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Research Article

On the Noncommutative Neutrix Product of Distributions

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Let f and g be distributions and let $g_n = (g * \delta_n)(x)$, where $\delta_n(x)$ is a certain sequence converging to the Dirac-delta function $\delta(x)$. The noncommutative neutrix product $f \circ g$ of f and g is defined to be the neutrix limit of the sequence $\{fg_n\}$, provided the limit h exists in the sense that $\text{N-lim}_{n\to\infty}\langle f(x)g_n(x), \phi(x)\rangle = \langle h(x), \phi(x)\rangle$, for all test functions in $\mathfrak{D}$. In this paper, using the concept of the neutrix limit due to van der Corput (1960), the noncommutative neutrix products $x_+^r \ln x_+ \circ x_-^{-r-1} \ln x_-$ and $x_-^{-r-1} \ln x_- \circ x_+^r \ln x_+$ are proved to exist and are evaluated for $r = 1, 2, \ldots$. It is consequently seen that these two products are in fact equal.

1. Introduction

Certain operations on smooth functions (such as addition, and multiplication by scalars) can be extended without difficulty to arbitrary distributions. Others (such as multiplication, convolution, and change of variables) can be defined only for particular distributions. We are obliged to impose certain restrictions on the distributions when we try to define a multiplicative operation for distributions.

The technique of neglecting appropriately defined infinite quantities was devised by Hadamard and the resulting finite value extracted from the divergent integral is usually referred to as the Hadamard finite part. In fact, Hadamard's method can be regarded as a particular application of the neutrix calculus developed by van der Corput, see [1, 2]. This is a very general principle for the discarding of unwanted infinite quantities from asymptotic expansions and has been widely exploited in the context of distributions, by Fisher in connection with the problem of distributional multiplication, see [3–6] or [7].

Recently, Jack Ng and van Dam applied the neutrix calculus, in conjuction with the Hadamard integral, developed by van der Corput, to quantum field theories, in particular, to obtain finite results for the cofficients in the perturbation series. They also applied neutrix calculus to quantum field theory, obtaining finite renormalization in the loop calculations, see [8, 9].

In the following we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions.

Definition 1.1. Let f be a distribution in $\mathcal{D}'$ and let α be an infinitely differentiable function. The product αf is defined by

$$\langle \alpha f, \phi \rangle = \langle f, \alpha\phi \rangle \tag{1.1}$$

for all functions ϕ in $\mathcal{D}$.

The first extension of the product of a distribution and an infinitely differentiable function is the following, see, for example, [10].

Definition 1.2. Let f and g be distributions in $\mathcal{D}'$ for which on the interval (a,b), f is the rth derivative of a locally summable function F in $L^p(a,b)$ and $g^{(r)}$ is a locally summable function in $L^q(a,b)$ with $1/p + 1/q = 1$. Then the product $fg = gf$ of f and g is defined on the interval (a,b) by

$$fg = \sum_{i=0}^{r} \binom{r}{i} (-1)^i \left[F g^{(i)}\right]^{(r-i)}. \tag{1.2}$$

Now let ρ be a fixed infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-1}^{1} \rho(x)dx = 1$.

We define the function $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$. It is obvious that $\{\delta_n\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta function $\delta(x)$. Now let f be an arbitrary distribution and define the function f_n by

$$f_n(x) = f * \delta_n = \int_{-1/n}^{1/n} f(x-t)\delta_n(t)dt. \tag{1.3}$$

Then $\{f_n\}$ is a sequence of infinitely differentiable functions converging to the distribution f.

The next definition for the product of two distributions, given in [11], is in general noncommutative and generalizes Definition 1.2.

Definition 1.3. let f and g be arbitrary distributions and let $g_n = g * \delta_n$. The product $f \cdot g$ of f and g exists and is equal to h on the open interval (a,b) if

$$\lim_{n\to\infty} \langle f g_n, \phi \rangle = \langle h, \phi \rangle \tag{1.4}$$

for all $\phi \in \mathcal{D}$.

It was proved that if the product exists by Definition 1.2, then it exists by Definition 1.3 and $fg = f \cdot g$.

However, there are still many pairs of distributions whose products do not exist by Definition 1.3.

We need the following definitions of the neutrix and the neutrix limit to define the product for a considerably larger class of pairs of distributions, see [1, 4, 2].

