Section 1 it is easy to check that $\gamma(\Phi, S)$ and $\kappa(F)$, where F is defined by (5.14), satisfy (5.11).

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E. Muhamadiev: Department of Information Systems Technologies, Vologda State Technical University, 15 Lenin Street, Vologda 160035, Russia
E-mail address: muhamerg@mail.vstu.edu.ru



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