Definition 1.4. A neutrix N is defined as a commutative additive group of functions $\nu(\xi)$ defined on a domain N' with values in an additive group N'', where further if for some ν in $N, \nu(\xi) = \gamma$ for all $\xi \in N'$, then $\gamma = 0$. The functions in N are called negligible functions.

Definition 1.5. Let N' be a set contained in a topological space with a limit point b which does not belong to N'. If $f(\xi)$ is a function defined on N' with values in N'' and it is possible to find a constant c such that $f(\xi) - c \in N$, then c is called the neutrix limit of f as ξ tends to b and write $\text{N-}\lim_{\xi \to b} f(\xi) = c$.

The following definition for the noncommutative product of two distributions was given in [4] and generalizes Definitions 1.2 and 1.3.

Definition 1.6. let f and g be arbitrary distributions and let $g_n = g * \delta_n$. The neutrix product $f \circ g$ of f and g exists and is equal to h on the open interval (a,b) $(-\infty \le a < b \le \infty)$ if

$$\text{N-}\lim_{n\to\infty} \langle f g_n, \phi \rangle = \langle h, \phi \rangle \tag{1.5}$$

for all $\phi \in \mathcal{D}$, where N is the neutrix having domain $N' = \{1,2,\dots,n,\dots\}$ and range N'' the real numbers, with negligible functions finite linear sums of the functions

$$n^{\lambda} \ln^{r-1} n, \quad \ln^{r} n \quad (\lambda > 0,\ r = 1,2,\dots) \tag{1.6}$$

and all functions which converge to zero in the usual sense as n tends to infinity, see [10, 11] or [2].

The next theorem shows that Definition 1.6 is really the extension of Definition 1.3 and the proof of theorem is immediate.

Theorem 1.7. *Let f and g be distributions for which the product $f \cdot g$ exists. Then the neutrix product $f \circ g$ exists and defines the same distribution (see [11]).*

2. Results on the neutrix product

In many elementary applications a relatively small class of distributions is sufficient. This consists of the regular distributions, delta functions, derivatives of delta functions, and linear combinations of these. This may give the impression that the delta and its derivatives are the only singular distributions which really matter. However, there are many examples of singular distributions other than these which are of immediate practical interest. We here define the neutrix product of a singular distribution $x_-^{-r-1} \ln x_-$ and the locally summable function $x_+^r \ln x_+$.

In the following theorem, which was proved in [12], the locally summable function $x_+^r \ln x_+$ and the distribution x_-^{-r} are defined by

$$x_+^r \ln x_+ = \begin{cases} x^r \ln x, & x > 0, \\ 0, & x < 0, \end{cases} \qquad x_-^{-r} = -\frac{1}{(r-1)!}(\ln x_-)^{(r)}, \tag{2.1}$$

where

$$\ln x_- = \begin{cases} \ln |x|, & x < 0, \\ 0, & x > 0. \end{cases} \tag{2.2}$$

Theorem 2.1. *The neutrix products $x_+^r \ln x_+ \circ x_-^{-r-1}$ and $x_-^{-r-1} \circ x_+^r \ln x_+$ exist and*

$$x_+^r \ln x_+ \circ x_-^{-r-1} = x_-^{-r-1} \circ x_+^r \ln x_+ = L_r \delta(x) \tag{2.3}$$

for $r = 1, 2, \ldots$, where $L_r = (-1)^r [c_2(\rho) - \pi^2/12] + (-1)^r c_1(\rho)\psi(r)$ and

$$c_1(\rho) = \int_0^1 \ln t \rho(t) dt, \qquad c_2(\rho) = \int_0^1 \ln^2 t \rho(t) dt. \tag{2.4}$$

Further the distribution $x_-^{-r} \ln x_-$, which is distinct from the definition given by Gel'fand and Shilov [13], is defined by (see [14])

$$x_-^{-r} \ln x_- = F(x_-, -r) \ln x_- - \frac{1}{(r-1)!}\psi_1(r-1)\delta^{(r-1)}(x) \tag{2.5}$$

for $r = 1, 2, \ldots$, where

$$\begin{gathered} \psi(r) = \begin{cases} 0, & r = 0, \\ \sum_{i=1}^{r} i^{-1}, & r \geq 1, \end{cases} \qquad \psi_1(r) = \begin{cases} 0, & r = 0, \\ \sum_{i=1}^{r} \frac{\psi(i)}{i}, & r \geq 1, \end{cases} \\ \langle F(x_-, -r) \ln x_-, \phi(x) \rangle \\ = \int_0^\infty x^{-r} \ln x \left[\phi(-x) - \sum_{i=0}^{r-2} \frac{\phi^{(i)}(0)}{i!}(-x)^i - \frac{\phi^{(r-1)}(0)}{(r-1)!} H(1-x)(-x)^{r-1} \right] dx, \end{gathered} \tag{2.6}$$

where $H(x)$ denotes the Heaviside function. It can be easily shown by induction that the distribution $x_-^{-r} \ln x_-$ is also defined by an equation

$$x_-^{-r} \ln x_- = \psi(r-1) x_-^{-r} - \frac{1}{2(r-1)!}(\ln^2 x_-)^{(r)}. \tag{2.7}$$

The following lemma is easily proved.

Lemma 2.2. *Let $\rho(x)$ be infinitely differentiable function with the properties given in the introduction. For positive integer r,*

$$\begin{aligned}
&\int_0^1 y^r \ln y\, dy = -\frac{1}{(r+1)^2},\\
&\int_0^1 y^r \ln(1-y)dy = -\frac{\psi(r+1)}{r+1},\\
&\int_0^1 y^r \ln y \ln(1-y)dy = \frac{\psi(r+1)}{(r+1)^2} - \frac{1}{r+1}[\xi(2)-\psi_2(r+1)],\\
&\int_0^1 y^r \ln^2(1-y)dy = \frac{2}{r+1}\psi_1(r+1),\\
&\int_0^1 y^r \ln y \ln^2(1-y)dy = -\frac{2}{(r+1)^2}\psi_1(r+1) + \frac{2}{r+1}\xi(2)\psi(r+1) - \frac{2}{r+1}\sum_{i=1}^{r+1}\frac{\psi_2(i)}{i}\\
&\qquad\qquad + \frac{2}{r+1}\sum_{i=1}^{\infty}\frac{\psi(i)}{(i+1)^2} - \frac{2}{r+1}\sum_{i=1}^{r+1}\frac{\psi(i)}{i^2} = \alpha_r,\\
&\int_0^1 u^{r+1}\rho^{(r+1)}(u)du = \frac{1}{2}(-1)^{r+1}(r+1)!,\\
&\int_0^1 u^{r+1}\ln u\rho^{(r+1)}(u)du = (-1)^{r+1}(r+1)!\left[c_1(\rho) + \frac{1}{2}\psi(r+1)\right],\\
&\int_0^1 u^{r+1}\ln^2 u\rho^{(r+1)}(u)du = (-1)^{r+1}(r+1)!\left[c_2(\rho) + 2c_1(\rho)\psi(r+1) + \psi_3(r+1)\right] = \beta_r,\\
&\int_0^1 u^{r+1}\ln^3 u\rho^{(r+1)}(u)du = (-1)^{r+1}(r+1)!\left[c_3(\rho) + 2c_2(\rho)\psi(r+1)\right.\\
&\qquad\qquad \left. + 6c_1(\rho)\psi_3(r+1) + 3\psi_4(r+1)\right] = \theta_r,
\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}
&\psi_2(r) = \begin{cases} 0, & r=0,\\ \displaystyle\sum_{i=1}^{r} i^{-2}, & r\geq 1,\end{cases} \qquad \xi(2) = \frac{\pi^2}{6}, \qquad c_3(\rho) = \int_0^1 \ln^3 t\rho(t)dt,\\
&\psi_3(r) = \begin{cases} 0, & r=0,\\ \displaystyle\sum_{i=1}^{r} \frac{\psi(i-1)}{i}, & r\geq 1,\end{cases} \qquad \psi_4(r) = \begin{cases} 0, & r=0,\\ \displaystyle\sum_{i=1}^{r} \frac{\psi_3(i-1)}{i}, & r\geq 1.\end{cases}
\end{aligned} \tag{2.9}$$

We now prove the following theorem.

Theorem 2.3. *The neutrix products $x_+^r \ln x_+ \circ x_-^{-r-1}\ln x_-$ and $x_-^{-r-1}\ln x_- \circ x_+^r \ln x_+$ exist and*

$$x_+^r \ln x_+ \circ x_-^{-r-1}\ln x_- = \Delta_r(\rho)\delta(x) \tag{2.10}$$

$$= x_-^{-r-1}\ln x_- \circ x_+^r \ln x_+ \tag{2.11}$$

for $r = 1, 2, \ldots,$ *where*

$$\begin{aligned}\Delta_r(\rho) = {} & (-1)^r (r+1)\alpha_r \left[c_1(\rho) + \frac{1}{2}\psi(r+1) + \frac{1}{4} \right] + L_r \psi(r) - \frac{\theta_r}{2(r+1)!} \\ & + \frac{\beta_r}{r!} \left[\frac{1}{2(r+1)^2} + \frac{\psi(r+1)}{r+1} \right] + (-1)^r \left[c_1(\rho) + \frac{1}{2}\psi(r+1) \right] \psi_1(r+1).\end{aligned} \tag{2.12}$$

Proof. We put

$$(x_-^{-r-1} \ln x_-)_n = x_-^{-r-1} \ln x_- * \delta_n(x) = \psi(r)(x_-^{-r-1})_n - \frac{1}{2r!}[\ln^2 x_-]_n^{(r+1)} \tag{2.13}$$

so that

$$(x_-^{-r-1} \ln x_-)_n = \frac{\psi(r)}{r!} \int_x^{1/n} \ln(t-x)\delta_n^{(r+1)}(t)dt - \frac{1}{2r!} \int_x^{1/n} \ln^2(t-x)\delta_n^{(r+1)}(t)dt \tag{2.14}$$

on the interval $[0, 1/n]$, the intersection of the supports of $x_+^r \ln x_+$ and $(x_-^{-r-1} \ln x_-)_n$. The neutrix limit of the sequence $x_+^r \ln x_+ \cdot (x_-^{-r-1})_n$ obviously converges, as n tends to infinity, to the neutrix product $x_+^r \ln x_+ \circ x_-^{-r-1}$ given as in (2.3). Thus it is sufficient to evaluate the neutrix product of the distributions $x_+^r \ln x_+$ and $[\ln^2 x_-]^{(r+1)}$ so as to complete the proof of the theorem. □

Now we have on the interval $[0, 1/n]$ that

$$\left\langle x_+^r \ln x_+ [\ln^2 x_-]_n^{(r+1)}, x^k \right\rangle = \int_0^{1/n} x^{r+k} \ln x \int_x^{1/n} \ln^2(t-x)\delta_n^{(r+1)}(t)dt\,dx. \tag{2.15}$$

Making the substitutions $nx = v$ and $nt = u$,

$$\begin{aligned}& \int_0^{1/n} x^{r+k} \ln x \int_x^{1/n} \ln^2(t-x)\delta_n^{(r+1)}(t)dt\,dx \\ & \quad = n^{-k} \int_0^1 v^{r+k} [\ln v - \ln n] \int_v^1 [\ln(u-v) - \ln n]^2 \rho^{(r+1)}(u)du\,dv.\end{aligned} \tag{2.16}$$

If $k > 0$, then

$$\operatorname{N-lim}_{n-\infty} \left\langle x_+^r + \ln x_+ [\ln^2 x_-]_n^{(r+1)}, x^k \right\rangle = 0. \tag{2.17}$$

For $k = 0$, we have

$$\begin{aligned}
&\mathrm{N}\text{-}\lim_{n-\infty} \int_0^{1/n} x^r \ln x \int_x^{1/n} \ln^2(t-x)\delta_n^{(r+1)}(t)\,dt\,dx \\
&\quad = \int_0^1 v^r \ln v \int_v^1 \ln^2(u-v)\rho^{(r+1)}(u)\,du\,dv \\
&\quad = \int_0^1 \rho^{(r+1)}(u) \int_0^u v^r \ln v \ln^2(u-v)\,dv\,du \\
&\quad = \int_0^1 u^{r+1} \ln^3 u \rho^{(r+1)}(u) \int_0^1 y^r\,dy\,du \\
&\qquad + 2\int_0^1 u^{r+1} \ln^2 u \rho^{(r+1)}(u) \int_0^1 y^r \ln(1-y)\,dy\,du \\
&\qquad + \int_0^1 u^{r+1} \rho^{(r+1)}(u) \int_0^1 y^r \ln y \ln^2(1-y)\,dy\,du \\
&\qquad + \int_0^1 u^{r+1} \ln^2 u \rho^{(r+1)}(u) \int_0^1 y^r \ln y\,dy\,du \\
&\qquad + 2\int_0^1 u^{r+1} \ln u \rho^{(r+1)}(u) \int_0^1 y^r \ln y \ln(1-y)\,dy\,du \\
&\qquad + \int_0^1 u^{r+1} \ln u \rho^{(r+1)}(u) \int_0^1 y^r \ln^2(1-y)\,dy\,du
\end{aligned} \tag{2.18}$$

on making the substitution $v = uy$.

It immediately follows from Lemma 2.2 and (2.15) that

$$\mathrm{N}\text{-}\lim_{n\to\infty} \left\langle x_+^r \ln x_+ \left[-\frac{1}{2r!} \left(\ln^2 x_-\right)_n^{(r+1)} \right], x^k \right\rangle = \Delta_r(\rho) - L_r \psi(r). \tag{2.19}$$

Further when $k = 1$, we have

$$\begin{aligned}
&\left\langle x_+^r \ln x_+ \left[\ln^2 x_-\right]_n^{(r+1)}, x \right\rangle \\
&\quad = n^{-1} \int_0^1 v^{r+1} [\ln v - \ln n] \int_v^1 [\ln(u-v) - \ln n]^2 \rho^{(r+1)}(u)\,du\,dv = O(n^{-1}\ln n).
\end{aligned} \tag{2.20}$$

Let $\phi(x)$ be an arbitrary function in $\mathcal{D}$. Then by the mean value theorem $\phi(x) = \phi(0) + x\phi'(\xi x)$ where $0 < \xi < 1$. It follows that

$$\begin{aligned}
&\left\langle x_+^r \ln x_+ \left(x_-^{-r-1} \ln x_-\right)_n, \phi(x) \right\rangle \\
&\quad = \psi(r) \left\langle x_+^r \ln x_+ \left(x_-^{-r-1}\right)_n, \phi(x) \right\rangle - \frac{1}{2r!} \left\langle x_+^r \ln x_+ \left[\ln^2 x_-\right]_n^{(r+1)}, \phi(x) \right\rangle \\
&\quad = \psi(r) \left\langle x_+^r \ln x_+ \left(x_-^{-r-1}\right)_n, \phi(x) \right\rangle - \frac{1}{2r!} \left\langle x_+^r \ln x_+ \left[\ln^2 x_-\right]_n^{(r+1)}, \phi(0) \right\rangle \\
&\qquad - \frac{1}{2r!} \left\langle x_+^r \ln x_+ \left[\ln^2 x_-\right]_n^{(r+1)}, x\phi'(\xi x) \right\rangle
\end{aligned} \tag{2.21}$$

and so

$$\text{N-}\lim_{n\to\infty}\langle x_+^r \ln x_+ (x_-^{-r-1}\ln x_-)_n, \phi(x)\rangle = \Delta_r(\rho)\phi(0) = \Delta_r\langle \delta(x), \phi(x)\rangle \tag{2.22}$$

on using (2.15), (2.17), and (2.19). Equation (2.11) follows.

Next, we consider the neutrix product of $x_-^{-r-1}\ln x_-$ and $x_+^r \ln x_+$. Similarly, it follows from (2.7) that the neutrix limit of the sequence $x_-^{-r-1}(x_+^r \ln x_+)_n$ converges to the neutrix product $x_-^{-r-1}\circ x_+^r \ln x_+$ as $n\to\infty$. As in the proof of (2.11) we evaluate the neutrix product of $[\ln^2 x_-]^{(r+1)}$ and $x_+^r \ln x_+$.

Now

$$\begin{aligned}\left\langle [\ln^2 x_-]^{(r+1)}(x_+^r \ln x_+)_n, x^k\right\rangle &= (-1)^{r+1}\left\langle \ln^2 x_-, [(x_+^r \ln x_+)_n x^k]^{(r+1)}\right\rangle \\ &= (-1)^{r+1}\sum_{j=0}^{k}\binom{r+1}{j}\frac{k!}{(k-j)!}\left\langle \ln^2 x_-, (x_+^r \ln x_+)_n^{(r-j+1)} x^{k-j}\right\rangle\end{aligned} \tag{2.23}$$

for $k = 0, 1, 2, \ldots$.

Then we have on the interval $[-1/n, 0]$, the intersection of the supports of $\ln^2 x_-$ and $(x_+^r \ln x_+)_n$, that

$$\begin{aligned}&\left\langle \ln^2 x_-, (x_+^r \ln x_+)_n^{(r-j+1)} x^{k-j}\right\rangle \\ &\quad= \int_{-1/n}^{0} x^{k-j}\ln^2(-x)\int_{-1/n}^{x}(x-t)^r \ln(x-t)\delta_n^{(r-j+1)}(t)\,dt\,dx \\ &\quad= (-1)^{r-k+1}n^{-k}\int_0^1 v^{k-j}[\ln v - \ln n]^2\int_v^1 (u-v)^r[\ln(u-v)-\ln n]\rho^{(r-j+1)}(u)\,du\,dv\end{aligned} \tag{2.24}$$

on making the substitutions $-nt = u$ and $-nx = v$.

Thus

$$\text{N-}\lim_{n\to\infty}\left\langle \ln^2 x_-, (x_+^r \ln x_+)_n^{(r-j+1)} x^{k-j}\right\rangle = 0 \tag{2.25}$$

for $k > 0$.

If $k = 0$, then

$$\begin{aligned}&\left\langle [\ln^2 x_-]^{(r+1)}, (x_+^r \ln x_+)_n\right\rangle \\ &\quad= \int_0^1 [\ln v - \ln n]^2\int_v^1 (u-v)^r[\ln(u-v)-\ln n]\rho^{(r+1)}(u)\,du\,dv.\end{aligned} \tag{2.26}$$

It immediately follows that

$$\text{N-}\lim_{n\to\infty}\left\langle [\ln^2 x_-]^{(r+1)}, (x_+^r \ln x_+)_n\right\rangle = \int_0^1 \ln^2 v\int_v^1 (u-v)^r \ln(u-v)\rho^{(r+1)}(u)\,du\,dv. \tag{2.27}$$

We have, on making the substitution $v = uy$,

$$\begin{aligned}
&\int_0^1 \ln^2 v \int_v^1 (u-v)^2 \ln(u-v)\rho^{(r+1)}(u)\,du\,dv \\
&\quad = \int_0^1 \rho^{(r+1)}(u) \int_0^u (u-v)^r \ln^2 v \ln(u-v)\,dv\,du \\
&\quad = \int_0^1 u^{r+1}\rho^{(r+1)}(u) \int_0^1 (1-y)^r [\ln u + \ln(1-y)][\ln u + \ln y]^2\,dy\,du, \\
&\int_0^1 \ln^2 v \int_v^1 (u-v)^2 \ln(u-v)\rho^{(r+1)}(u)\,du\,dv \\
&\quad = \int_0^1 u^{r+1}\rho^{(r+1)}(u) \int_0^1 (1-y)^r [\ln u + \ln(1-y)][\ln u + \ln y]^2\,dy\,du \\
&\quad = \int_0^1 u^{r+1}\ln^3 u\rho^{(r+1)}(u) \int_0^1 w^r\,dw\,du \\
&\qquad + 2\int_0^1 u^{r+1}\ln^2 u\rho^{(r+1)}(u) \int_0^1 w^r \ln(1-w)\,dw\,du \\
&\qquad + \int_0^1 u^{r+1}\rho^{(r+1)}(u) \int_0^1 w^r \ln w \ln^2(1-w)\,dw\,du \\
&\qquad + \int_0^1 u^{r+1}\ln^2 u\rho^{(r+1)}(u) \int_0^1 w^r \ln w\,dw\,du \\
&\qquad + 2\int_0^1 u^{r+1}\ln u\rho^{(r+1)}(u) \int_0^1 w^r \ln w \ln(1-w)\,dw\,du \\
&\qquad + \int_0^1 u^{r+1}\ln u\rho^{(r+1)}(u) \int_0^1 w^r \ln^2(1-w)\,dw\,du,
\end{aligned} \tag{2.28}$$

on making another substitution $w = 1 - y$.

And so we obtain the same integrals as in Lemma 2.2. Thus

$$\mathrm{N}\text{-}\lim_{n\to\infty} \left\langle -\frac{1}{2r!}[\ln^2 x_-]^{(r+1)}, (x_+^r \ln x_+)_n x^k \right\rangle = \Delta_r(\rho) - L_r\psi(r). \tag{2.29}$$

Further

$$\left\langle [\ln^2 x_-]^{(r+1)}, (x_+^r \ln x_+)_n x \right\rangle = O(n^{-1}\ln n). \tag{2.30}$$

Again let $\phi(x)$ be arbitrary function in $\mathcal{D}$ with $\phi(x) = \phi(0) + x\phi'(\xi x)$, then

$$\begin{aligned}
&\langle x_-^{-r-1}\ln x_- (x_+^r \ln x_+)_n, \phi(x)\rangle \\
&\quad = \psi(r)\langle x_-^{-r-1}(x_+^r \ln x_+)_n, \phi(x)\rangle - \frac{1}{2r!}\left\langle [\ln^2 x_-]^{(r+1)}(x_+^r \ln x_+)_n, \phi(0)\right\rangle \\
&\qquad - \frac{1}{2r!}\left\langle [\ln^2 x_-]^{(r+1)}(x_+^r \ln x_+)_n, x\phi(\xi x)\right\rangle
\end{aligned} \tag{2.31}$$

for $r = 1, 2, \ldots$, and so

$$\mathrm{N}\text{-}\lim_{n\to\infty} \langle x_-^{-r-1}\ln x_- (x_+^r \ln x_+)_n, \phi(x)\rangle = \Delta_r(\rho)\phi(0) = \Delta_r\langle \delta(x), \phi(x)\rangle \tag{2.32}$$

on using (2.23), (2.25), (2.27), and (2.29). Equation (2.11) follows and the proof is complete.

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References

[1] J. G. van der Corput, "Introduction to the neutrix calculus," *Journal d'Analyse Mathématique*, vol. 7, pp. 281–399, 1960.

[2] R. Hoskins and J. S. Pinto, *Distributions Ultradistributions and Other Generalized Functions*, Ellis Horwood, Chichester, UK, 1994.

[3] B. Fisher, "Neutrices and the product of distributions," *Studia Mathematica*, vol. 57, no. 3, pp. 263–274, 1976.

[4] B. Fisher, "A non-commutative neutrix product of distributions," *Mathematische Nachrichten*, vol. 108, no. 1, pp. 117–127, 1982.

[5] B. Fisher and K. Taş, "On the non-commutative neutrix product of the distributions x_+^{λ} and x_+^{μ}," *Acta Mathematica Sinica*, vol. 22, no. 6, pp. 1639–1644, 2006.

[6] B. Fisher and K. Taş, "On the non-commutative neutrix product of the distributions $x_+^{-r}\ln^p x_+$ and $x_+^{\mu}\ln^q x$," *Integral Transforms and Special Functions*, vol. 17, no. 7, pp. 513–519, 2006.

[7] E. Özçağ, B. Fisher, and H. Gürçay, "Some results on the non-commutative neutrix product of distributions," *Integral Transforms and Special Functions*, vol. 11, no. 1, pp. 49–60, 2001.

[8] Y. J. Ng and H. van Dam, "Neutrix calculus and finite quantum field theory," *Journal of Physics A*, vol. 38, no. 18, pp. L317–L323, 2005.

[9] Y. Jack Ng and H. van Dam, "An application of neutrix calculus to quantum theory," *International Journal of Modern Physics A*, vol. 21, no. 2, pp. 297–312, 2006.

[10] B. Fisher, "The product of distributions," *The Quarterly Journal of Mathematics*, vol. 22, no. 2, pp. 291–298, 1971.

[11] B. Fisher, "On defining the product of distributions," *Mathematische Nachrichten*, vol. 99, no. 1, pp. 239–249, 1980.

[12] B. Fisher and A. Kiliçman, "Neutrices and the non-commutative neutrix product of distributions," *Boletín Sociedad Matemática Mexicana*, vol. 3, no. 2, pp. 299–307, 1997.

[13] I. M. Gel'fand and G. E. Shilov, *Generalized Functions. Vol. I: Properties and Operations*, Academic Press, New York, NY, USA, 1964.

[14] E. Özçağ and B. Fisher, "On defining the distribution $x_+^{-r}\ln x_+$," *Rostocker Mathematisches Kolloquium*, no. 42, pp. 25–30, 1990.

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